An introduction to spectral theory of time-varying linear discrete-time systems

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Abstract

We present a tutorial introduction to the approach to spectral theory of time-varying linear systems via the analysis of the controllability properties of an associated projected system. The necessary concepts of universally regular controls, control sets and characteristic exponents are discussed. It is shown that to each main control set and control sequence u there exists a subspace with dimension equal to the index of the control set whose projected forward evolution does not leave the closure of the control set.

1 Introduction

The question as to which exponential growth rates a linear system

$$\dot{x}(t) = A(u(t))x(t)\,, \quad t \in \mathbb{R} \quad \text{or} \quad x(t+1) = A(u(t))x(t)\,, \quad t \in \mathbb{N}$$

$$x(0) = x_0 \in \mathbb{R}^n$$

$$A : U \to \mathbb{R}^{n\times n}$$

can exhibit has attracted much attention in recent years. For continuous time systems Floquet, Lyapunov and Morse spectra have been studied by Colonius and Kliemann [8],[9],[10].

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In this article we treat the discrete-time case. We will give an overview over the main results, without proving most of them. Nonetheless we attempt to present the geometric ideas that lead to proofs which have appeared in [19]. An exception to this rule is Section 6, where new results are presented.

In the following Section 2 we present the class of systems that are studied and state the problem that is to be investigated. Section 3 presents a brief review of the dynamics of the time-invariant case in order to motivate the following discussion of time-varying systems. In Section 4 we introduce accessibility and discuss the related regularity properties of control sequences. This leads to the introduction of universally regular controls. In Section 5 the dynamics of the projected system are analyzed. In order to do this control sets are introduced and their most important properties are discussed. This leads to the concept of main control sets and their associated indices. In Section 6 invariance principles of main control sets are studied. The section culminates in the result that justifies the figure of speech that control sets extend generalized eigenspaces: For each main control set and each infinite control sequence there exists a linear subspace of dimension equal to the index of the control set such that the evolution of this subspace remains in the closure of the control set, if at all possible, which is determined by the dimension of the kernel of the transition matrices given by the control sequence. In the final Section 7 it is explained how the results on control sets may be employed for the analysis of Floquet and Lyapunov exponents.

2 Problem statement

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and let $\tilde{U} \subset \mathbb{K}^m$ be open and connected. For an analytic map

$$A : \tilde{U} \to \mathbb{K}^{n\times n}\,, \tag{1}$$

we consider a family of time-varying linear system of the form

$$\begin{aligned} x(t+1) &= A(u(t))x(t)\,, \quad t \in \mathbb{N} \\ x(0) &= x_0 \quad \in \mathbb{K}^n, \end{aligned} \tag{2}$$

where $u : \mathbb{N} \to U \subset \tilde{U}$. For $t \in \mathbb{N}$ U^t denotes the set of admissible finite control sequences $u = (u(0), \ldots, u(t-1))$, while $U^{\mathbb{N}}$ is the set of infinite control sequences $u = (u(0), u(1), \ldots)$. It will always be clear from the context whether u denotes an element of U, U^t or $U^{\mathbb{N}}$.

The evolution operator generated by a control sequence $u \in U^{\mathbb{N}}$ is defined by

$$\Phi_u(s, s) = I\ , \quad \Phi_u(t+1, s) = A(u(t))\Phi_u(t, s)\,, \quad t \geq s \in \mathbb{N}. \tag{3}$$

With this notation $\Phi_u(t, 0)x_0$ is the solution of (2) corresponding to the initial value x_0 and the control u at time t.

For systems of the form (2) let $\lambda(x_0, u)$ denote the Lyapunov exponent corresponding to the initial value $(0, x_0) \in \mathbb{N} \times \mathbb{K}^n$ and the sequence $A(u(\cdot)) \in \ell^\infty(\mathbb{N}, \mathbb{K}^{n\times n})$ determined by $u \in U^{\mathbb{N}}$, i.e. the exponential growth rate of the corresponding solution:

$$\lambda(x_0, u) = \limsup_{t\to\infty} \frac{1}{t} \log \|\Phi_u(t, 0)x_0\|,$$

while $\beta(u)$ denotes the Bohl exponent determined by $u \in U^{\mathbb{N}}$:

$$\beta(u) = \limsup_{s,t-s\to\infty} \frac{1}{t-s} \log \|\Phi_u(t,s)\| .$$

It is known that in general $\max_{x_0 \neq 0} \lambda(x_0, u) \leq \beta(u)$ where strict inequality is possible, see [11]. The interest in the Bohl exponent stems from the fact that its negativity characterizes exponential stability of the time-varying system given by the matrix sequence $\{A(u(t))\}_{t\in\mathbb{N}}$.

Floquet exponents are the Lyapunov exponents corresponding to periodic sequences $u \in U^{\mathbb{N}}$. For $t \in \mathbb{N}$, $u \in U^t$ it is easy to see that the set of Floquet exponents determined by the t–periodic continuation of u is given by

$$\sigma_{Fl}(u) := \{\frac{1}{t} \log |\lambda|;\ \lambda \in \sigma(\Phi_u(t,0))\} , \tag{4}$$

where we continue to use the convention $\log 0 = -\infty$. For a system of form (2) determined by the map A and the set of admissible controls U the Floquet, Lyapunov and Bohl spectrum are the sets of all corresponding exponents, i.e.

$$\Sigma_{Fl}(A,U) := \bigcup_{t\geq 1, u\in U^t} \sigma_{Fl}(u) , \tag{5}$$

$$\Sigma_{Ly}(A,U) := \{\lambda(x_0,u)\,;\, x_0 \in \mathbb{K}^n\backslash\{0\}, u \in U^{\mathbb{N}}\} , \tag{6}$$

$$\Sigma_{Bo}(A,U) := \{\beta(u); u \in U^{\mathbb{N}}\} . \tag{7}$$

To the linear system (2) we associate a system on projective space whose controllability properties determine the sets of characteristic exponents in a qualitative way. In the sequel $\mathbb{P}^{n-1}_{\mathbb{K}}$ denotes the $n-1$ dimensional projective space, and for $W \subset \mathbb{K}^n$, $\mathbb{P}W$ denotes the natural projection of $W\backslash\{0\}$ onto the projective space $\mathbb{P}^{n-1}_{\mathbb{K}}$. With this notation the projected system is given by

$$\begin{aligned} \xi(t+1) &= \mathbb{P}A(u(t))\xi(t) , \quad t \in \mathbb{N} \\ \xi(0) &= \xi_0 \in \mathbb{P}^{n-1}_{\mathbb{K}} \\ u &\in U^{\mathbb{N}}(\xi_0). \end{aligned} \tag{8}$$

Naturally for each point in projective space this only makes sense if $\xi(t) \not\subset \mathrm{Ker}A(u(t))$. We therefore define the admissible control values for ξ by $U(\xi) := \{u \in U; A(u)x \neq 0, x \neq 0, \mathbb{P}x = \xi\}$. The sets of admissible control sequences are denoted by $U^t(\xi), U^{\mathbb{N}}(\xi)$. The solution of (8) corresponding to an initial value ξ_0 and a control sequence $u \in U^{\mathbb{N}}(\xi_0)$ is denoted by $\xi(\cdot;\xi_0,u)$.

