On stability radii of infinite dimensional time–varying discrete-time systems

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Abstract

This note introduces a stability radius for discrete-time linear time-varying systems on Banach spaces under structured time-varying perturbations of multi–output feedback type. Additive perturbations are considered which can be represented as a series of infinitely many perturbation terms. We derive counterparts to some results established for time-varying differentiable systems in [5], [7] and apply the results to periodic systems.

<u>Notation</u>

X, U_i, Y_i, U, Y	Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ with norms $\|\cdot\|_X$ etc.
$\mathcal{L}(X, Y)$	Banach space of bounded linear operators from X to Y provided with the operator norm $\|.\|_{\mathcal{L}(X,Y)}$
$\mathbb{N}_{t_0}$	$= \{t \in \mathbb{N};\ t_0 \leq t\}$
$l^2(t_0, \infty; Y)$	set of square summable Y-valued sequences $(y(t))_{t\in\mathbb{N}_{t_0}}$
$\|y(\cdot)\|_{l^2(t_1,t_2)}$	$= \left(\sum_{k=t_1}^{t_2} \|y(k)\|_Y^2\right)^{1/2}$ for $0 \leq t_1 \leq t_2 \leq \infty$
$l^\infty(t_0, \infty; Y)$	set of bounded Y-valued sequences $(y(t))_{t\in\mathbb{N}_{t_0}}$
$\mathbb{D}$	open unit disk of $\mathbb{C}$

1 Introduction

Let X be a Banach space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We consider time–varying linear discrete-time systems of the form

$$x(t+1) = A(t)x(t), \quad t \in \mathbb{N} \tag{1.1}$$

where $A(\cdot) = (A(t))_{t\in\mathbb{N}} \in \mathcal{L}(X)^{\mathbb{N}}$ is a sequence of linear operators on X. Such a system is called *exponentially stable,* if the associated evolution operator

$$\Phi(t,t) = I_X, \quad \Phi(t,s) = A(t-1)\cdots A(s), \qquad s,t \in \mathbb{N}, \quad t > s \tag{1.2}$$

satisfies

$$\|\Phi(t,s)\|_{\mathcal{L}(X)} \leq c\,\beta^{t-s}, \qquad s,t \in \mathbb{N}, \quad t \geq s \tag{1.3}$$

for some constants $c, \beta > 0$, $\beta < 1$. In this case $A(\cdot)$ is a bounded sequence in $\mathcal{L}(X)$, i.e. $A(\cdot) \in \ell^\infty(\mathbb{N}, \mathcal{L}(X))$.

Stability problems for (time–varying) discrete-time systems in Banach spaces have received some attention over the last few years, see [18],[16], [17], [24], [12] and references therein. Several authors have noted the possibility of studying functional differential systems, periodic systems and sampled-data systems in this context, see [13], [16], [4]. For these applications it is essential that discrete-time systems on infinite dimensional spaces are considered.

In this paper we will investigate the *robustness* of exponentially stable systems (1.1) under various types of perturbations acting on the generator $A(\cdot) \in \ell^\infty(\mathbb{N}, \mathcal{L}(X))$. Over the last decade, problems of robust stability and robust stabilization have been very prominent in control theory. Most of the literature deals with the time–invariant finite dimensional case. For state space systems of the form $\dot{x} = Ax$ or $x(t+1) = Ax(t) \quad (A \in \mathcal{L}(\mathbb{K}^n))$ linear perturbations of the form

(a) $A \rightsquigarrow A_\Delta = A + \Delta, \quad \Delta \in \mathcal{L}(\mathbb{K}^n) \quad$ (unstructured perturbations)

(b) $A \rightsquigarrow A_\Delta = A + D\Delta E, \quad \Delta \in \mathcal{L}(\mathbb{K}^q, \mathbb{K}^\ell) \quad$ (simple structured perturbation)

(c) $A \rightsquigarrow A_\Delta = A + \sum_{i=1}^{N} D_i \Delta_i E_i, \quad \Delta_i \in \mathcal{L}(\mathbb{K}^{q_i}, \mathbb{K}^{\ell_i}) \quad$ (multi-perturbations).

have been considered, where D, E, D_i, E_i are given operators defining the perturbation structure. A typical question is to determine the *stability radius* of a given system under these perturbations, i.e. the largest real number $r_\mathbb{K}$ such that no perturbation Δ of operator norm $\|\Delta\| < r_\mathbb{K}$ destabilizes the system. In the complex case ($\mathbb{K} = \mathbb{C}$) a fairly complete theory of the stability radius $r_\mathbb{C}$ is available for perturbations of the form (a), (b), see the survey [6] and references therein, where also results concerning stability radii of discrete-time systems can be found. For extensions to infinite dimensional systems described by semigroups of operators on a complex Banach space, see [15]. Time–varying and non-linear perturbations of time-invariant linear systems have been considered in [9], [11].

The problem of calculation of the real stability radius has been recently solved in the simple structured case [20]. However, when it comes either to real perturbations of more complicated structure, see [6], or in the complex case to perturbations of the form (c) of time–invariant systems or to any type of perturbations of *time–varying* systems the theory is far from complete. For time–varying (differentiable or discrete time) systems as well as for multi-perturbations of time–varying systems no computable formulae are available for the complex stability radius. There are lower bounds which can be significantly improved by scaling techniques, see [3] and [5], respectively. These scaling techniques were combined to analyze the robustness of time–varying differentiable systems under time–varying multi-perturbations in [8] for the finite and [7] for the infinite dimensional case. To our knowledge analogous questions have not yet been investigated for discrete-time systems.

Our motivation to consider infinite dimensional systems (1.1) comes partly from applications to periodic and delay systems and partly from recent results for continuous time evolution systems on Banach spaces [15]. Contrary to the finite–dimensional case there are basic differences between continuous time and discrete-time linear systems in the infinite dimensional context. Here an important special feature of the discrete-time case is that the generator $A(t)$ has the same domain of definition as the evolution operator $\Phi(t,s)$ whereas there is usually a gap between these domains of definition in the continuous time case. As a consequence it is not necessary to admit unbounded structure operators D_i, E_i in (c). Moreover exponential stability implies boundedness of $\{A(t); t \in \mathbb{N}\}$ in $\mathcal{L}(X)$ and so it is natural to assume that the structure operators $D_i(t), E_i(t)$ are bounded in time. Therefore the technical difficulties arising from the double unboundedness of $D_i(\cdot), E_i(\cdot)$ (see [7]) are not present in the discrete-time case. Most importantly, the solvability problem for the perturbed equation disappears so that – contrary to the continuous time case – the stability problem can be studied in its own right without intervention of the existence problem for the perturbed evolution operator.

We proceed as follows. In the next section we present a detailed problem formulation and collect some preliminary results concerning Bohl exponents and Bohl transformations. In Section 3 we derive a discrete-time counterpart of a lower estimate for $r_{\mathbb{C}}$ [5], we demonstrate by an example that this lower estimate is in general not tight and describe a scaling technique for improving the lower bounds. Once the right framework is set, the proofs are not more difficult than in the finite dimensional case and simpler than for infinite dimensional continuous time evolution systems [7]. In Section 4 we consider the exponential stability of time-invariant systems (1.1) under multi-perturbation of the form (c). A formula for $r_{\mathbb{C}}$ involving the μ-function [6] is extended to infinite–dimensions and to an infinite series of perturbation terms. In Section 5 some recent results concerning time–varying, nonlinear and dynamic perturbations [9] are also extended to the infinite dimensional context. In Section 6 the previous results are applied to obtain stability conditions for periodic perturbed discrete-time systems.

2 Problem Formulation and Preliminary Results

Throughout the paper we assume that $A(\cdot) \in l^\infty(\mathbb{N}; \mathcal{L}(X))$ is a given sequence of operators on the Banach space X such that the system (1.1) is exponentially stable. An important problem in robustness analysis is that of determining the extent to which exponential stability is preserved under various types of parameter perturbations. Let $U_i, Y_i, i \in \mathbb{N}$ be Banach spaces over $\mathbb{K}$ and suppose that $A(\cdot)$ is subjected to affine perturbations of the form

$$A(\cdot) \rightsquigarrow A(\cdot) + \sum_{i=1}^{\infty} D_i(\cdot)\Delta_i(\cdot)E_i(\cdot) \tag{2.1}$$

where for all $i \in \mathbb{N}$ we assume $D_i(\cdot) \in l^\infty(\mathbb{N}; \mathcal{L}(U_i, X))$, $E_i(\cdot) \in l^\infty(\mathbb{N}; \mathcal{L}(X, Y_i))$ are given and $\Delta_i(\cdot) \in l^\infty(\mathbb{N}; \mathcal{L}(Y_i, U_i))$ are unknown *uniformly bounded* sequences of perturbation operators. To ensure convergence we will assume that

$$\sup_{t\in\mathbb{N}} \sum_{i=1}^{\infty} \|D_i(t)\|^2_{\mathcal{L}(U_i,X)} < \infty, \quad \sup_{t\in\mathbb{N}} \sup_{x\in X, \|x\|=1} \sum_{i=1}^{\infty} \|E_i(t)x\|^2_{Y_i} < \infty, \quad \sup_{i,t\in\mathbb{N}} \|\Delta_i(t)\|_{\mathcal{L}(Y_i,U_i)} < \infty. \tag{2.2}$$

This assumption on the structure operators is automatically satisfied in the case of finitely many perturbation operators where $U_i = Y_i = \{0\}, i > N$ for some $N \in \mathbb{N}$. In that case, (2.1) has the form

$$A(\cdot) \rightsquigarrow A(\cdot) + \sum_{i=1}^{N} D_i(\cdot)\Delta_i(\cdot)E_i(\cdot). \tag{2.3}$$

In order to rewrite the perturbed system equation

$$x(t+1) = \left[A(t) + \sum_{i=1}^{\infty} D_i(t)\Delta_i(t)E_i(t)\right] x(t)\ , \quad t \in \mathbb{N} \tag{2.4}$$

we introduce the vector spaces

$$U = \{(u_i) \in \prod_{i\in\mathbb{N}} U_i; \quad \sum_{i=1}^{\infty} \|u_i\|^2_{U_i} < \infty\}, \quad Y = \{(y_i) \in \prod_{i\in\mathbb{N}} Y_i; \quad \sum_{i=1}^{\infty} \|y_i\|^2_{Y_i} < \infty\}. \tag{2.5}$$

