ON LIPSCHITZ CONTINUITY OF THE TOP LYAPUNOV EXPONENT OF LINEAR PARAMETER VARYING AND LINEAR SWITCHING SYSTEMS

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Abstract

We study families of time-varying linear systems, where time-variations have to satisfy restrictions on the dwell time, that is, on the minimum distance between discontinuities, as well as on the derivative in between discontinuities. For this class of systems we study continuity properties of the growth rate as a function of the systems' data. It is shown by example, that a straightforward topology on the space of systems does not yield the desired continuity result. A new natural metric is introduced and a continuity result is obtained. Furthermore, local Lipschitz continuity may be shown for the (generic) case of irreducible systems. The methods rely heavily on a recent converse Lyapunov theorem for the class under consideration.

Keywords: Linear switching system, linear parameter-varying system, growth rate, Lipschitz continuity, parameterized Lyapunov function

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1 Introduction

In this paper we study families of continuous time linear parameter-varying systems in $\mathbb{K}^n$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. We consider systems of the form

$$\dot{x}(t) = A(\theta(t))x(t)\,, \quad t \in \mathbb{R}\,, \tag{1}$$

where $\theta(\cdot) \in L^\infty(\mathbb{R}, \Theta)$ takes values in a nonempty compact set $\Theta \subset \mathbb{K}^m$ and $A : \Theta \to \mathbb{K}^{n\times n}$ is Lipschitz continuous.

We impose restrictions on the parameter variations θ by requiring a certain *dwell time* $h > 0$, that is, a minimal positive distance between discontinuities of θ. Furthermore, θ is assumed to be absolutely continuous in between discontinuities, satisfying a constraint on the derivative given by a compact convex set Θ_1. This setup encompasses many of the systems which can be found under the names of *linear parameter-varying systems* and *linear switching systems* in the literature. A discussion of the relation of these systems to our class can be found in [19]. For a discussion of the interest in these system classes and available results we refer to [1, 2, 8, 9, 11, 12, 13] and references therein.

In this paper we study continuity properties of the growth rate of the family of systems (1) as a function of the data (h, Θ, Θ_1, A). It is shown, that it is in general not reasonable to look for results with respect to the Hausdorff metric in the Θ and Θ_1 spaces. We propose a suitable new metric with respect to which a continuity result can be shown. For irreducible systems a local Lipschitz continuity result is obtained. The results extend similar results contained in the papers

[16, 17], where the same problem was studied for linear inclusions of the form $\dot{x} \in \{Ax \mid A \in \mathcal{M}\}$ and for linear parameter-varying systems.

We proceed as follows. In the ensuing Section 2 we present a rigorous definition of the class of systems that we are considering. Section 3 discusses some of the structure of the set of admissible parameter variations. For the case of irreducible systems, that is, systems for which there is no nontrivial invariant subspace, we review a construction of parameterized Lyapunov functions, which has recently been presented in [19]. In Section 4 we first discuss the metric on the space of systems that is seemingly the natural one in our case. Namely, the metric that is obtained by using the Hausdorff metric in the space of parameter sets Θ and restrictions on the derivative Θ_1. It is shown by example, that with this choice the growth rate is not a continuous function of the data. In Section 5 we introduce a new metric by considering the Hausdorff metric on the space of parameter variations induced by (Θ, Θ_1). With this topology the continuity and Lipschitz results are shown in Section 6. A vital step in these proofs consists of obtaining bounds on the eccentricity of the Lyapunov functions for irreducible systems. We conclude in Section 7.

2 Families of Linear Time-Varying Systems

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ denote the real or the complex field. We consider families of time-varying systems of the form

$$\dot{x}(t) = A(\theta(t))x(t)\,, \quad t \in \mathbb{R}\,, \tag{2}$$

where $\theta(\cdot) \in L^\infty(\mathbb{R}, \Theta)$ is an admissible parameter variation taking values in a nonempty compact set $\Theta \subset \mathbb{K}^m$ and $A : \Theta \to \mathbb{K}^{n\times n}$ is Lipschitz continuous. For fixed $\theta(\cdot)$ the evolution operator generated by (2) is denoted by $\Phi_\theta(t,s)$, $t, s \in \mathbb{R}$.

We are interested in constraining the parameter variations under consideration to be piecewise continuous and satisfying constraints given by the parameters $h > 0$, compact sets Θ, Θ_1 and the Lipschitz continuous map A. Before we precisely state the conditions on these four parameters defining the system $\Sigma = (h, \Theta, \Theta_1, A)$, we will introduce the notion of admissible parameter variations. In order to denote the discontinuities of parameter variations, which for the purposes of this paper are discrete sets, we consider (bounded or unbounded) index sets $\mathcal{I} \subset \mathbb{Z}$. In the following it will always be tacitly assumed, that these index sets are given as the intersection of a real interval with $\mathbb{Z}$, i.e. of the form $\mathcal{I} := [a, b] \cap \mathbb{Z}$, where $a, b \in \mathbb{R} \cup \{\pm\infty\}$.

Definition 1. *Let $\Sigma = (h, \Theta, \Theta_1, A)$ be given. If $h \in (0, \infty)$, a parameter variation $\theta : \mathbb{R} \to \Theta$ is called* admissible *(with respect to Σ), if there is a index set $\mathcal{I}_\theta \subset \mathbb{Z}$ and times $t_k, k \in \mathcal{I}_\theta$ such that*

(i) $h \le t_{k+1} - t_k$, for $k \in \mathcal{I}_\theta, k < \sup \mathcal{I}_\theta$,

(ii) for $k \in \mathcal{I}_\theta, k < \sup \mathcal{I}_\theta$ the function θ is absolutely continuous on the interval $[t_k, t_{k+1})$, and satisfies

$$\dot{\theta}(t) \in \Theta_1\,, \quad a.e. \tag{3}$$

(This condition also applies to $(-\infty, \inf \mathcal{I}_\theta)$, $(\sup \mathcal{I}_\theta, \infty)$ if $\inf \mathcal{I}_\theta$, resp. $\sup \mathcal{I}_\theta$, is finite.)

If $h = \infty$, the admissible parameter variations are given as the set of absolutely continuous functions $\theta : \mathbb{R} \to \Theta$ satisfying (3) almost everywhere on $\mathbb{R}$.

The set of admissible parameter variations is denoted by $\mathcal{U}$ or $\mathcal{U}(\Sigma)$, if dependence on the data needs to be emphasized. For simplicity it will always be assumed, that the time t_0 appearing in (i) satisfies $t_0 > 0$. By $t_0(u)$ we denote the smallest positive discontinuity of a parameter variation u.

We need the following notation. The spaces of nonempty compact subsets, respectively nonempty convex compact subsets of $\mathbb{K}^m$ are denoted by $\mathcal{K}(\mathbb{K}^m)$, respectively $\mathrm{Co}(\mathbb{K}^m)$. Both these spaces are complete metric spaces with respect to the Hausdorff metric, which we denote by $H(\cdot,\cdot)$. Recall that the relative interior of a convex set $\mathcal{M} \subset \mathbb{K}^m$, denoted by $\mathrm{ri}\,\mathcal{M}$, is the interior of $\mathcal{M}$ in the relative topology of the affine space space generated by $\mathcal{M}$. Or to put it another way, the interior of $\mathcal{M}$ with respect to the smallest affine space containing $\mathcal{M}$. With this we formulate the following conditions that are assumed throughout the paper. We note that the following conditions are more restrictive than the ones used in [19].

(A1) $h \in (0, \infty]$,

(A2) $\Theta \subset \mathbb{K}^m$ is a finite disjoint union of sets $\Omega_j \in \mathrm{Co}\,(\mathbb{K}^m)$, $j \in \{1, \dots, k\}$; if $h = \infty$ then $k = 1$, i.e. Θ is compact and convex,

(A3) $\Theta_1 \in \mathrm{Co}\,(\mathbb{K}^m)$,

(A4) $0 \in \mathrm{ri}\,\Theta_1$,

(A5) $\mathrm{span}\,\Theta_1 \supset \mathrm{span}\,(\Omega_j - \eta_j)$, $j = 1, \dots, k$ where $\eta_j \in \Omega_j$ is arbitrary.

(A6) $A : \Theta \to \mathbb{K}^{n \times n}$ is a Lipschitz continuous map from the parameter space to the space of matrices.

Using methods from [6] it is shown in Corollary 4.5 of [19], that under the above assumptions system (2) may be interpreted as a linear flow on a vector bundle. To be precise, $\mathcal{U}(\Sigma)$ is a compact metric space as a compact subset of $L^\infty(\mathbb{R}, \mathrm{conv}\,\Theta)$ (endowed with the weak-$*$-topology). The shift $u(\cdot) \mapsto u(t + \cdot)$ is continuous on that space. Now consider the trivial vector bundle $\pi : \mathcal{U} \times \mathbb{K}^n \to \mathcal{U}$, then the dynamical system given by

$$(u, x) \mapsto (u(t + \cdot), \Phi_u(t, 0)x) \tag{4}$$

is a continuous dynamical system. Note, in particular, that it is linear in the second component.

