A control design method for a class of SISO switched linear systems

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Abstract

We consider the control design for a class of single-input single-output (SISO) switched linear systems of arbitrary order. Our approach is motivated by applications in which the major design objective is to achieve similar behaviour of the closed-loop system in each system mode ensuring stability for arbitrary switching. We study the stability of the resulting closed-loop system and exploit the algebraic properties of a class of matrices to derive simple stability conditions for the switched system. Further we the implementation of the switched controller with integrators is discussed and conditions are derived to avoid transient dynamics induced by the switching. We show that closed-loop stability and transient-free switching can be achieved simultaneously.

I. Introduction

In this paper we consider the control design for classes of switched linear systems. Such system are characterised by dynamics that can be described by a combination of continuous differential equations and some discontinuous switching mechanism that orchestrates between them. Dynamical systems of this class can be found in various fields of engineering applications, such as aviation technology [1], [2], power electronics [3], automotive engineering [4] or power generation [5]. While the stability analysis of this system class has been subject of a great number of publications in the recent past (see [6], [7], [8] and references therein) many problems in this area remain unsolved. In particular, given the frequency with which one finds switched linear control systems in practice, one of the most pressing needs is for the development of analytic tools for the design of such systems.

The system class considered in this article is characterised by a process that exhibits arbitrary switching between its constituent linear system dynamics. However we shall assume that such mode switch is immediately detected such that appropriate control action can be applied. We choose a controller structure as depicted in Figure 1 where the controller for each mode is realised as a single LTI system. Such a controller structure is referred to as local-state controller [9]. At any switching instant of the process, the appropriate controller is deployed in the closed loop by switching the process input to the respective controller output. To aid analysis we make the assumption that there is no time-delay between the switching of the process and switching of the controller output. Further we dispense with any controller state reset as considered in [10], [11].

Our contribution in this paper is to develop tools for the design of classes of such systems where the design objectives are similar for each process mode. The stability analysis of the resulting switched closed-loop system stems from the ideas followed in [12], [13] and [14] where the same controller structure is considered. The stability analysis in these contributions is restricted to processes that can be described by a set of scalar linear ordinary differential equations and allows for controllers of only first order. In this contribution we extend the stability results to higher-order controllers and propose an extended controller structure for controllers with pure integrator action. This structure yields switched closed-loop systems that can be analysed with the tools derived in this paper. Further we give conditions that guarantee transient-free switching between the subsystems under certain conditions.

In the next section we give a precise problem statement followed by the introduction of the control design method proposed and some preliminary results in Section III. In Section IV we analyse the stability of the resulting closed-loop system. Our main results show that the complexity of the stability analysis can be significantly reduced. Since we choose a local-state controller structure and do not allow for

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any controller state resets, the controllers for each mode are required to be stable LTI systems. Thus such controller structure does not allow for controllers with integrators. In Section V we consider the implementation of such controllers and show that a simple variation of the controller structure allows the application of our stability results even when the controllers have integrators. In Section VI we address the problem of transient motion that occurs when switching between the system modes. We obtain a simple condition that guarantees transient-free switching, when switching in steady state, without the need of any controller state reset. We finish with an illustrating example and a discussion of our results. Note that parts of this paper have been presented at [15].

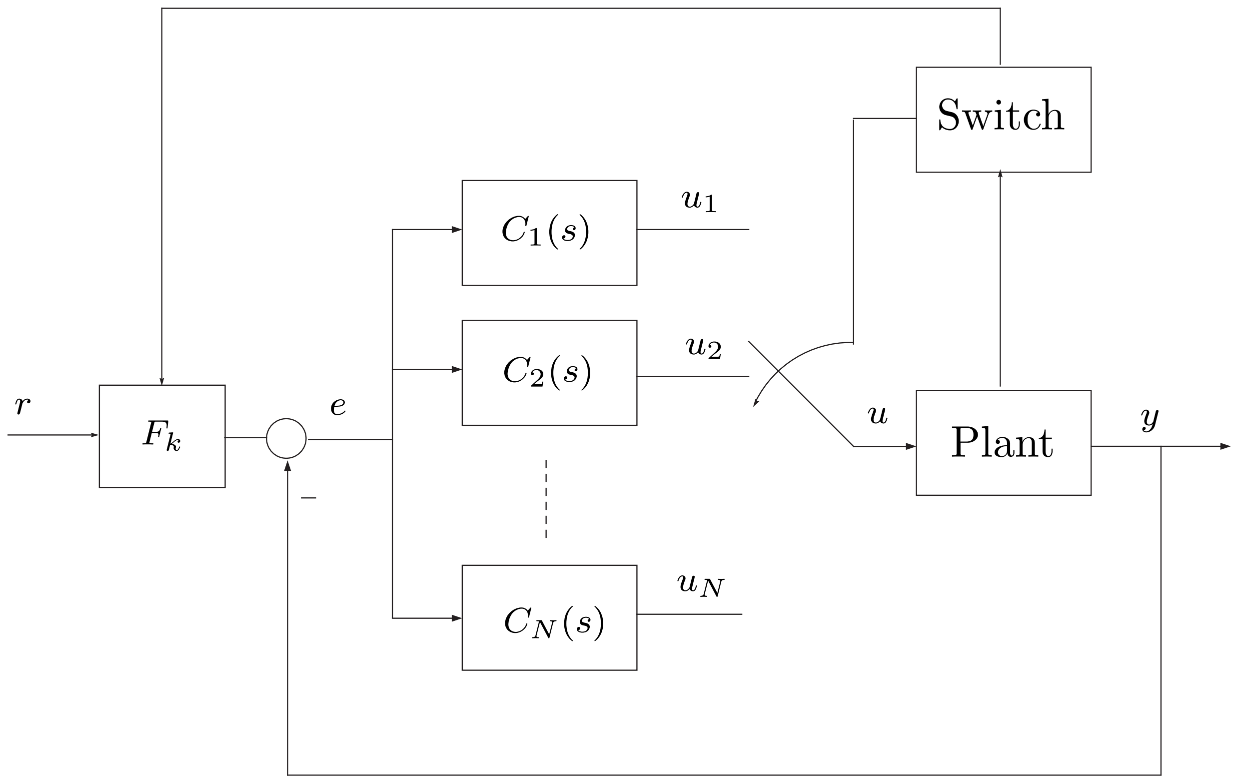


Fig. 1. Structure of the considered switched linear control system

II. Problem statement

The process dynamics are given by the linear time-varying scalar differential equation of the form

$$y^{(n_p)} = \sum_{l=0}^{n_p-1} q_l(t)y^{(l)} + p_0(t)u \tag{1}$$

where $y^{(n_p)}$ denotes the n_p'th derivative of $y(t)$ and $p_0(t)$, $q_l(t)$ are piecewise constant functions taking on values in the finite sets $p_0(t) \in \{p_{01}, \dots, p_{0N}\}$, and $q_l(t) \in \{q_{l1}, \dots, q_{lN}\}$ $\forall\, l = 0, \dots, n_p - 1$. Without loss of generality, we assume that the discontinuities occur simultaneously such that $p_0(t) = p_{0k}$ whenever $q_l(t) = q_{lk}$ for all $l = 0, \dots, n_p - 1$ where $k \in \mathcal{I} = \{1, \dots, N\}$ denotes the process mode.

