State Feedback Stabilization with Guaranteed Transient Bounds

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Abstract

We analyze under which conditions linear time-invariant state space systems can be stabilized by static linear state feedback such that prescribed transient bounds hold pointwise in time for the trajectories of the closed loop system. We introduce the concepts of (M,β)-stability and quadratic (M,β)-stability which guarantee a transient bound M along with a decay rate $\beta < 0$. Necessary and sufficient conditions for quadratic (M,β)-stabilizability are derived which can be expressed in terms of linear matrix inequalities. A full characterization of general (M,β)-stabilizability is still missing. However, we obtain necessary and sufficient criteria under which a given system can be transformed by linear state feedback into a closed loop system generating a strict contraction semigroup with respect to the spectral norm.

1 Introduction

Trajectories of asymptotically stable linear systems may move far away from the origin before ultimately approaching it. This transient behavior has recently been studied by several authors [2, 3, 6, 7, 8], in particular its relation to the behavior of the spectrum of the system matrix under perturbations. There are several motivations for the analysis of the transient behavior of linear systems. One of these concerns the relation between a nonlinear system and its linearization at an asymptotically stable fixed point. Although by Lyapunov theory asymptotic stability of the nonlinear system entails the asymptotic stability of the linearization at the fixed point, the domain of attraction may be very thin so that the fixed point becomes "practically unstable", see [2].

This reasoning suggests that if linear design is performed as a local design for a nonlinear system, then it might be desirable to construct feedbacks that generate small transients for the linearization. In this paper we investigate the achievable transitory behavior using static linear state feedback. The use of time-varying linear feedback to reduce transients has been studied in [3, 7]. The relation of this problem to the pole placement problem has been investigated by Izmailov, see [5] where some results are announced.

We proceed as follows. In the ensuing Section 2 we define the notion of (strict) (M,β)-stability and derive a sufficient condition in terms of quadratic Lyapunov functions. An example shows that this sufficient condition may be quite conservative. This motivates the

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introduction of the notion of (strict) quadratic (M,β)-stability and (M,β)-stabilizability. In Section 3 the main results of this note are presented. We derive necessary and sufficient conditions for strict quadratic (M,β)-stabilizability of linear systems. In Section 4 we briefly discuss how quadratic (M,β)-stabilizability may be characterized in terms of linear matrix inequalities. Whether this is possible for (M,β)-stabilizability in general is questionable.

2 Problem formulation

A linear system of the form $\dot{x} = Ax$ is called exponentially stable with a decay rate of $\beta < 0$, if there exists $M \geq 1$ such that $\|\exp(At)\| \leq M \exp(\beta t)$ holds for all $t \geq 0$. When both M and β are prescribed this imposes not only conditions on the long-term behavior, but also on the transient behavior of the system.

Definition 2.1. *Given the constants $M \geq 1$, $\beta < 0$, and a norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, a matrix $A \in \mathbb{C}^{n\times n}$ is called* (M,β)-stable *(resp.* strictly (M,β)-stable*) if*

$$\|e^{At}\| \leq Me^{\beta t}, \quad t \geq 0, \qquad (\textit{resp. } \|e^{At}\| < Me^{\beta t}, \quad t > 0). \tag{2.1}$$

A matrix pair $(A,B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ is called (strictly) (M,β)-stabilizable *if there exists a feedback matrix $F \in \mathbb{C}^{m\times n}$ such that the closed loop matrix $A-BF$ is (strictly) (M,β)-stable.*

Clearly, (M,β)-stability implies exponential stability. The set of (M,β)-stable matrices is closed and a subset of $\{A \in \mathbb{C}^{n\times n}; \max_{\lambda\in\sigma(A)} \operatorname{Re}\lambda \leq \beta\}$. On the other hand, the set of strictly (M,β)-stable matrices is not open. It can be proved that the interior of the set of (M,β)-stable matrices is given by the set of strictly (M,β)-stable matrices with the additional condition that the *spectral abscissa* $\alpha(A) := \max_{\lambda\in\sigma(A)} \operatorname{Re}\lambda$ is less than β.