We now define the object of interest in this paper, which is the top Lyapunov exponent or the (uniform) exponential growth rate associated to system (2), or (4). Given the map $A : \Theta \to \mathbb{K}^{n \times n}$ and the set of admissible parameter variations $\mathcal{U}(\Sigma) \subset L^\infty(\mathbb{R}, \Theta)$, define for $t \geq 0$ the sets of finite time evolution operators

$$\mathcal{S}_t(\Sigma) := \{\, \Phi_u(t, 0) \mid u \in \mathcal{U}(\Sigma) \}\,, \quad \mathcal{S}(\Sigma) := \bigcup_{t \geq 0} \mathcal{S}_t(\Sigma).$$

Throughout the paper $\| \cdot \|$ denotes a fixed norm on $\mathbb{K}^n$ and the corresponding operator norm on $\mathbb{K}^{n \times n}$. We now introduce for $t > 0$ finite time growth constants given by

$$\widehat{\rho}_t(\Sigma) := \sup \left\{ \frac{1}{t} \log \|S\| \mid S \in \mathcal{S}_t(\Sigma) \right\}.$$

It is easy to see, that because of the shift-invariance of $\mathcal{U}$ the function $t \longmapsto t\widehat{\rho}_t(\Sigma)$ is subadditive. This implies by a folklore result (see e.g. p. 27/28 of [10]), that the following limit exists (and is independent of $\| \cdot \|$)

$$\widehat{\rho}(\Sigma) := \lim_{t \to \infty} \widehat{\rho}_t(\Sigma) = \inf_{t \geq 0} \widehat{\rho}_t(\Sigma). \tag{5}$$

It is well known, that an alternative way to describe ρ is given by

$$\widehat{\rho}(\Sigma) = \inf\{\beta \in \mathbb{R} \mid \exists\, M \geq 1 \text{ such that } \|\Phi_u(t, 0)\| \leq Me^{\beta t} \text{ for all } u \in \mathcal{U}, t \geq 0\}\,. \tag{6}$$

For this reason the quantity $\rho(\Sigma)$ is called *uniform exponential growth rate* of the family of linear time-varying systems of the form (2) given by Σ. Another way to define exponential growth is to employ a trajectory-wise definition. In this case we define the Lyapunov exponent corresponding to an initial condition $x_0 \in \mathbb{K}^n \setminus \{0\}$ and $u \in \mathcal{U}$ by

$$\lambda(x_0, u) := \limsup_{t \to \infty} \frac{1}{t} \log \|\Phi_u(t, 0)x_0\|\,, \tag{7}$$

and define as exponential growth rate $\kappa(\Sigma) := \sup\{\lambda(x, u) \mid 0 \neq x \in \mathbb{K}^n\,,\, u \in \mathcal{U}\}$. As our system may be interpreted as a linear flow on a vector bundle, we have that $\rho(\Sigma) = \kappa(\Sigma)$, see [6].

3 Concatenation, Irreducibility and Lyapunov norms

In this section we briefly describe the main results from [19], which are essential in order to obtain the desired Lipschitz continuity results. We assume the system $\Sigma = (h, \Theta, \Theta_1, A)$ to be given. For ease of notation we will therefore suppress the dependence on these data of $\rho(\Sigma)$, $\mathcal{S}_t(\Sigma)$ and other objects we intend to define. The problem with our system class is, that simple concatenation of admissible parameter variations does in general not result in an admissible parameter variation. In contrast for every admissible parameter variation $u \in \mathcal{U}$ and $t \geq 0$ there is a certain subset of $\mathcal{U}$ of admissible parameter variations w for which the following concatenation is also admissible

$$(u \diamond_t w)(s) := \begin{cases} u(s)\,, & s < t \\ w(s-t)\,, & t \leq s \end{cases} . \tag{8}$$

It is easy to see that this subset depends on the continuous extension of u at t from the left and on the difference between the time instance t and the largest discontinuity of u smaller than t. To denote these quantities we define

$$u(t^-) := \lim_{s \nearrow t} u\,(s) \tag{9}$$

and

$$\tau^-\,(u,t) := \min\{h, t - \max\{t_k \mid t_k < t \text{ where } t_k \text{ is a discontinuity of } u\}\}\,. \tag{10}$$

We first treat the case $h \in (0, \infty)$ and define for $(\theta, \tau) =: \omega \in \Theta \times [0, h)$ the set of concatenable parameter variations by

$$\mathcal{U}\,(\omega) := \mathcal{U}\,(\theta, \tau) := \{u \in \mathcal{U} \,|\, u\,(0) = \theta \text{ and } h \leq t_0\,(u) + \tau\}\ ,$$

and for $\tau = h$ and $\omega = (\theta, h)$

$$\mathcal{U}\,(\omega) := \mathcal{U}\,(\theta, h) := \{u \in \mathcal{U} \,|\, u\,(0) = \theta \text{ or } h \leq t_0\,(u)\}\ .$$

Note that with this definition we clearly have $\mathcal{U} = \cup_{\omega \in \Theta \times [0,h]} \mathcal{U}(\omega)$ as by definition every admissible parameter variation is continuous on some interval of the form $[0, \tau]$.

The interpretation of the set $\mathcal{U}\,(\theta, \tau)$ is the following. If a parameter variation u is defined on the interval $(-\infty, t)$, then for $w \in \mathcal{U}$ the concatenation (8) defines an admissible parameter variation if and only if

$$w \in \mathcal{U}(u(t^-), \tau^-\,(u,t))\,.$$

In the case $h = \infty$ there is no need to account for discontinuities. We thus define for $\theta \in \Theta$ the set

$$\mathcal{U}(\theta) := \{u \in \mathcal{U} \mid u(0) = \theta\}\,.$$

For the sake of a unified notation, we define

$$\Pi(\Theta, h) := \begin{cases} \Theta \times [0, h]\,, & \text{if } h \in (0, \infty)\,, \\ \Theta\,, & \text{if } h = \infty\,. \end{cases}$$

For each $\omega \in \Pi(\Theta, h)$ and $t \geq 0$ we consider the set of evolution operators "starting in ω" given by

$$\mathcal{S}_t(\omega) := \{\ \Phi_u(t, 0) \mid u \in \mathcal{U}\,(\omega)\}. \tag{11}$$

Similarly, we define for $\omega, \zeta \in \Pi(\Theta, h)$ and for $t \geq 0$ the sets of evolution operators "starting in ω and ending at ζ" by

$$\mathcal{R}_t(\omega, \zeta) := \{\Phi_u(t, 0) \,|\, u \in \mathcal{U}\,(\omega) \ \text{ and } u \diamond_t w \in \mathcal{U}(\omega)\,, \forall w \in \mathcal{U}(\zeta)\}\ . \tag{12}$$

Thus by definition if $R \in \mathcal{R}_s(\omega,\zeta)$ and $S \in \mathcal{S}_t(\zeta)$, then $SR \in \mathcal{S}_{t+s}(\omega)$. We now define

$$\begin{aligned} \mathcal{S}_{\le T}(\omega) \ &: \ = \bigcup_{0\le t\le T} \mathcal{S}_t(\omega) \ \text{ and } \ \mathcal{S}(\omega) := \bigcup_{t\ge 0} \mathcal{S}_t(\omega)\,, \quad \text{respectively} \\ \mathcal{R}_{\le T}(\omega,\zeta) \ &: \ = \bigcup_{0\le t\le T} \mathcal{R}_t(\omega,\zeta) \ \text{ and } \ \mathcal{R}(\omega,\zeta) := \bigcup_{t\ge 0} \mathcal{R}_t(\omega,\zeta)\,. \end{aligned}$$

Note that the definition entails, that for every $\omega \in \Pi(\Theta,h)$ the set $\mathcal{R}(\omega,\omega)$ is a semigroup.

Remark 2. *It is useful to keep in mind the following remark on parameter variations connecting two points $\omega,\zeta \in \Pi(\Theta,h)$. If $h \in (0,\infty)$, then for all $\omega,\zeta \in \Pi(\Theta,h)$ the set $\mathcal{R}_{2h}(\omega,\zeta)$ is not empty. For if $\omega = (\theta,\tau), \zeta = (\eta,\sigma)$, then it suffices to define $u(s) = \theta, 0 \le s < h$ and $u(s) = \eta, h \le s \le 2h$, which defines an admissible parameter variation with $\Phi_u(2h,0) \in \mathcal{R}(\omega,\zeta)$. Similarly, if $h = \infty$ then using (A5) there exists a constant $\bar{c}$ such that $\mathcal{R}_{\bar{c}}(\omega,\zeta) \neq \emptyset$ for all $\omega,\zeta \in \Theta$.*

If we want to describe the exponential growth rate within the subsets of evolution operators with given initial and end condition, this leads to the definitions

$$\widehat{\rho}_t(\omega) := \max\left\{ \frac{1}{t}\log\|S\| \mid S \in \mathcal{S}_t(\omega) \right\}\,,\ \widehat{\rho}_t(\omega,\zeta) := \max\left\{ \frac{1}{t}\log\|S\| \mid S \in \mathcal{R}_t(\omega,\zeta) \right\}\,.$$

It is shown in Lemma 4.9 of [19] that

$$\widehat{\rho} = \lim_{t\to\infty} \widehat{\rho}_t(\omega,\zeta) = \lim_{t\to\infty} \widehat{\rho}_t(\omega). \tag{13}$$

In the following the most important assumption is that of irreducibility of $A(\Theta)$. Recall that a set of matrices $\mathcal{M} \subset \mathbb{K}^{n\times n}$ is called irreducible, if only the trivial subspaces $\{0\}$ and $\mathbb{K}^n$ are invariant under all $A \in \mathcal{M}$ and reducible otherwise.

