The generalized spectral radius is strictly increasing

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Abstract

Using a result linking convexity and irreducibility of matrix sets it is shown that the generalized spectral radius of a compact set of matrices is a strictly increasing function of the set in a very natural sense. As an application some consequences of this property in the area of time-varying stability radii are discussed. In particular, using the implicit function theorem sufficient conditions for Lipschitz continuity are derived. An example is presented of a linearly increasing family of matrix polytopes for which the proximal subgradient of the generalized spectral radius at a certain polytope contains 0, so that the implicit function theorem is not applicable in all cases.

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1 Introduction

The generalized or joint spectral radius describes the exponential growth rate of a linear inclusion given by a (compact) set of matrices, or equivalently the growth rate of the possible products of matrices of that set. The topic has attracted the attention of numerous researchers in recent years partly due to the wide range of applications where this number characterizes some quantity of interest. For reasons of space we do not give an overview of the relevant literature and on the history of the results on the generalized spectral radius, referring to [1, 2] instead.

In a recent paper [1] it was shown that the generalized spectral radius is strictly increasing on the set of compact matrices in a sense that will be made precise in a moment. Unfortunately, the result was incomplete in that some

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particular cases were excluded by the assumptions. Also the proof was quite involved and had every appearance of being too complicated for the statement.

In this paper we complete the proof of the previous paper by showing that the cases that were left open previously actually do not exist at all. The argument we give for this also allows for a significant reduction of the complexity of the proof of strict monotonicity so that we present this easier proof as well. We also discuss implications of the result for the theory of time-varying stability radii. Some of these involve assumptions about the proximal subgradient of the generalized spectral radius and we conclude the paper with an example showing that these assumptions are not always satisfied.

The basic setup of the problem we are interested in can be described as follows. Given a nonempty, compact set of matrices $\mathcal{M} \subset \mathbb{K}^{n\times n}$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ we consider the discrete linear inclusion

$$x(t+1) \in \{Ax(t) \mid A \in \mathcal{M}\}\,. \tag{1}$$

A solution of (1) is a sequence $\{x(t)\}_{t\in\mathbb{N}}$, such that for every $t \in \mathbb{N}$ there is an $A(t) \in \mathcal{M}$ with $x(t+1) = A(t)x(t)$. The quantity we want to investigate is the exponential growth rate of this system, which is frequently called generalized spectral radius, joint spectral radius of the matrix set $\mathcal{M}$ or maximal Lyapunov exponent of (1). In order to define these different versions of the same number we define first the sets of products of length t

$$\mathcal{S}_t := \{A(t-1)\cdots A(0) \mid A(s) \in \mathcal{M}\,, s = 0,\dots,t-1\}\,,$$

and the semigroup given by

$$\mathcal{S} := \bigcup_{t=0}^{\infty} \mathcal{S}_t\,.$$

Let $r(A)$ denote the spectral radius of A and let $\|\cdot\|$ be some operator norm on $\mathbb{K}^{n\times n}$. Define for $t \in \mathbb{N}$

$$\overline{\rho}_t(\mathcal{M}) := \sup\{r(S_t)^{1/t} \mid S_t \in \mathcal{S}_t\}\,, \qquad \hat{\rho}_t(\mathcal{M}) := \sup\{\|S_t\|^{1/t} \mid S_t \in \mathcal{S}_t\}\,. \tag{2}$$

The *joint spectral radius*, respectively the *generalized spectral radius* are now defined as

$$\overline{\rho}(\mathcal{M}) := \limsup_{t\to\infty} \overline{\rho}_t(\mathcal{M})\,, \quad \hat{\rho}(\mathcal{M}) := \lim_{t\to\infty} \hat{\rho}_t(\mathcal{M})\,.$$

The maximal Lyapunov exponent of (1) can be defined as

$$\kappa(\mathcal{M}) := \max \limsup_{t\to\infty} \frac{1}{t} \log \|x(t)\|\,,$$

where the maximum is taken over all solutions $x(t)$ of (1), and where we use the convention $\log 0 = -\infty$. By now, it is well known [3, 2], that

$$\overline{\rho}(\mathcal{M}) = \hat{\rho}(\mathcal{M}) = e^{\kappa(\mathcal{M})}\,,$$

and we denote this quantity by $\rho(\mathcal{M})$ in the sequel.

A basic observation is that the analysis of linear inclusions becomes much easier, if we restrict ourselves to the case of irreducible sets. Recall that $\mathcal{M} \subset \mathbb{K}^{n\times n}$ is called *irreducible*, if only the trivial subspaces $\{0\}$ and $\mathbb{K}^n$ are invariant under all matrices $A \in \mathcal{M}$. Otherwise $\mathcal{M}$ is called *reducible*. It is clear, that the semigroup $\mathcal{S}$ is irreducible if and only if the set $\mathcal{M}$ is.

It is easy to see, that if the set $\mathcal{M}$ is reducible, then the set may be brought simultaneously to upper block triangular form, where the blocks on the diagonal are irreducible or 0. The generalized spectral radius is then given by the maximum of the generalized spectral radii of the blocks on the diagonal.