In this article we want to characterize those systems that are strictly (M,β)-stable or that may be stabilized by state feedback such that the closed loop system is strictly (M,β)-stable. However, an easily checkable criterion that characterizes strict (M,β)-stability is currently unknown. But there is a well-known and easily checkable sufficient condition in terms of quadratic Lyapunov functions, which is the basis of the results in this paper. Let us first introduce some relevant notations.

Throughout the paper $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product on $\mathbb{C}^n$, $\langle x,y\rangle = x^*y$, where * denotes conjugate transpose. The associated vector norm on $\mathbb{C}^n$ and the corresponding operator norm on $\mathbb{C}^{n\times n}$ are both denoted by $\|\cdot\|$. The notation $P > 0$ will mean that $P = P^*$ is positive definite with respect to the standard inner product.

Given the $n \times n$ matrix $P > 0$, the *inner product weighted with* P is defined by $\langle x,y\rangle_P := x^*Py$, with the associated $P-$norm $\|x\|_P := \langle x,x\rangle_P^{1/2}$. A matrix is then Hermitian with respect to the inner product $\langle\cdot,\cdot\rangle_P$ (or P-Hermitian for short) if the P-Hermitian adjoint equals A, i.e. $P^{-1}A^*P = A$. The P-Hermitian part of $A \in \mathbb{C}^{n\times n}$ is defined by $\frac{1}{2}(P^{-1}A^*P+A)$, and we see that the P-Hermitian part of A is negative definite with respect to $\langle\cdot,\cdot\rangle_P$ if and only if the familiar Lyapunov inequality $\mathcal{H}_P(A) := \frac{1}{2}(A^*P + PA) < 0$ holds.

Standard Lyapunov theory tells us that for any asymptotically stable matrix A there exists a matrix $P > 0$ such that $\mathcal{H}_P(A) < 0$. For this choice of P the map $x \mapsto V(x) = \langle x, x \rangle_P$ is a *quadratic Lyapunov function* for A. The derivative of V along the solutions of $\dot{x} = Ax$ is given by

$$\dot{V}(x) = \frac{d}{dt}\langle x(t), x(t) \rangle_P = 2x^*(t)\mathcal{H}_P(A)x(t) < 0, \; x(0) \neq 0.$$

The matrix A is said to generate a P-contraction semigroup if for all $t \geq 0$ we have $\|e^{At}\|_P \leq 1$ and we speak of a strong P-contraction semigroup if there exists $\beta < 0$ such that for all $t \geq 0$ we have $\|e^{At}\|_P \leq e^{\beta t}$. Thus $\dot{x} = Ax$ is asymptotically stable if and only if A generates a strong P-contraction semigroup for some $P > 0$.

Remark 2.1. The matrix A generates a (resp. strong) P-contraction semigroup if the closed unit ball $\bar{B}_P = \{x \in \mathbb{C}^n; \; \|x\|_P \leq 1\}$ is forward invariant under the flow of $\dot{x} = Ax$ (resp. e^{At} maps $\bar{B}_P$ into the open unit ball B_P for $t > 0$).

We now present a sufficient condition for (M, β)-stability. Recall that for regular matrices P the *condition number* is defined by $\kappa(P) := \|P\| \, \|P^{-1}\|$. This quantity measures the deformation introduced by changing the Euclidean norm into a P-norm.

Lemma 2.1. *Let $A \in \mathbb{C}^{n \times n}$ and $M \geq 1, \beta < 0$. If there exists a matrix $P > 0$ which satisfies $\kappa(P) \leq M^2$ and*

$$\mathcal{H}_P(A) \leq \beta P < 0, \qquad (\textit{resp. } \mathcal{H}_P(A) < \beta P), \tag{2.2}$$

then A is (M, β)-stable (resp. strictly (M, β)-stable).