Let U_{inv} be the set $\{u \in U;\ \det A(u) \neq 0\}$, which is clearly the complement of a set defined by analytic equations in U. In the sequel we will have to make use of the existence of invertible matrices $A(u)$, so that we have to assume that $U_{inv} \neq \emptyset$.

The following general assumption will be made throughout the remainder of this article.

Assumption 2.1 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and consider system (2). We assume that the map A in (1) and the sets $U \subset \tilde{U} \subset \mathbb{K}^m$ are such that:*

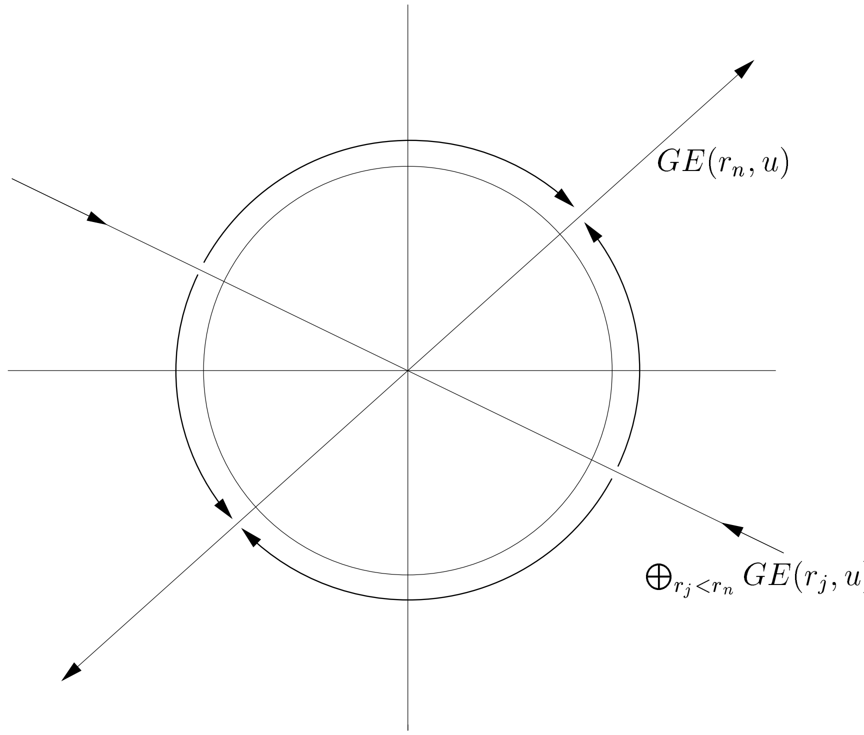

Figure 1: Time-invariant dynamics

(i) $U_{inv} \neq \emptyset$.

(ii) $intU$ *is connected,* U *is bounded.*

(iii) $U \subset clintU \subset \tilde{U}$.

3 The Time-Invariant Case

To understand the dynamics of the projected system let us briefly recall how the projection of a time-invariant system behaves on $\mathbb{P}_{\mathbb{K}}^{n-1}$: Eigenspaces and generalized eigenspaces are invariant and points whose spectral decomposition consists of several (generalized) eigenvectors are attracted to those generalized eigenspaces whose corresponding eigenvalues have the greatest modulus. From now on we use the following notation for $t \in \mathbb{N}$, $u \in U^t$, $\lambda \in \sigma(\Phi_u(t,0))$ and $r = |\lambda|$ let $GE(r,u) := \bigoplus_{\mu\in\sigma(\Phi_u(t,0)),|\mu|=r} GE(\mu,u)$, where GE denotes the generalized eigenspace. Fix $t \in \mathbb{N}$, $u \in U^t$ and consider the time-invariant system

$$x(t+1) = \Phi_u(t,0)x(t)\,.$$

The qualitative behavior of this system is depicted in Figure 1, for the projection onto the sphere. Identification of opposite points yield the behavior on $\mathbb{P}_{\mathbb{K}}^{n-1}$.

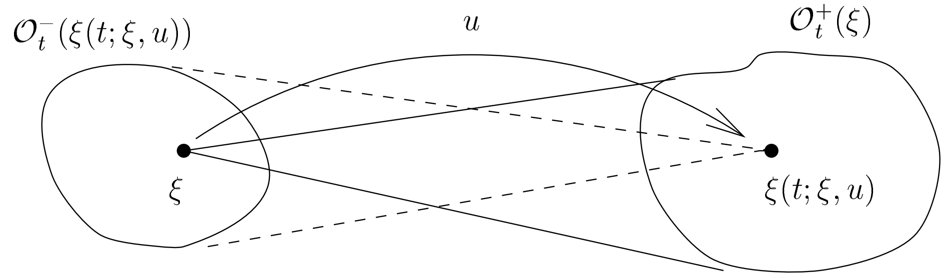

Figure 2: The effect of a regular control.

4 Forward Accessibility and Regularity

A fundamental assumption in our approach is that the projected system is forward accessible: For $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$ and a time $t \in \mathbb{N}$ we define the *forward orbit a time* t to be the set of point that can be reached from ξ, i.e.

$$\mathcal{O}_t^+(\xi) := \{\eta \in \mathbb{P}_{\mathbb{K}}^{n-1}; \exists\, u \in U^t(\xi) \text{ with } \eta = \xi(t;\xi,u)\} \,.$$

The forward orbit of ξ is then defined by $\mathcal{O}^+(\xi) := \bigcup_{t\in\mathbb{N}} \mathcal{O}_t^+(\xi)$. The backward orbit of ξ at time t is given by $\mathcal{O}_t^-(\xi) := \{\eta \in \mathbb{P}_{\mathbb{K}}^{n-1}; \exists\, u \in U^t(\eta) \text{ with } \xi = \xi(t;\eta,u)\}$ which leads to a definition of $\mathcal{O}^-(\xi)$ analogous to that of the positive forward orbit.

Definition 4.1 (Accessibility) *The system (8) is called* forward accessible *from* ξ *if* $int\mathcal{O}^+(\xi) \neq \emptyset$ *and* forward accessible *if it is forward accessible from all* $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$.

If we assume forward accessibility this means that for each $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$ there exists a $t \in \mathbb{N}$ such that $\text{int}\mathcal{O}_t^+(\xi) \neq \emptyset$. If we consider the the map

$$F_t(\xi,\cdot) : U^t(\xi) \to \mathbb{P}_{\mathbb{K}}^{n-1}\,, \quad F_t(\xi,u) := \xi(t;\xi,u)\,, \tag{9}$$

it follows by Sard's theorem that for some $u_0 \in U^t(\xi)$ the rank of the linearization of $F_t(\xi,\cdot)$ in u_0 must be full, i.e. equal to $n-1$. We denote this rank by $r(t;\xi,u_0)$ and call a pair $(\xi,u) \in \mathbb{P}_{\mathbb{K}}^{n-1} \times \text{int}U^t$ *regular*, if $u \in \text{int}U^t(\xi)$ and $r(t;\xi,u) = n-1$. A control $u \in \text{int}U^t$ is called *universally regular*, if (ξ,u) is a regular pair for all $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$. We denote regular forward orbits that is the set of points reachable from ξ such that ξ and the applied control sequence form a regular pair by $\hat{\mathcal{O}}_t^+(\xi)$ etc. Some properties of the regular forward orbits are studied in [2] for analytic invertible systems and similar arguments are applicable here. In particular the regular forward and backward orbits are open and if the regular forward orbit at time t is nonempty then its closure coincides with that of the orbit at time t.