Thus at any time instant the process dynamics in Figure 1 correspond to exactly one of the N linear systems

$$\dot{x}(t) = A_k x(t) + b_k u(t) \tag{2a}$$

$$y(t) = c^T x(t) \tag{2b}$$

where

$$A_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -q_{0k} & -q_{1k} & \cdots & \cdots & -q_{n-1k} \end{pmatrix}$$

$$c^T = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}, \quad b_k = \begin{pmatrix} 0 & \cdots & 0 & p_{0k} \end{pmatrix}^T.$$

With each mode $k \in \mathcal{I}$ we associate the proper transfer function

$$P_k(s) = c^T(sI - A_k)^{-1}b_k.$$

We shall assume that the mode-switches of the process are immediately detectable such that the switching instances can be assumed to be known for the controller. Given these assumptions, our objective is to design a controller such that the closed-loop system

- is asymptotically stable for arbitrary switching signals,
- has the poles $\Lambda_t \subset \mathbb{C}_-$, specified independently of the process mode $k \in \mathcal{I}$,
- and has little or no transients induced by the switching action.

III. Preliminary discussion: Basic ideas

In order to achieve the design objectives we associate an individual controller for each process mode $k \in \mathcal{I}$. We choose a controller architecture where each controller is realised as an LTI system as depicted in Figure 1.

The dynamics of the individual controllers are given by

$$\dot{x}_k(t) = K_k x_k(t) + l_k e(t) \quad (3a)$$
$$u_k(t) = m_k^T x_k(t) + j_k e(t) \quad (3b)$$

where $x_k(t) \in \mathbb{R}^{n_c}$ is the state-vector of the controller associated with mode $k \in \mathcal{I}$; the input $e(t) \in \mathbb{R}$ is shared by all controllers and each controller has an individual control signal $u_k(t) \in \mathbb{R}$. For the realisation of the controllers we choose the control canonical form as above with $K_k \in \mathbb{R}^{n_c \times n_c}$, $l_k, m_k^T \in \mathbb{R}^{n_c}$ and $j_k \in \mathbb{R}$. The respective transfer functions are given by

$$C_k(s) = m_k^T(sI - K_k)^{-1}l_k.$$

As design-law for the controllers we choose a set of stable target-poles Λ_t and design the controllers using standard pole-placement techniques such that the closed-loop system in each mode has the specified target-poles Λ_t. Our results throughout this paper base on the following assumption.

Assumption 3.1 (Pole-placement): For each process mode $k \in \mathcal{I}$ the controller $C_k(s)$ is designed such that the poles of the closed-loop transfer function

$$\frac{C_k(s)P_k(s)}{1 + C_k(s)P_k(s)}$$

are simple and lie in the open left half-plane and are constant for all $k \in \mathcal{I}$. We denote the set of those target poles by $\Lambda_t = \{\lambda_1, \ldots, \lambda_{n_p+n_c}\}$. The resulting controllers $C_k(s)$ have poles in the open left half-plain.

The state x of the switched closed-loop system consists of the process states x_p and the controller-states x_k

$$x = \begin{pmatrix} x_p^T & x_1^T & \ldots & x_N^T \end{pmatrix}^T$$

where $x \in \mathbb{R}^n$, $n = n_p + Nn_c$. For the switched closed-loop system we obtain

$$\dot{x}(t) = H(t)x(t)\,, \tag{4}$$

where $H(\cdot)$ is an arbitrary piecewise constant function $H : \mathbb{R} \to \mathcal{H} = \{H_1, \ldots, H_N\} \subset \mathbb{R}^{n\times n}$. The constituent system matrices in each mode $k \in \mathcal{I}$ are given by

$$H_k = \begin{pmatrix} A_k - b_k j_k c_k & b_k m_1^T \delta_{k1} & \cdots & b_k m_N^T \delta_{kN} \\ -l_1 c^T & K_1 & & 0 \\ \vdots & & \ddots & \\ -l_N c^T & 0 & & K_N \end{pmatrix} \tag{5}$$

where δ_{kj} denotes the Kronecker symbol.

Before we present the main results we note some preliminary observations. Given the process (2) and controllers (3) in control canonical form, all closed-loop system matrices H_k are identical except for the n_p-th row, Furthermore as all but one of the sets of Kronecker symbols are equal to 0, we have that $\sigma(H_k) \supset \sigma(K_l)$ for $l \neq k$. By design (Assumption 3.1) the remaining eigenvalues are given by Λ_t for all $k \in \mathcal{I}$. Thus the spectrum of H_k is given by

$$\sigma(H_k) = \Lambda_t \cup \bigcup_{l\neq k} \sigma(K_l)\,,$$

accounting for multiplicities. Therefore the matrices H_k have pairwise $n_p + (N-1)n_c$ common eigenvalues.

A useful consequence of this approach is that the subspace corresponding to the target poles do not depend on k given some mild conditions. This fact shall be useful in the following discussion and we state it formally as the following lemma.

Lemma 3.1: Let $\lambda \in \Lambda_t$ be a simple eigenvalue of each H_k, then there exists a vector $v \neq 0$ such that for all $k \in \mathcal{I}$

$$H_k v = \lambda v\,. \tag{6}$$

Proof: As $\lambda \in \sigma(H_k)$ we have that the rows $\tilde{h}_{jk}$ of $\lambda I - H_k$ are linearly dependent for each k. On the other hand, all the rows, but the n_p'th are independent of k. By inspection the set of $n-1$ rows obtained by omitting the n_p'th row is linearly independent, since by assumption λ is not an eigenvalue of one of the controllers K_k, $k \in \mathcal{I}$. Thus for each k there are constants γ_{jk} such that

$$\tilde{h}_{n_p k} = \sum_{j\neq n_p} \gamma_{jk} \tilde{h}_{jk}\,. \tag{7}$$

Now by definition an eigenvector v of H_1 corresponding to the eigenvalue λ satisfies $\tilde{h}_{j1} v = 0, j = 1, \ldots, n$. This implies that $\tilde{h}_{jk} v = 0, j = 1, \ldots, n, j \neq n_p$ for each $k \in \mathcal{I}$. This, however, implies by (7) that also $\tilde{h}_{n_p k} v = 0$, so that we have $(\lambda I - H_k)v = 0$. ■

Hence, if the eigenvalues $\lambda \in \Lambda_t$ are simple, all closed-loop system matrices H_k have $n_p + n_c$ eigenvectors in common. This fact can be exploited to derive simple conditions for stability as we shall discuss in the following section.

IV. Stability

Assume that we are given N matrices of the form (5) and that the poles of the individual systems have been placed so that Lemma 3.1 is applicable.