Proof. Let $P > 0$ such that $\kappa(P) \leq M^2$ and (2.2) is satisfied. Then for all $x \in \mathbb{C}^n$ and $t \geq 0$

$$\frac{d}{dt}\|e^{At}x\|_P^2 = (e^{At}x)^* \, (PA + A^*P) \, (e^{At}x) \leq 2\beta\|e^{At}x\|_P^2. \tag{2.3}$$

By Gronwall's Lemma we obtain

$$\|e^{At}x\|_P^2 \leq e^{2\beta t}\|x\|_P^2, \quad t \geq 0. \tag{2.4}$$

The Euclidean norm and the P-norm may be compared by means of the following inequalities

$$\lambda_{\min}(P)\langle x, x \rangle \leq \langle x, x \rangle_P \leq \lambda_{\max}(P)\langle x, x \rangle, \; x \in \mathbb{C}^n \tag{2.5}$$

where $\lambda_{\min}(P)$, $\lambda_{\max}(P)$ denote the minimal and maximal eigenvalue of P, respectively. Now, for a positive definite matrix P the condition number is $\kappa(P) = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$. Equation (2.4) leads to

$$\|e^{At}\| \leq \sqrt{\kappa(P)}e^{\beta t} \quad \text{for } t \geq 0. \tag{2.6}$$

This proves that A is (M, β)-stable. If $\mathcal{H}_P(A) < \beta P$ the strict versions of inequalities (2.3) and (2.4) are valid for $t > 0$ and $x \neq 0$ so that in this case A is strictly (M, β)-stable. $\square$

Lemma 2.1 yields only a sufficient condition for (M,β)-stability. A matrix A may be (M,β)-stable although all $P > 0$ with $\mathcal{H}_P(A) \leq \beta P$ have a condition number greater than M^2. This is illustrated in the following example where we make use of the fact that $\mathcal{H}_P(A - \beta I) = \mathcal{H}_P(A) - \beta P$, i.e. a change of the bound for the decay rate, β, is equivalent to a shift of A by a multiple of the identity.

Example 2.1. Consider the matrix $A = \left(\begin{smallmatrix} -1 & 150 \\ 0 & -51 \end{smallmatrix}\right)$. The spectral norm of the matrix exponential for a real 2×2 matrix in upper triangular form $A = \left(\begin{smallmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{smallmatrix}\right)$, $\lambda_1 \neq \lambda_2$, is given by

$$\|e^{At}\| = \frac{1}{2}\left|e^{\lambda_1 t} - e^{\lambda_2 t}\right| \left(\sqrt{\coth(\tfrac{1}{2}(\lambda_1 - \lambda_2)t)^2 + (\tfrac{\alpha}{\lambda_1 - \lambda_2})^2} + \sqrt{1 + (\tfrac{\alpha}{\lambda_1 - \lambda_2})^2}\right).$$

For $\beta = -1$ we get the monotone increasing function

$$\|e^{(A-\beta I)t}\| = \tfrac{1}{2}(1 - e^{-50t})\left(\sqrt{\coth(25t)^2 + 9} + \sqrt{10}\right) \xrightarrow{t\to\infty} \sqrt{10} \quad \text{as } \lim_{x\to\infty}\coth(x) = 1.$$

Hence, $M = \sqrt{10}$ is the smallest possible bound for strict (M,β)-stability with $\beta = -1$. Now let us examine which bound can be obtained using Lemma 2.1. The strict Lyapunov inequality $PA + AP + 2P > 0$ is unsolvable, but there exist matrices $P > 0$ which solve $P(A - \beta I) + (A - \beta I)^* P \leq 0$ for $\beta = -1$. The matrix $P = \left(\begin{smallmatrix} p_1 & p_3 \\ p_3 & p_2 \end{smallmatrix}\right)$ is a solution of this inequality if and only if

$$150p_1 - 50p_3 = 0, \qquad 150p_3 - 50p_2 < 0.$$

If we fix $p_1 = 1$ then necessarily $p_3 = 3$ and $p_2 > 9$. With this choice P is positive definite. Other solutions are positive scalar multiples of solutions representable in such a manner. The condition number for a 2×2 real matrix $P > 0$ is given by

$$\kappa(P) = \frac{\operatorname{trace} P}{2\det P}\left(\operatorname{trace} P + \sqrt{(\operatorname{trace} P)^2 - 4\det P}\right) - 1.$$