U and Y are Banach spaces with respect to the norms $\|(u_i)\|_U = (\sum_{i=1}^{\infty} \|u_i\|_{U_i}^2)^{1/2}$ and $\|(y_i)\|_Y = (\sum_{i=1}^{\infty} \|y_i\|_{Y_i}^2)^{1/2}$ respectively. We define for all $t \in \mathbb{N}$ bounded linear operators

$$\begin{aligned} D(t) : U \to X \quad &; \qquad (u_i)_{i\in\mathbb{N}} \mapsto \sum_{i=1}^{\infty} D_i(t) u_i \\ E(t) : X \to Y \quad &; \qquad x \mapsto (E_i(t)x)_{i\in\mathbb{N}} \\ \Delta(t) :\ Y \to U \quad &; \qquad (y_i)_{i\in\mathbb{N}} \mapsto (\Delta_i(t) y_i)_{i\in\mathbb{N}}. \end{aligned} \tag{2.6}$$

Then (2.4) takes the simpler form

$$x(t+1) = [A(t) + D(t)\Delta(t)E(t)]\, x(t)\ , \quad t \in \mathbb{N}. \tag{2.7}$$

By (2.2) we have $D(\cdot) \in l^\infty(\mathbb{N}, \mathcal{L}(U,X)), E(\cdot) \in l^\infty(\mathbb{N}, \mathcal{L}(X,Y)), \Delta(\cdot) \in l^\infty(\mathbb{N}, \mathcal{L}(Y,U))$. The size of the "block diagonal" perturbation operator $\Delta(\cdot)$ is measured by the l^∞–norm

$$\|\Delta(\cdot)\|_\infty = \sup_{t\in\mathbb{N}} \sup_{i\in\mathbb{N}} \|\Delta_i(t)\|_{\mathcal{L}(Y_i,U_i)}. \tag{2.8}$$

Note that

$$\|\Delta(t)\| := \sup_{i\in\mathbb{N}} \|\Delta_i(t)\|_{\mathcal{L}(Y_i,U_i)} = \|\Delta(t)\|_{\mathcal{L}(Y,U)} \tag{2.9}$$

is the operator norm of $\Delta(t) \in \mathcal{L}(Y,U)$.

Our aim is to determine the minimal size $\|\Delta(\cdot)\|_\infty$ of a block diagonal disturbance $\Delta(\cdot)$ of the form (2.6) which destabilizes the system (2.7).

Definition 2.1 *The stability radius of (1.1) with respect to time-varying perturbations of structure $(D_i(\cdot), E_i(\cdot))_{i\in\mathbb{N}}$ is defined by*

$$r_{\mathbb{K}}^1(A; (D_i, E_i)) = \inf\{\|\Delta(\cdot)\|_\infty; \Delta_i(\cdot) \in l^\infty(\mathbb{N}; \mathcal{L}(Y_i, U_i)) \text{and (2.4) is not exponentially stable}\} \tag{2.10}$$

where we set $\inf \emptyset = \infty$.

Suppose $A(t) \equiv A \in \mathcal{L}(X)$, $D_i(t) \equiv D_i \in \mathcal{L}(U_i, X)$, $E_i(t) \equiv E_i \in \mathcal{L}(X, Y_i)$, $i \in \mathbb{N}$. *The* *stability radius of (1.1) with respect to constant perturbations of structure* $(D_i, E_i)_{i\in\mathbb{N}}$ *is defined by*

$$r^1_{\mathbb{K},c}(A;(D_i,E_i)) = \inf\{\|\Delta\|;\, \Delta_i \in \mathcal{L}(Y_i,U_i) \text{ and (2.4) is not exponentially stable}\}. \qquad (2.11)$$

In the unstructured case ($N = 1$ in (2.3) and $D_1 = E_1 = I_X$) $r^1_{\mathbb{C},c}(A) = r^1_{\mathbb{C},c}(A;(I_X,I_X))$ is the distance of A from the set of not exponentially stable operators in the normed space $\mathcal{L}(X)$.

The following time-invariant examples show that a wide class of Hilbert-Schmidt operators of fixed structure can be represented in the form $\sum\limits_{i\in\mathbb{N}} D_i\Delta_i E_i$ where $D_i, E_i, i \in \mathbb{N}$ are rank one operators satisfying assumption (2.2). A similar construction is applicable for nuclear operators in Banach spaces.

Example 2.2 Assume X is a Hilbert space over $\mathbb{C}$ and $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of X. A family $(\eta_{i,j})_{i,j\in\mathbb{N}}$ with $\sum\limits_{i,j\in\mathbb{N}} |\eta_{i,j}|^2 < \infty$ defines a Hilbert-Schmidt operator via

$$A = \sum_{i,j\in\mathbb{N}} \eta_{i,j} e_i \otimes e_j$$

where $e_i \otimes e_j \in \mathcal{L}(X)$ is the rank one operator defined by $x \mapsto (x|e_j)e_i$ [14]. Consider a time-invariant linear system

$$x(t+1) = Ax(t)\,, \quad t \in \mathbb{N}.$$

A can be interpreted as an infinite-dimensional matrix with entries $\eta_{i,j}$. Suppose we need to model independent uncertainties for certain entries of A, i.e. there is a subset $K \subset \mathbb{N}\times\mathbb{N}$ containing the indices of the uncertain entries. To each $k \in K$ we associate a scaling factor c_k describing the relative uncertainty of the entry k of A. We define for all $k = (i_k, j_k) \in K$

$$D_k : \mathbb{C} \to X\ ,\ \alpha \mapsto \alpha c_k e_{i_k}$$

where $c_k \in \mathbb{C}$ with $\sum\limits_{k\in K} |c_k|^2 < \infty$.

$$E_k : X \to \mathbb{C}\ ,\ x \mapsto (x|e_{j_k}).$$

The unknown perturbations δ_k are linear operators on $\mathbb{C}$, i.e. complex numbers. If we set $\delta_{i,j} = c_{i,j} = 0$ if $(i,j) \notin K$ the perturbed operator is of the form

$$A + \sum_{k \in K} D_k \delta_k E_k = \sum_{i,j \in \mathbb{N}} (\eta_{i,j} + c_{i,j}\delta_{i,j}) e_i \otimes e_j \ , \quad (\delta_k) \in l^\infty(K, \mathbb{C}). \tag{2.12}$$

Hence the perturbed operators will be Hilbert-Schmidt operators. The set of constant perturbations Δ with $\|\Delta\| < \rho$ consists of all Hilbert-Schmidt operators of the form $\sum_{k \in K} \delta_k c_k e_{i_k} \otimes e_{j_k}$ satisfying

$$\sup_{k \in K} |\delta_k| < \rho.$$

□

In the general time-varying case it is very difficult to determine the stability radius $r_{\mathbb{K}}$. We will only be able to derive lower bounds for $r_{\mathbb{K}}$. In the remaining part of this section we present some preliminary results which will help us to improve the bounds by scaling techniques.

The largest exponential growth rate of system (1.1) is given by the discrete time version of the (upper) *Bohl* exponent [2] (named generalized spectral radius in [17]). In the following definition we do not assume (1.1) to be exponentially stable and let $(A(t))_{t \in \mathbb{N}}$ be an arbitrary sequence in $\mathcal{L}(X)$.

Definition 2.3 (Bohl exponent) *Given a sequence $(A(t))_{t \in \mathbb{N}}$ in $\mathcal{L}(X)$ the (upper) Bohl exponent of the system (1.1) is*

$$\beta(A(\cdot)) = \inf\{\beta ; \exists c_\beta \geq 1 : t \geq s \geq 0 \Rightarrow \|\Phi(t,s)\| \leq c_\beta \beta^{t-s}\}. \tag{2.13}$$

$\beta(A(\cdot))$ may be infinite, but if $\|A(t)\| \leq \gamma$ for all $t \in \mathbb{N}$ then it follows from (1.2) that

$$\|\Phi(t,s)\| \leq \|A(t-1)\| \cdots \|A(s)\| \leq \gamma^{t-s}, \tag{2.14}$$

hence $\beta(A(\cdot)) \leq \gamma$.Thus $\beta(A(\cdot))$ is finite if and only if $(A(t))_{t \in \mathbb{N}}$ is bounded. The Bohl exponent $\beta : l^\infty(\mathbb{N}; \mathcal{L}(X)) \to \mathbb{R}_+$ has the following properties.

Proposition 2.4 *Let $(A(t))_{t\in\mathbb{N}} \in l^\infty(\mathbb{N};\mathcal{L}(X))$. Then*

(i) $\beta(A(\cdot)) = \limsup_{s,t-s\to\infty} \|\Phi(t,s)\|^{\frac{1}{t-s}}$

(ii) The function $\beta : (l^\infty(\mathbb{N};\mathcal{L}(X)), \|\cdot\|_\infty) \to \mathbb{R}_+$ *is upper semicontinuous.*

(iii) If $A(t) = A \in \mathcal{L}(X)$ *is constant in* $t \in \mathbb{N}$ *then*

$$\beta(A) = \lim_{t\to\infty} \|A^t\|^{1/t} = r(A)$$

is the spectral radius of A.

(iv) The following statements are equivalent:

(a) (1.1) is exponentially stable.

(b) $\beta(A(\cdot)) < 1$

(c) $\exists \gamma > 0\ \forall s \in \mathbb{N}\ \forall x_0 \in X:\ \sum_{t=s}^{\infty} \|\Phi(t,s)x_0\|^2 \le \gamma^2 \|x_0\|^2.$

Proof: The proof of the above statements can be found in [17], except for the equivalence of (iv)(c) to the exponential stability of (1.1), which can be found in [18]. □

The Bohl exponent of (1.1) is said to be *strict* if "lim sup" in (i) can be replaced by "lim".