In the irreducible case the following theorem is extremely useful.

Theorem 1. *[4, 1] If $\mathcal{M}$ is compact and irreducible, then there exists a norm v on $\mathbb{K}^n$, such that*

(i) for all $x \in \mathbb{K}^n, A \in \mathcal{M}$ it holds that

$$v(Ax) \leq \rho(\mathcal{M})v(x)\,,$$

(ii) for all $x \in \mathbb{K}^n$ there exists an $A \in \mathcal{M}$ such that

$$v(Ax) = \rho(\mathcal{M})v(x)\,.$$

For an irreducible set of matrices $\mathcal{M}$ we call a norm v a *Barabanov norm* (with respect to $\mathcal{M}$), if it satisfies the assertions (i) and (ii) of Theorem 1.

In order to characterize irreducibility the following easy lemma, that has been noted by several authors, is useful in the sequel.

Lemma 2. *[4] Let $\mathcal{S} \subset \mathbb{K}^{n\times n}$ be a semigroup, then the following statements are equivalent.*

(i) $\mathcal{S}$ is reducible,

(ii) there exist vectors $x, l \in \mathbb{K}^n \setminus \{0\}$ such that

$$\langle l, Sx\rangle = 0\,, \quad \textit{for all } S \in \mathcal{S}\,.$$

Our main tool in the proof of monotonicity properties of the generalized spectral radius is the following observation, that gives an interesting link between irreducibility of a convex set and the behavior of the spectral radius on the generated semigroup.

Proposition 3. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $n > 1$. Let $\mathcal{M} \subset \mathbb{K}^{n\times n}$ be a convex set containing more than one point. If the semigroup $\mathcal{S}$ generated by $\mathcal{M}$ satisfies*

$$\sigma(S) \subset \{0\} \cup \{z \in \mathbb{C} \mid |z| = 1\}\,, \quad \forall S \in \mathcal{S}(\mathcal{M})\,, \tag{3}$$

then $\mathcal{M}$ is reducible.

Remark 4. *If the assumption of convexity is dropped in the previous Proposition 3 the assertion is false, as can be seen by examples given in [5], where irreducible semigroups of matrices satisfying* (3) *are presented. The assumption that $\mathcal{M}$ should contain more than one point is made to exclude the case that* $, \mathbb{K} = \mathbb{R}, n = 2$ *and $\mathcal{M}$ contains just one rotation matrix. In this case* (3) *is satisfied and $\mathcal{M}$ is irreducible. Otherwise, if $n > 1$ and $\mathbb{K} = \mathbb{C}$ or if $n > 2$ and $\mathbb{K} = \mathbb{R}$ singleton matrix sets are of course always reducible.*

Proof. If $\mathcal{M}$ is reducible there is nothing to show. So let us assume that $\mathcal{M}$ is irreducible. It is well known that this implies in particular, that not all the matrices $S \in \mathcal{S}$ are nilpotent, see [2].

For $S \in \mathcal{S}$ denote by P_S the reducing projection of S corresponding to the eigenvalues of modulus 1; that is, $P_S^2 = P_S, P_S S = S P_S$ and $\operatorname{Im} P_S$ is the sum of the generalized eigenspaces corresponding to the eigenvalues of modulus 1. Note that $\operatorname{rank} P_S$ is constant on $\mathcal{M}$ due to the convexity of $\mathcal{M}$, because if there were $A, B \in \mathcal{M}$ with $\operatorname{rank} P_A < \operatorname{rank} P_B$, then by continuity of the spectrum some convex combination $\lambda A + (1-\lambda)B$, $\lambda \in (0,1)$ has an eigenvalue of modulus different from 0 and 1 which is excluded by the assumption. Then of course $\operatorname{rank} P_A = \operatorname{rank} P_{A^2}$ and as $\mathcal{S}_2$ is pathwise connected, we obtain that $\operatorname{rank} P_S$ is constant on $\mathcal{M} \cup \mathcal{S}_2$. By induction $\operatorname{rank} P_S$ is constant on $\mathcal{S}$. As not all $S \in \mathcal{S}$ are nilpotent, we obtain that $\operatorname{rank} P_S \geq 1$ for all $S \in \mathcal{S}$.

In particular, $r(S) = 1$ for all $S \in \mathcal{S}$ and using the definition of $\bar{\rho}$ we see that $\rho(\mathcal{M}) = 1$. Thus by irreducibility and Theorem 1 there is a norm v on $\mathbb{K}^n$, so that for the induced operator norm we have

$$v(S) = 1\,, \quad \forall S \in \mathcal{S}(\mathcal{M})\,. \tag{4}$$

In the remainder of the proof we frequently make use of the fact, that by [5, Lemma 2.1] $r(S) = 1 = v(S)$ for all $S \in \mathcal{S}$ implies, that the restriction of S to $\operatorname{Im} P_S$ is diagonalizable for all $S \in \operatorname{cl} \mathcal{S}$.