By writing $p_2 = 9 + \alpha$ we get $\kappa(\alpha) = \frac{10+\alpha}{2\alpha}\left((10+\alpha) + \sqrt{(10+\alpha)^2 - 4\alpha}\right) - 1$ which has a minimum of $19 + 6\sqrt{10}$ at $\tilde{\alpha} = 10$. Therefore the best bound obtainable by Lemma 2.1 is $\sqrt{\kappa(\tilde{\alpha})} = 3 + \sqrt{10}$. In this example there is a gap of 3 between the Lyapunov bound and the minimal bound M.

Remark 2.2. By extending Lemma 2.1 to a larger class of norms the gap illustrated in Example 2.1 may be closed. We generalize the condition number estimate for the P-norm by defining the eccentricity of a given norm $\nu(\cdot)$ on $\mathbb{C}^n$ by

$$\operatorname{ecc}(\nu) := \frac{\max_{\|x\|=1}\nu(x)}{\min_{\|x\|=1}\nu(x)}, \tag{2.7}$$

which yields $\operatorname{ecc}(\|\cdot\|_P) = \kappa(P)^{\frac{1}{2}}$. Following Remark 2.1 we replace the quadratic Lyapunov inequality (2.2) by the condition that $A - \beta I$ generates a contraction semigroup with respect to ν, i.e.

$$e^{(A-\beta I)t}B_\nu \subset B_\nu, \quad t \geq 0, \tag{2.8}$$

where $B_\nu = \{x \in \mathbb{C}^n;\ \nu(x) < 1\}$ denotes the open unit ball defined by ν. Then (2.7) and (2.8) yield

$$\|e^{At}\| \le (\operatorname{ecc}\nu)\nu(e^{At}) \le (\operatorname{ecc}\nu)e^{\beta t}, \quad t \ge 0. \tag{2.9}$$

It is easily verified that for $\beta \in (\alpha(A), 0)$ the minimal M for which A is (M,β)-stable is given by $M = \operatorname{ecc}\nu$ for the norm $\nu(x) := \sup_{t\ge 0} e^{-\beta t}\|e^{At}x\|$. Hence, the minimal M is

$$M_{\text{inf}} := \inf\left\{\operatorname{ecc}\nu;\ \nu \text{ is a norm on } \mathbb{C}^n \text{ and } \forall t \ge 0 : e^{(A-\beta I)t}B_\nu \subset B_\nu\right\}.$$

Lemma 2.1 motivates the following definition, which leads to an easily checkable condition.

Definition 2.2. *Given the constants $M \ge 1, \beta < 0$, a matrix $A \in \mathbb{C}^{n\times n}$ is called* (strictly) quadratically $(M,\beta)-$stable *if there exists $P > 0$ such that*

$$\kappa(P) \le M^2 \quad \textit{and} \quad \mathcal{H}_P(A) \le \beta P\,, \qquad (\textit{resp. } \mathcal{H}_P(A) < \beta P)\,.$$

A pair $(A,B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ is called (strictly) quadratically (M,β)-stabilizable *if there exists a matrix $F \in \mathbb{C}^{m\times n}$ such that $A - BF$ is (strictly) quadratically $(M,\beta)-$stable.*

We will establish necessary and sufficient criteria for strict quadratic $(M,\beta)-$stabilizability.

3 Quadratic (M,β)-stabilizability and -stabilization

Let $M \ge 1$, $\beta < 0$ be given and consider the system

$$\dot{x} = Ax + Bu, \tag{3.1}$$

where $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$.