Regular pairs are interesting because if (ξ,u) is a regular pair then using local surjectivity we obtain that $\xi(t;\xi,u) \in \text{int}\mathcal{O}_t^+(\xi)$ and on the other hand by the implicit function theorem $\xi \in \text{int}\mathcal{O}_t^-(\xi(t;\xi,u))$, see Figure 2. In the case when ξ is a fixed point under a control u

such that (ξ, u) is a regular pair this implies that $\xi \in V := \text{int}\mathcal{O}_t^+(\xi) \cap \text{int}\mathcal{O}_t^-(\xi)$, from which follows that V is a region of complete controllability, i.e. it is possible to steer from any $\eta_1 \in V$ to any $\eta_2 \in V$ by first steering from η_1 to ξ and then to η_2.

Proposition 4.2 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$ there exist $u_\xi \in intU^t, t \in \mathbb{N}$ such that (ξ, u_ξ) is a regular pair and*

$$\xi = \xi(t; \xi, u_\xi) \tag{10}$$

if and only if there exists an open neighborhood V of ξ such that $V \subset \hat{\mathcal{O}}_t^+(\xi) \cap \hat{\mathcal{O}}_t^-(\xi)$.

Condition (10) interpreted for the linear system states that ξ is an eigenspace of $\Phi_u(t, 0)$. The statement of the previous proposition may be extended to sums of generalized eigenspaces corresponding to eigenvalues of the same modulus once it is defined when such a subspace should be called regular. For two finite control sequences $u_1 \in U^{t_1}$, $u_2 \in U^{t_2}$ we define the concatenation (u_1, u_2) to be the sequence in $U^{t_1+t_2}$ given by $(u_1, u_2) = (u_1(0), \ldots, u_1(t_1 - 1), u_2(0), \ldots, u_2(t_2 - 1))$.

Definition 4.3 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $t \in \mathbb{N}$, $u \in U^t$, $\lambda \in \sigma(\Phi_u(t, 0)) \setminus \{0\}$, $|\lambda| = r$. We call the sum of generalized eigenspaces $\mathbb{P}GE(r, u)$* regular *if u can be partitioned as $u = (u_1, u_2)$ with $u_1 \in U^{t_1}$, $u_2 \in intU^{t_2}$, $t = t_1 + t_2$ and it holds that*

$$(\xi, u_2) \text{ is a regular pair for every } \xi \in \mathbb{P}\Phi_{u_1}(t_1, 0)GE(r, u)\,. \tag{11}$$

Proposition 4.4 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $t \in \mathbb{N}$, $u \in U^t$. If for $\lambda \in \sigma(\Phi_u(t, 0))$, $|\lambda| = r > 0$ the space $\mathbb{P}GE(r, u)$ is regular then there exists an open set W such that*

$$\mathbb{P}GE(r, u) \subset W \subset \bigcap_{\xi \in \mathbb{P}GE(r,u)} \hat{\mathcal{O}}^+(\xi) \cap \hat{\mathcal{O}}^-(\xi)\,. \tag{12}$$

Thus regular generalized eigenspaces are important for the analysis of the projected system. As every eigenspace to a universally regular control is regular the following consequence of results shown in [16] is of interest.

Proposition 4.5 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For the projected system (8) the following statements are equivalent.*

(i) System (8) is forward accessible.

(ii) There exists a $t^* \in \mathbb{N}$ such that for all $t > t^*$ the set of universally regular control sequences is generic in $intU^t$.

The set of universally regular $u \in U^t$ will be denoted by U_{reg}^t. It follows from the results in [16] that if $\text{int}\mathcal{O}_t^+(\xi) \neq \emptyset$ for all $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$ then $t^* \leq tn$. Note that U_{reg}^t is open for all $t \in \mathbb{N}$.

Remark 4.6 Let us point out that we use the term *generic* for sets that are the complement of closed subanalytic sets of lower dimension in the real case or proper analytic subsets in the complex case, see [17], [15]. ■

5 Control Sets

As we have seen in the previous section there exists a region of complete controllability around a regular generalized eigenspace. On the other hand if ξ lies in a region of complete controllability then it is an eigenspace for an appropriate control sequence. This is why the study of exponential growth rates in particular Floquet exponents leads to the study of controllability questions for the projected system. For this we introduce the following definition.

Definition 5.1 (Control set) *Let* $\mathbb{K} = \mathbb{R}, \mathbb{C}$. *Consider system (8). A set* $\emptyset \neq D \subset \mathbb{P}^{n-1}_{\mathbb{K}}$ *is called a* precontrol set, *if*

(i) $D \subset cl\mathcal{O}^+(\xi),\ \forall \xi \in D$.

(ii) For every $\xi \in D$ *there exists a* $u \in U^{\mathbb{N}}(\xi)$ *and an increasing sequence* $(t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ *such that* $\xi(t_k; \xi, u) \in D$ *for all* $k \in \mathbb{N}$.

A precontrol set D *is called* control set, *if furthermore*

(iii) D *is a maximal set with respect to inclusion satisfying (i).*

A control set C *is called* invariant control set, *if*

$$clC = cl\mathcal{O}^+(\xi),\ \forall \xi \in C\,. \tag{13}$$

In the continuous time case complete controllability holds in the interior of control sets. This is false for forward accessible discrete time systems as shown in an example in [2]. For this reason the concept of the core of a control set has been introduced, which is a set that behaves as the interior of a control set of a continuous time system.
We give a definition of core that slightly differs from the original definition in that we require a regularity condition to hold. So to contrast it it might be called *regular core* of a control set. It should, however, be noted that for the systems studied in [2] core and regular core of a control set coincide. An example of a discrete time systems of the form (8) for which core and interior of a control set differ is presented in [19].

Definition 5.2 (Regular core) *Let* $\mathbb{K} = \mathbb{R}, \mathbb{C}$. *Let* $D \subset \mathbb{P}^{n-1}_{\mathbb{K}}$ *be a control set with* $intD \neq \emptyset$. *The* (regular) core *of* D *is defined as*

$$core(D) := \{\xi \in D;\ \hat{\mathcal{O}}^+(\xi) \cap D \neq \emptyset \text{ and } \hat{\mathcal{O}}^-(\xi) \cap D \neq \emptyset\}\,. \tag{14}$$

In view of the terminology we have just introduced Proposition 4.4 states that if a sum of generalized eigenspaces $\mathbb{P}GE(r, u)$ is regular, then there exists a control set D such that $\mathbb{P}GE(r, u) \subset \text{core}(D)$. Now we know by Proposition 4.5 that generically a finite control sequence u is regular, and thus most Floquet exponents correspond to eigenspaces that project to the core of some control set. It should be noted that it cannot be concluded that the projection of an arbitrary eigenspace corresponding to any control is contained in the closure of a control set with nonempty interior. In fact, it is possible that any point of the projective space may be a precontrol set, but the control sets with nonempty interior do not cover the whole projective space. For an example see [19] or in the continuous time case [8]. Of particular interest are the invariant and the open control set. These are especially easy to characterize.