Pick an arbitrary $A \in \mathcal{M}$ and assume that it is in Jordan canonical form so that

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & N \end{bmatrix}, \tag{5}$$

where $\sigma(A_{11}) \subset \{z \in \mathbb{C} \mid |z| = 1\}$ and N is a nilpotent matrix. Also $A_{11} \in \mathbb{K}^{n_1 \times n_1}$ has no defective eigenvalues by [5, Lemma 2.1]. Pick an arbitrary $T \in \mathcal{S}$ different from A and partition it into the same block structure

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}. \tag{6}$$

Then for $A^k T \in \mathcal{S}, k \in \mathbb{N}$ and for $k \geq n$ we obtain

$$A^k T = \begin{bmatrix} A_{11}^k T_{11} & A_{11}^k T_{12} \\ 0 & 0 \end{bmatrix}, \tag{7}$$

and so $\sigma(A_{11}^k T_{11}) \subset \{z \in \mathbb{C} \mid |z| = 1\}, k \geq n$ as by the previous remark the multiplicity of the eigenvalues different from zero has to add up to a constant.

As for a certain sequence $n_k \to \infty$ we have $A_{11}^{n_k} \to I$, we see in particular, that $\sigma(T_{11}) \subset \{z \in \mathbb{C} \mid |z| = 1\}$ and so also

$$1 = |\det(T_{11})| .$$

Furthermore, we see that T_{11} is diagonalizable by applying [5, Lemma 2.1] again. As $T \in \mathcal{S}$ was arbitrary this argument applies to all products of the form $((1-\lambda)A + \lambda B)S$, where $B \in \mathcal{M}, S \in \mathcal{S}$ are arbitrary, and $\lambda \in [0,1]$. We partition B and S in the same way as A. Then the upper left block of $((1-\lambda)A + \lambda B)S$ is given by

$$((1-\lambda)A_{11} + \lambda B_{11})S_{11} + \lambda B_{12}S_{21} . \tag{8}$$

As we have seen, that the modulus of the determinant of this expression is always equal to 1, we obtain that the polynomial

$$p(\lambda) := \det(A_{11}S_{11} + \lambda(B_{11}S_{11} + B_{12}S_{21} - A_{11}S_{11}))$$

has a constant modulus on the interval $[0,1]$ and is therefore constant. Then the following polynomial is also constant

$$\tilde{p}(\lambda) := \det(A_{11}^{-1})p(\lambda)\det(S_{11}^{-1}) = \det(I + \lambda(A_{11}^{-1}(B_{11} + B_{12}S_{21}S_{11}^{-1}) - I)) ,$$

which implies that for every $B \in \mathcal{M}, S \in \mathcal{S}$ the matrix

$$N_1(B,S) := A_{11}^{-1}(B_{11} + B_{12}S_{21}S_{11}^{-1}) - I \tag{9}$$

is nilpotent. (We suppress the dependence of $N_1(B,S)$ on A, as this matrix is fixed in our argument).

In particular, if we set $S = I$ in the previous calculation, that is if we set $S_{11} = I$ and $S_{21} = 0$, then we have $B_{11} = A_{11}(I + N_1(B,I))$, where $N_1(B,I)$ is a nilpotent matrix. Using again the sequence $\{n_k\}$ from above we obtain from $A_{11}^{n_k} \to I$ that $A_{11}^{n_k-1}B_{11} \to I + N_1(B,I)$. As every $A_{11}^{n_k-1}B_{11}$ is the upper left block of an element of $\mathcal{S}$ of the form (7), it follows that $I + N_1(B,I)$ is the upper left block of an element of $\operatorname{cl}\mathcal{S}$ of the form (7). This implies that $N_1(B,I) = 0$ using [5, Lemma 2.1]. As $N_1(B,I) = 0$ and $B \in \mathcal{M}$ was arbitrary all matrices in $\mathcal{M}$ are of the form

$$\begin{bmatrix} A_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} .$$

In particular, as $\mathcal{M}$ is not a singleton set, this shows that $1 \leq n_1 < n$. Furthermore, it follows, that the upper left block of $(\lambda A + (1-\lambda)B)S$, which we wrote down in (8), is actually given by

$$A_{11}S_{11} + \lambda B_{12}S_{21} .$$

Again this shows, that $A_{11}^n S_{11} + \lambda A_{11}^{n-1}B_{12}S_{21}$ is the upper left block of an upper block triangular matrix and by multiplying with powers of $A_{11}^n S_{11}$ we obtain that

$$(A_{11}^n S_{11})^m + \lambda (A_{11}^n S_{11})^{m-1} A_{11}^{n-1} B_{12} S_{21}$$

is the upper left block of an upper block triangular $T \in \mathcal{S}$. Note that from (7) we see that $A_{11}^n S_{11}$ is diagonalizable and has its spectrum in the unit circle. Thus for a suitable subsequence of $\{(A_{11}^n S_{11})^m\}_{m\in\mathbb{N}}$ the previous expression converges to

$$I + \lambda (A_{11}^n S_{11})^{-1} A_{11}^{n-1} B_{12} S_{21} = I + \lambda S_{11}^{-1} A_{11}^{-1} B_{12} S_{21} \,. \tag{10}$$