By Lemma 2.1 this system is strictly quadratically (M,β)-stabilizable if and only if we can find a matrix F such that the following matrix inequality has a positive definite solution $P > 0$ with $\kappa(P) \le M^2$

$$\mathcal{H}_P(A - BF) = \frac{1}{2}\left(P(A-BF) + (A-BF)^*P\right) = \mathcal{H}_P(A) - \mathcal{H}_P(BF) < \beta P\,. \tag{3.2}$$

It is therefore no big surprise that the following set is crucial for our considerations of (M,β)-stabilizability. For every $P \ge 0$ let

$$\mathcal{M}_P(\beta) := \{v \in \mathbb{C}^n;\ \frac{1}{2}v^*(A^*P + PA)v \ge \beta v^*Pv\}. \tag{3.3}$$

The set $\mathcal{M}_P(\beta)$ consists of those initial values for which the solutions have an initial growth of at least β, i. e. $\frac{d}{dt}\|e^{At}v\|_P\big|_{t=0} \ge \beta\|v\|_P$. Thus $\mathcal{M}_P(\beta)$ is the subset of state space where the dynamics of the uncontrolled system has to be modified in order to meet the control aim.

Theorem 3.1. *Consider the pair (A, B) and constants $M \geq 1, \beta < 0$. The following statements are equivalent:*

(i) The system $\dot{x} = Ax + Bu$ is strictly quadratically (M, β)-stabilizable.

(ii) There exist $\gamma \in \mathbb{R}$ and a matrix $P > 0$ with $\kappa(P) \leq M^2$ such that

$$\mathcal{H}_P(A - \gamma BB^*P) < \beta P. \tag{3.4}$$

In this case, (3.4) *holds for all $\gamma' > \gamma$.*

(iii) There exists a matrix $P > 0$ with $\kappa(P) \leq M^2$ such that

$$\mathcal{M}_P(\beta) \cap \ker B^*P = \{0\} . \tag{3.5}$$

Note that there is no counterpart for the equivalence $(i) \iff (ii)$ for the case of non-strict quadratic (M, β)-stabilizability.

Proof. $(ii) \Rightarrow (i)$ is trivial (choose $F = \gamma B^*P$). If γ satisfies (3.4) so does $\gamma' = \gamma + \alpha > \gamma$ since PBB^*P is positive semidefinite and

$$\mathcal{H}_P(A - \gamma' BB^*P) = \mathcal{H}_P(A - \gamma BB^*P) - \alpha PBB^*P < 0.$$

$(i) \Rightarrow (iii)$ Let F be a feedback such that $A - BF$ is strictly quadratically (M, β)-stable. Then there exists $P > 0$ with $\kappa(P) \leq M^2$ such that $\mathcal{H}_P(BF) > \mathcal{H}_P(A) - \beta P$ holds. Therefore for all $v \in \mathcal{M}_P(\beta)$ we have $v^*(PBF + F^*B^*P)v > 0$ and it follows that $B^*Pv \neq 0$.

$(iii) \Rightarrow (ii)$ Let (3.5) hold for some $P > 0$ with $\kappa(P) \leq M^2$. We show that for a suitable choice of $\gamma > 0$ we have $\mathcal{H}_P(A - \gamma BB^*P) < \beta P$. Our first step is to simplify the expression

$$\mathcal{H}_P(A - \gamma BB^*P) = \frac{1}{2}(A^*P + PA) - \gamma PBB^*P .$$

Choose a unitary matrix U such that $UPBB^*PU^*$ is a diagonal matrix with the entries of the diagonal sorted in decreasing order. We then perform the change of coordinates $\tilde{x} = Ux$ and continue our analysis with respect to the transformed matrices $\tilde{A} = UAU^*, \tilde{B} = UB, \tilde{P} = UPU^*$. It is easy to see that (3.5) implies $\mathcal{M}_{\tilde{P}}(\beta) \cap \ker \tilde{B}^*\tilde{P} = \{0\}$ and since $\kappa(P) = \kappa(\tilde{P})$ we may assume without loss of generality that PBB^*P is of the form $\left(\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $D > 0$ diagonal. Partitioning $\mathcal{H}_P(A - \beta I)$ similarly to PBB^*P in the form

$$\mathcal{H}_P(A - \beta I) = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} ,$$

we see that the closed loop $\dot{x} = (A - \gamma BB^*P)x$ is strictly quadratically (M, β)-stable if and only if

$$\mathcal{H}_P(A - \beta I - \gamma BB^*P) = \begin{bmatrix} H_{11} - \gamma D & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} < 0 . \tag{3.6}$$