Theorem 5.3 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that system (8) is forward accessible, then*

(i) There exists a unique invariant control set $C \subset \mathbb{P}^{n-1}_{\mathbb{K}}$. It is given by

$$C := \bigcap_{\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}} cl\mathcal{O}^+(\xi)\,. \tag{15}$$

(ii) There exists a unique open control set $C^- \subset \mathbb{P}^{n-1}_{\mathbb{K}}$. It satisfies

$$clC^- = C^* := \bigcap_{\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}} cl\mathcal{O}^-(\xi)\,. \tag{16}$$

Moreover is holds that core$(C^-) = C^-$.

The idea of the proof of part (i) is the following. We know there exists universally regular controls and that their generalized eigenspaces project to the cores of control sets. So in particular for a universally regular u the (sum of) generalized eigenspace(s) $\mathbb{P}GE(|\lambda_n|, u) \subset$ core(C) for a control set C. By the dynamics depicted in Figure 3 C intersects the forward orbit of every $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$ that is not contained in the projection of the sum of eigenspaces corresponding to smaller eigenvalues. As the projected system is forward accessible we can always steer out of this sum and thus C intersects every forward orbit. Using the definition of control sets it is then easy to show that (15) does not define an empty set and that the set described in this manner is an invariant control set. The proof of part (ii) follows a similar pattern.

The representation (15), (16) show that any point may be steered to the invariant control set, respectively may be reached from the open control set. This motivates the following definition. Let D_1, D_2 be control sets in $\mathbb{P}^{n-1}_{\mathbb{K}}$ for the system (8). We define

$$D_1 \leq D_2 :\Leftrightarrow \text{ There exist } \xi \in D_1,\;\; t \in \mathbb{N},\;\; u \in U^t \text{ such that } \xi(t;\xi,u) \in D_2\,. \tag{17}$$

A priori this defines only a partial order on the control sets. What is however evident at this point is the following.

Proposition 5.4 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that system (8) is forward accessible.*

(i) C is the unique maximal control set with respect to the order "$\leq$" on the control sets.

(ii) C^- is the unique minimal control set with respect to the order "$\leq$" on the control sets.

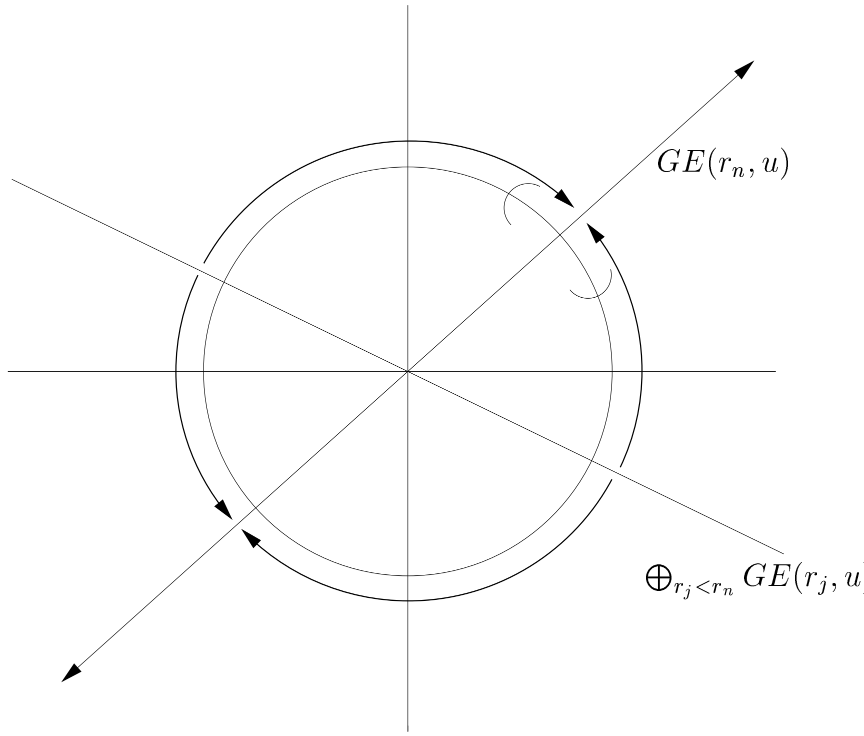

Figure 3: Construction of the invariant control set.

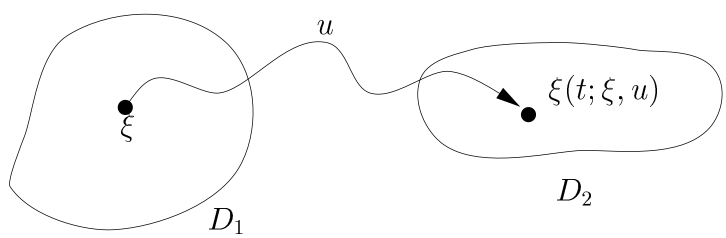

Figure 4: Control ordered control sets.

In order to obtain further results on the control set structure of the projected system the following definition is useful. For every $t \in \mathbb{N}$, $u \in U^t$, we will from now on consider the set $|\sigma(\Phi_u(t,0))| := \{r_1, \dots, r_\nu\}$, where $r_i = |\lambda_i|$ for $\lambda_i \in \sigma(\Phi_u(t,0))$, $r_1 < \dots < r_\nu$. Consider the sequence $|\lambda_1| \leq \dots \leq |\lambda_n|$ where each λ_i occurs as often as its the algebraic multiplicity. We define for $i = 1, \dots, n$

$$Q_i(t) := \bigcup_{u \in U^t_{reg}} \mathbb{P}GE(|\lambda_i|, u), \quad Q_i := \bigcup_{t=1}^{\infty} Q_i(t). \tag{18}$$

Furthermore for a map $A : \tilde{U} \to \mathbb{R}^{n \times n}$ we introduce the following index which is a measure of what sets of rank deficient matrices separate $A(\mathrm{int} U)$. Define the sets

$$U_i := \{u \in U; \dim \mathrm{Ker} A(u) \leq i\}, \tag{19}$$

and the singularity index

$$\bar{\imath}(A, U) := \min\{i; \mathrm{int} U_i \text{ is pathwise connected}\}. \tag{20}$$

Note that all the sets U_i are generic in U, as $U_i \supset U_{inv} \neq \emptyset$. Moreover, $\mathbb{K} = \mathbb{C}$ implies that $\bar{\imath}(A, U) = 0$ as proper analytic subsets are nowhere separating in the complex case, see [13] Proposition 7.4. The significance of the indices $i > \bar{\imath}(A, U)$ is explained in the following proposition.