Let the columns of $V_t \in \mathbb{C}^{n\times(n_p+n_c)}$ form a basis of the common subspace of all matrices $H_k \in \mathcal{H}$ and consider the matrix

$$T \;:=\; \begin{pmatrix} V_t & e_{n_p+n_c+1} & \cdots & e_n \end{pmatrix}\,. \tag{8}$$

Note that T is invertible as the vectors $e_{(n_p+n_c+1)}, \dots, e_n$ form a basis of an invariant subspace of H_1, which does not intersect $\operatorname{span} V_t$ as $\Lambda_t \cap \sigma\big(K_k\big) = \emptyset \ \forall\, k \in \mathcal{I}$.

Applying the similarity transformation T we obtain

$$T^{-1} H_1 T \;=\; \operatorname{diag}\left(D_t, K_2, \dots, K_N\right),$$
$$T^{-1} H_2 T \;=\; \operatorname{diag}\left(D_t, K_2, \dots, K_N\right) + T^{-1} e_n \tilde{h}_2^T T\,,$$

up to

$$T^{-1} H_N T \;=\; \operatorname{diag}\left(D_t, K_2, \dots, K_N\right) + T^{-1} e_n \tilde{h}_N^T T\,,$$

where $\sigma(D_t) = \Lambda_t$ and $\tilde{h}_k := h_{kn_p} - h_{1n_p}$ denotes the differences between the n_p'th rows of H_k and H_1. As implied by our construction the differences between the matrices are all multiples of the same columns. Furthermore inspection of the n_p'th rows of the matrices H_k shows that $\tilde{h}_k$ can only have nonzero entries in its first $n_p + n_c$ positions and in the positions $n_p + (k-1)n_c + 1, \dots, n_p + kn_c$. Hence, in the lower block corresponding to the controllers only the controller K_k is perturbed. So that for $k = 2, \dots, N$ the matrices after similarity transformation are of the form

$$T^{-1} H_k T = \begin{pmatrix} D_t & 0 & \dots & U_{1k} & 0 \\ 0 & K_2 & 0 & U_{2k} & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ & & & K_k + U_{kk} & \\ & & & & 0 \\ & & & U_{Nk} & K_N \end{pmatrix} \tag{9}$$

where $U_k = \begin{pmatrix} U_{1k}^T & U_{2k}^T & \dots & U_{Nk}^T \end{pmatrix}^T \in \mathbb{R}^{n \times n_c}$ denotes the perturbation term of the k'th system. Since $rank\{H_j - H_k\} = 1$ for all $j \neq k$ and $j, k \in \mathcal{I}$ the perturbation term U_k has rank 1. We denote

$$R_1 := \operatorname{diag}\left(K_2, \dots, K_N\right),$$

and for $k = 2, \dots, N$ the lower right $(N-1)n_c \times (N-1)n_c$-block of $T^{-1} H_k T$ by

$$R_k := \begin{pmatrix} K_2 & 0 & U_{2k} & & 0 \\ & \ddots & \vdots & & \vdots \\ & & K_k + U_{kk} & & \\ & & \dots & \ddots & 0 \\ 0 & & U_{Nk} & & K_N \end{pmatrix}.$$

It follows that the closed-loop system is exponentially stable if and only if the switched system formed by the matrices R_k, $k \in \mathcal{I}$ is exponentially stable.

Theorem 4.1: Consider the switched process (2) and let Assumption 3.1 be satisfied. Then the following statements are equivalent:

1) The switched linear system (4) with $H(t) \in \mathcal{H}$ is exponentially stable.
2) The switched linear system $\dot{x} = R(t)x$ with $R : \mathbb{R} \to \{R_1, \dots, R_N\}$ is exponentially stable.

Proof: The transformed system matrices $T^{-1} H_k T$ in (9) are in block triangular form with homogeneous dimensions for all $k \in \mathcal{I}$. It is well know that switched systems of this structure are exponentially stable if and only if the switched systems formed from the diagonal blocks are stable. Since D_t is a Hurwitz matrix by Assumption 3.1, the switched system with system matrices (9), $k \in \mathcal{I}$ is exponentially stable if and only if the switched system $\{R_1, \dots, R_N\}$ is exponentially stable. By congruence the switched system $\{H_1, \dots, H_N\}$ is exponentially stable. ■

In view of quadratic stability we can render our statement more precisely. Let $V(x) = x^T P x, P = P^T > 0$, then V is called a common quadratic Lyapunov function (CQLF) for the switched system (4) if P satisfied the Lyapunov inequality simultaneously for all constituent system matrices in $\mathcal{H}$.

Theorem 4.2: Consider the switched process (2) and let Assumption 3.1 be satisfied. Then the following statements are equivalent:

1) The switched linear system (4) with $H(t) \in \mathcal{H}$ has a common quadratic Lyapunov function.
2) The switched linear system $\dot{x} = R(t)x$ with $R : \mathbb{R} \to \{R_1, \dots, R_N\}$ has a common quadratic Lyapunov function.

Proof: Let $V(x) = x^T P_1 x$ be a quadratic Lyapunov function for D_t and $V(x) = x^T P_2 x$ be a common Lyapunov function for $\{R_1, \dots, R_N\}$. Then for suitable $\gamma > 0$ the symmetric matrix $P = \operatorname{diag}(P_1, \gamma P_2)$ defines a common quadratic Lyapunov function for $\{T^{-1}H_1T, \dots, T^{-1}H_NT\}$. Thus a CQLF for $\{H_1, \dots, H_N\}$ exists as well by congruence. Conversely, if a CQLF exists for $\{H_1, \dots, H_N\}$ and so also for $\{T^{-1}H_1T, \dots, T^{-1}H_NT\}$, then by applying a scaling argument it is easy to see that $\{R_1, \dots, R_N\}$ have a CQLF. This completes the proof. ■

The above theorems reduces the stability analysis of the switched system of dimension $n_p + Nn_c$ to the stability of a system of dimension $(N-1)n_c$. In the following we consider two special cases and show that Theorem 4.1 can be used to obtain very elegant stability conditions.

A. N first order controllers

We begin our analysis with the case where the controllers C_k are of first order. Thus for Assumption 3.1 to hold, the process dynamics have to be of order strictly less than three. We now employ Theorem 3.1 in [12]. Essentially, the theorem establishes asymptotic stability of the class of switched systems (4) with the following properties:

- every matrix in $\mathcal{H}$ is Hurwitz and diagonalisable;
- the eigenvectors of any matrix in $\mathcal{H}$ are real;
- every pair of matrices in $\mathcal{H}$ share at least $n-1$ linearly independent common eigenvectors.

Let the target poles Λ_t be distinct and real. With the assumption that the pole-placement is feasible for all modes $k \in \mathcal{I}$, the resulting closed-loop system matrices H_k have pairwise $n-1$ real distinct eigenvalues. By Lemma 3.1 the matrices H_k, $k \in \mathcal{I}$, have $n_p + 1$ common eigenvectors. Moreover, since each pair of closed-loop system matrices H_k share $N-2$ of the remaining inactive controllers they have pairwise $n-1$ common eigenvectors.