Remark 2.5 a) To verify the exponential stability of (1.1) via Proposition 2.4 (iv) it is sufficient to prove (iv)(c) for all $s \in \mathbb{N}_{t_0}$, where $t_0 \in \mathbb{N}$ is any given initial time.

b) If $A(\cdot) = A \in \mathcal{L}(X)$ is constant we obtain from Proposition 2.4 the well-known spectral characterization of exponential stability: $x(t+1) = Ax(t)$ *is exponentially stable if and only if* $r(A) < 1$, i.e. the spectrum of A lies in the open unit disk $\mathbb{D}$. □

Definition 2.6 *Given Banach spaces X, Y over $\mathbb{K}$, two sequences $(A(t))_{t\in\mathbb{N}}$ and $(B(t))_{t\in\mathbb{N}}$ in $\mathcal{L}(X,Y)$ are called asymptotically equivalent if* $\lim_{t\to\infty} \|A(t) - B(t)\| = 0$.

Proposition 2.7 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume (1.1) is exponentially stable and $(D_i(\cdot))_{i\in\mathbb{N}}, (E_i(\cdot))_{i\in\mathbb{N}}$ satisfy (2.2). Let $(B(t))_{t\in\mathbb{N}}, (F_i(t))_{t\in\mathbb{N}}, (G_i(t))_{t\in\mathbb{N}}$ be sequences in $\mathfrak{L}(X), \mathfrak{L}(U_i, X)$ $\mathfrak{L}(X, Y_i)$, respectively, such that (2.2) is satisfied. The operators constructed via (2.6) will be denoted by $D(\cdot), E(\cdot), F(\cdot), G(\cdot)$, respectively. If $(A(t))_{t\in\mathbb{N}}$,$(D(t))_{t\in\mathbb{N}}$ and $(E(t))_{t\in\mathbb{N}}$ are asymptotically equivalent to $(B(t))_{t\in\mathbb{N}}, (F(t))_{t\in\mathbb{N}}, (G(t))_{t\in\mathbb{N}}$, respectively then*

$$x(t+1) = B(t)x(t), \quad t \in \mathbb{N}$$

is exponentially stable and

$$r^1_{\mathbb{K}}(A; (D_i, E_i)) = r^1_{\mathbb{K}}(B; (F_i, G_i)).$$

Proof: In [17] it is shown that the Bohl exponents of asymptotically equivalent systems coincide. Hence the first assertion follows from Proposition 2.4(iv). For any fixed block diagonal perturbation $\Delta(\cdot)$, $(D(t)\Delta(t)E(t))_{t\in\mathbb{N}}$ is asymptotically equivalent to $(F(t)\Delta(t)G(t))_{t\in\mathbb{N}}$, because

$$\|D(t)\Delta(t)E(t) - F(t)\Delta(t)G(t)\| \leq$$

$$\|D(t) - F(t)\|\|\Delta(t)E(t)\| + \|F(t)\Delta(t)\|\|E(t) - G(t)\|.$$

Thus the Bohl exponents of the perturbed systems are equal, if the perturbations are equal. This implies the second assertion. □

Note that in the preceding Proposition (2.7) it is not sufficient to assume that for every $i \in \mathbb{N}$ $(D_i(t))_{t\in\mathbb{N}}$, $(E_i(t))_{t\in\mathbb{N}}$ are asymptotically equivalent to $(F_i(t))_{t\in\mathbb{N}}$ resp. $(G_i(t))_{t\in\mathbb{N}}$. Simple counterexamples can be constructed using the perturbed Hilbert-Schmidt operators of Example 2.2.

Definition 2.8 *A sequence $(T(t))_{t\in\mathbb{N}}$ of invertible transformations $T(t) \in \mathfrak{L}(X)$ is said to be a Bohl transformation if $(T(t))_{t\in\mathbb{N}}$ and $(T(t)^{-1})_{t\in\mathbb{N}}$ are both uniformly bounded in $\mathfrak{L}(X)$.*

Let us briefly discuss the effect of Bohl transformations on the stability radius. If we set $\hat{x}(t) = T(t)^{-1}x(t)$ the nominal system equation (1.1) can be rewritten as

$$\hat{x}(t+1) = T(t+1)^{-1}A(t)T(t)\hat{x}(t).$$

The associated evolution operator is

$$\hat{\Phi}(t,s) = T(t)^{-1}\Phi(t,s)T(s). \tag{2.15}$$

The perturbed system equation can be rewritten

$$\hat{x}(t+1) = T(t+1)^{-1}A(t)T(t)\hat{x}(t) + \sum_{i=1}^{\infty}[T(t+1)^{-1}D_i(t)]\Delta_i(t)[E_i(t)T(t)]\hat{x}(t)\ , \quad t \in \mathbb{N}. \tag{2.16}$$

The following proposition summarizes some elementary properties of discrete-time Bohl transformations.

Proposition 2.9 *(i) The Bohl transformations form a group with respect to (pointwise) multiplication.*

(ii) The Bohl exponent is invariant with respect to Bohl transformations.

(iii) If $\alpha_i = (\alpha_i(t))_{t\in\mathbb{N}} \in (\mathbb{K}^*)^{\mathbb{N}},\ i \in \mathbb{N}$ are scalar Bohl transformations and we denote

$$D_i(\alpha)(t) = D_i(t)\alpha_i(t)^{-1},\ E_i(\alpha)(t) = \alpha_i(t)E_i(t),\ t \geq 0 \tag{2.17}$$

for $i \in \mathbb{N}$, then

$$r^1_{\mathbb{K}}(A; (D_i(\alpha), E_i(\alpha))) = r^1_{\mathbb{K}}(A; (D_i, E_i)).$$

(iv) If $T(\cdot) \in l^\infty(\mathbb{N}, \mathcal{L}(X))$ is a Bohl transformation and

$$\hat{A}(t) = T(t+1)^{-1}A(t)T(t), \quad \hat{D}_i = T(t+1)^{-1}D_i(t), \quad \hat{E}_i(t) = E_i(t)T(t),\ t \in \mathbb{N}$$

then

$$r^1_{\mathbb{K}}(A; (D_i, E_i)) = r^1_{\mathbb{K}}(\hat{A}; (\hat{D}_i, \hat{E}_i)). \tag{2.18}$$

Proof: (i) follows applying Proposition 2.4 (ii).(ii) is obvious. (iii) is an immediate consequence of $\Delta_i(t) = \alpha_i(t)^{-1}\Delta_i(t)\alpha_i(t)$. To prove (iv) note that $T(\cdot)$ being a Bohl transformation, (2.16) is exponentially stable if and only if (2.4) is. This implies (2.18). □

Remark 2.10 (a) Note that $(D_i(\alpha), E_i(\alpha))$ defined in the above Proposition 2.9 does not necessarily satisfy (2.2) even though the initial data did, and vice versa. This is because for $t \in \mathbb{N}$ the set $\{\alpha_i(t); i \in \mathbb{N}\}$ need not be bounded. However, the notion of the stability radius is still appliquable for such a system, as a perturbation Δ will either destabilize both the original and the transformed system or not.

(b) The Bohl transformations described in Proposition 2.9(iv) transform the whole state space system (A, D, E), whereas the scalar Bohl transformations in Proposition 2.9(iii) act on the structure operators (D,E) only, which is a quite different concept. Arbitrary Bohl transformations applied to the structure operators only, will in general change the stability radius. □

3 Complex stability radius for time-varying systems

Throughout this section we assume that $\mathbb{K} = \mathbb{C}$, (1.1) is exponentially stable with evolution operator $\Phi(t,s)$ and $D_i(\cdot) \in l^\infty(\mathbb{N}; \mathcal{L}(U_i, X))$, $E_i(\cdot) \in l^\infty(\mathbb{N}; \mathcal{L}(X, Y_i))$, $i \in \mathbb{N}$ are given bounded operator sequences satisfying (2.2). The solution of (2.4) with initial value $x(t_0) = x_0$ is

$$x(t) = \Phi(t,t_0)x_0 + \sum_{k=t_0}^{t-1} \Phi(t,k+1)D(k)\Delta(k)E(k)x(k) \ , \ t \in \mathbb{N}_{t_0} \tag{3.1}$$

where $D(k)$, $\Delta(k)$, $E(k)$ are defined by (2.6).

If $y(t) = y(t, t_0, x_0)$ is the associated output then

$$y(t) = y_0(t) + \sum_{k=t_0}^{t-1} E(t)\Phi(t,k+1)D(k)\Delta(k)E(k)x(k) \ , \quad t \in \mathbb{N}_{t_0} \tag{3.2}$$

where $y_0(\cdot) = E(\cdot)\Phi(\cdot, t_0)x_0$ is in $l^2(\mathbb{N}_{t_0}, Y)$ (because of (1.3)). Consider the system

$$x(t+1) \quad = \quad A(t)x(t) + D(t)u(t) \, , \quad t \in \mathbb{N}_{t_0}$$

$$y(t) \quad = \quad E(t)x(t). \tag{3.3}$$

The *input–state* and *input–output* operators of this system at time t_0 are given, respectively, by

$$(\mathbb{M}_{t_0} u)(t) = \sum_{k=t_0}^{t-1} \Phi(t, k+1) D(k) u(k), \quad t \in \mathbb{N}_{t_0} \tag{3.4}$$

$$(\mathbb{L}_{t_0} u)(t) = \sum_{k=t_0}^{t-1} E(t) \Phi(t, k+1) D(k) u(k), \quad t \in \mathbb{N}_{t_0}. \tag{3.5}$$

In particular $(\mathbb{M}_{t_0} u)(t_0) = 0, (\mathbb{L}_{t_0} u)(t_0) = 0$.