Proposition 5.5 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that (8) is forward accessible. If $i > \bar{\imath}(A, U)$ then Q_i is contained in a precontrol set. If $\bar{\imath}(A, U) = 0$ then for each $i = 1, \dots, n$ the set Q_i is contained in a connected component of core(D) for some control set D.*

Remark 5.6 An example of a control set with core that is not connected is given in [19]. ■

The following theorem summarizes the main properties of control sets with nonempty core.

Theorem 5.7 *Assume that (8) is forward accessible and that $\bar{\imath}(A, U) \leq 1$, then the following statements hold:*

(i) The number κ of control sets $D_1, \dots, D_\kappa$ with nonempty interior satisfies

$$1 \leq \kappa \leq n. \tag{21}$$

(ii) For every $t > 0$, $u \in U^t_{reg}$, $r \in |\sigma(\Phi_u(t,0))|$ there exists a control set D_i $1 \leq i \leq \kappa$ such that

$$\mathbb{P}GE(r, u) \subset \mathit{core}(D_i). \tag{22}$$

(iii) The core of the control sets $D_1, \dots, D_\kappa$ consists of exactly those elements $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$ which are eigenvectors to a nonzero eigenvalue of some $\Phi_u(t,0)$ where (ξ, u) is a regular pair. If $U = U_{inv}$ the control may be chosen to be universally regular.

(iv) For every $t > 0$, $u \in U^t$, $r \in |\sigma(\Phi_u(t,0))|$ there exists an $j \in \{1,\dots,\kappa\}$ with $\mathbb{P}GE(r,u) \cap clD_j \neq \emptyset$. Also for every $t \in \mathbb{N}$, $u \in U^t$ and every $j = 1,\dots,\kappa$ there exists an $r \in |\sigma(\Phi_u(t,0))|$ with $\mathbb{P}GE(r,u) \cap clD_j \neq \emptyset$.

It has been shown that under the assumption of the previous theorem for every $i \in \{1,\dots,n\}$ there exists a control set D such that $Q_i \subset D$.. From now on the following terminology is used.

Definition 5.8 (Main control set) *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that (8) is forward accessible. A control set D is called* main control set *if for every index $1 \leq i \leq n$ it holds that*

$$Q_i \cap D \neq \emptyset \;\; \Rightarrow \;\; Q_i \subset D\,.$$

The result of the previous theorem may then be paraphrased by saying that in the case when $\bar{\imath}(A,U) \leq 1$, i.e. in particular in complex or real invertible case, the only control sets with nonempty core are main control sets. Let us now examine further properties of main control sets. Let $n(\lambda, u)$ denote the dimension of the generalized eigenspace of the eigenvalue λ of $\Phi_u(t,0)$.

Theorem 5.9 *Assume that (8) is forward accessible, then the following holds.*

(i) If $\bar{\imath}(A,U) = 0$ then the core of every main control set is connected.

(ii) The main control sets are completely ordered with respect to the order "$\leq$".

(iii) For each main control set D the number

$$m(D) = \sum_{\mathbb{P}GE(\lambda,u) \subset core(D)} n(\lambda, u) \tag{23}$$

is independent of $u \in U^t_{reg}$ and $t \in \mathbb{N}$.

Definition 5.10 (Index of a main control set) *Assume that (8) is forward accessible. For a main control set $D \subset \mathbb{P}^{n-1}_{\mathbb{K}}$ the number $m(D)$ is called the* index of the control set D.

6 Invariance Principles

Let us now investigate which points do not leave a control set under the application of a control sequence u. To this end we recall the definition of ω-limit sets.

Definition 6.1 (Limit sets) *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $u \in U^{\mathbb{N}}$, $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$. The positive ω-limit set is defined by*

$$\omega^+(\xi, u) := \left\{ \eta \in \mathbb{P}^{n-1}_{\mathbb{K}} ; \exists \{t_k\}_{k\in\mathbb{N}} \subset \mathbb{N}, \lim_{k\to\infty} t_k = \infty \text{ such that } \eta = \lim_{k\to\infty} \xi(t_k; \xi, u) \right\}. \tag{24}$$

The negative ω-limit set is defined by

$$\begin{aligned} \omega^-(\xi, u) \;:=\; & \Big\{\eta \in \mathbb{P}^{n-1}_{\mathbb{K}}; \exists\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{N}, \lim_{k\to\infty} t_k = \infty,\ \exists\{\eta_k\} \subset \mathbb{P}^{n-1}_{\mathbb{K}}, \\ & \xi = \xi(t_k; \eta_k, u) \text{ such that } \eta = \lim_{k\to\infty} \eta_k\Big\} \,. \end{aligned} \tag{25}$$

For $t \in \mathbb{N}$, $u \in U^t$, $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$ $\omega^+(\xi, u)$ $(\omega^-(\xi, u))$ denotes the positive (resp. negative) ω-limit set that is obtained by applying the t-periodic continuation of u.

Note that with this definition we do not exclude the possibility that ω-limit sets may be empty, e.g. if $u \notin U^{\mathbb{N}}(\xi)$. For a discussion of the concept of ω-limit sets we refer the reader to [1], Chapter 1.

Proposition 6.2 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Let $u \in U^t_{reg}$. Then for every control set D with nonempty interior we have that*

$$\mathbb{P}(\bigoplus_r GE(r, u)) \subset core(D)\,, \tag{26}$$

where the sum is taken over all $r \in |\sigma(\Phi_u(t, 0))|$ such that $\mathbb{P}GE(r, u) \subset core(D)$.

Proof. Let $0 \neq x \in \bigoplus_r GE(r, u)$ as in (26). We represent x as

$$x = x_1 + \ldots + x_\nu\,,$$

where $x_j \in GE(r_j, u)$ and $r_1 < \ldots < r_\nu$. Let $\underline{j}, \overline{j}$ be the smallest and largest indices such that $x_j \neq 0$. Then $\omega^+(\mathbb{P}x, u) \subset \mathbb{P}GE(r_{\overline{j}}) \subset \mathrm{core}(D)$, and $\omega^-(\mathbb{P}x, u) \subset \mathbb{P}GE(r_{\underline{j}}) \subset \mathrm{core}(D)$. Using the fact that $u \in U^t_{reg}$ it follows in a straightforward way that $\mathbb{P}x \in \mathrm{core}(D)$. ∎