Thus the requirements for Theorem 3.1 in [12] are met and the closed-loop system is exponentially stable for arbitrary switching sequences. In other words, the switched system (4) is stable for arbitrary switching if we choose arbitrary real negative target-poles Λ_t such that the design-law in Assumption 3.1 is satisfied by first-order controllers [13].

Theorem 4.1 can be used to extend this result for systems with non-real target poles Λ_t. Choosing a modal-basis for V_t in (8) we obtain a transformation matrix T with real entries. It follows that the system matrices R_k of the reduced system are in $\mathbb{R}^{N-1 \times N-1}$. Further, $\sigma(R_k) = \cup_{l \neq k} \sigma(K_l)$. Since the controllers are of first order, it follows that the matrices R_k also satisfy the requirement of Theorem 3.1 in [12].

Corollary 4.1: The switched system (4) with system matrices (5) where Assumption 3.1 is satisfied using N stable first-order controllers is asymptotically stable.

B. Two subsystems of arbitrary order

Consider now the special case where $N = 2$ and the controllers are of arbitrary order n_c. Due to the pole-placement requirement (Assumption 3.1) we obtain for the respective spectra

$$\begin{aligned} \sigma\big(H_1\big) &= \Lambda_t \cup \sigma\big(K_2\big) \\ \sigma\big(H_2\big) &= \Lambda_t \cup \sigma\big(K_1\big). \end{aligned}$$

Applying the similarity transformation T of (8) to our two system matrices we obtain

$$T^{-1}H_1T = \begin{pmatrix} D_t & 0 \\ 0 & K_2 \end{pmatrix} \tag{10a}$$

$$T^{-1}H_2T = \begin{pmatrix} D_t & 0 \\ 0 & K_2 \end{pmatrix} + \begin{pmatrix} 0 & U_1 \\ 0 & U_2 \end{pmatrix} \tag{10b}$$

where $\begin{pmatrix} U_1^T & U_2^T \end{pmatrix}^T \in \mathbb{R}^{2n_c \times n_c}$ and $\sigma\big(D_t\big) = \Lambda_t$. Note that $\operatorname{rank}\{ U_2\} = 1$ as we have $\operatorname{rank}\{H_1 - H_2\} = 1$. Further it follows from the spectrum of H_2 that $\sigma\big(K_2 + U_2\big) = \sigma\big(K_1\big)$.

The following theorem reduces the stability problem of the switched system defined by $\{H_1, H_2\}$ to a stability problem only involving the controllers.

Theorem 4.3: Consider the matrices H_1, H_2 in (5) and let Assumption 3.1 be satisfied such that $\sigma\big(H_k\big) = \Lambda_t \cup \sigma(K_l)$ for $k, l = 1, 2,\ \ k \neq l$. Assume furthermore that $\Lambda_t \cap \sigma\big(K_k\big) = \emptyset,\ k = 1, 2$. Then the following statements are equivalent:

1) The switched system given by the set of matrices $\{H_1, H_2\}$ is asymptotically stable for arbitrary switching signals;
2) The switched system given by the set of matrices $\{K_2, K_2 + U_2\}$ is asymptotically stable for arbitrary switching signals;
3) The switched system given by the set of matrices $\{K_1, K_2\}$ is asymptotically stable for arbitrary switching signals.

Proof: The equivalence of (i) and (ii) can be seen as follows. Firstly, the matrices in (5) and (10) are obtained from one another by simultaneous similarity. Thus the set $\{H_1, H_2\}$ defines an asymptotically stable switched system if and only if $\{T_1^{-1}H_1T_1,\, T_1^{-1}H_2T_1\}$ does. On the other hand $\sigma(D_t) = \Lambda_t \subset \mathbb{C}_-$, so that the exponential stability of $\{T_1^{-1}H_1T_1, T_1^{-1}H_2T_1\}$ is equivalent to that of the lower diagonal block $\{K_2, K_2 + U_2\}$.

The equivalence (ii) $\Leftrightarrow$ (iii) follows if we find a similarity transformation that transforms K_2 and $K_2 + U_2$ into K_2 and K_1 respectively. Note first, that since $\operatorname{rank}\{H_2 - H_1\} = 1$, the perturbation $(U_1^T, U_2^T)^T$ is also of rank one. Further, the block $K_2 + U_2$ is similar to K_1 since the eigenvalues in Λ_t in H_2 are generated by the closed loop system of A_2 and K_2.

Consider now the matrices K_2^T and $K_2^T + U_2^T$ and define

$$x_m := \big(K_2^T\big)^m x = \big(K_2^T + U_2^T\big)^m x\,, \qquad m = 0, \ldots, n_c - 1 \tag{11}$$

for some $x \in \mathbb{R}^{n_c}$. If we can find a vector x such that the sequence $x_m, m = 0, \ldots, n_c - 1$ is well-defined and linearly independent, then the similarity transformation

$$S = \begin{pmatrix} x_0 & \cdots & x_{n_c - 1} \end{pmatrix}$$

yields

$$S^{-1}K_2^TS = K_2^T\,, \text{ and } S^{-1}(K_2^T + U_2^T)S = K_1^T\,.$$

This assertion follows since the assumption (11) guarantees that both matrices are brought simultaneously in transposed companion form (sometimes also known as second companion form) and because the companion form of $K_2 + U_2$ is K_1 by similarity. By taking transposes of the previous equations we have found the desired transformation that concludes the proof in case that (11) holds.

Consider the sequence of conditions in (11):

$$\begin{aligned} K_2^T x &= \big(K_2^T + U_2^T\big)\, x\,, \quad (m = 1) \\ \big(K_2^T\big)^2 x &= \Big(\big(K_2^T\big)^2 + K_2^TU_2^T + U_2^TK_2^T + \big(U_2^T\big)^2\Big)\, x\,, \quad (m = 2) \\ \vdots\ &= \quad \vdots \end{aligned}$$

By induction these conditions require that

$$U_2^T \left(K_2^T\right)^m x = 0\,, \quad \text{for} \quad m = 0, \ldots, n_c - 2\,.$$

Consider now the intersection of the kernels of $U_2^T \left(K_2^T\right)^m$ for $m = 0, \ldots, n_c - 2$

$$V \;:=\; \bigcap_{m=0}^{n_c-2} \ker U_2^T \left(K_2^T\right)^m \;.$$

As $\operatorname{rank}\left\{U_2^T\right\} = 1$, the kernel of $U_2^T \left(K_2^T\right)^m$ has dimension $n_c - 1$ for $m = 0, \ldots, n_c - 2$ and so by dimensionality reasons we find that $\dim V \;\geq\; 1$. Choose an $x \in V$, $x \neq 0$. If the set of vectors $\{x_m, m = 0, \ldots, n_c - 1\}$ is linearly independent, then (11) holds and we are done. If this is not the case this means that the lower-dimensional subspace

$$W := \operatorname{span}\left\{x_m \mid m = 0, \ldots, n_c - 1\right\}$$

is K_2^T-invariant and by definition is contained in the kernel of U_2^T. Hence on this lower dimensional subspace K_2^T is not perturbed by U_2^T. We may then repeat the argument on the restriction of K_2^T to an invariant subspace complementary to W. This procedure can be iterated until (11) holds on one any of these lower dimensional complementary subspaces. For reasons of dimensionality this procedure terminates and the assertion follows. ■

Comment: Theorem 4.3 reduces the complexity of the stability analysis of the switched system considerably. To guarantee asymptotic stability of the switched system (4) with $N = 2$ we only need to consider the asymptotic stability of the switched system given by

$$\dot{x} = K(t)x, \quad K(t) \in \{K_1, \ldots, K_N\} \subset \mathbb{R}^{n_c \times n_c} \tag{12}$$

for arbitrary switching signals. Thus, the stability problem of the switched system (4) of order $n_p + 2n_c$ is reduced to the stability problem of a switched system of order n_c.