Because of (1.3), $\mathbb{M}_{t_0}$, $\mathbb{L}_{t_0}$ define bounded linear operators from $l^2(\mathbb{N}_{t_0}, U)$ into $l^2(\mathbb{N}_{t_0}, X)$ and $l^2(\mathbb{N}_{t_0}, Y)$, respectively. Using these operators we can rewrite (3.1), (3.2) as

$$x(\cdot, t_0, x_0) = \Phi(\cdot, t_0) x_0 + (\mathbb{M}_{t_0} \Delta_{t_0} y(\cdot, t_0, x_0))(\cdot) \tag{3.6}$$

$$y(\cdot, t_0, x_0) = y_0(\cdot, t_0, x_0) + (\mathbb{L}_{t_0} \Delta_{t_0} y(\cdot, t_0, x_0))(\cdot) \tag{3.7}$$

where

$$\Delta_{t_0} : l^2(\mathbb{N}_{t_0}; Y) \mapsto l^2(\mathbb{N}_{t_0}; U) \tag{3.8}$$

is the multiplication operator defined by $\Delta(\cdot)$. Clearly the operator norm of Δ_{t_0} is uniformly bounded in $t_0 \in \mathbb{N}$

$$\|\Delta_{t_0}\| \leq \|\Delta\|_\infty = \sup_{t \in \mathbb{N}} \|\Delta(t)\|_{\mathcal{L}(Y,U)}. \tag{3.9}$$

Theorem 3.1 *Let (1.1) be exponentially stable and* $(D_i(t))_{t\in\mathbb{N}}, (E_i(t))_{t\in\mathbb{N}}$ *bounded sequences in* $\mathcal{L}(U_i, X)$*, respectively* $\mathcal{L}(X, Y_i)$*,* $i \in \mathbb{N}$ *such that (2.2) is satisfied. Then*

$$r^1_{\mathbb{C}}(A; (D_i, E_i)) \geq r^1_{\mathbb{C}}(A; D, E) \geq \|\mathbb{L}_{t_0}\|^{-1} \quad \textit{for all} \quad t_0 \in \mathbb{N}. \tag{3.10}$$

Proof: The first inequality follows since the perturbed system (2.4) can be written in the form (2.7) with block diagonal $\Delta(t)$ using (2.6) where (2.9) holds for the norm. In order to prove the second inequality, let $t_0 \in \mathbb{N}$ be given and $(\Delta(t))_{t\in\mathbb{N}}$ any bounded sequence in $\mathcal{L}(Y, U)$

such that $\|\Delta\|_\infty < \|\mathbb{L}_{t_0}\|^{-1}$. We have to show that the system (2.7) is exponentially stable. Let $\Phi_\Delta(\cdot,\cdot)$ be the evolution operator generated by

$$x_\Delta(t+1) = [A(t) + D(t)\Delta(t)E(t)]x_\Delta(t).$$

By Proposition 2.4(iv)(c) and Remark 2.5 it is sufficient to show

$$\exists \gamma_\Delta > 0 \; \forall t_0' \in \mathbb{N}_{t_0} \; \forall x_0 \in X: \; \|x_\Delta(\cdot; t_0', x_0)\|_{l^2(\mathbb{N}_{t_0'}, X)} \le \gamma_\Delta \|x_0\|. \tag{3.11}$$

Now $\|\mathbb{L}_{t_0}\|$, $\|\mathbb{M}_{t_0}\|$, $\|\Delta_{t_0}\|$ are nonincreasing in $t_0 \in \mathbb{N}$, hence $\mathbb{L}_{t_0'}\Delta_{t_0'}$ is a contraction on $L^2(t_0', \infty; Y)$ and

$$\|\mathbb{L}_{t_0'}\Delta_{t_0'}\| \le \|\mathbb{L}_{t_0}\|\|\Delta_{t_0}\| =: \alpha < 1 \quad , \; t_0' \ge t_0.$$

By Banach's fixed point theorem there exists a solution $y(\cdot) \in l^2(\mathbb{N}_{t_0'}, Y)$ of

$$y(\cdot) = y_0(\cdot; t_0', x_0) + \left(\mathbb{L}_{t_0'}\Delta_{t_0'} y(\cdot)\right)(\cdot). \tag{3.12}$$

Moreover

$$\|y(\cdot)\|_{l^2(\mathbb{N}_{t_0'}, Y)} \le (1-\alpha)^{-1} \|y_0(\cdot; t_0', x_0)\|_{l^2(\mathbb{N}_{t_0'}, Y)} \le \frac{\|E(\cdot)\|_\infty \gamma \|x_0\|}{1-\alpha} \tag{3.13}$$

where γ is chosen as in Proposition 2.4(iv)(c). Since (3.7) has a *unique* solution $y(\cdot; t_0, x_0)$ in $Y^{\mathbb{N}_{t_0}}$ (for all $t_0 \in \mathbb{N}$) we have $y(\cdot) = y(\cdot; t_0', x_0)$. It follows from (3.6) (with t_0' instead of t_0) and (3.13) that

$$\begin{aligned} \|x_\Delta(\cdot; t_0', x_0)\|_{l^2(\mathbb{N}_{t_0'}, X)} &\le \gamma\|x_0\| + \|\mathbb{M}_0\|\|\Delta\|_\infty \|y(\cdot; t_0', x_0)\|_{l^2(\mathbb{N}_{t_0'}, Y)} \\ &\le \left[\gamma + \|\mathbb{M}_0\|\|\Delta\|_\infty \|E(\cdot)\|_\infty \frac{\gamma}{1-\alpha}\right] \|x_0\| \quad , \; \forall t_0' \ge t_0, \; x_0 \in X. \end{aligned}$$

Thus (3.11) is satisfied.

□

Even for time–varying systems with a *single* perturbation, equality does *in general not* hold in (3.10) as the following example illustrates.

Example 3.2 Consider a scalar system of period two,

$$x(t+1) = [a(t) + d(t)\delta(t)e(t)]x(t) \tag{3.14}$$

where

$$a(t) = \begin{cases} 1 & t \text{ even} \\ a_0^2 a_1^2 & t \text{ odd} \end{cases}, \qquad d(t) = \begin{cases} {a_0}^{-1} & t \text{ even} \\ a_0 & t \text{ odd} \end{cases}, \qquad e(t) = \begin{cases} {a_1}^{-1} & t \text{ even} \\ a_1 & t \text{ odd} \end{cases}$$

and $a_0, a_1 \in \mathbb{C}^*$, $|a_0 a_1| < 1$. The scalar Bohl transformation defined by

$$\alpha(t) = \begin{cases} a_1 & t \text{ even} \\ {a_0}^{-1} & t \text{ odd} \end{cases} \tag{3.15}$$

transforms (3.14) to the perturbed system

$$\hat{x}(t+1) = (a_0 a_1 + \hat{d}(t)\delta(t)\hat{e}(t))\hat{x}(t) \tag{3.16}$$

where

$$\hat{d}(t) = \begin{cases} 1 & t \text{ even} \\ {a_1}^{-1} a_0 & t \text{ odd} \end{cases}, \qquad \hat{e}(t) = \begin{cases} 1 & t \text{ even} \\ {a_0}^{-1} a_1 & t \text{ odd} \end{cases}.$$

As $\hat{d}(t)\hat{e}(t) \equiv 1$ the stability radius of (3.16) satisfies $r^1_{\mathbb{C}}(a_0 a_1; \hat{d}, \hat{e}) = r^1_{\mathbb{C}}(a_0 a_1; 1, 1) = 1 - |a_0 a_1|$. It follows from (2.18) that $r^1_{\mathbb{C}}(a; d, e) = 1 - |a_0 a_1|$. On the other hand, the input-output operator of (3.14) has norm $\|\mathbb{L}_{t_0}\| \geq |a_1/a_0|$. Note that because of periodicity $\|\mathbb{L}_{t_0}\|$ is constant in t_0. Choosing $a_0 = 1/(2a_1)$ where $a_1 \to \infty$ we obtain a family of periodic systems (3.14) with fixed stability radius $r^1_{\mathbb{C}} = 1/2$ and $\|\mathbb{L}_{t_0}\| \to \infty$. Thus the lower bound $\lim_{t_0 \to \infty} \|\mathbb{L}_{t_0}\|^{-1} = \|\mathbb{L}_0\|^{-1}$ becomes arbitrarily bad. □

We will now apply *scaling techniques* to improve the lower bound obtained in Theorem 3.1, see [3], [5],[6]. Let

$$\alpha(\cdot) = (\alpha_i(\cdot)) \in \mathcal{B}^{\mathbb{N}}$$

where $\mathcal{B} \subset (\mathbb{C}^*)^{\mathbb{N}}$ is the set of all scalar Bohl transformations. By Proposition 2.9 (iii)

$$r^1_{\mathbb{C}}(A; (D_i, E_i)) = r^1_{\mathbb{C}}(A; (D_i(\alpha), E_i(\alpha))). \tag{3.17}$$

We denote by $\mathbb{L}^{\alpha}_{t_0}$, $\mathbb{M}^{\alpha}_{t_0}$ the operators given by (3.4), (3.5) with $D(\cdot), E(\cdot)$ replaced by $D(\alpha)(\cdot)$, $E(\alpha)(\cdot)$, see (2.17). For example,

$$(\mathbb{L}^{\alpha}_{t_0} u)(t) = \sum_{k=0}^{t-1} E(t)\alpha(t)\Phi(t,k+1)D(k)\alpha(k)^{-1}u(k).$$

As a direct consequence of Theorem 3.1 and (3.17) we obtain the following result.

Corollary 3.3 *Under the assumptions and with the notations of Theorem 3.1, if $\alpha(\cdot) \in \mathcal{B}^{\mathbb{N}}$ is such that $D(\alpha)(\cdot)$, $E(\alpha)(\cdot)$ satisfy (2.2) then*

$$r^1_{\mathbb{C}}(A;(D_i,E_i)) \geq \|\mathbb{L}^{\alpha}_{t_0}\|^{-1} \quad \textit{for all} \quad t_0 \geq 0. \tag{3.18}$$

In order to illustrate the improvement achievable by using the scaling technique we return to Example 3.2.

Example 3.4 Applying the scalar Bohl transformation

$$\alpha(t) = \begin{cases} {a_0}^{-1} & t \text{ even} \\ {a_1}^{-1} & t \text{ odd} \end{cases} \tag{3.19}$$

to the structure operators of system (3.14) as described in (2.17) we obtain the input output operator of the time invariant system $(a_o a_1, 1, 1)$. A short computation yields

$$(\mathbb{L}^{\alpha}_0 u)(t) = \sum_{k=0}^{t-1} e(t)\alpha(t)\Phi(t,k+1)d(k)\alpha(k)^{-1}u(k) = \sum_{k=0}^{t-1}(a_0 a_1)^{t-k-1}u(k)$$

and so $\|\mathbb{L}^{\alpha}_0\|^{-1} = 1 - |a_0 a_1| = r^1_{\mathbb{C}}(a;d,e)$. □

We have not found a counterexample to disprove the following conjecture, which is the discrete-time counterpart to a conjecture in [5].