Using the Schur complement [4] the inequality (3.6) is equivalent to

$$H_{22} < 0, \quad (H_{11} - \gamma D) - H_{12}H_{22}^{-1}H_{12}^{*} < 0\,. \tag{3.7}$$

The kernel condition (3.5) is equivalent to the submatrix $H_{22} < 0$ as H_{22} represents the restriction of $\mathcal{H}_P(A - \beta I)$ to the kernel of B^*P. The second inequality in (3.7) can always be fulfilled choosing γ large enough since $D > 0$. □

The proof shows that for a given $P > 0$ there exists a state feedback matrix F such that $A - \beta I - BF$ generates a strict P-contraction semigroup if and only if the submatrix H_{22} of (3.7) is negative definite.

We note two consequences of Theorem 3.1 which simplify the situation for the case that minimizing the M is more important than guaranteeing a certain rate of decay $\beta < 0$. The following corollary presents conditions for quadratic (M, β)-stabilization for arbitrary $\beta < 0$.

Corollary 3.1. *Consider the pair (A, B), and let $M \geq 1$. The following statements are equivalent.*

(i) For some $\beta < 0$ the system $\dot{x} = Ax + Bu$ is strictly quadratically (M, β)-stabilizable.

(ii) There exist $\gamma > 0$ and a matrix $P > 0$ with $\kappa(P) \leq M^2$ such that

$$\mathcal{H}_P(A - \gamma BB^*P) < 0. \tag{3.8}$$

(iii) There exists a matrix $P > 0$ with $\kappa(P) \leq M^2$ such that

$$\{v \in \mathbb{C}^n;\, v^*(A^*P + PA)v \geq 0\} \cap \ker B^*P = \{0\}\,. \tag{3.9}$$

(iv) There exists a matrix $P > 0$ with $\kappa(P) \leq M^2$ such that

$$x \in \ker B^* \setminus \{0\} \implies x^*(AP^{-1} + P^{-1}A^*)x < 0. \tag{3.10}$$

Proof. This is immediate from Theorem 3.1. □

In the previous result condition (3.10) is remarkable as it allows for a nice geometric interpretation of the problem. Namely, given the pair (A, B) the question is if we can find a matrix $P > 0$ with $\kappa(P) \leq M^2$ such that for all $x \in \ker B^*$, $x \neq 0$ the condition $\Re\langle A^*x, P^{-1}x\rangle < 0$ holds. In other words, the system $\dot{x} = A^*x$ is dissipative in the weighted inner product $\langle \cdot, \cdot \rangle_{P^{-1}}$ on the subspace $\ker B^*$. A case of particular interest is that of feedback matrices F such that the closed loop system matrix $A - BF$ generates a strict contraction semigroup for the spectral norm, that is, if we specialize to the case $M = 1$ with $\beta < 0$ arbitrary. Then we obtain

Corollary 3.2. *Consider the pair (A, B). The following statements are equivalent.*

(i) there exists a feedback matrix F such that $(A - BF)$ generates a strong contraction semigroup with respect to the spectral norm,

(ii) there exists $\gamma > 0$ such that $(A - \gamma BB^*)$ generates a strong contraction semigroup with respect to the spectral norm,

(iii) it holds that

$$x \in \ker B^* \setminus \{0\} \implies x^*(A + A^*)x < 0. \tag{3.11}$$

Proof. In fact, $A - BF$ generates a strong contraction semigroup with respect to the spectral norm, if there exists $\beta < 0$ such that for all $t \geq 0$

$$\|e^{(A-BF)t}\| \leq e^{\beta t}.$$

Therefore Corollary 3.1 is applicable with $M = 1$. In this case, the positive definite matrices $P > 0$ with $\kappa(P) \leq M^2 = 1$ occurring in the statements of Corollary 3.1 are necessarily multiples of the identity. □

4 Quadratic (M, β)-stabilization and LMIs

In this section we briefly discuss how the geometric characterizations for strict quadratic (M, β)-stabilizability obtained in Section 3 can be reformulated in terms of linear matrix inequalities (LMIs). We refer to [1] for an overview of applications of LMIs in control.