Comment: It should be emphasised that the proof of Theorem 4.3 relies on the fact that the controller-matrices are in companion form. At this point it is not clear what role the specific realisation chosen for the controllers plays for the result. However, it is obvious that the equivalence $(ii) \Leftrightarrow (iii)$ can only be true when $\operatorname{rank}\{K_1 - K_2\} = \operatorname{rank}\{U_2\} = 1$.

Comment: The equivalence of the asymptotic stability of the system (4) and (12) is less obvious than intuition might suggest. As we shall see in the next section, the result does not generalise for systems with more than two subsystems. In this context it is worth noting that the switched system (12) is not explicitly part of the closed-loop system (4). For the switched system (12) the controller dynamics K_k act on the same state-space; however the controllers in the closed-loop system (4) are realised as individual LTI systems and therefore do not share the states.

Note, that the transformed system matrices (10) are in block diagonal form. Applying Theorem 4.2 we also find an extension of Theorem 4.3 on common quadratic Lyapunov functions.

Corollary 4.2: Consider the matrices H_1, H_2 in (5) and let Assumption 3.1 be satisfied such that $\sigma\big(H_k\big) = \Lambda_t \cup \sigma(K_l)$ for $k, l = 1, 2,\ k \neq l$. Assume furthermore that $\Lambda_t \cap \sigma\big(K_k\big) = \emptyset,\ k = 1, 2$. Then there exists a common quadratic Lyapunov function for the switched system given by the set of matrices $\{H_1, H_2\}$ if and only if the switched system given by the set of matrices $\{K_1, K_2\}$ has a common quadratic Lyapunov function.

C. The case of $N = 3$

The above findings suggest that the switched closed-loop system (4) is stable if and only if the switched system (12) consisting of the controllers form a stable system. Unfortunately that is not true as the following example shows.

Example 4.1: Consider the switched process (2) with $N = 3$, where

$$A_1 = \begin{pmatrix} 0 & 1 \\ -11.84 & -2.4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -34.28 & -11.6 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ -29.7 & -11 \end{pmatrix},$$

and $b_k = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$, $c_k^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$ for $k = 1, 2, 3$, and let the requested target-poles be given by $\Lambda_t = \{-1 \pm 3i, -1.8, -8\}$.

It can be verified that the pole-placement requirement is satisfied by the following set of controllers (3) with

$$K_1 = \begin{pmatrix} 0 & 1 \\ -9.6 & -9.4 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 1 \\ -7.4 & -0.2 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 1 \\ -5.5 & -0.8 \end{pmatrix},$$

and $m_1^T = \begin{pmatrix} 30.34 & -7.536 \end{pmatrix}$, $m_2^T = \begin{pmatrix} -109.7 & 34.1 \end{pmatrix}$, $m_3^T = \begin{pmatrix} -19.35 & 42.54 \end{pmatrix}$, and $l_k = \begin{pmatrix} 0 & 1 \end{pmatrix}^T, j_k = 0$ for $k = 1, 2, 3$.

It can be numerically verified that $V(x) = x^T P x$ with

$$P = \begin{pmatrix} 3.0745 & 0.0671 \\ 0.0671 & 0.4356 \end{pmatrix}$$

is a common quadratic Lyapunov function for the switched system (12) with $K(t) \in \{K_1, K_2, K_3\}$. Hence, the switched system (12) consisting of the controllers is asymptotically stable for arbitrary switching.

However we can find a switching sequence for which the closed-loop switched system (4) is unstable. Consider the periodic switching signal associated with the monodromy matrix $\Phi(t + T, t) = e^{H_3 T_3} e^{H_2 T_2} e^{H_1 T_1}$ where $T = T_1 + T_2 + T_3$ and $T_1 = 0.72, T_2 = 0.32, T_3 = 0.22$. The spectral radius $\varrho(\Phi(T, 0)) = 1.024 > 1$. Hence, there exists a periodic switching sequence for which the closed-loop system is unstable [16].

The above example shows that Theorem 4.3 and Corollary 4.2 cannot be generalised for systems with an arbitrary number of subsystems since we have found a CQLF for the controllers and cannot conclude stability, let alone the existence of a CQLF. Thus we have to resort to Theorem 4.1 for the analysis of switched systems (5) with $N \geq 3$.

V. Controllers with integrators

Many control problems require the use of controllers with integrators to meet the design specifications. A straight forward implementation of controllers with integrators using the controller architecture proposed in this paper will lead to closed-loop system matrices (5) with eigenvalues at zero. These eigenvalues are inherited by the zero-poles of the controllers that are not active in the loop. However, the stability results derived in the previous sections require that the constituent closed-loop systems $\dot{x} = H_k x$, $k \in \mathcal{I}$ are stable LTI systems and thus we cannot apply the stability results when the controllers have integrators. In this section we show that this problem can be resolved by choosing a variation of the local-state controller architecture such that the stability results derived earlier in this paper can be applied.

In the following we shall assume that the controllers have the same number of integrators for each mode $k \in \mathcal{I}$. Then we can choose a controller-architecture such that these integrators are shared by the controllers and therefore are always active in the closed loop. For this purpose we choose a controller architecture with a joint integrator in front of the controller bank as shown in Figure 2 such that the local controllers $C_k(s)$ have no pure integrator.

Choosing the state-vector of the closed-loop system as $x = \begin{pmatrix} x_p^T & v^T & x_1^T & \cdots & x_N^T \end{pmatrix}^T$ yields the system matrices

$$H_k = \begin{pmatrix} A_k & b_k j_k & b_1 m_1^T \delta_{k1} & \cdots & b_N m_N^T \delta_{kN} \\ -c^T & 0 & 0 & \cdots & 0 \\ 0 & l_1 & K_1 & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & \\ 0 & l_N & & 0 & K_N \end{pmatrix}$$