We cite the following simple lemma from [16]. It is crucial in the following construction of norms that are Lyapunov functions for our system and will be used in the sequel.

Lemma 3. *Let $\mathbb{K} = \mathbb{R},\mathbb{C}$ and let $\mathcal{S} \subset \mathbb{K}^{n\times n}$ be an irreducible semigroup. For any family of sets $\mathcal{S}_t$, $t \in \mathbb{R}_+$ such that*

$$\mathcal{S} = \bigcup_{t\ge 0} \mathcal{S}_t\,,$$

there are $\varepsilon > 0$ and $T \in \mathbb{R}_+$ such that for all $z \in \mathbb{K}^n, A \in \mathbb{K}^{n\times n}$ there is an $S \in \bigcup_{0\le t\le T} \mathcal{S}_t$ with

$$\|ASz\| \ge \varepsilon \|A\|\|z\|\,.$$

We note the following immediate corollary for further reference.

Corollary 4. *Let $\mathbb{K} = \mathbb{R},\mathbb{C}$. Let $\mathcal{R} \subset \mathbb{K}^{n\times n}$ be an irreducible set of matrices and let $\mathcal{S} \subset \mathbb{K}^{n\times n}$ be an irreducible semigroup. For arbitrary families of sets $\mathcal{R}_t, \mathcal{S}_t, t \in \mathbb{R}_+$, such that*

$$\mathcal{R} = \bigcup_{t\ge 0} \mathcal{R}_t\,, \quad \mathcal{S} = \bigcup_{t\ge 0} \mathcal{S}_t\,,$$

and with the property that

$$SR \in \mathcal{R}_{s+t}\,, \quad \text{for all } S \in \mathcal{S}_s, R \in \mathcal{R}_t\,, \quad s,t \ge 0\,,$$

there are $\varepsilon > 0$ and $\tau \ge 0$ such that for all $z \in \mathbb{K}^n, A \in \mathbb{K}^{n\times n}$ there is an $R \in \bigcup_{0\le t\le \tau} \mathcal{R}_t$ with

$$\|ARz\| \ge \varepsilon \|A\|\|z\|\,.$$

Proof. By irreducibility for each $0 \neq z \in \mathbb{K}^n$ there is an $R \in \mathcal{R}$ such that $Rz \neq 0$. Using a standard compactness argument for the unit sphere in $\mathbb{K}^n$, it follows, that there are an $\varepsilon_1 > 0$ and finitely many $R_1, \dots, R_m \in \mathcal{R}$, such that for every $z \in \mathbb{K}^n$ there is a $1 \leq j \leq m$ with

$$\|R_j z\| \geq \varepsilon_1 \|z\| .$$

Choose τ_1 such that $R_j \in \bigcup_{0 \leq t \leq \tau_1} \mathcal{R}_t$, $j = 1, \dots, m$. Now let $\varepsilon_2 > 0, \tau_2 \geq 0$ be the constants guaranteed for $\mathcal{S}$ by Lemma 3. Then for every $z \in \mathbb{K}^n$, $A \in \mathbb{K}^{n \times n}$, there exists a $1 \leq j \leq m$ and $S \in \bigcup_{0 \leq t \leq \tau_2} \mathcal{S}_t$ such that

$$\|ASR_j z\| \geq \varepsilon_2 \|A\| \|R_j z\| \geq \varepsilon_1 \varepsilon_2 \|A\| \|z\| .$$

The assertion follows by setting $\varepsilon := \varepsilon_1 \varepsilon_2$ and $\tau := \tau_1 + \tau_2$. □

By Proposition 5.3 in [19], if $A(\Theta)$ is irreducible and our standing assumptions hold, then the sets $\mathcal{R}(\omega, \zeta), \mathcal{S}(\omega)$ are irreducible for all $\omega, \zeta \in \Pi(\Theta, h)$. We define limit sets of the form

$$\mathcal{S}_\infty(\omega) \quad : \quad = \{ S \in \mathbb{K}^{n \times n} \,|\, \exists\, t_k \to \infty \,, S_k \in \mathcal{S}_{t_k}(\omega) : \, e^{-\rho t_k} S_k \to S \}, \tag{14}$$

$$\mathcal{R}_\infty(\omega, \zeta) \quad : \quad = \{ S \in \mathbb{K}^{n \times n} \,|\, \exists\, t_k \to \infty \,, S_k \in \mathcal{R}_{t_k}(\omega, \zeta) : \, e^{-\rho t_k} S_k \to S \}. \tag{15}$$

By Lemma 6.1 in [19] the set $\cup_{\omega \in \Pi(\Theta, h)} \mathcal{S}_\infty(\omega)$ is bounded, the sets $\mathcal{R}_\infty(\omega, \zeta), \mathcal{S}_\infty(\omega)$ are each a compact, nonempty set not equal to $\{0\}$, and the sets $\mathcal{R}_\infty(\omega, \omega)$, $\mathcal{S}_\infty(\omega)$ are irreducible.

Thus the functions $v_\omega : \mathbb{K}^n \to \mathbb{R}_+$ defined for $\omega \in \Pi(\Theta, h)$ by setting

$$v_\omega(x) := \max \, \{ \, \|Sx\| \,\,|\, S \in \mathcal{S}_\infty(\omega) \} \,, \tag{16}$$

can easily seen to be norms on $\mathbb{K}^n$. They form a family of parameterized Lyapunov functions for our system in the sense described in the following result, which recalls Theorem 6.4 of [19].

Theorem 5. *Consider system (2) with (A1)-(A6). Assume that $A(\Theta)$ is irreducible and let $\omega \in \Pi(\Theta, h)$ be arbitrary. Then*

(i) For all $u \in \mathcal{U}(\omega), t \geq 0$ and all $x \in \mathbb{K}^n$ it holds that

$$v_\zeta(\Phi_u(t, 0)x) \leq e^{\rho t} \, v_\omega(x) \,, \tag{17}$$

whenever $\Phi_u(t, 0) \in \mathcal{R}_t(\omega, \zeta)$ for $\zeta \in \Pi(\Theta, h)$. In particular, for all $t \geq s \geq 0$ it holds that

$$v_{u(t^-), \tau^-(u,t)}(\Phi_u(t, 0)x) \leq e^{\rho(t-s)} \, v_{u(s^-), \tau^-(u,s)}(\Phi(s, 0)x) \,. \tag{18}$$

(ii) For every $x \in \mathbb{K}^n$, $\omega \in \Pi(\Theta, h)$, and every $t \geq 0$,, there exist $u \in \mathcal{U}(\omega)$ and a piecewise continuous map $\zeta : [0, t] \to \Pi(\Theta, h)$, with $\zeta(0) = \omega$, and such that for all $s \in [0, t]$ we have

$$v_{\zeta(s)}(\Phi_u(s, 0)x) = e^{\rho s} \, v_\omega(x) \,.$$

If $h = \infty$, then ζ may be chosen to be continuous. If $h < \infty$ and $\omega = (\theta, \tau) \in \Theta \times [0, h)$, the function ζ may be chosen, so that its discontinuities on $[0, t)$ coincide with those of u. Otherwise, ζ may have one further discontinuity at 0.

Finally, we will use a continuity result for the norms v_ω, which is contained in Proposition 6.6 and Corollary 6.7 of [19]. To state the result, we introduce the space of continuous, positively homogeneous functions on $\mathbb{K}^n$ defined by

$$\mathrm{Hom}\,(\mathbb{K}^n, \mathbb{R}) := \{ \mathrm{f} : \mathbb{K}^n \to \mathbb{R} \,|\, \forall \alpha \geq 0 : \mathrm{f}(\alpha \mathrm{x}) = \alpha \mathrm{f}(\mathrm{x}) \text{ and f is continuous on } \mathbb{K}^n \} \,.$$

Clearly, all norms on $\mathbb{K}^n$ are elements of $\mathrm{Hom}\,(\mathbb{K}^n, \mathbb{R})$. This space becomes a Banach space if equipped with the norm

$$\|f\|_{\infty, \mathrm{hom}} := \max \{ |f\,(x)| \,\mid\, \|x\|_2 = 1 \} \,.$$

Proposition 6. *Consider system (2) with (A1)-(A6). Assume that $A(\Theta)$ is irreducible. Then the map*

$$\omega \longmapsto v_\omega \,, \tag{19}$$

is Lipschitz continuous from $\Pi(\Theta, h)$ to $\mathrm{Hom}\,(\mathbb{K}^n, \mathbb{R})$.

Furthermore, there exists a constant $1 \le C \in \mathbb{R}$, such that for all $\omega, \zeta \in \Pi(\Theta, h)$ and all $x \in \mathbb{K}^n$ we have

$$C^{-1} v_\omega(x) \le v_\zeta(x) \le C v_\omega(x)\,. \tag{20}$$

4 Counterexamples to Continuity

In this section we give some examples showing that the assumptions on the data made thus far do not guarantee continuity of the growth rate. In the first example we show that an amalgamation of several of the sets Ω_j may lead to a discontinuity. The ensuing example shows that convergence of the sets Θ_1 may also be a problem.