Corollary 6.3 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Let $u \in U^t_{reg}$ and $|\sigma(\Phi_u(t, 0))| = \{r_1, \ldots, r_\nu\}$ where $r_1 < \ldots < r_\nu$. If there exists a control set D and indices i, l such that*

$$\mathbb{P}GE(r_i, u) \subset core(D)\,,$$

$$\mathbb{P}GE(r_{i+l}, u) \subset core(D)\,.$$

Then

$$\mathbb{P}(\bigoplus_{j=0}^{l} GE(r_{i+j}, u)) \subset core(D)\,. \tag{27}$$

Proof. Let $0 \neq x_i \in GE(r_i, u)$ and $0 \neq x_{i+l} \in GE(r_{i+l}, u)$. For $0 < j < l$ it holds by Proposition 4.4 that $\mathbb{P}GE(r_{i+j}, u) \subset \mathrm{core}(D')$ for some control set D'. Hence for $0 \neq x_{i+j} \in GE(r_{i+j}, u)$ there exists an $\varepsilon > 0$ such that $\mathbb{P}(x_{i+j} + \varepsilon x_i) \in \mathrm{core}(D')$ and $\mathbb{P}(x_{i+j} + \varepsilon x_{i+l}) \in \mathrm{core}(D')$. Now $\omega^-(\mathbb{P}(x_{i+j} + \varepsilon x_i), u) \subset \mathbb{P}GE(r_i, u) \subset \mathrm{core}(D)$ and thus $D \leq D'$. On the other hand $\omega^+(\mathbb{P}(x_{i+j} + \varepsilon x_{i+l}), u) \subset \mathbb{P}GE(r_{i+l}, u) \subset \mathrm{core}(D)$ and thus $D \geq D'$. Hence $D = D'$ by maximality of control sets and the assertion follows from Proposition 6.2. ∎

An interesting subset in a control set D is the set of those points, whose trajectory does not leave the closure of the control set forward and backward in time.

Definition 6.4 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For $t \in \mathbb{N}$, $u \in U^t_{inv}$ a control set D define*

$$D(u) = \{\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}; \omega^+(\xi, u), \omega^-(\xi, u) \subset clD\}. \tag{28}$$

For universally regular control sequences the set $D(u)$ is particularly easy to characterize.

Theorem 6.5 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $t \in \mathbb{N}$, $u \in U^t_{reg}$. Then*

$$D(u) = \mathbb{P}(\bigoplus_r GE(r, u)), \tag{29}$$

where the sum is taken over all $r \in |\sigma(\Phi(t, u))|$ with $\mathbb{P}GE(r, u) \subset core(D)$.

Proof. Clearly, if $\xi \in \mathbb{P}(\bigoplus_r GE(r, u))$ defined by equation (29) then $\omega^+(\xi, u), \omega^-(\xi, u) \subset \mathbb{P}(\bigoplus_r GE(r, u)) \subset \mathrm{core}(D)$ by Proposition 6.2. Let $0 \neq x \in \mathbb{K}^n$ be such that $\mathbb{P}x \in D(u) \backslash \mathbb{P}(\bigoplus_r GE(r, u))$. Let

$$x = \sum_{j=1}^{\nu} x_j ,$$

where $x_j \in GE(r_j, u)$. Now if j_0 is such that $x_{j_0} \neq 0$ and $\mathbb{P}x_{j_0} \notin \mathrm{core}(D)$, then by Corollary 6.3 it holds that $r_{j_0} > \max \overline{r} := \{r_j; \mathbb{P}GE(r_j, u) \subset \mathrm{core}(D)\}$ or $r_{j_0} < \underline{r} := \min\{r_j; \mathbb{P}GE(r_j, u) \subset \mathrm{core}(D)\}$. In the first case we obtain

$$\omega^+(\mathbb{P}x, u) \subset \mathbb{P}(\bigoplus_{r > \overline{r}} GE(r, u)),$$

but

$$\mathbb{P}(\bigoplus_{r > \overline{r}} GE(r, u)) \cap \mathrm{cl}(D) = \emptyset .$$

Hence $\mathbb{P}x \notin D(u)$. In the second case

$$\omega^-(\mathbb{P}x, u) \subset \mathbb{P}(\bigoplus_{r < \underline{r}} GE(r, u)),$$

and also $\mathbb{P}x \notin D(u)$. ■

To summarize we have obtained the following picture of the control structure of the system on projective space. For a map A and a set of admissible controls U such that the system on $\mathbb{P}^{n-1}_{\mathbb{K}}$ is forward accessible and $\bar{i}(A, U) \leq 1$, there exist a sequences of control sets $D_1 \ldots . D_\kappa$ and associated indices $m(D_1), \ldots, m(D_\kappa)$, with $\sum_{j=1}^{\kappa} m(D_j) = n$. More specifically if we write

$$\mu_j = \sum_{l=1}^{j} m(D_l)$$

for $j = 1, \ldots, \kappa$ then

$$\bigcup_{i=\mu j-1+1}^{\mu_j} Q_i \subseteqq \mathrm{core}(D_j),$$

where equality holds if $U = U_{inv}$. So the numbers from 1 to n are partitioned into κ non-interlacing subsequences which represent the indices i such that $Q_i \subset \mathrm{core}(D_j)$:

$$\underbrace{1,\dots,\mu_1}_{D_1},\ \underbrace{\mu_1+1,\dots,\mu_2}_{D_2},\ \underbrace{\mu_2+1,\dots,}_{\cdots\cdots},\ \dots,\ \underbrace{\dots,\mu_{\kappa-1}}_{\cdots\cdots},\ \underbrace{\mu_{\kappa-1}+1,\dots,n}_{D_\kappa}.$$

The order between the main control sets is simply reflected in the order of the subsequences. In case there are control sets with nonempty core that are not main control sets this can be extended in a natural way by considering indices that do not correspond to main control sets, but to control set clusters, see [18].

With this notation we may formulate the the following invariance principle which also motivates the interpretation of control sets and their indices as an extension of eigenspaces and their dimension.

Theorem 6.6 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $u \in U^{\mathbb{N}}$ and $d(u) := \max_{t\in\mathbb{N}} \dim\ker \Phi_u(t,0)$. Let $\mu_1, \dots, \mu_\kappa$ be the indices for the control set structure as described above.*

(i) For every main control set D_j with $\mu_{j-1} > d(u)$ there exists a linear subspace $X_j(u)$ satisfying

$$\dim X_j(u) = m(D_j) = \mu_j - \mu_{j-1}\,,$$

$$\text{for all } t \in \mathbb{N} \text{ it holds that } \mathbb{P}\Phi_u(t,0)X_j(u) \subset cl D_j\,.$$

(ii) If $d(u) > 0$ and a main control set D_j exists such that $\mu_{j-1} < d(u) < \mu_j$ then there exists a linear subspace $X_j(u)$ satisfying

$$\dim X_j(u) = \mu_j - d(u)\,,$$

$$\text{for all } t \in \mathbb{N} \text{ it holds that } \mathbb{P}\Phi_u(t,0)X_j(u) \subset cl D_j\,.$$

Proof. Fix $u \in U^{\mathbb{N}}$. Let $T \in \mathbb{N}$ be large enough such that $\dim\ker \Phi_u(t,0) = d(u)$ for all $t \geq T$. Choose $u^* \in U^{t^*}_{reg}$, for some $t^* \in \mathbb{N}$. For any $t \geq T$ consider the control sequence $v_t := (u_{[0,t-1]}, u^*)$, where $u_{[0,t-1]}$ denotes the first t entries of the sequence u. Denote $|\sigma(\Phi_{v_t}(t+t^*,0))| = \{r_1, \dots, r_\nu\}$, with $0 \leq r_1 < \dots < r_\nu$. For those indices k with $r_k > 0$ consider the sums of generalized eigenspaces $\mathbb{P}GE(r_k, v_t)$. By Proposition 4.4 it follows that $\mathbb{P}GE(r_k, v_t) \subset \mathrm{core}(D)$ for some control set D. Hence we obtain for D that

$$Q_i \cap D \neq \emptyset \text{ for } i = \sum_{l=1}^{k-1} r_l + 1, \dots, \sum_{l=1}^{k} r_l\,.$$