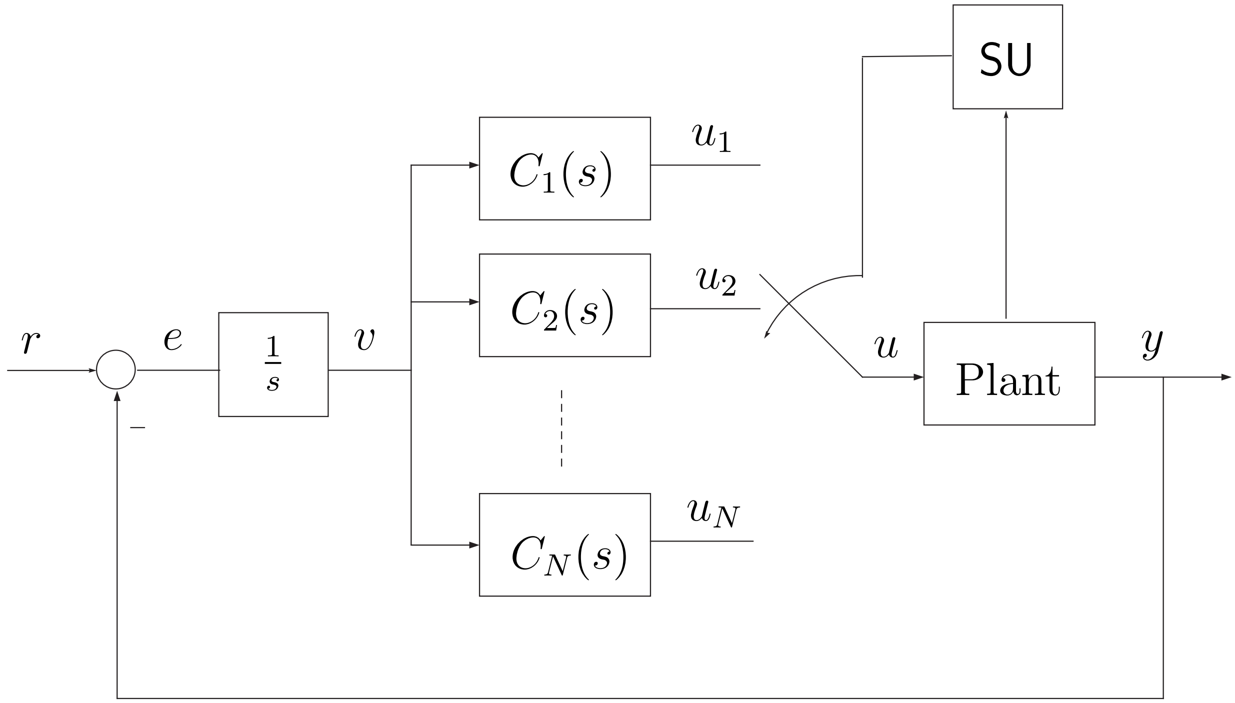


Fig. 2. Controller architecture with joint integrator.

for all $k \in \mathcal{I}$.

Positioning the joint integrator in front of the controller bank preserves the property of the system matrices $\mathrm{rank}\{H_k - H_l\} = 1$ for $k \neq l\ \forall\ k, l \in \mathcal{I}$, since the matrices H_k again only differ in the n_p'th row. Since the integrator is constantly active in the closed loop the eigenvalues of $H_k\ \forall\ k \in \mathcal{I}$ lie in the open left half-plane if Assumption 3.1 is satisfied. Hence, the results of the previous section are applicable to controllers with integrators when choosing a controller architecture with a joint integrator in front of the controller bank.

VI. Transient-free switching

The switched control scheme can induce undesirable transient dynamics. In this section we show that these transients can be avoided using the local-state controller architecture described above. In particular we shall consider transient dynamics that occur due to switching when the subsystems have reached steady-state. To analyse the transient dynamics of the switched system we introduce the input-output description of the switched closed-loop system. For each mode $k \in \mathcal{I}$ the dynamics are given by:

$$\dot{x}(t) = H_k x(t) + F_k r(t) \tag{13}$$

$$y(t) = G x(t) \tag{14}$$

where $F_k \in \mathbb{R}^{n\times 1}$ is some pre-filter gain and $G \in \mathbb{R}^{1\times n}$ is the constant output matrix for all modes.

The following example motivates the closer investigation of the transient dynamics of switched systems.

Example 6.1: Consider the process (1) with switched linear dynamics and two modes. The dynamics of the constituent modes are described by the transfer functions

$$P_1(s) = \frac{1}{s+14}, \quad P_2(s) = \frac{1}{s+3}.$$

We choose first-order controllers for the control of this process. The closed-loop poles are given by $\Lambda_t = \{-2, -20\}$. The controllers with transfer functions

$$C_1(s) = \frac{-72}{s+8}, \quad C_2(s) = \frac{s-14}{s+18}$$

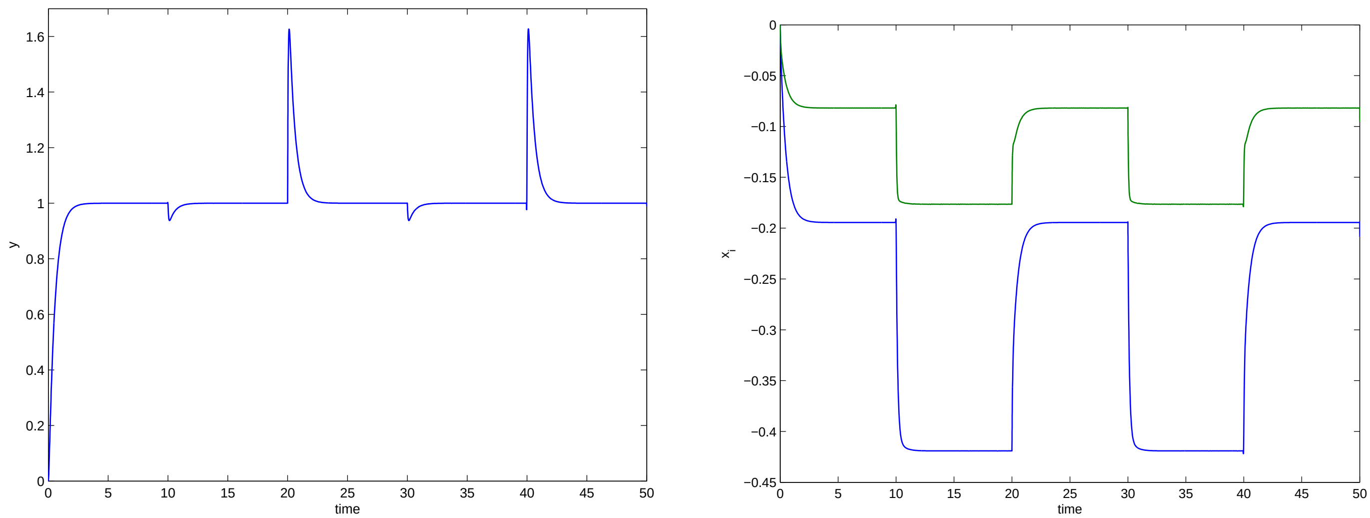
satisfy this pole assignment and Assumption 3.1. For compensation of the static steady-state error we use the pre-filter gains $F_1 = -0.5556$ and $F_2 = -2.3529$.