In order to investigate continuity properties of ρ we first have to define a suitable topology. As all the maps A are Lipschitz continuous functions on a compact set, each A may be extended to a globally bounded, globally Lipschitz continuous function on $\mathbb{K}^m$. We denote by $\mathrm{Lip}\,(\mathbb{K}^m, \mathbb{K}^{n\times n})$ the space of such functions. This is a Banach space, if considered with the norm

$$\|A\|_{\mathrm{Lip}} := \|A\|_\infty + L(A)\,, \quad A \in \mathrm{Lip}\,(\mathbb{K}^m, \mathbb{K}^{n\times n})\,,$$

where $\|A\|_\infty$ denotes the standard supremum norm of A and $L(A) \ge 0$ is the smallest global Lipschitz constant for A.

We consider two spaces of systems

$$\mathcal{L} := \{\Sigma := (h, \Theta, \Theta_1, A) \mid h \in (0,\infty), \Sigma \text{ satisfies (A1)} - \text{(A6)}\}\,,$$

$$\mathcal{L}(\infty) := \{\Sigma := (\infty, \Theta, \Theta_1, A) \mid \Sigma \text{ satisfies (A1)} - \text{(A6)}\}\,.$$

These may be endowed with the metric inherited from $(0,\infty)\times\mathcal{K}(\mathbb{K}^{n\times n})\times\mathrm{Co}(\mathbb{K}^{n\times n})\times\mathrm{Lip}\,(\mathbb{K}^m, \mathbb{K}^{n\times n})$, resp. $\mathcal{K}(\mathbb{K}^{n\times n}) \times \mathrm{Co}(\mathbb{K}^{n\times n}) \times \mathrm{Lip}\,(\mathbb{K}^m, \mathbb{K}^{n\times n})$, which may be defined as the sum of the metrics in the individual components. The following examples show in a sense, that these "natural" metrics are not the appropriate ones when considering the exponential growth rate. First however, we recall the following result from [19]. To this end we introduce a topology on $\mathcal{L} \cup \mathcal{L}(\infty)$ by using the standard interpretation of the symbol $h_k \to \infty$.

Proposition 7. *The map*

$$(h, \Theta, \Theta_1, A) \mapsto \rho(h, \Theta, \Theta_1, A)$$

is upper semicontinuous on $\mathcal{L} \cup \mathcal{L}(\infty)$.

The following examples are modifications of an example given in [19] showing discontinuity of the growth rate as a function of the data. However, the example in [19] does not satisfy the assumptions (A1)-(A6), so that we prefer to present different examples here.

Example 8. *Let $0 < h < \infty$, $\Theta_1 := [-1, 1] \subset \mathbb{R}$ and $\Theta(0) = [0, 2\pi]$. Define furthermore*

$$A_1(\theta) = \begin{bmatrix} -1 + 3/2\cos^2\theta & 1 - 3/2\sin\theta\cos\theta \\ -1 - 3/2\sin\theta\cos\theta & -1 + 3/2\sin^2\theta \end{bmatrix}.$$

We recall the well known fact that the characteristic polynomial of $A_1(\theta)$ is equal to $p(z) = z^2 + 1/2z + 1/2$ with zeros $-1/4 \pm i\sqrt{7}/4$ independent of θ, [7, 15].

We begin by giving a lower bound for the exponential growth rate of $(h, \Theta(0), \Theta_1, A_1)$*. Define the admissible parameter variation*

$$\theta(t) = \begin{cases} t, & t \in [0, 2\pi]\,, \\ 4\pi - t, & t \in [2\pi, 4\pi]\,, \end{cases}$$

and continue this function periodically. Then on the interval $[0, 2\pi]$ *we are in the situation of the classical example and it is well known that*

$$\Phi_\theta(2\pi, 0) = \begin{bmatrix} e^{\pi} & 0 \\ 0 & e^{-2\pi} \end{bmatrix}.$$

For the calculation of $\Phi_\theta(4\pi, 2\pi)$ *numerical evaluation yields*

$$\Phi_\theta(4\pi, 2\pi) \approx \begin{bmatrix} 0.0597 & -0.178 \\ 0.178 & 0.1932 \end{bmatrix}.$$

And by calculating the spectral radius $r(\Phi_\theta(4\pi, 0)) = r(\Phi_\theta(4\pi, 2\pi)\Phi_\theta(2\pi, 0)) \approx 1.3799$*, we see, that the exponential growth rate corresponding to* $(h, \Theta(0), \Theta_1, A_1)$ *is approximately lower bounded by* $\log 1.3799/4\pi \approx 0.0256$*.*

On the other hand, as all matrices in $A_1(\Theta(0))$ *are Hurwitz, with spectral abscissa equal to* $-1/4$*, there exists a* $\delta > 0$*, such that for any* $y \in [0, 2\pi]$ *the differential inclusion*

$$\dot{x} \in \{Bx \mid B \in A_1([y, y + \delta])\} \tag{21}$$

is exponentially stable. (This uses the fact that the exponential growth rate of linear differential inclusions is a continuous function of the set of matrices with respect to the Hausdorff metric, see e.g. [4]). In particular, any solution of (21) *satisfies a bound of the type* $\|x(t)\| \leq M_y \|x(0)\|$*, with* M_y *possibly depending on the point* y *defining the inclusion.*

Partition the interval $[0, 2\pi]$ *in the form*

$$t_0 = 0 < t_1 = t_0 + \delta < \ldots < t_j = t_{j-1} + \delta < \ldots < t_k = 2\pi\,,$$

where k *is the smallest integer bigger or equal to* $2\pi/\delta$*.*

For $0 < \varepsilon < \delta$ *define* $\Theta(\varepsilon)$ *as the finite union of sets* $\Omega_j(\varepsilon)$*,* $j = 0, \ldots, k-1$ *given by*

$$\Omega_j(\varepsilon) = [t_j, t_{j+1} - \varepsilon]\,.$$

Now the exponential growth rates given by (21) yield an upper bounds for the growth rates of solutions of $(h, \Theta(\varepsilon), \Theta_1, A_1)$*. In particular, defining* $M := \max_{j=0,\ldots,k-1} M_{t_j}$ *we have for any solution of the system defined by* $\Theta(\varepsilon)$ *the (very rough) bound*

$$\|x(t)\| \leq M^{t/h+1} \|x(0)\|\,. \tag{22}$$

This shows, that for all $\varepsilon > 0$ *the exponential growth rate is at most* $h^{-1} \log M$*. Hence for* $h > 50 \log M$ *we have*

$$\limsup_{\varepsilon \to 0} \rho(h, \Theta(\varepsilon), \Theta_1, A_1) < 0.02 \leq \rho(h, \Theta(0), \Theta_1, A_1)\,,$$

whereas on the other hand $H(\Theta(\varepsilon), \Theta(0)) = \varepsilon$*. This shows that the growth rate is not a continuous function of the parameter set* Θ*.*

The following example is a modification of the previous one.

Example 9. *We embed the data of the previous example into* $\mathbb{R}^2$ *by setting* $\Theta := [0, 2\pi] \times \{0\}$*,* $\Theta_1(0) := [-1, 1] \times \{0\}$ *and defining* $A_2(\theta_1, \theta_2) := A_1(\theta_1)$*.*

For $\varepsilon, \delta > 0$ define

$$\Theta_{1,\delta}(\varepsilon) := \text{conv}\left\{ \begin{bmatrix} 0 & \varepsilon \end{bmatrix}^T, \begin{bmatrix} 0 & -\varepsilon \end{bmatrix}^T, \begin{bmatrix} 1 & \frac{\varepsilon(1-\delta)}{\delta} \end{bmatrix}^T, \begin{bmatrix} -1 & -\frac{\varepsilon(1-\delta)}{\delta} \end{bmatrix}^T \right\}.$$

In the following δ will be a fixed parameter and we investigate the behavior of the growth rate as $\varepsilon \to 0$. Note that for fixed $\delta > 0$ and all $\varepsilon > 0$ the sets $\Theta_{1,\delta}(\varepsilon)$ intersect the horizontal axis exactly in the interval $[-\delta, \delta]$.

We will keep the set Θ fixed in the following argument. At first glance, this appears to be implausible, as then there is not point in introducing a two-dimensional set $\Theta_1(\varepsilon)$. It is, however, straightforward though a bit more cumbersome to apply the same arguments to the sets $\Theta(\varepsilon) := [0, 2\pi] \times [-\varepsilon, \varepsilon]$.

By Proposition 7 ρ is upper semicontinuous, and therefore we have for all $h \in (0, \infty]$ that

$$\limsup_{\delta \to 0} \rho(h, \Theta, \Theta_{1,\delta}(\varepsilon), A_2) \leq \rho(h, \Theta, \{0\}, A_2).$$

For the previous inequality we have used that the set of derivative constraints that can be effectively used is equal to $[-\delta, \delta]$. Applying the same argument as in (22), it may be seen, that the quantity on the right is nonpositive, if we fix an h large enough. Thus by choosing $\delta > 0$ small enough and $h > 0$ large enough, we can ensure, that for all $\varepsilon > 0$

$$\rho(h, \Theta, \Theta_{1,\delta}(\varepsilon), A_2) < 0.01.$$

Now for fixed $\delta > 0$ it holds that $\lim_{\varepsilon \to 0} \Theta_{1,\delta}(\varepsilon) = \Theta_1(0)$. On the other hand

$$\limsup_{\varepsilon \to 0} \rho(h, \Theta, \Theta_{1,\delta}(\varepsilon), A_2) < 0.01 < 0.02 \leq \rho(h, \Theta, \Theta_1(0), A_2).$$

Thus the growth rate does not depend continuously on Θ_1.