Conjecture 3.5 *Given an exponentially stable system with simple structured perturbation*

$$x(t+1) = [A(t) + D(t)\Delta(t)E(t)]x(t)$$

then

$$r^1_{\mathbb{C}}(A;(D,E)) = \sup\{\|\mathbb{L}^{\alpha}_{t_0}\|^{-1}\,;\ \alpha(\cdot) \in \mathcal{B}, D(\alpha),\ E(\alpha) \textit{satisfy (2.2)},\ t_o \in \mathbb{N}\}.$$

Remark 3.6 (a) For single perturbation time-invariant systems in Hilbert spaces, this is trivially true because in this case $r^1_{\mathbb{C}}(A;(D,E)) = \|\mathbb{L}_0\|^{-1}$, see Corollary 5.4 in Section 5.

(b) In the case $\mathbb{K} = \mathbb{R}$ this conjecture cannot hold as even in finite dimensions it is possible that, $r^1_{\mathbb{R}}(A;(D,E)) > r^1_{\mathbb{C}}(A;(D,E)) = \|\mathbb{L}_0\|^{-1} \geq \|\mathbb{L}_0^{\alpha}\|^{-1}$ for all $\alpha(\cdot) \in \mathcal{B}$ (see [6]). □

4 Complex stability radius of time-invariant systems

We now study stability radii for time-invariant systems. We will start by studying stability radii of time-invariant systems for complex structured multi-perturbations and then turn to the simpler case of perturbations which can be described via a single additive term. If not stated otherwise we will assume $\mathbb{K} = \mathbb{C}$. As before, X, U_i, Y_i are Banach spaces and $A \in \mathcal{L}(X), D_i \in \mathcal{L}(U_i, X), E_i \in \mathcal{L}(X, Y_i), i \in \mathbb{N}$ are given linear operators. We define the Banach spaces U, Y by (2.5) and the time-invariant linear operators $D \in \mathcal{L}(U, X), E \in \mathcal{L}(X, Y)$ by (2.6).

For finite-dimensional systems Doyle [3] has introduced the μ-function, which can be used to characterize the complex stability radius for structured multi-perturbations in the finite dimensional case [6]. To extend this result to Banach spaces we use the following definition:

Definition 4.1 *An operator $\Delta \in \mathcal{L}(Y, U)$ is said to be $((Y_i, U_i))_{i\in\mathbb{N}}$-block diagonal if there are $\Delta_i \in \mathcal{L}(Y_i, U_i)$ such that*

$$\Delta((y_i)_{i\in\mathbb{N}}) = (\Delta_i y_i)_{i\in\mathbb{N}}\,, \quad (y_i)_{i\in\mathbb{N}} \in Y.$$

The μ-function is defined on $\mathcal{L}(U, Y)$ by:

$$\mu(M) = \left(\inf\{\|\Delta\|\,;\ \Delta \textit{ is } ((Y_i, U_i))_{i\in\mathbb{N}}\textit{-block diagonal, } (I + M\Delta) \textit{ is not invertible in } \mathcal{L}(Y)\}\right)^{-1}.$$

Remark 4.2 In our notation we suppress the dependency of μ on the block-diagonal structure, as it is always clear from the context, which structure is considered. □

Proposition 4.3 *Consider the time-invariant system*

$$x(t+1) = \left[A + \sum_{i\in\mathbb{N}} D_i\Delta_i E_i\right] x(t)\,, \quad t \in \mathbb{N} \tag{4.1}$$

where $r(A) < 1$. *Define the operator valued function* G *on the resolvent set* $\rho(A) = \mathbb{C}\backslash\sigma(A)$ *by*

$$G(s) = E(sI - A)^{-1}D\,, \quad s \in \rho(A).$$

Then (4.1) *is stable if and only if* $(I - G(s)\Delta)$ *is invertible for all* $s \in \mathbb{C}\backslash\mathbb{D}$. *Furthermore*

$$r^1_{\mathbb{C},c}(A;(D_i,E_i)) = \left[\sup_{\omega\in[0,2\pi]} \mu(G(e^{i\omega}))\right]^{-1}.$$

Proof: In the proof Δ will always denote a $((U_i,Y_i))_{i\in\mathbb{N}}$-block diagonal operator.

Note that for operators $S \in \mathcal{L}(Y,X), T \in \mathcal{L}(X,Y)$ the operator $I_X - ST$ is invertible in $\mathcal{L}(X)$ iff $I_Y - TS$ is invertible in $\mathcal{L}(Y)$. To prove this assume $I_X - ST$ is invertible and set $R = (I_X - ST)^{-1}$. Then check that $(I_Y - TS)(I_Y + TRS) = I_Y = (I_Y + TRS)(I_Y - TS)$.

1. Assume $(I - G(s)\Delta)$ is invertible for all $s \in \mathbb{C}\backslash\mathbb{D}$. By the above remark this implies $(I - (sI - A)^{-1}D\Delta E)$ is invertible for all $s \in \mathbb{C}\backslash\mathbb{D}$, and so $\sigma(A + D\Delta E) \subset \mathbb{D}$, i.e. $A + D\Delta E$ is exponentially stable. As $\|\Delta\| < \left[\sup_{s\in\mathbb{C}\backslash\mathbb{D}} \mu(G(s))\right]^{-1}$ implies $(I - G(s)\Delta)$ is invertible for all $s \in \mathbb{C}\backslash\mathbb{D}$ we conclude that

$$r^1_{\mathbb{C},c}(A;(D_i,E_i)) \geq \left[\sup_{s\in\mathbb{C}\backslash\mathbb{D}} \mu(G(s))\right]^{-1}.$$

2. Fix $\varepsilon > 0$ and choose $s_0 \in \mathbb{C}\backslash\mathbb{D}$ such that $[\mu(G(s_0))]^{-1} \leq [\sup_{s\in\mathbb{C}\backslash\mathbb{D}} \mu(G(s))]^{-1} + \varepsilon/2$. By definition of μ there is a Δ is such that $\|\Delta\| \leq [\sup_{s\in\mathbb{C}\backslash\mathbb{D}} \mu(G(s))]^{-1} + \varepsilon$ and $(I - G(s_0)\Delta)$ is not invertible. Then $(I - (s_0 I - A)^{-1}D\Delta E)$ is not invertible. Hence $s_0 \in \sigma(A + D\Delta E)$ and the perturbed system is not exponentially stable.

We conclude that

$$r^1_{\mathbb{C},c}(A;(D_i,E_i)) \leq \left[\sup_{s\in\mathbb{C}\backslash\mathbb{D}} \mu(G(s))\right]^{-1}.$$

3. It remains to show

$$\sup_{s\in\mathbb{C}\setminus\mathbb{D}} \mu(G(s)) = \sup_{\omega\in[0,2\pi]} \mu(G(e^{i\omega})).$$

To this end assume there is a Δ and a $s \in \mathbb{C}\setminus\mathbb{D}$ such that $(I - G(s)\Delta)$ is not invertible. By 2. this implies $s \in \sigma(A+D\Delta E)$. The function defined on $[0,1]$ by $\tau \to A+D\tau\Delta E$ is continuous in the norm, and so the spectrum of $A+D\tau\Delta E$ depends upper semicontinuously on τ, see [10]. So the set $\{\tau; r(A+D\tau\Delta E) < 1\}$ is open in $[0,1]$, but by assumption not equal to $[0,1]$. If we set $\tau_0 = \sup\{\tau; r(A+D\tau\Delta E) < 1\}$, then $\sigma(A+D\tau_0\Delta E) \cap \partial\mathbb{D} \neq \emptyset$ by [10] Thm. IV.3.16, because $r(A+D\tau_0\Delta E) \geq 1$. Hence there is an $\omega \in \mathbb{R}$ such that $e^{i\omega} \in \sigma(A+D\tau_0\Delta E)$. This means $(e^{i\omega}I - A - D\tau_0\Delta E)$ is not invertible or equivalently $(I - (e^{i\omega}I - A)^{-1}D\tau_0\Delta E)$ is not invertible in $\mathcal{L}(X)$. From this we conclude that $(I - G(e^{i\omega})\tau_0\Delta)$ is not invertible. Since $\|\tau_0\Delta\| \leq \|\Delta\|$ we see that

$$\sup_{s\in\mathbb{C}\setminus\mathbb{D}} \mu(G(s)) \leq \sup_{\omega\in[0,2\pi]} \mu(G(e^{i\omega})).$$

But then the two expressions must be equal. □

For time–invariant systems with a *single* perturbation ($N = 1$) on complex Banach spaces the stability radius can be expressed by the norm of the transfer function, whereas if X, Y and U are Hilbert spaces it can also be characterized via the input-output operator. To prove this we use the following lemma, which is an extension of a result in [3] to Banach spaces.

Lemma 4.4 *In the single perturbation case ($N = 1$ in (2.3), i.e. $U = U_1, Y = Y_1, \Delta = \Delta_1 \in \mathcal{L}(Y_1, U_1)$)*

$$\mu(M) = \|M\|. \tag{4.2}$$

Proof: By assumption there is no constraint in the structure, i.e. Δ can be any operator in $\mathcal{L}(Y,U)$. Now suppose Δ is such that $(I + M\Delta)$ is not invertible. Then $\|M\|\|\Delta\| \geq \|M\Delta\| \geq 1$ and hence $\|\Delta\| \geq \|M\|^{-1}$. But this implies $\mu(M)^{-1} \geq \|M\|^{-1}$ or $\mu(M) \leq \|M\|$.