The resulting closed-loop system dynamics are identical for each mode and given by the transfer function

$$T_k(s) \;=\; \frac{C_k(s)P_k(s)}{1 + C_k(s)P_k(s)} \;=\; \frac{40}{s^2 + 22s + 40}\,, \quad k \in \{1, 2\}.$$

Since we apply first-order controllers the autonomous switched closed-loop system is asymptotically stable for arbitrary switching signals (Corollary 4.1) and thus the switched input-output system (13)-(14) is bounded-input bounded-output stable [16].

Figure 3a shows a step response of the closed-loop system. Here a switching signal $\sigma(t)$ is chosen that changes the mode every 10 time-units. At every switching instant we can observe a significant transient response before the output reaches its reference value again. Note that the switching



(a) (b)

Fig. 3. Step-response of the closed-loop system in Example 6.1 with the switching instances $10, 20, 30, \ldots$. Part (a) shows the evolution of the output $y(t)$, Part (b) shows the two controller-states $x_1(t)$ and $x_2(t)$.

is slow enough to allow the system to reach practically steady-state between two consecutive switching instances, i.e. the observed transient-peaks are not due to unsettled control signals or states. □

Loosely speaking, these transients occur due to state-transitions after the switch until the system reaches steady-state again. However, if we design the switched system such that the steady-state is equal for all modes, these transients are avoided.

Theorem 6.1: The switched input-output system (13)-(14) has no transient responses when switching at steady-state, if and only if

$$H_i^{-1}F_i \;=\; H_j^{-1}F_j, \quad \forall\, i, j \in \mathcal{I}\,.$$

Proof: Let t_{i_k} be the switching instant in which the system switches from mode i to mode j and let $\lim_{t\to t_{i_k}} x(t) = \hat{x}^{(i)} = -H_i^{-1}F_i r$ where $\hat{x}^{(i)}$ denotes the steady-state of the LTI system in mode i for the constant input r. During the interval $t_{i_k} \le t < t_{i_{k+1}}$ the dynamics of the switched system are given by $\left(H_j, F_j, \bar{C}_j\right)$. Hence,

$$\begin{aligned} \dot{x}(t_{i_k}) &= H_j\hat{x}^{(i)} + F_j r \\ &= -H_jH_i^{-1}F_i r + F_j r\,. \end{aligned}$$

The switched system shows no transient response if and only if $\dot{x}(t_{i_k}) = 0$. Thus

$$\begin{aligned} 0 &= -H_jH_i^{-1}F_i r + F_j r \\ H_i^{-1}F_i r &= H_j^{-1}F_j r \end{aligned}$$

Since r is a constant scalar, the proof is complete. ■

The above theorem is a necessary and sufficient condition for the transient-free switching at steady state. This condition can be easily translated into a transfer function condition for the control design. Loosely speaking, these transients occur due to the controller state-transition after the switch. While controller $C_k(s)$ is active in the loop the control-output u_l of $C_l(s)$ evolves according to

$$U_l(s) = \frac{C_l(s)}{1 + C_k(s)P_k(s)} R(s).$$

Assume now that switching only occurs when the closed-loop system reached (practically) steady-state, i.e. $\dot{x} \approx 0$. Since we consider systems in control-canonical form, the condition of Theorem 6.1 is equivalent to demanding that the controller-outputs u_l at steady-state is constant for each mode $k \in \mathcal{I}$. Thus we require that for each mode $l \in \mathcal{I}$ there exists a $\gamma_l \in \mathbb{R}$ such that

$$\lim_{s \to 0} \frac{C_l(s)}{1 + C_k(s)P_k(s)} = \gamma_l \quad \forall\, k \in \mathcal{I}. \tag{15}$$

Comment: In case the process dynamics have integral action for each mode condition (15) is always satisfied. When the open-loop system does not contain an integrator and also for the case where all integrators belong to the controllers, the transient-free condition reduces to:

$$K_{C_k} K_{P_k} = \gamma \quad \forall\, k \in \mathcal{I}\,,$$

where K_{C_k}, K_{P_k} are the steady-state gains of the controller and process in mode k, respectively. A more detailed discussion of these cases can be found in [17].

Comment: Condition (15) is formulated in the frequency-domain such that it can be easily incorporated into the control-design procedure (Assumption 3.1). In fact, by adding one extra degree of freedom we can achieve both, stability of the switched system and transient-free switching when the system is in steady-state.

VII. Summarising example

The following example illustrates the results in this paper. We consider a switched process with two modes and demonstrate that the proposed design procedure yields a closed-loop switched system that is stable for arbitrary switching. Moreover, we choose controllers with integrators using the controller architecture described in Section V and demonstrate that transient-free switching can be achieved simultaneously.

Example 7.1: Given the process (2) with transfer functions

$$P_1(s) = \frac{2}{s+9}\,, \quad P_2(s) = \frac{1}{s+8}$$

and let the target poles be $\Lambda_t = \{-0.5 \pm i, -10\}$. We shall require a controller with integrator. Using the standard pole-placing method yields the two controllers

$$C_1(s) = \frac{-12.75s + 12.5}{s(s+8)}\,, \quad C_2(s) = \frac{-3.38s + 6.25}{s(s+2)}\,.$$

If we implement the controller using a joint integrator in front of the controller bank, we obtain for the controller matrices $K_1 = -8$ and $K_2 = -2$. Since K_1, K_2 are scalar Corollary 4.1 guarantees exponential stability of the closed-loop system for arbitrary switching.

Consider now the transient performance of the switched system. For a unit step the closed-loop system reaches the steady-state $\hat{x}_1 = \begin{pmatrix}1 & 1.4 & 0.7 & 0.18\end{pmatrix}^T$ in mode 1 and $\hat{x}_2 = \begin{pmatrix}1 & 1.92 & 0.96 & 0.24\end{pmatrix}^T$ in mode 2. Thus the condition in Theorem 6.1 is not satisfied and we expect to see transient dynamics at the switching instances. Figure 4a shows the step-response of the switched closed-loop system when the process mode switches every 20 time-units. Even though the system reaches steady-state in every switching interval, we can observe considerable transients at the switching instances.