Note that in this example there is a discontinuity hidden in the "effective" Θ_1. While $\Theta_{1,\delta}(\varepsilon)$ converges to $\Theta_1(0)$ in the Hausdorff metric as $\varepsilon \to 0$, the set of effectively usable derivatives are equal to $[-\delta, \delta]$ independently of ε. This set obviously does not converge to $\Theta_1(0)$. This shows that the Hausdorff metric does not provide the correct notion of distance for the problem treated in this paper.

5 An alternative to the Hausdorff Topology

We have seen in the previous section, that using the product topology on $\mathcal{K}(\mathbb{K}^m) \times \text{Co}(\mathbb{K}^m)$ the growth rate ρ is not a continuous function. One way of resolving this problem is to list a number of special cases in which continuity results can be obtained. A more satisfying approach, however, is to introduce a new topology, for which a continuity result can be shown and to give special subsets on which the topology induced by the classical Hausdorff topology coincides with the new one. In this section we introduce the required new metric, which is defined in terms of the space of parameter variations.

Our construction will be performed in two steps. We first consider the space $\widetilde{\mathcal{N}}$ defined by

$$\widetilde{\mathcal{N}} := \{(\Theta, \Theta_1) \in \mathcal{K}(\mathbb{K}^m) \times \text{Co}(\mathbb{K}^m) \mid (\Theta, \Theta_1) \text{ satisfy (A2)–(A5) } \}. \tag{23}$$

On this space we have so far used the usual Hausdorff metric H inherited from the space $\mathcal{K}(\mathbb{K}^{2m})$. We introduce a pseudo-metric as follows. For $(\Theta, \Theta_1) \in \widetilde{\mathcal{N}}$ define the set of admissible parameter variations as

$$\mathcal{U}(\Theta, \Theta_1) := \{u : \mathbb{R} \to \Theta \mid u \text{ is absolutely continuous and } \dot{u}(t) \in \Theta_1 \text{ a.e. } \}. \tag{24}$$

Note that by the Arzela-Ascoli theorem $\mathcal{U}(\Theta, \Theta_1)$ is a compact subset of $L^\infty(\mathbb{R}, \mathbb{K}^m)$ with respect to the strong topology. We denote the Hausdorff metric in $L^\infty(\mathbb{R}, \mathbb{K}^m)$ by H_∞ and define a pseudo-metric (also called H_∞) on $\widetilde{\mathcal{N}}$ by pulling this metric back, i.e. by setting

$$H_\infty((\Theta, \Theta_1), (\Theta', \Theta_1')) := H_\infty(\mathcal{U}(\Theta, \Theta_1), \mathcal{U}(\Theta', \Theta_1')) \,. \tag{25}$$

We collect some properties of this pseudo-metric in the following remark.

Remark 10. *(i) On $\widetilde{\mathcal{N}}$ it always holds that*

$$H(\Theta, \Theta') \leq H_\infty((\Theta, \Theta_1), (\Theta', \Theta_1')) \,,$$

because by (A4) the constant functions $u_\theta \equiv \theta$ are contained in $\mathcal{U}(\Theta, \Theta_1)$ and clearly it holds that $\operatorname{dist}_\infty(u_\theta, \mathcal{U}(\Theta', \Theta_1')) \geq \operatorname{dist}(\theta, \Theta')$.
(ii) As we see from Example 9, it may happen that $\mathcal{U}(\Theta, \Theta_1) = \mathcal{U}(\Theta', \Theta_1')$ for $(\Theta, \Theta_1) \neq (\Theta', \Theta_1') \in \widetilde{\mathcal{N}}$.

By the previous remark, to obtain a metric we just have to introduce suitable equivalence classes with respect to Θ_1. Given Θ, Θ_1 we define the set $\Theta_{1,\min}(\Theta, \Theta_1)$ as the smallest convex set with the property

$$\mathcal{U}(\Theta, \Theta_{1,\min}) = \mathcal{U}(\Theta, \Theta_1) \,.$$

Note that $\Theta_{1,\min}(\Theta, \Theta_1)$ is simply defined by

$$\Theta_{1,\min}(\Theta, \Theta_1) = \bigcap \Theta_1' \,,$$

where the intersection is taken over all convex sets Θ_1' with the property that

$$\mathcal{U}(\Theta, \Theta_1) = \mathcal{U}(\Theta, \Theta_1') \,.$$

We then define the space

$$\mathcal{N} := \{(\Theta, \Theta_1) \in \mathcal{K}(\mathbb{K}^m) \times \operatorname{Co}(\mathbb{K}^m) \mid \quad (\Theta, \Theta_1) \text{ satisfy (A2)–(A5)} \text{ and } \Theta_1 = \Theta_{1,\min}(\Theta, \Theta_1)\} \,.$$

On $\mathcal{N}$ it holds that H_∞ is a metric. However, the convergence $(\Theta_k, \Theta_{1k}) \to (\Theta', \Theta_1')$ with respect to H_∞ is neither necessary nor sufficient for the convergence $(\Theta_k, \Theta_{1k}) \to (\Theta, \Theta_1)$ with respect to the Hausdorff metric.

In the following we restrict the spaces $\mathcal{L}$ and $\mathcal{L}(\infty)$ to subsets of $(0, \infty) \times \mathcal{N} \times \operatorname{Lip}(\mathbb{K}^m, \mathbb{K}^{n\times n})$, resp. $\mathcal{N} \times \operatorname{Lip}(\mathbb{K}^m, \mathbb{K}^{n\times n})$ and use the metric

$$d(\Sigma, \Sigma') := |h - h'| + H_\infty((\Theta, \Theta_1), (\Theta', \Theta_1')) + \|A - A'\|_{\mathrm{Lip}} \,, \tag{26}$$

on $\mathcal{L}$ and the same expression omitting the $|h - h'|$ on $\mathcal{L}(\infty)$.

We note the following two subsets of $\mathcal{N}$ on which the topologies induced by the standard Hausdorff metric and H_∞ coincide locally. This is only a simple example and certainly more elaborate cases where this property holds may be constructed.

Lemma 11. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\Theta, \Theta' \in \operatorname{Co}(\mathbb{K}^{n\times n})$ and let Θ_1 be the unit ball of a strictly convex norm w on $\mathbb{K}^m$. Denote by C a constant such that $C^{-1}w(\cdot) \leq \|\cdot\| \leq Cw(\cdot)$. Then*

(i) $H_\infty((\Theta, \Theta_1), (\Theta', \Theta_1)) \leq CH(\Theta, \Theta')$,

(ii) For $0 < \alpha, \beta \in \mathbb{R}$ we have

$$H_\infty((\Theta, \alpha\Theta_1), (\Theta, \beta\Theta_1)) \leq (\min\{\alpha, \beta\})^{-1} \operatorname{diam}(\Theta) |\alpha - \beta| \,.$$

Proof. (i) We denote by π_Θ the projection from $\mathbb{K}^{n\times n}$ onto Θ with respect to the norm w. That is, $\pi_\Theta(\eta)$ is the unique $\theta \in \Theta$ with $w(\eta-\theta) = \mathrm{dist_w}(\eta,\Theta)$. It is well known that by convexity of Θ the map π_Θ is globally Lipschitz continuous with respect to the norm w with constant $L = 1$. Thus for any $u \in \mathcal{U}(\Theta', \Theta_1)$ we have that $\pi_\Theta \circ u \in \mathcal{U}(\Theta,\Theta_1)$, where

$$\max\{w(u(t) - \pi_\Theta \circ u(t)) \mid t \in \mathbb{R}\} \le \max\{\mathrm{dist_w}(\eta,\Theta) \mid \eta \in \Theta'\} \le \mathrm{CH}(\Theta,\Theta')\,.$$

The assertion follows by symmetry.

(ii) Fix $\theta_0 \in \Theta$ and assume $\alpha < \beta$. For $u \in \mathcal{U}(\Theta, \beta\Theta_1)$ we define a function $w \in \mathcal{U}(\Theta,\alpha\Theta_1)$ by

$$w(t) := \theta_0 + \frac{\alpha}{\beta}\left(u(t) - \theta_0\right)\,.$$

It is easy to see that $\|u-w\|_\infty \le \mathrm{diam}\,(\Theta)(\min\{\alpha,\beta\})^{-1}|\alpha-\beta|$. As $\mathcal{U}(\Theta,\alpha\Theta_1) \subset \mathcal{U}(\Theta,\beta\Theta_1)$ this shows the assertion.

□

A consequence of the previous lemma is that if we have compact sets $P \subset \mathrm{Co}\,(\mathbb{K}^m)$ and $[a,b] \subset (0,\infty)$ and a strictly convex norm w with unit ball Θ_1, then the metrics H and H_∞ are equivalent on the set $\{(\Theta,\alpha\Theta_1) \mid \Theta \in P, \alpha \in [a,b]\}$. Consequently, the continuity results of the next section with respect to the metric d also apply to the more classical metric. We repeat that one might think of further special cases for which this is true, but we choose not to dwell on this.