For every main control set D as specified in (i) it follows that there exists a linear subspace $X(u,t)$ satisfying

$$\dim X(u,t) = m(D)\,,$$

$$\mathbb{P}X(u,t) \subset \mathrm{core}(D)\,,$$

$$\Phi_{v_t}(t+t^*,0)X(u,t) = X(u,t)\,.$$

Using the definition of control sets it is easy to see that for all $0 \le s \le t$

$$\Phi_u(s,0)X(u,t) \subset D\,. \tag{30}$$

As the Grassmannian $G^{n,m(D)}_{\mathbb{K}}$, i.e. the space of $m(D)$-dimensional subspaces of $\mathbb{K}^n$ is compact we may choose a subsequence $(t_k)_{k\in\mathbb{N}}$ such that

$$X(u) := \lim_{k\to\infty} X(u,t_k) \tag{31}$$

exists. It follows that $X(u) \subset \mathrm{cl}D$. Assume there exists a $\bar{t} \in \mathbb{N}$ such that

$$\mathbb{P}\Phi_u(\bar{t},0)X(u) \not\subset \mathrm{cl}D\,.$$

Then for all $m(D)$-dimensional subspaces X in an appropriate neighborhood W of $X(u)$ (in the Grassmannian $G^{n,m(D)}_{\mathbb{K}}$) it follows that $\mathbb{P}\Phi_u(\bar{t},0)X \not\subset \mathrm{cl}D$. But for all $t > \bar{t}$ it holds by (30) that $\Phi_u(\bar{t},0)X(u,t) \subset D$. This contradicts (31) so that the proof of (i) is completed.

(ii) now follows as (i) while making the appropriate change of dimension of $X_j(u)$. ■

7 Characteristic exponents

Up to now we have described the control structure of a system on projective space. With the insight that has been gained let us now discuss properties of the set of characteristic exponents that may be deduced from our knowledge about the control sets. For a control set D with nonempty core we define the associated Floquet exponents to be

$$\Sigma_{Fl}(D) := \bigcup_{t\ge 1, u\in U^t} \{\frac{1}{t}\log r;\ r \in |\sigma(\Phi_u(t,0))|,\ \mathbb{P}GE(r,u) \subset \mathrm{core}(D)$$

$$\text{and } \mathbb{P}GE(r,u) \text{ is regular } \}. \tag{32}$$

Let us begin by explaining how to obtain the Lyapunov exponent $\lambda(x_0,u)$ from the trajectory $\xi(\cdot;\mathbb{P}x_0,u)$ of the projected system. For $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$, $u \in U(\xi)$ define

$$q(\xi,u) := \log\frac{\|A(u)x\|}{\|x\|}, \quad \text{where} \quad x \neq 0,\ \mathbb{P}x = \xi. \tag{33}$$

This is well defined as multiplication of x with a non-zero scalar does not alter the value of $q(\xi,u)$. For $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$, $t \in \mathbb{N}$, $u \in U^t(\xi)$ define

$$J(t;\xi,u) = \sum_{s=0}^{t-1} q(\xi(s;\xi,u),u(s)). \tag{34}$$

Then we obtain the following expression for Lyapunov exponents:

Lemma 7.1 *Let $\mathbb{K} = \mathbb{R},\mathbb{C}$. For $x_0 \in \mathbb{K}^n \setminus \{0\}$, $u \in U^{\mathbb{N}}$ it holds that*

$$\lambda(x_0,u) = \begin{cases} \limsup_{t\to\infty} \frac{1}{t}J(t;\mathbb{P}x_0,u), & \text{if} \quad u \in U^{\mathbb{N}}(x_0). \\ -\infty, & \text{otherwise.} \end{cases} \tag{35}$$

The previous lemma shows that we may speak of the Lyapunov exponent corresponding to $(\xi_0, u) \in \mathbb{P}_{\mathbb{K}}^{n-1} \times U^{\mathbb{N}}$ which we denote by $\lambda(\xi_0, u)$. The Floquet spectrum is closely related to the structure of the control sets examined up to now. In order to explore this relationship we need a controllability property in the cores of control sets. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and consider system (8) on $\mathbb{P}_{\mathbb{K}}^{n-1}$. Consider the functions

$$h : \mathbb{P}_{\mathbb{K}}^{n-1} \times \mathbb{P}_{\mathbb{K}}^{n-1} \to \mathbb{N} \cup \{\infty\} \tag{36}$$

$$h(\xi, \eta) := \min\{t \in \mathbb{N};\ \text{there is a } u \in U^t \text{ such that } \xi(t; \xi, u) = \eta\},$$

$$H : M \times M \to \mathbb{R}_+ \cup \{\infty\} \tag{37}$$

$$H(\xi, \eta) := \inf\{\max_{1 \le s \le t} |J(s; \xi, u)|; \quad t \in \mathbb{N};\ u \in U^t \text{ such that } \xi(t; \xi, u) = \eta\},$$

where $\min \emptyset = \inf \emptyset = \infty$.
The previous definition is the discrete–time analogue of the *first–time hitting map*, as defined for instance in [6], [7]. As we treat non–invertible systems as well it is important for us to obtain information not only on the time that elapses to steer from ξ to η, but also on the "cost" incurred in doing so. For the projected system (8) and the function q interpreted as a cost $|q(\xi, u)|$ may be arbitrarily large if u is chosen such that $A(u)$ is almost singular. The essential point is that both these values may be simultaneously bounded if one tries to reach a compact subset of the core of a control set.

Lemma 7.2 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and assume that system (8) is forward accessible. Let $D \subset \mathbb{P}_{\mathbb{K}}^{n-1}$ be a control set. Assume there are two non-void compact sets K_1, K_2 with $K_1 \subset D$ and $K_2 \subset core(D)$, then there are constants $\overline{h} \in \mathbb{N}, \overline{H} \in \mathbb{R}_+$ such that*

$$h(\xi, \eta) \le \overline{h} \text{ for all } \xi \in K_1,\ \eta \in K_2, \tag{38}$$

$$H(\xi, \eta) \le \overline{H} \text{ for all } \xi \in K_1,\ \eta \in K_2. \tag{39}$$

This observation may be used in the proof of the following statement.