To prove the converse inequality we may assume $M \neq 0$. Let $\{u_n\}_{n\in\mathbb{N}} \subset U, \|u_n\| = 1$ be a sequence such that $\|Mu_n\| \to \|M\|$. For each $n \in \mathbb{N}$ define a functional on the linear subspace generated by Mu_n by

$$\alpha M u_n \mapsto \alpha \quad , \alpha \in \mathbb{C}$$

and extend this to a functional $f : Y \to \mathbb{C}$ with the same norm. Then define $\Delta_n \in \mathfrak{L}(Y, U)$ by $\Delta_n(y) = f(y)u_n$ so that we have $\|\Delta_n\| = \|Mu_n\|^{-1}$. Setting $y_n = Mu_n$ we have by construction $y_n = M\Delta_n y_n$ hence $(I + M(-\Delta_n))$ is not invertible and $\|\Delta_n\| \geq \mu(M)^{-1}$. Since $\|\Delta_n\| \to \|M\|^{-1}$ this proves $\mu(M) \geq \|M\|$. □

Corollary 4.5 *Suppose* $A(t) \equiv A \in \mathfrak{L}(X)$, $D(t) \equiv D \in \mathfrak{L}(U, X)$, $E(t) \equiv E \in \mathfrak{L}(X, Y)$, *where* $r(A) < 1$ *and consider the associated transfer operator*

$$G(s) = E(sI_X - A)^{-1}D, \qquad s \in \mathbb{C} \setminus \sigma(A)$$

and the input output operator $\mathbb{L} : l^2(\mathbb{N}, U) \to l^2(\mathbb{N}, Y)$, *defined by*

$$(\mathbb{L}u)(t) = \sum_{k=0}^{t-1} EA^{t-1-k}Du(k), \quad t \in \mathbb{N}. \tag{4.3}$$

Then

$$r^1_{\mathbb{C},c}(A; D, E) = \max_{|s|=1} \|G(s)\|^{-1}_{\mathfrak{L}(U,Y)}. \tag{4.4}$$

If X, Y, U *are Hilbert spaces then*

$$r^1_{\mathbb{C},c}(A; D, E) = \max_{|s|=1} \|G(s)\|^{-1}_{\mathfrak{L}(U,Y)} = \|\mathbb{L}\|^{-1}. \tag{4.5}$$

Proof: The assertion for the transfer function follows from Proposition 4.3 and Lemma 4.4. For Hilbert spaces U and Y it is well known that the norm of a shift-invariant Operator $\mathbb{L} : l^2(\mathbb{N}, U) \to l^2(\mathbb{N}, Y)$ is equal to the H^∞ norm of the associated transfer function, see [22], or for a detailed exposition [23]. In our case this means $\max_{|s|=1} \|G(s)\|_{\mathfrak{L}(U,Y)} = \|\mathbb{L}\|$. □

This result can be used to extend a known result for normal matrices [6] to normal operators on a Hilbert space. If A is normal perturbation of spectra results for normal operators imply $r^1_{\mathbb{K},c}(A) \geq 1 - r(A)$, see [19] or [10] Thm. IV.3.18, Problem V.4.8. It is then easy to show that equality holds.

Corollary 4.6 *Suppose X is a Hilbert space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), $A(t) \equiv A \in \mathcal{L}(X)$, $D(t) \equiv I_X$ $E(t) \equiv I_X$ where A is normal and $r(A) < 1$. Then*

$$r^1_{\mathbb{K},c}(A) = 1 - r(A).$$

Proof: If $\mathbb{K} = \mathbb{R}$, regard A as an operator on the complexification $X^{\mathbb{C}}$ of X (see [21]). As $\|A\| = r(A)$ and $\|A^n\| = \|A\|^n$ [14] we have:

$$\|G(e^{i\omega})\| = \|(e^{i\omega}I_X - A)^{-1}\| = \|\sum_{n=o}^{\infty} e^{-in\omega}A^n\| \leq \sum_{n=o}^{\infty} \|A\|^n = \frac{1}{1 - r(A)}$$

and so by Proposition 4.5: $r^1_{\mathbb{C},c}(A) \geq 1 - r(A)$. On the other hand if we set $\Delta = \frac{1-r(A)}{r(A)} A \in \mathcal{L}(X)$ then $\|\Delta\| = 1 - r(A)$ and $A + \Delta = \frac{1}{r(A)} A$ is unstable. □

5 Wider Perturbation Classes

In the last section we have restricted ourselves to the consideration of constant linear disturbances. In this section we will again investigate robust stability of *time-invariant* systems but with respect to wider perturbation classes. We consider perturbed system equations of the following kind:

$$\begin{aligned}
\Sigma_N \quad &: \quad x(t+1) = Ax(t) + \sum_{i=1}^{\infty} D_i N_i(E_i x(t)); \\
\Sigma_{\Delta(t)} \quad &: \quad x(t+1) = Ax(t) + \sum_{i=1}^{\infty} D_i \Delta_i(t) E_i x(t); \\
\Sigma_{N(t)} \quad &: \quad x(t+1) = Ax(t) + \sum_{i=1}^{\infty} D_i N_i(E_i x(t), t); \\
\Sigma_{\mathcal{N}} \quad &: \quad x(t+1) = Ax(t) + \sum_{i=1}^{\infty} D_i \mathcal{N}_i(E_i x(\cdot))(t).
\end{aligned}$$

As before we assume that (2.2) holds for $((D_i, E_i))_{i\in\mathbb{N}}$. Furthermore U, Y and D, E, Δ denote the Banach spaces resp. the operators constructed in (2.5),(2.6). Note that the disturbances $\Sigma_{\Delta(t)}$ are exactly those which we considered in Sections 1 to 3.

The unknown perturbation operators N_i, $\Delta_i(\cdot)$, $N_i(\cdot)$, $\mathcal{N}_i$ have the following properties

(i) $N_i : Y_i \to U_i$, $N_i(0) = 0$, N_i is continuous, uniformly Fréchet-differentiable at 0, i.e. for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|y\| < \delta$ implies $\|N_i(y) - N_i'(0)y\|/\|y\| < \varepsilon$ for all $i \in \mathbb{N}$ and of uniformly finite gain, i.e. there exists $\gamma \geq 0$ such that $\|N_i(y)\| \leq \gamma\|y\|$ for all $y \in Y_i\,, i \in \mathbb{N}\,;$

(ii) $\Delta_i(\cdot) = (\Delta_i(t))_{t\in\mathbb{N}} \in l^\infty(\mathbb{N}, \mathcal{L}(Y_i, U_i))$ and there exists $\gamma > 0$ such that $\|\Delta_i(t)\| \leq \gamma$ for all $t \in \mathbb{N}$, $i \in \mathbb{N}$;

(iii) $N_i : Y_i \times \mathbb{N} \to U_i$, $N_i(0, t) = 0$ for all $t \in \mathbb{N}$, $i \in \mathbb{N}$, and there exists $\gamma \geq 0$ such that $\|N_i(y, t)\| \leq \gamma\|y\|$ for all $y \in Y_i$, $t \in \mathbb{N}$, $i \in \mathbb{N}$;

(iv) $\mathcal{N}_i : l^2(\mathbb{N}; Y_i) \longrightarrow l^2(\mathbb{N}; U_i)$, $\mathcal{N}_i(0) = 0$, $\mathcal{N}_i$ is causal and of uniformly finite gain, i.e. there exists $\gamma \geq 0$ such that $\|\mathcal{N}_i(y(\cdot))\|_{l^2} \leq \gamma\|y(\cdot)\|_{l^2}$ for all $y(\cdot) \in l^2(\mathbb{N}; Y_i)$, $i \in \mathbb{N}$.

The operator $\mathcal{N}_i$ is said to be *causal* if for any $y(\cdot)$, $w(\cdot) \in l^2(\mathbb{N}, Y_i)$ and $t \in \mathbb{N}$:

$$w(\tau) = y(\tau),\ \tau = 0, \cdots, t \quad \Rightarrow \quad \mathcal{N}_i(y(\cdot))(t) = \mathcal{N}_i(w(\cdot))(t). \tag{5.1}$$

For any subset $T \subset \mathbb{N}$, $t \in T$ and $f : T \to Y$, we denote by $\pi_t f : T \to Y$ the map defined by

$$\pi_t f(\tau) = \begin{cases} f(\tau) & \text{if} \quad \tau \leq t \\ 0 & \text{if} \quad \tau > t \end{cases} \qquad \tau \in T.$$

The causality of $\mathcal{N}_i$ is equivalent to

$$\pi_t \mathcal{N}_i(\pi_t y(\cdot)) = \pi_t \mathcal{N}_i(y(\cdot)) \quad \text{for all} \quad y(\cdot) \in l^2(\mathbb{N}, Y_i),\ t \in \mathbb{N}.$$

As in (2.5),(2.6) we define U, Y and introduce the operators

(i) $N : Y \to U,\ N((y_i)_{i\in\mathbb{N}}) = (N_i(y_i))_{i\in\mathbb{N}}$

(ii) $\Delta : Y \times \mathbb{N} \to U,\ \Delta((y_i)_{i\in\mathbb{N}}, t) = (\Delta_i(t)y_i)_{i\in\mathbb{N}}$

(iii) $N : Y \times \mathbb{N} \to U,\ N((y_i)_{i\in\mathbb{N}}, t) = (N_i(y_i, t))_{i\in\mathbb{N}}$

(iv) $\mathcal{N} : l^2(\mathbb{N}; Y) \longrightarrow l^2(\mathbb{N}; U),\ \mathcal{N}((y_i(\cdot))_{i\in\mathbb{N}}) = (\mathcal{N}_i(y_i(\cdot)))_{i\in\mathbb{N}}.$

The perturbed systems Σ_N, $\Sigma_{\Delta(t)}$, $\Sigma_{N(t)}$, $\Sigma_{\mathcal{N}}$ can be interpreted as feedback systems with unknown feedback operators N, $\Delta(t)$, $N(t)$, $\mathcal{N}$ resp. , see Figure 1.