6 Continuity of the Exponential Growth Rate

One of the basic questions in stability theory concerns the continuity of the exponential growth rate as a function of the data. We now study this question with respect to the metric d on $\mathcal{L}$ and on $\mathcal{L}(\infty)$, which we defined in (26). In [19] it has been shown that the Gelfand formula allows for an easy criterion of continuity. Our first aim is to prove that this approach yields a continuity statement in our setup. Before doing so, however, we introduce a small technical trick, that will allow us to assume that the dwell time h is fixed.

Lemma 12. *For $\Sigma = (h,\Theta,\Theta_1,A) \in \mathcal{L}$ and $\tilde h > 0$ define $\tilde\Sigma := (\tilde h, \Theta, h/\tilde h\Theta_1, h/\tilde h A)$. Then for all $t \ge 0$ by setting $\tau = \tilde h t/h$ we have*

$$\mathcal{S}_\tau(\tilde\Sigma) = \mathcal{S}_t(\Sigma)\,, \tag{27}$$

and it holds that

$$\rho(\tilde\Sigma) = \frac{h}{\tilde h}\,\rho(\Sigma)\,. \tag{28}$$

Proof. Introduce the time-transformation $t(\tau) = (h/\tilde h)\tau$. Then given any admissible parameter variation $(t \mapsto u(t)) \in \mathcal{U}(\Sigma)$ we see that the discontinuities of $\tau \mapsto u(t(\tau))$ have a minimal distance of $\tilde h$ and

$$\frac{du(t(\tau))}{d\tau} \in \frac{h}{\tilde h}\,\Theta_1\,,\quad \text{a.e.}$$

Furthermore for the evolution operator $\Phi_u(t,0)$ we obtain that

$$\frac{d\Phi_{u(t(\cdot))}(t(\tau),0)}{d\tau} = \frac{h}{\tilde h} A(u(t(\tau)))\Phi_{u(t(\cdot))}(t(\tau),0)\,.$$

This means that every $S \in \mathcal{S}_t(\Sigma)$ is transformed into an $\tilde S \in \mathcal{S}_{\tau(t)}(\tilde\Sigma) = \mathcal{S}_{\tilde h t/h}(\tilde\Sigma)$, and vice versa. This implies in particular the equality $t\rho_t(\Sigma) = (\tilde h t/h)\rho_{(\tilde h t/h)}(\tilde\Sigma)$ and thus

$$\rho(\tilde\Sigma) = \frac{h}{\tilde h}\,\rho(\Sigma)\,,$$

as desired.

□

Lemma 13. *The maps* $\mathbb{R}_+ \times \mathcal{L} \to \mathcal{K}(\mathbb{K}^{n\times n})$, $\mathbb{R}_+ \times \mathcal{L}(\infty) \to \mathcal{K}(\mathbb{K}^{n\times n})$ *defined by*

$$(t, h, \Theta, \Theta_1, A) \mapsto \mathcal{S}_t(h, \Theta, \Theta_1, A)$$

are locally Lipschitz continuous.

Proof. Fix $t_1 > 0$ and compact sets $[h_0, h_1] \subset (0, \infty)$, $P \subset \mathcal{N}$ and $Q \subset \mathrm{Lip}\,(\mathbb{K}^m, \mathbb{K}^{n\times n})$. It is sufficient to prove the claim for compact sets of the type $[0, t_1] \times [h_0, h_1] \times P \times Q$. The proof is based on the bound

$$\begin{aligned} H(\mathcal{S}_t(h,\Theta,\Theta_1,A), \mathcal{S}_\tau(h',\Theta',\Theta_1',A')) &\leq \\ H(\mathcal{S}_t(h,\Theta,\Theta_1,A), \mathcal{S}_\tau(h,\Theta,\Theta_1,A)) &+ \\ H(\mathcal{S}_\tau(h,\Theta,\Theta_1,A), \mathcal{S}_\tau(h,\Theta',\Theta_1',A')) &+ \quad H(\mathcal{S}_\tau(h,\Theta',\Theta_1',A'), \mathcal{S}_\tau(h',\Theta',\Theta_1',A')) \,. \end{aligned} \tag{29}$$

If we can for each term on the right find a Lipschitz constant in the variable that is uniform with respect to remaining data over our compact set, then we are done.

In order to obtain a simple upper bound on the evolution operators consider any measurable function

$$B : [0, t_1] \to \bigcup_{(\Theta,\Theta_1)\in P, A\in Q} A(\Theta)\,.$$

By compactness the set on the right is bounded. Thus there is a constant C such that for any measurable function B the evolution operator of $\dot{x}(t) = B(t)x(t)$ is bounded by

$$\|\Phi_B(t,0)\| \leq e^{Ct}\,, \quad \text{for all } t \in [0, t_1]\,, \tag{30}$$

and so that we also have

$$\|\Phi_B(t,0) - I\| \leq (e^{Ct} - 1)\,, \quad \text{for all } t \in [0, t_1]\,.$$

With this we obtain an easy uniform bound for the first term in (29), because for any Σ in the compact set we have

$$H(\mathcal{S}_t(\Sigma), \mathcal{S}_\tau(\Sigma)) \leq (e^{C|\tau - t|} - 1)\,.$$

We now derive the Lipschitz continuity result for the second term in (29). i.e. with the assumption that h is fixed. Define $L_Q := \max\{L(A) \mid A \in Q\}$ and fix $t \in [0, t_1]$ and $\Sigma = (h, \Theta, \Theta_1, A), \Sigma' = (h, \Theta', \Theta_1', A') \in [h_0, h_1] \times P \times Q$. For $u \in \mathcal{U}(\Theta, \Theta_1)$ we may choose $v \in \mathcal{U}(\Theta', \Theta_1')$ such that $\|u(s) - v(s)\| \leq H_\infty((\Theta, \Theta_1), (\Theta', \Theta_1'))$ for all $s \in [0, t]$, because we may choose v, such that its discontinuities coincide with those of u. Then using the variation of constants formula we obtain

$$\begin{aligned} & \|\Phi_{A\circ u}(t,0) - \Phi_{A'\circ v}(t,0)\| \\ = \; & \left\| \int_0^t \Phi_{A\circ u}(t,s)[A(u(s)) - A'(v(s))]\Phi_{A'\circ v}(s,0)ds \right\| \\ \leq \; & e^{Ct} \int_0^t \|A(u(s)) - A'(u(s))\| + \|A'(u(s)) - A'(v(s))\| ds \\ \leq \; & te^{Ct}(\|A - A'\|_\infty + L(A')H_\infty((\Theta,\Theta_1),(\Theta',\Theta_1'))) \\ \leq \; & t_1 e^{Ct_1} \max\{1, L_Q\}(\|A - A'\|_\infty + H_\infty((\Theta,\Theta_1),(\Theta',\Theta_1')))\,. \end{aligned} \tag{31}$$

With this we have found the desired uniform Lipschitz constant. To treat the third expression in (29), note that by Lemma 12 we have

$$\begin{gathered} H(\mathcal{S}_t(h,\Theta,\Theta_1,A), \mathcal{S}_t(h',\Theta,\Theta_1,A)) = \\ H(\mathcal{S}_t(h,\Theta,\Theta_1,A), \mathcal{S}_\tau(h,\Theta,h/h'\Theta_1,h/h'A))\,, \end{gathered} \tag{32}$$

where $\tau = h'/ht$.

Assume without loss of generality, that $h' < h$, then $\mathcal{U}(\Theta, \Theta_1) \subset \mathcal{U}(\Theta, h/h'\Theta_1)$. To obtain a converse estimate fix $\theta_0 \in \Theta$ and for $u \in \mathcal{U}(\Theta, h/h'\Theta_1)$ define

$$v(t) := \theta_0 + \frac{h'}{h}(u(t) - \theta_0)\,,$$

from which it is obvious that $v \in \mathcal{U}(\Theta, \Theta_1)$ and

$$\|u - v\|_\infty = \left\| \left(1 - \frac{h'}{h}\right)(\theta_0 - u(\cdot)) \right\|_\infty \le \frac{\operatorname{diam}\Theta}{h_0}|h - h'|\,.$$

This implies

$$H_\infty((\Theta, \Theta_1), (\Theta, h/h'\Theta_1)) \le \frac{\operatorname{diam}\Theta}{h_0}|h - h'|\,.$$

Furthermore, $\|A - h/h'A\|_{\mathrm{Lip}} \le \frac{1}{h_0}\max\{\|A\|_{\mathrm{Lip}} \mid A \in Q\}|h - h'|$.

By the compactness of $[h_0, h_1]$ and $P \times Q$ the previous considerations show that the set

$$\{(\Theta, h/h'\Theta_1, h/h'A) \mid h, h' \in [h_0, h_1], (\Theta, \Theta_1) \in P, A \in Q\}$$

is also compact. Thus from the first steps of this proof, we see that the Hausdorff distance on the right of (32) may be bounded by $C(|t - \tau| + H_\infty((\Theta, \Theta_1), (\Theta, h/h'\Theta_1)) + \|A - h/h'A\|_{\mathrm{Lip}})$, for a suitable constant C. As we have seen, all the distances in this expression are linearly bounded in $|h - h'|$ with constants only depending on our compact set. This completes the proof. □

Corollary 14. *The maps*

$$\rho : \mathcal{L} \mapsto \mathbb{R}\,, \quad \rho :\ \mathcal{L}(\infty) \mapsto \mathbb{R}\,,$$

are continuous.