Proposition 7.3 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and assume that (8) is forward accessible. For a control set D with $core(D) \neq \emptyset$ the set $cl\Sigma_{Fl}(D)$ is an interval.*

The idea for a proof of this is simple: Take two Floquet exponents in $\Sigma_{Fl}(D)$ and corresponding regular eigenspaces $E(\lambda_1, u_1)$, $E(\lambda_2, u_2)$. In order to show that $[1/t_1 \log|\lambda_1|, 1/t_2 \log|\lambda_2|] \subset \mathrm{cl}\Sigma_{Fl}(D)$ we may choose $\xi_0 \in \mathbb{P}E(\lambda_1, u_1)$ apply the control u l times then choose a control v_1 that steers to $\mathbb{P}E(\lambda_2, u_2)$, apply the control u_2 k times and steer back to ξ_0 via a control v_2, see Figure 5. Using Lemma 7.2 we can bound the time and the perturbation of the Floquet exponent due to the effect of v_1 and v_2. Thus for k, l large it is possible to approximate all rational convex combinations of $1/t_1 \log|\lambda_1|$ and $1/t_2 \log|\lambda_2|$.

The following theorem summarizes the properties of the Floquet spectrum.

Theorem 7.4 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and assume that (8) is forward accessible. Let κ be equal to the number of main control sets.*

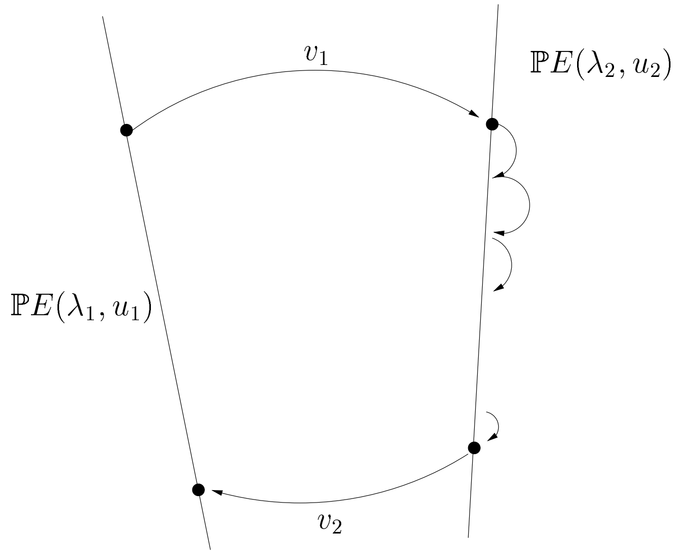

Figure 5: Construction of a dense set of Floquet exponents in the interval $[|\lambda_1|, |\lambda_2|]$.

(i) For each main control set D_j $j = 1, \ldots, \kappa$ the closed Floquet spectrum is an interval. We define

$$cl\Sigma_{FL}(D_j) =: [\alpha_j, \beta_j], \quad \alpha_j \leq \beta_j \,. \tag{40}$$

(ii) If all control sets with nonempty interior are main control sets then

$$cl\Sigma_{FL}(A, U) = \bigcup_{j=1}^{\kappa} [\alpha_j, \beta_j]. \tag{41}$$

(iii) If there exist control sets with nonempty interior that are not main control sets then there exists a constant $\bar{\beta} \in \mathbb{R}$ such that

$$cl\Sigma_{FL}(A, U) = \bigcup_{j=1}^{\kappa} [\alpha_j, \beta_j] \cup [-\infty, \bar{\beta}]. \tag{42}$$

(iv) If for two main control sets $D_{j_1} < D_{j_2}$ then

$$\alpha_{j_1} \leq \alpha_{j_2}, \quad \beta_{j_1} \leq \beta_{j_2}. \tag{43}$$

(v) For $j = 1, \ldots, \kappa$ it holds that

$$\# cl\Sigma_{FL}(D_j) \setminus \Sigma_{FL}(D_j) \leq m(D_j) + 1, \tag{44}$$

It should be noted, that the spectral intervals corresponding to different main control sets may overlap, i.e. that the statement $\alpha_i \leq \alpha_j$, $\beta_i \leq \beta_j$ in Theorem 7.4 does in no way exclude the possibility that $\beta_i > \alpha_j$. In fact, it is even possible that $\alpha_i = \alpha_j$ and $\beta_i = \beta_j$ for $i \neq j$. This phenomenon is shown in an example in [19].

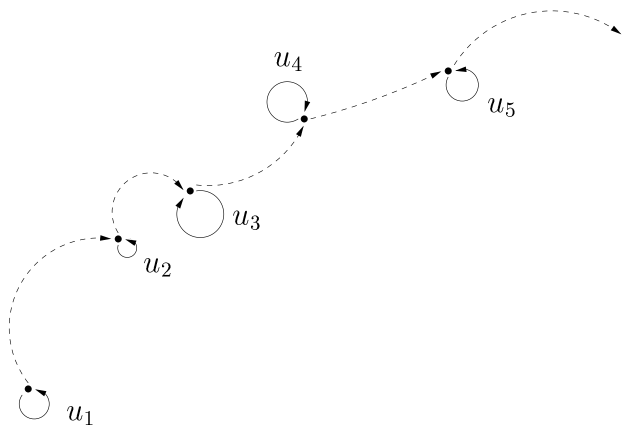

Figure 6: Approximation of a Lyapunov exponent in the closure of the Floquet exponents.

Theorem 7.5 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and assume that (8) is forward accessible.*

(i) Let D be a control set, with core$(D) \neq \emptyset$. Assume that $(\xi_0, u) \in \mathbb{P}_{\mathbb{K}}^{n-1} \times U^{\mathbb{N}}(\xi_0)$ are given with $\omega^+(\xi_0, u) \subset D$. If there exists a $t_0 \in \mathbb{N}$ with $\xi(t_0; \xi_0, u) \in$ core(D) then $\lambda(\xi_0, u) \in cl\Sigma_{Fl}(D)$.

(ii) Let D be a control set, with core$(D) \neq \emptyset$, then

$$cl\Sigma_{Fl}(D) \subset \Sigma_{Ly}(A, U). \tag{45}$$

For the proof of assertion (ii) take a sequence $\{\lambda_k\} \subset \Sigma_{Fl}(D)$ such that $\lambda_k \to \lambda^*$. Denote the associated control sequences and eigenspaces by u_k and E_k. Then a trajectory with Lyapunov exponent λ^* may be constructed by first following the control u_1 on the eigenspace E_1 then steering to E_2 and applying the control u_2 such that the Lyapunov exponent is up to ε_2 close to λ_2, then steering to E_3 and so forth, see Figure 6. If the sequence ε_k tends to zero then it is possible to show that the constructed trajectory has indeed the desired Lyapunov exponent. A further question of interest, especially if stabilization and robust stability questions are considered, concerns the lower and upper bounds of the spectral sets that we have defined. For upper bounds this has been studied in [3], [5], [12], [14] in a more general case then the one presented here. A consequence of the results in these references is that suprema of Floquet, Lyapunov and Bohl exponents coincide, from which it is possible to infer that also infima of Floquet and Lyapunov exponent coincide. Barabanov [4] proved that to each discrete inclusion given by a bounded set of matrices there exists a trajectory that realizes the maximal Lyapunov exponent. As should be expected trajectories realizing extremal Lyapunov exponents can be chosen to evolve in the invariant, respectively open control set, corresponding to the realization of the largest and the smallest exponent.

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