As this matrix occurs as an upper left block we may argue as before: First of all $S_{11}^{-1} A_{11}^{-1} B_{12} S_{21}$ is nilpotent as the modulus of the determinant of (10) is independent of λ and furthermore

$$S_{11}^{-1} A_{11}^{-1} B_{12} S_{21} = 0 \tag{11}$$

by yet another application of [5, Lemma 2.1]. Now (11) implies that $B_{12} S_{21} = 0$ for all $B \in \mathcal{M}, S \in \mathcal{S}$. This implies that

$$\begin{bmatrix} 0 & B_{12} \end{bmatrix} S e_1 = 0$$

for all $B \in \mathcal{M}, S \in \mathcal{S}$. This shows, that either $B_{12} = 0$ for all $B \in \mathcal{M}$, in which case $\mathcal{M}$ is reducible, or by taking a nonzero row of B_{12} that $\mathcal{S}$ is reducible by Lemma 2. This contradicts the assumption, that $\mathcal{M}$ is irreducible. □ ■

2 Strict Monotonicity

In this section we first give a simple proof of a strict monotonicity property of the generalized spectral radius on the set of compact matrices. The statement is essentially the same as in [1], but here we are able to give a shorter proof that also covers cases that were left open in [1]. It should be noted, that the corresponding statement for the continuous time case is false, as shown by a counterexample in [6].

In order to formulate the statement recall, that the *relative interior* of a convex set K is the interior of K in the relative topology of the smallest affine subspace containing K. The relative interior of K is denoted by $\operatorname{ri} K$, see [7] for further details. We denote the convex hull of a set $M \subset \mathbb{K}^m$ by $\operatorname{conv} M$.

Theorem 5. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Consider two compact sets $\mathcal{M}_1 \neq \mathcal{M}_2 \subset \mathbb{K}^{n\times n}$ and assume that $\mathcal{M}_2$ is irreducible. If*

$$\mathcal{M}_1 \subset \operatorname{ri} \operatorname{conv} \mathcal{M}_2 \,, \tag{12}$$

then

$$\rho(\mathcal{M}_1) < \rho(\mathcal{M}_2) \,.$$

Proof. Assume the assertion is false, so that $\rho(\mathcal{M}_1) = \rho(\mathcal{M}_2) = 1$ can be assumed without loss of generality. Also we will assume, that $\mathcal{M}_1, \mathcal{M}_2$ are convex, as $\rho(\mathcal{M}) = \rho(\operatorname{conv} \mathcal{M})$, [4]. Finally, we may assume that $\mathcal{M}_1$ is irreducible, for if this is not the case, then we may take a slightly bigger irreducible $\mathcal{M}_1' \supset \mathcal{M}_1$, while ensuring that (12) is still satisfied for $\mathcal{M}_1'$.

As $\mathcal{M}_1$ is convex and irreducible there exists a Barabanov norm v_1 with respect to $\mathcal{M}_1$ by Theorem 1. Hence $v_1(S) \leq 1$ for all $S \in \mathcal{S}(\mathcal{M}_1)$. If $v_1(S) = 1$ for all $S \in \mathcal{S}(\mathcal{M}_1)$ then by [5, Theorem 2.5] we have that (3) holds for all $S \in \mathcal{S}(\mathcal{M}_1)$, which by Proposition 3 contradicts convexity and irreducibility of $\mathcal{M}_1$. Thus there exists an $S \in \mathcal{S}(\mathcal{M}_1)$ with $v_1(S) < 1$, which we now consider to be fixed. Fix $x \in \mathbb{K}^n$ with $v_1(x) = 1$. Factorizing $S = A_k \cdots A_0$, $A_j \in \mathcal{M}_1, j = 0, \ldots, k$ there is some $l \in \{0, \ldots, k\}$, such that

$$v_1(\prod_{j=0}^{l-1} A_j x) = 1 \quad \text{and} \quad v_1(\prod_{j=0}^{l} A_j x) < 1\,.$$

Denoting $y = A_{l-1} \cdots A_0 x$ it follows, that $v_1(y) = 1, v_1(A_l y) < 1$. By definition of Barabanov norms, however, there is some $B \in \mathcal{M}_1$, such that $v_1(By) = 1$. As $v_1(By) = 1$ and $v_1(A_l y) < 1$, it follows by the triangle inequality, that for all $\varepsilon > 0$ we have

$$v_1(By + \varepsilon(B - A_l)y) \geq 1 + \varepsilon(1 - v(A_l y)) > 1\,.$$

Now by (12) we have $B + \varepsilon(B - A_l) \in \mathcal{M}_2$ for some $\varepsilon > 0$ small enough. Thus for some $C \in \mathcal{M}_2$ we have $v_1(Cy) > 1$, and so we obtain the inequality $v_1(CA_{l-1} \cdots A_0 x) > 1$. Using a standard compactness argument it follows, that there exists a constant $c > 1$, such that for every $x \in \mathbb{K}^n$ with $v_1(x) = 1$, there is an $T \in \mathcal{S}(\mathcal{M}_2)$ satisfying

$$v_1(Tx) > c v_1(x)\,.$$

By induction we obtain an unbounded solution of the discrete inclusion defined by $\mathcal{M}_2$. Using Theorem 1 (i) this contradicts $\rho(\mathcal{M}_2) = 1$, as $\mathcal{M}_2$ is irreducible. This completes the proof. $\square$ $\blacksquare$