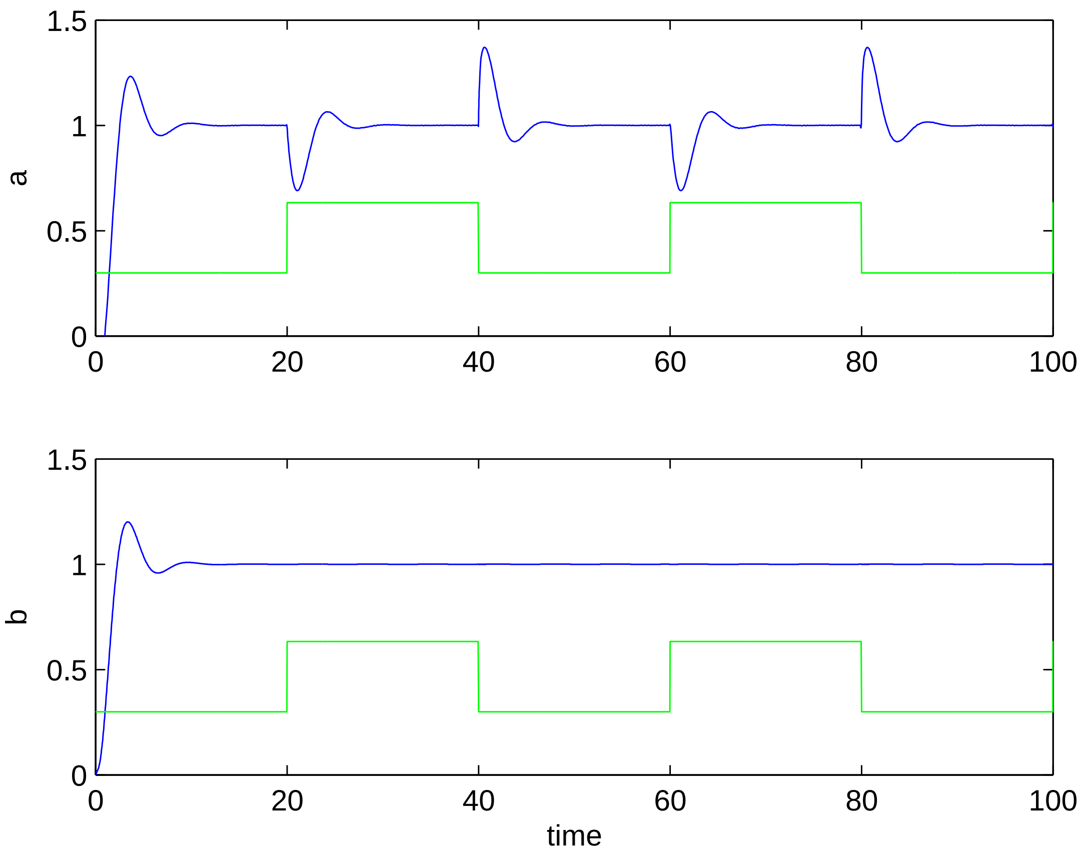


Fig. 4. (a) stable switched system with transients, (b) stable switched system without transients using condition (15); the lower line indicates the scaled switching signal, respectively.

In order to meet the additional condition (15) for transient-free switching we need a controller with an extra degree of freedom. As additional target-pole we choose $\lambda_t = -20$. Using pole-placement we obtain the controller transfer functions

$$C_1(s) = \frac{2.7s^2 - 6.25s + 125}{s^3 + 22s^2 + 27.78s}, \quad C_2(s) = \frac{63.9s^2 - 12.5s + 250}{s^3 + 28s^2 + 83.33s}.$$

Choosing again a controller structure with joint integrator, we can realise the controllers using

$$K_1 = \begin{pmatrix} 0 & 1 \\ -27.78 & -22 \end{pmatrix}, m_1 = \begin{pmatrix} 49.00 \\ -66.44 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 1 \\ -83.33 & -28 \end{pmatrix}, m_1 = \begin{pmatrix} -5076.39 \\ -1802.17 \end{pmatrix} \tag{16}$$

and $l_1 = l_2 = (1 \ \ 0)^T$.

Since the switched system has two modes and the Assumption 3.1 is satisfied we can analyse the stability of the close-loop switched system by only considering the matrices K_1 and K_2, Theorem 4.3. Both matrices $K_1, K2$ are in companion form and the product K_1K_2 has the positive eigenvalues $\lambda_1 = 4.6, \lambda_2 = 500.3$. Thus by Theorem 3.1 in [18] a CQLF exists for (12). It follows by Theorem 4.3 that the closed-loop system (5) is exponentially stable for arbitrary switching and there exists a CQLF for $\{H_1, H_2\}$ by Corollary 4.2. Explicit numerical expressions for the closed-loop system matrices and the respective Lyapunov functions are given in the Appendix.

The steady-state of the close-loop system is constant $\hat{x}_k = \begin{pmatrix} 1 & 1 & 0.036 & 0 & 0.012 & 0 \end{pmatrix}^T$ for all $k \in \mathcal{I}$. Thus the condition of Theorem 6.1 is satisfied and we expect to see no transient dynamics when switching in steady-state. Indeed the simulation of the step-response of the closed-loop system shows no transients at the switching instances (Figure 4b).

VIII. Conclusions and discussion

In this paper we considered a typical control problem for switched linear systems. It is shown that the stability analysis can be considerably simplified by using the proposed design-law and the local-state controller architecture. Further, for systems with first-order controllers we have shown that stability for arbitrary switching is always guaranteed. In the case that the switched system has only two modes, the stability of the switched closed-loop system is equivalent to the stability of the switched system defined by the controller-matrices. Thus the stability analysis degenerates from a switched system of order $n_p + 2n_c$ to that of a switched system of order n_c.

Furthermore, we have shown that transients at the switching instances can be avoided when satisfying condition (15). By adding an extra degree of freedom both stability of the closed-loop system and transient-free switching at steady-state can be achieved.

The stability analysis in this paper depends fundamentally on the assumption that the poles of the closed-loop transfer function are invariant while switching. This requires that the respective controller outputs are instantaneously activated whenever the process mode changes. From a practical point of view this is an unrealistic assumption. In most applications there will be a certain time-delay between the mode-switch of the process and the switching of the control signal. The impact of such delays on the stability of the closed-loop system are an important problem and are subject of future research.

An open question is also how the realisations of the transfer functions effect the results in this paper (*c.f.* [10]). Throughout this paper we assume that the individual controllers are realised in control canonical form. While this is a realistic approach a different choice of the realisation might provide better performance or stability properties. Since we can choose the controller realisations independently of each other, it might be possible to find conditions on the realisations that simplify the stability analysis.

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Appendix

The closed-loop system matrices of the switched system in Example 7.1 with the controller realisations (16) are given by

$$H_1 = \begin{pmatrix} -9 & 5.5 & 98.0 & -132.9 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -27.8 & -22 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -83.3 & -28 \end{pmatrix}, \quad H_2 = \begin{pmatrix} -3 & 63.9 & 0 & 0 & -5076.4 & -1802.2 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -27.8 & -22 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -83.3 & -28 \end{pmatrix}.$$

Indeed it can be verified that $V(x) = x^T P_K x$ with

$$P_k = \begin{pmatrix} 0.9492 & 0.0113 \\ 0.0113 & 0.0163 \end{pmatrix}$$

is a CQLF for the switched system (12) and $V(x) = x^T P_H x$ with

$$P_H = \begin{pmatrix} 0.0006 & 0.0007 & -0.0279 & -0.0027 & -0.0067 & -0.0129 \\ 0.0007 & 0.0132 & -0.0228 & -0.0688 & -0.9193 & -0.0543 \\ -0.0279 & -0.0228 & 13.3803 & 1.9447 & -29.3872 & 0.9925 \\ -0.0027 & -0.0688 & 1.9447 & 1.5944 & -0.1306 & -0.2960 \\ -0.0067 & -0.9193 & -29.3872 & -0.1306 & 149.6891 & 3.0613 \\ -0.0129 & -0.0543 & 0.9925 & -0.2960 & 3.0613 & 3.0188 \end{pmatrix}$$

is a CQLF for the closed-loop switched system (5).