Proof. This follows from an application of Corollary 8.1 in [19] using Lemma 13. □

We will now study cases in which the growth rate ρ is even a locally Lipschitz continuous function of the data. This result will be obtained for compact subsets $\mathcal{P}$ of $\mathcal{L}$ or $\mathcal{L}(\infty)$, with the property that each $\Sigma \in \mathcal{P}$ satisfies (A1)-(A6) and that furthermore for each $\Sigma \in \mathcal{P}$ the set $A(\Theta)$ is irreducible.

Remark 15. *(i) Note that the set of systems Σ for which $A(\Theta)$ is irreducible is open and dense in $\mathcal{L}$ and in $\mathcal{L}(\infty)$.*

(ii) If $A(\Theta)$ is reducible we can find a similarity transformation T such that for all $\theta \in \Theta$ the tranformed matrix $TA(\theta)T^{-1}$ is of the form

$$\begin{bmatrix} A_{11}(\theta) & A_{12}(\theta) & \dots & A_{1d}(\theta) \\ 0 & A_{22}(\theta) & \dots & A_{2d}(\theta) \\ & \ddots & \ddots & \vdots \\ 0 & & 0 & A_{dd}(\theta) \end{bmatrix}, \tag{33}$$

where the sets $A_{ii}(\Theta)$ are irreducible or $\{0\}$. It is an easy exercise to show, that in this case $\rho(A, \mathcal{U}) = \max_{i=1,\dots,d} \rho(A_i, \mathcal{U})$, where $A_i : \theta \to \mathbb{K}^{n_i \times n_i}$ is the map $\theta \mapsto A_{ii}(\theta)$.

Assuming irreducibility we define for the system $\Sigma = (h, \Theta, \Theta_1, A)$ the norms v_ω as in (16). We denote by $v_{\omega,\eta}$ the operator norm on $\mathbb{K}^{n \times n}$ induced on the linear maps from $(\mathbb{K}^n, v_\eta)$ to $(\mathbb{K}^n, v_\omega)$. Given a fixed norm $\|\cdot\|$ define the constants

$$\begin{aligned} c^+(\omega, \Sigma) \quad &: \quad = \max\{v_\omega(x) \mid \|x\| = 1\}\,, \\ c^-(\omega, \Sigma) \quad &: \quad = \min\{v_\omega(x) \mid \|x\| = 1\}\,. \end{aligned}$$

Note that with respect to the operator norms $v_{\omega,\eta}$ introduced above we have for arbitrary $B \in \mathbb{K}^{n \times n}$ that

$$\frac{c^-(\omega, \Sigma)}{c^+(\eta, \Sigma)} v_{\omega,\eta}(B) \le \|B\| \le \frac{c^+(\omega, \Sigma)}{c^-(\eta, \Sigma)} v_{\omega,\eta}(B)\,. \tag{34}$$

Lemma 16. *Let $\mathcal{P} \subset \mathcal{L}$ (or $\mathcal{P} \subset \mathcal{L}(\infty)$) be a compact set of systems, such that (A1)-(A6) hold and $A(\Theta)$ is irreducible for all $\Sigma \in \mathcal{P}$. Consider a sequence $\{\Sigma_k = (h_k, \Theta_k, \Theta_{1k}, A_k)\}_{k\in\mathbb{N}} \subset \mathcal{P}$ with $\Sigma_k \to \Sigma \in \mathcal{P}$. Let $\omega_k, \zeta_k \in \Pi(\Theta_k, h_k)$ be such that the sequences $\omega_k \to \omega \in \Pi(\Theta, h), \zeta_k \to \zeta \in \Pi(\Theta, h)$ are convergent. Then for all $T > 0$ and all $\varepsilon > 0$ it holds that*

$$\mathcal{R}_{\leq T}(\omega, \zeta, \Sigma) \subset \liminf_{k\to\infty} \mathcal{R}_{\leq T+\varepsilon}(\omega_k, \zeta_k, \Sigma_k)\,, \tag{35}$$

where as usual $\liminf$ *of a sequence of sets C_k denotes the set of all existing limits of sequences $\{c_k \in C_k\}_{k\in\mathbb{N}}$, see [3, Chapter 1].*

Proof. Consider the limit system (h, Θ, Θ_1, A) and let $u \in \mathcal{U}(\omega, \Sigma)$ generate the transition matrix $S \in \mathcal{R}_t(\omega, \zeta, \Sigma)$, $0 \leq t \leq T$. Fix $\varepsilon > 0$ and let k be large enough. By the convergence of $\Sigma_k \to \Sigma$ we may by Lemma 13 construct $u_k \in \mathcal{U}(\Sigma_k)$ defined on the interval $[0, t+\varepsilon_k]$, such that $\varepsilon_k \to 0$ as $k \to \infty$ and $\Phi_{u_k}(t+\varepsilon_k, 0) \to S$. As $u \in \mathcal{U}(\omega, \Sigma)$, it is an easy exercise to see that we may assume there are $\tilde{\omega}_k, \tilde{\zeta}_k$ such that $\Phi_{u_k}(t + \exists_k, 0) \in \mathcal{R}(\tilde{\omega}_k, \tilde{\zeta}_k, \Sigma_k)$, where $\tilde{\omega}_k \to \omega, \tilde{\zeta}_k \to \zeta$.

Now by construction $\|\tilde{\omega}_k - \omega_k\| \to 0$. By assumption (A4) there are thus times $\sigma_k \to 0$ and $R_k \in \mathcal{R}_{\sigma_k}(\omega_k, \tilde{\omega}_k, \Sigma_k)$ generated by parameter variations that are continuous on $[0, \sigma_k]$. By the same argument there are times $\tau_k \to 0$ and $T_k \in \mathcal{R}_{\tau_k}(\tilde{\zeta}_k, \zeta_k, \Sigma_k)$. In all we have $T_k S_k R_k \in \mathcal{R}_{t+\varepsilon_k+\sigma_k+\tau_k}(\omega_k, \zeta_k, \Sigma_k)$. By construction $R_k, T_k \to I$ and thus $T_k S_k R_k \to S$ as desired. □

The following property is essential for our results. The point is that a uniform bound for the constants occurring in (34) can be obtained over $\mathcal{P}$.

Theorem 17. *Let $\mathcal{P} \subset \mathcal{L}$ (or $\mathcal{P} \subset \mathcal{L}(\infty)$) be a compact set of systems, such that (A1)-(A6) hold and $A(\Theta)$ is irreducible for all $\Sigma \in \mathcal{P}$. Then there exist constants $C_-, C_+ > 0$ such that*

$$C_- \leq \frac{c^+(\omega, \Sigma)}{c^-(\zeta, \Sigma)} \leq C_+\,, \quad \textit{for all } \Sigma = (h, \Theta, \Theta_1, A) \in \mathcal{P},\ \omega, \zeta \in \Pi(\Theta, h)\,.$$

Proof. We begin by showing the existence of C_+. Assume that there exist sequences of the form $\{\Sigma_k = (h_k, \Theta_k, \Theta_{1k}, A_k)\}_{k\in\mathbb{N}} \subset \mathcal{P}$, $\{\omega_k, \zeta_k \in \Pi(\Theta_k, h_k)\}_{k\in\mathbb{N}}$, such that

$$\frac{c^+(\omega_k, \Sigma_k)}{c^-(\zeta_k, \Sigma_k)} \to \infty\,.$$

Without loss of generality we may assume that $\Sigma_k \to \Sigma = (h, \Theta, \Theta_1, A) \in \mathcal{P}$, $\omega_k \to \omega \in \Pi(\Theta, h)$ and $\zeta_k \to \zeta \in \Pi(\Theta, h)$. For all $k \in \mathbb{N}$ choose $S_k \in \mathcal{S}_\infty(\omega_k, \Sigma_k)$ such that $\|S_k\| = c^+(\omega_k, \Sigma_k)$ and define $\tilde{S}_k := S_k / \|S_k\|$. Then we may assume that $\tilde{S}_k \to \tilde{S}$, $\left\|\tilde{S}\right\| = 1$. Let $\varepsilon > 0$ and $T > 0$ be the constants for $\mathcal{R}(\zeta, \omega, \Sigma)$ guaranteed by Corollary 4 (with respect to the semigroup $\mathcal{R}(\omega, \omega, \Sigma)$). Fix an arbitrary $x_0 \in \mathbb{K}^n$, $\|x_0\| = 1$. Then by convergence and Lemma 16 for all k large enough there exists an $R_k \in \mathcal{R}_{t_k}(\zeta_k, \omega_k, \Sigma_k)$ with $t_k \leq T + \varepsilon$ such that

$$\left\|\tilde{S} R_k x_0\right\| \geq \frac{\varepsilon}{2}\,.$$

Define $T_k := \exp(-\rho(\Sigma_k) t_k) S_k R_k \in \mathcal{S}_\infty(\zeta_k, \Sigma_k)$. Then we obtain

$$v_{\zeta_k}(x_0) \geq \|T_k x_0\| = \frac{\|S_k\|}{\exp(\rho(\Sigma_k) t_k)} \left\|\tilde{S}_k R_k x_0\right\| \geq$$
$$\frac{\|S_k\|}{\exp(\rho(\Sigma_k) t_k)} \left(\left\|\tilde{S} R_k x_0\right\| - \left\|\tilde{S} - \tilde{S}_k\right\| \|R_k x_0\|\right)\,.$$

Thus for all k large enough we have

$$\begin{aligned}\frac{c^+(\omega_k, \Sigma_k)}{v_{\zeta_k}(x_0)} &\leq \exp(\rho(\Sigma_k) t_k) \left(\left\|\tilde{S} R_k x_0\right\| - \left\|\tilde{S} - \tilde{S}_k\right\| \|R_k x_0\|\right)^{-1} \\ &\leq \exp(\rho(\Sigma_k) t_k) \frac{4}{\varepsilon}.\end{aligned}$$

Where we have used that the sequence $\{R_k\}_{k\in\mathbb{N}}$ is bounded, so that the last term on the right converges to zero by construction. This completes the proof, because $t_k \leq T+\varepsilon$ and $\rho(\Sigma_k)$ is bounded by compactness of $\mathcal{P}$.