3 Stability radii

Using the results of the previous sections we can discuss some properties of time-varying stability radii. We generalize several results from [6]. Stability radii quantify robustness of a stable system with respect to a given class of uncertainties. The problem arises whenever due to incomplete modelling, neglect of dynamics or measurement uncertainty the model can be expected to behave differently than the process of interest. In this case it is of interest to know, whether a stability result for the model has implications for the real system assuming that bounds on the uncertainty can be given. Here we only treat the case of stability radii with respect to time-varying perturbations.

Assume we are given a *nominal* discrete-time system

$$x(t+1) = A_0 x(t)\,, \tag{13}$$

which is exponentially stable, i.e. $r(A_0) < 1$. We assume that *structure matrices* $A_1, \ldots, A_m$ are given, which may be used to include some knowledge of the nature of the uncertainty into the model, and consider the perturbed system

$$x(t+1) = \left(A_0 + \sum_{i=1}^{m} d_i(t) A_i \right) x(t)\,, \quad t \in \mathbb{N}\,. \tag{14}$$

Here the perturbations $d(t) = (d_1(t), \ldots, d_m(t))$ are unknown. Bounds on the size of the perturbation are given by a convex set $D \subset \mathbb{K}^m$, with $0 \in \operatorname{int} D$. Then the set αD, $\alpha > 0$ represents the set of perturbations with respect to an uncertainty level α.

Note that the uncertain system (14) can be reformulated as a linear inclusion of the form (1), where given the finite collection $A_0, \ldots, A_m \in \mathbb{K}^{n\times n}$, the compact, convex set $D \subset \mathbb{R}^m$ with $0 \in \operatorname{int} D$ and a real parameter $\alpha \geq 0$ we define

$$\mathcal{M}(A_0, \ldots, A_m, \alpha) := \left\{ A_0 + \sum_{i=1}^{m} d_i A_i \;\middle|\; d \in \alpha D \right\}\,,$$

and consider $x(t+1) \in \{Ax(t) \mid A \in \mathcal{M}(A_0, \ldots, A_m, \alpha)\}$. If the matrices A_i are fixed, we denote the generating set by $\mathcal{M}(\alpha)$ and the corresponding growth rate by $\rho(\alpha)$ for the sake of succinctness. The dependence of the set on D is suppressed, because we do not intend to vary D.

We now define stability radii by

$$\begin{aligned} r_{Ly}(A_0, (A_i)) &:= \inf\{\alpha \geq 0 \mid \rho(\alpha) \geq 1\}\,, \\ \bar{r}_{Ly}(A_0, (A_i)) &:= \inf\{\alpha \geq 0 \mid \rho(\alpha) > 1\}\,. \end{aligned}$$

The interpretation of these quantities is that $r_{Ly}(A_0, (A_i))$ is the largest uncertainty level below which no time-varying perturbation will destabilize the system, whereas $\bar{r}_{Ly}(A_0, (A_i))$ is the infimum of uncertainty levels for which there are exponentially destabilizing perturbations. These two quantities and their possible difference play a decisive role in the robustness analysis of nonlinear systems at a fixed point, see [6] for details.

It is clear, that the family $\mathcal{M}(A_0, \ldots, A_m, \alpha), \alpha \geq 0$ is an increasing family of matrices satisfying (12) for all pairs $0 < \alpha_1 < \alpha_2$. Thus the map $\alpha \to \rho(\mathcal{M}(\alpha))$ is strictly increasing for all $A_0, \ldots, A_m \in \mathbb{K}^{n\times n}$, if $\mathcal{M}(\alpha)$ is irreducible for some (and then for all) $\alpha > 0$.

We now give a complete description of the set of systems for which the two stability radii differ. In the formulation of this result an exceptional set will play a role, which we now define. To this end let $(A_0, \ldots, A_m) \in (\mathbb{K}^{n\times n})^{m+1}$ be fixed and note that we may modulo a similarity transformation assume, that the matrices in $\mathcal{M}(\alpha)$ are of the upper block triangular form

$$\begin{bmatrix} A_{0,11}+\sum_{i=1}^m d_iA_{i,11} & A_{0,12}+\sum_{i=1}^m d_iA_{i,12} & \dots & A_{0,1d}+\sum_{i=1}^m d_iA_{i,1d} \\ 0 & A_{0,22}+\sum_{i=1}^m d_iA_{i,22} & \dots & A_{0,2d}+\sum_{i=1}^m d_iA_{i,2d} \\ 0 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & A_{0,dd}+\sum_{i=1}^m d_iA_{i,dd} \end{bmatrix}, \tag{15}$$

where each of the diagonal blocks

$$\mathcal{M}_j(\alpha) := \{A_{0,jj} + \sum_{i=1}^m d_iA_{i,jj} \mid d \in \alpha D\}$$

is either irreducible or equal to $\{0\}$ for $j = 1, \dots, d$. In this case $1 \le d \le n$ depends uniquely on the matrices $A_0, \dots, A_m$ and $\mathcal{M}(\alpha), \alpha > 0$ is irreducible if and only if $d = 1$.