By Theorem 3.1 the system $\dot{x} = Ax + Bu$ is strictly quadratically (M, β)-stabilizable if and only if the following set is nonempty

$$\mathcal{N} := \{P \in \mathbb{C}^{n \times n} \mid P > 0, \kappa(P) \leq M^2 \text{ and } \mathcal{M}_P(\beta) \cap \ker B^* P = \{0\}\}.$$

It is easy to see that $\mathcal{N}$ is a cone, so that if $\mathcal{N}$ is nonempty, then there is a $P \in \mathcal{N}$ with $\sigma(P) \subset [M^{-2}, 1]$ which implies $\|P\| \leq 1$.

Furthermore, the set $\mathcal{N}$ is nonempty if and only if there are $P > 0, \kappa(P) \leq M^2$ and $F \in \mathbb{C}^{m \times n}$ such that (3.2) is satisfied. This inequality has the disadvantage that the unknowns P and F do not appear linearly, but by setting $Q = P^{-1}$ and $F = XP$ we obtain an LMI from (3.2) by pre- and post-multiplying Q. So all our conditions can be summarized by the LMI

$$\begin{aligned} I \leq Q \leq M^2 I, \\ AQ + QA^* - (BX + X^*B^*) < 2\beta Q. \end{aligned} \tag{4.1}$$

where the first inequality ensures that the eigenvalues of Q are contained in the interval $[1, M^2]$ which implies that $\kappa(Q) = \kappa(P) \leq M^2$. By this simple reformulation we obtain another condition for quadratic (M, β)-stabilizability as an immediate corollary to Theorem 3.1.

Corollary 4.1. *Consider the pair* (A, B) *and constants* $M \geq 1, \beta < 0$. *The following statements are equivalent:*

(i) The system $\dot{x} = Ax + Bu$ *is strictly quadratically* (M, β)*-stabilizable.*

(ii) The LMI (4.1) *is feasible, that is, there exists a solution* $(Q, X) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{m\times n}$ *of* (4.1).

(iii) There exists a solution $(Q, \varrho) \in \mathbb{C}^{n\times n} \times \mathbb{R}$ *of the LMI*

$$\begin{aligned} I \leq Q \leq M^2 I, \\ AQ + QA^* - 2\varrho BB^* < 2\beta Q. \end{aligned} \tag{4.2}$$

Proof. The equivalence of (i) and (ii) was shown in the derivation of (4.1). For the equivalence (ii)⇔(iii) note first that (Q, ϱ) solves (4.2) if and only if $(Q, \varrho B^*)$ solves (4.1), so that (iii)⇒(ii). Furthermore, by Theorem 3.1 strict quadratic (M, β)-stabilizability is equivalent to the existence of a stabilizing feedback of the form $F = \varrho B^* P$. In this case $(Q, \varrho B^*)$ solves (4.1), which implies (iii). □

The advantage of (iii) compared to (ii) in Corollary 4.1 is that the dimension of the parameter space is significantly reduced, depending on the dimension of B.

Remark 4.1. Using Corollary 4.1 we can add further design objectives depending on the specific problem since quadratic optimization problems may be solved on solution sets of LMIs. For example, if a feedback F of small norm is desirable, then it is advantageous to minimize $\gamma \geq 0$ under the constraints (4.1) and

$$\begin{pmatrix} \gamma I & X \\ X^* & \gamma I \end{pmatrix} \geq 0. \tag{4.3}$$

Using the Schur complement it may be seen that (4.3) is equivalent to $\gamma^2 I - XX^* \geq 0$, i.e. $\gamma \geq \|X\|$. As the solution set of (4.1) is not necessarily closed, there may not be an optimal solution, but at least the optimization problem yields matrices X with norm close to optimal and for the corresponding stabilizing feedback F we have $\|F\| \leq \|X\|\|P\| \leq \|X\|$.

Similarly, (4.2) may be used to minimize ρ.

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