The proof for the existence of C_- follows the same lines. □

The previous statement allows for a conclusion on the Lipschitz continuity of $\rho(\Sigma)$ as in the case of linear inclusions studied in [16].

Theorem 18. *Let $\mathcal{P}\subset\mathcal{L}$ (or $\mathcal{P}\subset\mathcal{L}(\infty)$) be a compact set of systems, such that (A1)-(A6) hold and $A(\Theta)$ is irreducible for all $\Sigma\in\mathcal{P}$. Then there exists a constant C such that*

$$|\rho(\Sigma_1)-\rho(\Sigma_2)| \leq Cd(\Sigma_1,\Sigma_2)\ ,\ \forall(\Sigma_1,\Sigma_2)\in\mathcal{P}\,.$$

In particular, the map $\Sigma\to\rho(\Sigma)$ is locally Lipschitz continuous on the spaces of irreducible systems given by $\{\Sigma\in\mathcal{L}\mid A(\Theta)$ is irreducible$\}$ and $\{\Sigma\in\mathcal{L}(\infty)\mid A(\Theta)$ is irreducible $\}$.

Proof. Let $L_{\mathcal{P}} := \max\{L(A)\mid (h,\Theta,\Theta_1,A)\in\mathcal{P}\}$. Let $\Sigma_1,\Sigma_2\in\mathcal{P}$ and pick $u\in\mathcal{U}(\Sigma_1)$. Assume for the moment that the dwell times h_1,h_2 of Σ_1 and Σ_2 coincide and let $h := h_1 = h_2$. By the argument in (31) there exists a $w\in\mathcal{U}_2$ such that

$$\|A_1(u(\cdot)) - A_2(w(\cdot))\|_\infty \leq \max\{1,L_{\mathcal{P}}\}d(\Sigma_1,\Sigma_2)\ .$$

Denote the evolution operator corresponding to $A_1(u(\cdot))$ by $\Phi(t,s)$ and the one corresponding to $A_2(w(\cdot))$ by $\Psi(t,s)$. For $t\geq 0$ denote $\zeta(t) := (w(t^-),\tau^-(w,t))$ and let $v_{\zeta(t)}$ be the parameterized Lyapunov function with respect to Σ_2 defined in (18). Furthermore denote by $v_{\zeta(t),\zeta(s)}$ the operator norms induced by the parameterized Lyapunov functions $v_{\zeta(t)},v_{\zeta(s)}$. Note that by a standard property of induced norms it holds for $0\leq r\leq s\leq t$ that $v_{\zeta(r),\zeta(t)}(TR)\leq v_{\zeta(s),\zeta(t)}(T)v_{\zeta(r),\zeta(s)}(R)$. Then we have for $t=k\in\mathbb{N}$ that

$$\begin{aligned} v_{\zeta(0),\zeta(k)}(\Phi(k,0)) &\leq v_{\zeta(0),\zeta(k)}(\Psi(k,k-1)\Phi(k-1,0)) \\ &\quad + v_{\zeta(0),\zeta(k)}\left((\Phi(k,k-1)-\Psi(k,k-1))\Phi(k-1,0)\right) \\ &\leq \left[e^{\rho(\Sigma_2)} + C\,\|\Phi(k,k-1)-\Psi(k,k-1)\|\right] v_{\zeta(0),\zeta(k-1)}(\Phi(k-1,0)) \end{aligned}$$

where C is a constant independent of $\Sigma\in\mathcal{P}$. Here we have used (18) to obtain the bound on the first term. The constant C exists by (34) and Theorem 17. Furthermore, using (31) we obtain a bound for the difference $\|\Phi(k,k-1)-\Psi(k,k-1)\|$ in terms of $d(\Sigma_1,\Sigma_2)$ so that for a suitable constant C_2 we obtain the inequality

$$v_{\zeta(0),\zeta(k)}(\Phi(k,0)) \leq \left[e^{\rho(\Sigma_2)} + CC_2 d(\Sigma_1,\Sigma_2)\right] v_{\zeta(0),\zeta(k-1)}(\Phi(k-1,0))\,,$$

which implies by induction and another application of Theorem 17 that for all $k\in\mathbb{N}$ we have

$$\|\Phi(k,0)\| \leq C_3 v_{\zeta(0),\zeta(k)}(\Phi(k,0)) \leq C_3\left[e^{\rho(\Sigma_2)} + CC_2 d(\Sigma_1,\Sigma_2)\right]^k\ .$$

Now the operators $\Phi(t,0), t\neq k$ are only small perturbations of some $\Phi(k,0)$; the constants C,C_2,C_3 were chosen independently of $\Sigma_1,\Sigma_2\in\mathcal{P}$ and $u\in\mathcal{U}_1$ was arbitrary. This shows that for all $t\geq 0$ and suitable constants C_4,C_5 we have

$$\rho_t(\Sigma_1) \leq \frac{\log C_4}{t} + \rho(\Sigma_2) + C_5 d(\Sigma_1,\Sigma_2)\ .$$

Hence

$$\rho(\Sigma_1) \leq \rho(\Sigma_2) + C_5 d(\Sigma_1,\Sigma_2)\,.$$

By symmetry we obtain

$$|\rho(\Sigma_1)-\rho(\Sigma_2)| \leq C_5 d(\Sigma_1\Sigma_2)\,.$$

This completes the proof for the case of common dwell time h.

To consider arbitrary dwell time note that for $\Sigma_1 = (h, \Theta, \Theta_1, A)$, $\Sigma_2 = (h', \Theta', \Theta_1', A')$ we have by Lemma 12 that

$$|\rho(\Sigma_1) - \rho(\Sigma_2)| = \left|\rho(\Sigma_1) - \frac{h}{h'}\rho(h, \Theta', h'/h\Theta_1', h'/hA'))\right|$$

$$\leq |\rho(\Sigma_1) - \rho(h, \Theta', h'/h\Theta_1', h'/hA'))| + \frac{\max\{\rho(\Sigma) \mid \Sigma \in \mathcal{P}\}}{\min\{h \mid (h, \Theta, \Theta_1, A) \in \mathcal{P}\}}|h - h'|\,.$$

The desired Lipschitz constant now follows by applying the first part of the proof to the first term on the right hand side. □

Finally, let us point out that from Theorem 17 we can conclude locally uniform convergence of ρ_t with a linear convergence rate on the set of irreducible systems.

Corollary 19. *Let $\mathcal{P} \subset \mathcal{L}$ (or $\mathcal{P} \subset \mathcal{L}(\infty)$) be a compact set of systems, such that (A1)-(A6) hold and $A(\Theta)$ is irreducible for all $\Sigma \in \mathcal{P}$. Then there exists a constant $C > 0$ such that for all $t \geq 1$ and all $\Sigma \in \mathcal{P}$ it holds that*

$$|\rho_t(\Sigma) - \rho(\Sigma)| < Ct^{-1}\,.$$

Proof. Let $\Sigma \in \mathcal{P}$ and $t \geq 0$ be arbitrary. Now choose $\omega, \zeta \in \Pi(\Theta, h)$ and $S_t \in \mathcal{R}_t(\omega, \zeta, \Sigma)$ such that $\|S_t\|^{1/t} = \rho_t(\Sigma)$. Then we have by Proposition 5 that $v_{\omega,\zeta}(S_t) \leq e^{\rho(\Sigma)t}$. Let $C > 0$ be the constant for $\mathcal{P}$ given by Theorem 17 and we obtain for $t > 0$ that

$$0 \leq \frac{1}{t}\log \sup_{S \in \mathcal{S}_t(\Sigma)} \|S\| - \rho(\Sigma) \leq \frac{1}{t}\log C v_{\omega,\zeta}(S_t) - \rho(\Sigma) \leq \frac{1}{t}\log C\,. \tag{36}$$

This proves the assertion. □

7 Conclusions

In this paper we have shown some continuity properties of a fairly general class of families of time-varying systems, that encompasses linear parameter-varying and linear switching systems. The main results were that of global continuity of the exponential growth rate obtained in Corollary 14 on the one hand and Theorem 17 showing local Lipschitz continuity on the space of irreducible systems; a set that is open and dense in the space of all systems.

A remaining open question concerns continuity properties of the exponential growth rate on the set $\mathcal{L} \cup \mathcal{L}(\infty)$, which has been avoided in this paper. In fact, this question is part of a larger question dealing with the behavior of the growth rate, as $h \to 0$, or $h \to \infty$ and also as $\Theta_1 \to \{0\}$ or $\Theta_1 \to \mathbb{K}^m$. This is a classical question in the area of linear time-varying systems. For the class of systems presented here, this limiting question is the subject of ongoing investigations. Partial results have been obtained in [17].

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