With respect to the structure (15) the problematic perturbation structures $(A_0, \dots, A_m) \in (\mathbb{K}^{n\times n})^{m+1}$ can be characterized by the following two properties

(i) $r(A_0) = 1$,

(ii) whenever $r(A_{0,jj}) = 1$ for some $j \in \{1, \dots, d\}$ then $\alpha \mapsto \mathcal{M}_j(\alpha)$ is constant,

The exceptional set is then defined by

$$E := \left\{(A_0, \dots, A_m) \in (\mathbb{K}^{n\times n})^{m+1} \mid \text{ items (i) and (ii) are satisfied.}\right\}. \tag{16}$$

Theorem 6. *Let $(A_0, \dots, A_m) \in (\mathbb{K}^{n\times n})^{m+1}$ and consider the perturbed discrete-time system* (14)*. Then*

$$r_{Ly}(A_0, (A_i)) = \bar{r}_{Ly}(A_0, (A_i)),$$

if and only if $(A_0, \dots, A_m) \in (\mathbb{K}^{n\times n})^{m+1} \setminus E$. Furthermore, E is equal to the set discontinuities of r_{Ly}, respectively $\bar{r}_{Ly}$.

Proof. If $\mathcal{M}(\alpha)$ is irreducible for all $\alpha > 0$ and $\alpha \mapsto \mathcal{M}(\alpha)$ is not constant (and therefore strictly increasing) then the map $\alpha \mapsto \rho(\alpha)$ is strictly increasing on $[0, \infty)$ by Proposition 5. This implies $r_{Ly}(A_0, (A_i)) = \bar{r}_{Ly}(A_0, (A_i))$.

If $\alpha \mapsto \mathcal{M}(\alpha)$ is constant, then of course $\rho(\alpha)$ is constant. Thus we have $r_{Ly}(A_0, (A_i)) = \bar{r}_{Ly}(A_0, (A_i))$, except for the case in which $1 = \rho(0) = r(A_0)$, as then we have $r_{Ly}(A_0, (A_i)) = 0 < \infty = \bar{r}_{Ly}(A_0, (A_i))$.

If $\mathcal{M}(\alpha)$ is reducible for $\alpha > 0$, then we may assume that $\mathcal{M}(\alpha)$ is of the form (15). Now for $j = 1, \dots, d$ either the map $\alpha \mapsto \mathcal{M}_j(\alpha)$ is constant or it is strictly increasing hence $\alpha \mapsto \rho(\mathcal{M}_j(\alpha))$ is either constant or strictly increasing. As we have

$$\rho(\mathcal{M}(\alpha)) = \max_{j=1,\dots,d} \rho(\mathcal{M}_j(\alpha))$$

this implies that the map $\alpha \mapsto \rho(\mathcal{M}(\alpha))$ is constant on an interval of the form $[0, \alpha^*)$ and strictly increasing on (α^*, ∞), where $\alpha^* \in [0, \infty]$ is given by

$$\min\{\alpha \mid \exists j \in \{1, \dots, d\}, \varepsilon > 0 : \mathcal{M}_j(\alpha + \varepsilon) \text{ is irreducible and } \rho(\mathcal{M}_j(\alpha)) = r(A_0)\}\,.$$

This implies the first assertion.

For the remaining statement note that it is not difficult to see that $r_{Ly}, \bar{r}_{Ly}$ are upper respectively lower semicontinuous on $(\mathbb{K}^{n\times n})^{m+1}$. Thus, whenever these two functions coincide on an open set they are continuous and furthermore they are discontinuous where they do not coincide. The arguments are not difficult and detailed in a similar situation in [8] so that we omit them here. □

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An alternative view on the previous results is, that (except for a few cases described by E) the map $\alpha \mapsto \rho(\mathcal{M}(\alpha)) - 1$ has exactly one zero or possibly no zero at all and that if such a zero exists, then the corresponding α is locally a continuous function of the data $A_0, \dots, A_m$. This is very reminiscent of an application of the implicit function theorem, and we will now use the Lipschitz version of it to obtain further regularity properties of the stability radii.

Our considerations are based on the fact that the generalized spectral radius is locally Lipschitz continuous on $I(\mathbb{K}^{n\times n})$, the set of irreducible, compact sets in $\mathbb{K}^{n\times n}$, see [1]. An easy consequence of this is the following observation.