Let $\mathrm{P}_c(\mathbb{K})$ denote the $\mathbb{K}$-linear space of constant perturbations which are block diagonal with respect to (U_i, Y_i) and satisfy (2.2). The sets of perturbations (i) — (iv) are $\mathbb{K}$-linear spaces and are denoted by $\mathrm{P}_n(\mathbb{K})$, $\mathrm{P}_t(\mathbb{K})$, $\mathrm{P}_{nt}(\mathbb{K})$, $\mathrm{P}_d(\mathbb{K})$ respectively. As perturbation norms we choose

(i) $\|N\|_n = \inf\{\gamma \in \mathbb{R}_+; \forall y \in Y : \|N(y)\| \le \gamma\|y\|\}, \quad N \in \mathrm{P}_n(\mathbb{K}),$

(ii) $\|\Delta\|_t = \sup_{t\in\mathbb{N}} \|\Delta(t)\|, \quad \Delta \in \mathrm{P}_t(\mathbb{K}),$

(iii) $\|N\|_{nt} = \inf\{\gamma \in \mathbb{R}_+; \forall y \in Y\ \forall t \in \mathbb{N} : \|N(y,t)\| \le \gamma\|y\|\}, \quad N \in \mathrm{P}_{nt}(\mathbb{K}),$

(iv) $\|\mathcal{N}\|_d = \inf\{\gamma \in \mathbb{R}_+; \forall y(\cdot) \in l^2(\mathbb{N}; Y) : \|\mathcal{N}(y(\cdot))\|_{l^2} \le \gamma\|y(\cdot)\|_{l^2}\}, \mathcal{N} \in \mathrm{P}_d(\mathbb{K}).$

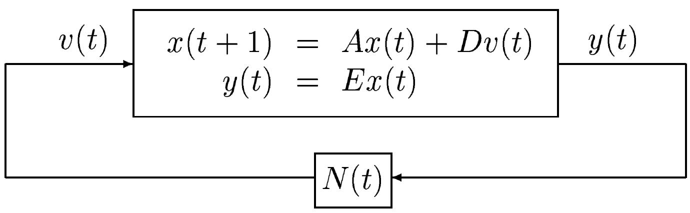

Figure 1: Feedback interpretation of the system $\Sigma_{N(t)}$

There is an obvious norm preserving embedding of the space $\mathrm{P}_c(\mathbb{K})$ into the normed spaces $\mathrm{P}_n(\mathbb{K})$ and $\mathrm{P}_t(\mathbb{K})$:

$$\mathrm{P}_c(\mathbb{K}) \subset \mathrm{P}_n(\mathbb{K}) \ , \quad \mathrm{P}_c(\mathbb{K}) \subset \mathrm{P}_t(\mathbb{K}). \tag{5.2}$$

Similarly, there are *norm preserving inclusions*

$$\mathrm{P}_t(\mathbb{K}) \subset \mathrm{P}_{nt}(\mathbb{K}), \quad \mathrm{P}_n(\mathbb{K}) \subset \mathrm{P}_{nt}(\mathbb{K}), \quad \mathrm{P}_{nt}(\mathbb{K}) \subset \mathrm{P}_d(\mathbb{K}). \tag{5.3}$$

We explain this briefly for the latter embedding. Every $N \in \mathrm{P}_{nt}(\mathbb{K})$ defines a causal operator $\mathcal{N} \in \mathrm{P}_d(\mathbb{K})$ via

$$\mathcal{N}(y(\cdot))(t) = N(y(t), t)\ , \quad t \in \mathbb{N},\ y(\cdot) \in l^2(\mathbb{N}, Y). \tag{5.4}$$

In fact we have

$$\|\mathcal{N}(y(\cdot))\|_{l^2}^2 = \sum_{t\in\mathbb{N}} \|N(y(t), t)\|^2 \leq \|N\|_{nt}^2 \|y(\cdot)\|_{l^2}^2 \tag{5.5}$$

so that $\mathcal{N}$ maps $l^2(\mathbb{N}, Y)$ to $l^2(\mathbb{N}, U)$ and is of finite gain $\|\mathcal{N}\|_d \leq \|N\|_{nt}$. To prove the converse inequality, let $\varepsilon > 0$ and choose $t_0 \in \mathbb{N}$, $y_0 \in Y$, $y_0 \neq 0$ such that

$$\|N(y_0, t_0)\| \geq (\|N\|_{nt} - \varepsilon)\|y_0\|\ .$$

Define $y(t_0) = y_0$ and $y(t) = 0$ for $t \in \mathbb{N}$, $t \neq t_0$. Then $\|y(\cdot)\|_{l^2} = \|y_0\|$ and

$$\|\mathcal{N}(y(\cdot))\|_{l^2} = \|N(y_0, t_0)\| \geq (\|N\|_{nt} - \varepsilon)\|y(\cdot)\|_{l^2}$$

so that $\|\mathcal{N}\|_d = \|N\|_{nt}$.

For each of the perturbed equations Σ_N, $\Sigma_{\Delta(t)}$, $\Sigma_{N(t)}$ the initial condition $x(t_0) = x_0$ determines a unique solution $x(t; t_0, x_0)$, $t \geq t_0$, $t \in \mathbb{N}$. In the case of equation $\Sigma_{\mathcal{N}}$ it is necessary in addition to specify an initial sequence $\varphi_0(\tau)$, $0 \leq \tau < t_0$. If $\mathcal{N} \in \mathrm{P}_d(\mathbb{K})$, $t_0 \in \mathbb{N}$, $x_0 \in X$ and $\varphi_0 = (\varphi_0(0), \cdots, \varphi_0(t_0 - 1)) \in Y^{t_0}$ then $x(\cdot) = (x(t))_{t \geq t_0}$ is called a solution of $\Sigma_{\mathcal{N}}$ with initial value (x_0, φ_0) at time $t_0 \in \mathbb{N}$ if it satisfies

$$\begin{aligned} x(t+1) &= Ax(t) + D\mathcal{N}(y_{\varphi_0}(\cdot))(t)\ , \quad t \geq t_0 \\ x(t_0) &= x_0 \end{aligned} \tag{5.6}$$

where

$$y_{\varphi_0}(\tau) = \begin{cases} \varphi_0(\tau) & \text{if} \quad 0 \leq \tau < t_0 \\ Ex(\tau) & \text{if} \quad \tau \geq t_0 \end{cases}.$$

This solution is uniquely determined and can be constructed recursively from (5.6) by causality of $\mathcal{N}$. We denote it by $x(\cdot\,; t_0, x_0, \varphi_0)$.

Definition 5.1 *$\Sigma_{\mathcal{N}}$ is said to be* globally asymptotically stable *(g.a.s.) if it satisfies the two conditions*

(i) The origin is stable *for $\Sigma_{\mathcal{N}}$, i.e. for any $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that for $x_0 \in X$, $\varphi_0 \in Y^{t_0}$*

$$\|x_0\| < \delta\ ,\ \|\varphi_0\|_{l^2(0,t_0-1)} < \delta \ \Rightarrow\ \|x(t; t_0, x_0, \varphi_0)\| < \varepsilon \quad \text{for all } t \geq t_0.$$

(ii) The origin $0 \in X$ is globally attractive, *i.e. for any $(t_0, x_0, \varphi_0) \in \mathbb{N} \times X \times Y^{t_0}$ we have $\lim_{t\to\infty} x(t; t_0, x_0, \varphi_0) = 0$.*

The global asymptotic stability of Σ_N, $\Sigma_{\Delta(t)}$, $\Sigma_{N(t)}$ is defined as usual.

Definition 5.2 *The stability radius of A with respect to perturbations of the structure $((D_i, E_i))_{i\in\mathbb{N}}$ and the class P_d is defined by*

$$r^1_{\mathbb{K},d}(A; (D_i, E_i)) = \inf\{\|\mathcal{N}\|_d; \mathcal{N} \in P_d(\mathbb{K}), \Sigma_{\mathcal{N}} \text{ is not g.a.s. }\}.$$

Stability radii with respect to perturbations of the classes $\mathrm{P}_n(\mathbb{K})$, $\mathrm{P}_t(\mathbb{K})$, $\mathrm{P}_{nt}(\mathbb{K})$ are defined in an analogous way; they are denoted by $r^1_{\mathbb{K},n}(A; (D_i, E_i))$, $r^1_{\mathbb{K},t}(A; (D_i, E_i))$, $r^1_{\mathbb{K},nt}(A; (D_i, E_i))$, respectively. Note that $r^1_{\mathbb{K},t}(A; (D_i, E_i)) = r^1_{\mathbb{K}}(A; (D_i, E_i))$ as defined in Definition 2.1.

Proposition 5.3 *Suppose $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $A(t) \equiv A \in \mathcal{L}(X)$ is time invariant with spectral radius $r(A) < 1$. Then*

$$r^1_{\mathbb{K},c}(A; (D_i, E_i)) \geq r^1_{\mathbb{K},n}(A; (D_i, E_i)) \geq r^1_{\mathbb{K},t}(A; (D_i, E_i)) \geq r^1_{\mathbb{K},nt}(A; (D_i, E_i)) \geq \tag{5.7}$$

$$r^1_{\mathbb{K},d}(A; (D_i, E_i)) \geq \|\mathbb{L}\|^{-1}.$$

where $\mathbb{L}$ is defined by (4.3).

Proof: We first prove $r^1_{\mathbb{K},n} \geq r^1_{\mathbb{K},t}$. Assume $N \in \mathrm{P}_n(\mathbb{K})$, $\|N\|_n < r^1_{\mathbb{K},t}$. We must prove that Σ_N is globally asymptotically stable. By assumption N is Fréchet-differentiable at 0 and because of (5.2) we have $\|N'(0)\| < r^1_{\mathbb{K},t} \leq r^1_{\mathbb{K},c}$. Thus $B = A + DN'(0)E$ is exponentially stable and $R(x) = D(N(Ex) - N'(0)Ex)$ satisfies $\|R(x)\|/\|x\| \to 0$ as $x \to 0$. Then Theorem 5.6.1 and Corollaries in [1] imply that the origin is stable.

Now suppose $x(\cdot) = x(\cdot\,; t_0, x_0)$ is any trajectory of Σ_N with $x(t_0) = x_0$ and set $y(t) = Ex(t)$, $t \geq t_0$. Define for $t \in \mathbb{N}, i \in \mathbb{N}$

$$\Delta_i(t)(\alpha y(t)) := \alpha N_i(y(t))\,, \quad \alpha \in \mathbb{K}$$

and extend this to an operator in $\mathcal{L}(Y_i, U_i)$ as in Lemma 4.4. This defines a $\Delta \in \mathrm{P}_t(\mathbb{K})$, $\|\Delta\|_t \leq \|N\|_n < r^1_{\mathbb{K},t}$ such that $x(\cdot)$ is a trajectory of $\Sigma_{\Delta(t)}$. It follows that $\lim_{t\to\infty} x(t) = 0$. Hence Σ_N is g.a.s. .