Lemma 7. *The map*

$$(A_0, \dots, A_m, \alpha) \mapsto \rho(A_0, \dots, A_m, \alpha) := \rho\left(\left\{A_0 + \sum_{i=1}^m d_i A_i \;\middle|\; d \in \alpha D\right\}\right)$$

is locally Lipschitz continuous on the set $I(\mathbb{K}^{n\times n}) \times \mathbb{R}_{>0}$.

Proof. Note that the map

$$(A_0, \dots, A_m, \alpha) \mapsto \left\{A_0 + \sum_{i=1}^m d_i A_i \;\middle|\; d \in \alpha D\right\}$$

is Lipschitz continuous with respect to the Hausdorff metric. As the composition of Lipschitz continuous maps is again Lipschitz continuous the claim follows from [1, Corollary 4.2]. □ ■

In order to use the implicit function theorem we need to recall a few concepts from nonsmooth analysis for the convenience of the reader, for further details we refer to [9, Chapter 1] and [10].

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. A vector $\zeta \in \mathbb{R}^n$ is called a *proximal subgradient* of f in x, if there are constants $\sigma, C > 0$, such that for all y with $\|x - y\| \le C$ we have

$$f(x) - f(y) + \langle \zeta, y - x\rangle \le \sigma \|x - y\|^2\,.$$

The set of proximal subgradients of f in x is denoted by $\partial_P f(c)$. Using [9, Theorem 2.6.1] the *Clarke subdifferential* of a locally Lipschitz continuous function f in x, may then be defined by

$$\partial_{\mathrm{Cl}} f(x) := \operatorname{conv}\left\{ \zeta \in \mathbb{R}^n \;\middle|\; \exists x_k \to x, \zeta_k \in \partial_P f(x_k) : \zeta = \lim_{k\to\infty} \zeta_k \right\}$$

Note that if $f : \mathbb{R}^p \to \mathbb{R}$ is locally Lipschitz continuous then

$$\partial_{\mathrm{Cl}} f(x) = \operatorname{conv}\left\{ c \in \mathbb{R}^p \;\middle|\; \exists x_k \to x : c = \lim_{k\to\infty} \bigtriangledown f(x_k) \right\}, \tag{17}$$

see [10, Theorem II.1.2], where we tacitly assume that the gradient $\bigtriangledown f$ exists in x_k if we write $\bigtriangledown f(x_k)$. Recall that Lipschitz continuity of f implies that it is differentiable almost everywhere by Rademacher's theorem. If we consider functions $f : \mathbb{C}^n \to \mathbb{R}$ then we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ in order to define proximal subgradients.

Proposition 8. *Let $n, m \in \mathbb{N}$. Fix $\{A_0, \ldots, A_m\} \in I(\mathbb{K}^{n\times n})$ and let*

$$r_{Ly}(A_0, (A_i)) < \infty .$$

Consider the map $\kappa : (\mathbb{K}^{n\times n})^{m+1} \times \mathbb{R}_+ \to \mathbb{R}, (B_0, \ldots, B_m, \alpha) \mapsto \rho(\mathcal{M}(\alpha))$ and denote

$$\partial_{\mathrm{Cl},\alpha}\kappa(z) := \left\{ c \in \mathbb{R} \mid \exists p' \in (\mathbb{K}^{n\times n})^{m+1} : (p', c) \in \partial_{\mathrm{Cl}}\, \kappa(z) \right\} .$$

If

$$\inf \partial_{\mathrm{Cl},\alpha}\, \kappa(A_0, \ldots, A_m, r_{Ly}(A_0, (A_i))) > 0 \,, \tag{18}$$

then $r_{Ly} = \bar{r}_{Ly}$ on a neighborhood of $(A_0, \ldots, A_m) \in (\mathbb{K}^{n\times n})^{m+1}$ and on this neighborhood r_{Ly} is locally Lipschitz continuous.

Proof. By Lemma 7 and (18) we may apply the implicit function theorem for Lipschitz continuous maps [10, Theorem VI.3.1] to κ in the point $(A_0, \ldots, A_m, r_{Ly}(A_0, (A_i)))$. This states that for every $(B_0, \ldots, B_m)$ in a suitable open neighborhood of $(A_0, \ldots, A_m) \in (\mathbb{K}^{n\times n})^{m+1}$ the map

$$\alpha \mapsto \rho(\mathcal{M}(B_0, \ldots, B_m, \alpha))$$

has a unique root and this root is a locally Lipschitz continuous function of $(B_0, \ldots, B_m)$. In other words, this means that on this neighborhood the functions r_{Ly} and $\bar{r}_{Ly}$ coincide and are locally Lipschitz continuous. □

■

It is now interesting to note, that the previous result is not applicable in all points of continuity of r_{Ly}, so that we cannot conclude local Lipschitz continuity of r_{Ly} outside of the set E. The following example shows, that $0 \in \partial_{\mathrm{Cl},\alpha}\kappa(z)$ is possible.

Example 9. *For $\mathbb{K} = \mathbb{R}$ and $n = 2$ we consider the matrices*

$$A_0 := \begin{bmatrix} 1/2 & 1/3 \\ 1/2 & 1/3 \end{bmatrix}, \quad A_1 := \begin{bmatrix} 1/2 & 0 \\ -1/2 & 0 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & 1/3 \\ 0 & 0 \end{bmatrix}, \quad A_3 := \begin{bmatrix} 0 & 0 \\ 0 & 1/3 \end{bmatrix}.$$

And for $\delta \in (0, 1]$ we define the (irreducible) set

$$\mathcal{M}_\delta := \{A_0 + \alpha A_1 + \beta A_2 + \gamma A_3 \mid \alpha \in [-\delta, \delta], -\delta \le \beta, \gamma, \text{ and } \beta + \gamma \le \delta\}$$

Note that for $\delta = 1$ we have that $\|A\|_1 \le 1$ for all $A \in \mathcal{M}_1$, where $\|\cdot\|_1$ denotes the usual 1-norm. Hence $\rho(\mathcal{M}_1) \le 1$. On the other hand we have $I_2 \in \mathcal{M}_1$, so that $\rho(\mathcal{M}_1) \ge 1$, and hence equality holds. Also this shows that $\|\cdot\|_1$ is a Barabanov norm for $\mathcal{M}_1$.

Now if we consider

$$A(\delta) := \begin{bmatrix} \dfrac{1+\delta}{2} & \dfrac{2+\delta}{6} \\ \dfrac{1-\delta}{2} & \dfrac{2+\delta}{6} \end{bmatrix},$$

then $A(\delta) \in \mathcal{M}_\delta$ for $\delta \in [0, 1]$. This may be seen by setting $\alpha(\delta) = \delta, \beta(\delta) = \gamma(\delta) = \delta/2$. Consequently, we have $\rho(\mathcal{M}_\delta) \ge r(A(\delta))$. A short calculation reveals, that

$$r(A(\delta)) = \frac{5}{12} + \frac{1}{3}\delta + \frac{1}{12}\sqrt{25 - 8\delta - 8\delta^2}$$

and so $r(A(1)) = 1 = \rho(\mathcal{M}_1)$. Also we have $d/d\delta\, r(A(\delta))_{|\delta=1} = 0$. Using that $r(A(\delta)) \le \rho(\mathcal{M}_\delta) < \rho(\mathcal{M}_1) = r(A(1))$ for $0 < \delta < 1$, it is easy to see, that 0 is a proximal subgradient of $\delta \mapsto \rho(\mathcal{M}_\delta)$ in $\delta = 1$.

Interpreting this observation in context of stability radii, this means, that (A_0, A_1, A_2, A_3) is a good candidate for a perturbation structure, at which the time-varying stability radius is not locally Lipschitz continuous, as the assumptions of Proposition 8 are violated. We will see, that this is indeed the case.

Consider the map $c \mapsto (cA_0, cA_1, cA_2, cA_3)$, for $c \in [1, 1.2]$. We denote the stability radius $r_{Ly}(cA_0, cA_1, cA_2, cA_3) =: r_{Ly}(c)$ for brevity. Denoting furthermore by $\delta(c)$ the smallest positive solution of $r(cA(\delta)) = 1$ it is clear, that

$$r_{Ly}(c) \le \delta(c),$$

because $1 = r(cA(\delta(c))) \le \rho(c\mathcal{M}_{\delta(c)})$. Solving the equation $r(cA(\delta)) = 1$ for $c \ge 1$ leads to

$$\delta(c) = \frac{2 - c - \sqrt{c^2 + c - 2}}{c}.$$

Summarizing this shows that $r_{Ly}(1) = \delta(1) = 1$ and $r_{Ly}(c) \le \delta(c)$ for $c > 1$. As $\delta(c)$ is not Lipschitz on $[1, 1.2]$, the same is true for $r_{Ly}(c)$. Consequently, the function $r_{Ly} : (\mathbb{R}^{2\times 2})^4 \to \mathbb{R} \cup \{\infty\}$ is not locally Lipschitz at (A_0, A_1, A_2, A_3).

The intuition behind the construction of the previous example is the following: Note that from the first unit vector e_1 using matrices in $\mathcal{M}_1$ we can only reach vectors of 1-norm equal to 1. On the other hand the sequence $x(t) \equiv e_1$ is a solution of the linear inclusion given by $\mathcal{M}_1$ that realizes the exponential growth rate and is optimal with respect to the Barabanov norm $\| \cdot \|_1$ (in the sense that condition (ii) of Theorem 1 is satisfied for every t). Using the variation of constants formulas for derivatives with respect to parameters of solutions of differential equations as "guidelines" this would suggest, that the derivative from the left with respect to δ of the growth rate of that particular solution is 0, because infinitesimally in one step $\|Ae_1\|_1 = 1$, even if A is restricted to $\mathcal{M}_\delta, \delta < 1$. It seems plausible that the proximal subgradients of ρ can be characterized by the proximal subgradients of solutions, that are always optimal with respect to some Barabanov norm. This question remains to be investigated.

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