In view of (5.2), (5.3), it only remains to show that $r^1_{\mathbb{K},d}(A; (D_i, E_i)) \geq \|\mathbb{L}\|^{-1}$. Suppose $\mathcal{N} \in \mathrm{P}_d(\mathbb{C})$ and $\|\mathcal{N}\|_d < \|\mathbb{L}\|^{-1}$. If $x(\cdot) = x(\cdot\,; t_0, x_0, \varphi)$ with $t_0 \in \mathbb{N}$, $x_0 \in X$, $\varphi \in Y^{t_0}$ is any trajectory of $\Sigma_{\mathcal{N}}$ and $y_\varphi(t) = Ex(t)$, $t \geq t_0$, we have

$$x(t) = A^{t-t_0}x_0 + \sum_{k=t_0}^{t-1} A^{t-1-k} D\mathcal{N}(y_\varphi(\cdot))(k)\,, \quad t \geq t_0 \tag{5.8}$$

and, for any $\tau \in \mathbb{N}$, $\tau > t_0$

$$y_\varphi(t) = EA^{t-t_0}x_0 + [\mathbb{L}\mathcal{N}(\pi_\tau y_\varphi(\cdot))]\,(t)\,, \quad t_0 \leq t \leq \tau.$$

By Proposition 2.4 (iv) there exists a constant $c > 0$ such that for $\tau \in \mathbb{N}$, $\tau > t_0$

$$\begin{aligned} \|\pi_\tau y_\varphi\|_{l^2} &\leq \|\varphi\| + \left(\sum_{t=t_0}^{\tau} \|y_\varphi(t)\|^2\right)^{1/2} \\ &\leq \|\varphi\| + c\|x_0\| + \|\mathbb{L}\mathcal{N}(\pi_\tau y_\varphi(\cdot))\|_{l^2(t_0,\tau)} \end{aligned}$$

where $\|\varphi\| = \|\varphi\|_{l^2(0,t_0-1)}$. Now $\alpha = \|\mathbb{L}\|\|\mathcal{N}\|_d < 1$ and so the above inequality implies

$$\|\pi_\tau y_\varphi\|_{l^2} \leq \frac{c\|x_0\| + \|\varphi\|}{1-\alpha}, \qquad \tau > t_0.$$

Hence $y_\varphi(\cdot) \in l^2(\mathbb{N}_{t_0}; Y)$ and

$$\|y_\varphi\|_{l^2} \leq \frac{c\|x_0\| + \|\varphi\|}{1-\alpha}. \tag{5.9}$$

Taking norms in (5.8) we obtain

$$\|x(t)\| \leq \|A^{t-t_0}\|\|x_0\| + \sum_{k=t_0}^{t-1} \|A^{t-1-k}\|\|D\mathcal{N}(y_\varphi(\cdot))(k)\|\,, \quad t \geq t_0.$$

The right hand side of this inequality is a convolution of the sequences

$$z_1(t) := \|A^{t-t_0}\|\,, \quad t \geq t_0$$

$$z_2(t_0) := \|x_0\|\,, \quad z_2(t+1) = \|D\mathcal{N}(y_\varphi(\cdot))(t)\|\,, \quad t \geq t_0.$$

We have $(z_1(t))_{t\in\mathbb{N}_{t_0}} \in l^1(\mathbb{N}_{t_0}, \mathbb{R})$ and by (5.9) $(z_2(t))_{t\in\mathbb{N}_{t_0}} \in l^2(\mathbb{N}_{t_0}, \mathbb{R})$. Hence the convolution $z_1 \star z_2$ is in $l^2(\mathbb{N}_{t_0}, \mathbb{R})$ with norm $\|z_1 \star z_2\|_{l^2} \leq \|z_1\|_{l^1}\|z_2\|_{l^2}$ (see [21]). Using (5.9) it follows that there exists $K > 0$ such that

$$\|x(t)\| \leq K(\|x_0\| + \|\varphi\|)\,, \quad t \geq t_0.$$

Here K only depends upon (A, D, E) and $\|\mathcal{N}\|_d$. This shows the stability of $\Sigma_\mathcal{N}$. Furthermore we have $x(\cdot) \in l^2(\mathbb{N}_{t_0}; X)$ and so $\lim_{t\to\infty} \|x(t)\| = 0$. This proves that $\Sigma_\mathcal{N}$ is g.a.s. . We conclude that $r^1_{\mathbb{K},d} \geq \|\mathbb{L}\|^{-1}$. □

The *complex* stability radius is invariant under the above extensions of the perturbation class, if the perturbation is of simple structure and the spaces $X, U = U_1, Y = Y_1$ are Hilbert spaces. In this case we have equality in (5.7).

Corollary 5.4 *Suppose X, Y, U are Hilbert spaces over $\mathbb{C}$, $A \in \mathcal{L}(X)$, $D \in \mathcal{L}(U, X)$, $E \in \mathcal{L}(X, Y)$, $r(A) < 1$. Then*

$$r^1_{\mathbb{C},c}(A; (D, E)) = r^1_{\mathbb{C},n}(A; (D, E)) = r^1_{\mathbb{C},t}(A; (D, E)) = r^1_{\mathbb{C},nt}(A; (D, E)) = r^1_{\mathbb{C},d}(A; (D, E)). \tag{5.10}$$

Proof: This is a consequence of the preceding Theorem 5.3 and Corollary 4.5. □

6 Application: Periodic Systems

In this section we will examine how the previous results can be applied to periodic discrete-time infinite dimensional systems. We will restrict ourselves to simple structured perturbations, to avoid overburdened notations. More complicated situations could be examined using the same procedures, obtaining the same results.

For periodic discrete-time systems on a Banach space X there is a natural way to construct a time-invariant system. We will study the following situation: Assume X is a Banach space and $A(\cdot) \in l^\infty(\mathbb{N}, \mathcal{L}(X))$ is periodic with period $p \in \mathbb{N}_1$

$$A(t + p) = A(t) \quad t \in \mathbb{N}.$$

Consider the periodic system

$$x(t + 1) = A(t)x(t) , \quad t \in \mathbb{N} \tag{6.1}$$

and the perturbed system

$$x(t + 1) = [A(t) + D(t)\Delta(t)E(t)]\, x(t) , \quad t \in \mathbb{N} \tag{6.2}$$

where $D(\cdot) \in l^\infty(\mathbb{N}, \mathcal{L}(U, X)), E(\cdot) \in l^\infty(\mathbb{N}, \mathcal{L}(X, Y))$ are again supposed to be periodic with period p.

Introduce the Banach space $\hat{X} := X^p$ and the operator $\hat{A}$ defined on $\hat{X}$ by

$$\hat{A} = \begin{bmatrix} 0 & 0 & \dots & \dots & A(p-1) \\ A(0) & 0 & \dots & \dots & 0 \\ 0 & A(1) & 0 & \dots & \dots \\ \dots & \dots & \ddots & \dots & \dots \\ \dots & \dots & 0 & A(p-2) & 0 \end{bmatrix}. \tag{6.3}$$

The corresponding time-invariant discrete-time system is given by

$$\hat{x}(t+1) = \hat{A}\hat{x}(t)\,, \quad t \in \mathbb{N}. \tag{6.4}$$

The relation between A and $\hat{A}$ is the following: Given any trajectory $\{x(0), x(1), \dots\}$ of (6.1) the sequence

$$\begin{bmatrix} x(0) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x(1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ x(p-1) \end{bmatrix}, \begin{bmatrix} x(p) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots$$

is a trajectory of (6.4). Likewise trajectories of (6.1) with initial time $1 \leq t_0 \leq p-1$ can be represented in a similar manner, where the entry in the initial vector of the trajectory of (6.4) moves downward as t_0 increases. On the other hand any trajectory of system (6.4) is nothing else but a vector of p trajectories of (6.1) with initial times from 0 to $p-1$. Thus (6.1) is exponentially stable if and only if (6.4) is. To redefine the structure operators $D(t), E(t)$ accordingly we define

$$\hat{D} : U^p \to \hat{X}\,. \quad \hat{E} : \hat{X} \to Y^p$$

$$\hat{D} = \begin{bmatrix} 0 & 0 & \dots & \dots & D(p-1) \\ D(0) & 0 & \dots & \dots & 0 \\ 0 & D(1) & 0 & \dots & \dots \\ \dots & \dots & \ddots & \dots & \dots \\ \dots & \dots & 0 & D(p-2) & 0 \end{bmatrix}, \quad \hat{E} = \mathrm{diag}(E(0), \dots, E(p-1))$$

Now for every $t \in \mathbb{N}$ let $\hat{\Delta}(t) \in \mathcal{L}(Y^p, U^p)$ and assume that Δ is block-diagonal with respect to the natural decomposition of Y^p and U^p. For $k = 0, ..., p-1$ we denote the k-th column of $\hat{D}$, the k-th row of $\hat{E}$ and the $k-th$ entry in the diagonal of $\hat{\Delta}$ by $\hat{D}_k$, $\hat{E}_k$ and $\hat{\Delta}_k$ respectively.

With these notations the perturbed equation corresponding to (6.3) is

$$\hat{x}(t+1) = \left[\hat{A} + \hat{D}\hat{\Delta}(t)\hat{E}\right]\hat{x}(t) = \left[\hat{A} + \sum_{k=0}^{p-1}\hat{D}_k\hat{\Delta}_k(t)\hat{E}_k\right]\begin{bmatrix}\hat{x}_0(t)\\ \vdots \\ \hat{x}_{p-1}(t)\end{bmatrix} =$$

$$\begin{bmatrix}\left[A(p-1) + D(p-1)\hat{\Delta}_{p-1}(t)E(p-1)\right]\hat{x}_{p-1}(t)\\ \left[A(0) + D(0)\hat{\Delta}_0(t)E(0)\right]\hat{x}_0(t)\\ \vdots \\ \left[A(p-2) + D(p-2)\hat{\Delta}_{p-2}(t)E(p-2)\right]\hat{x}_{p-2}(t)\end{bmatrix}. \tag{6.5}$$

Any perturbation $\Delta(\cdot)$ corresponding to (6.2) leads to a perturbation $\hat{\Delta}$ as in (6.5) via

$$\hat{\Delta}_k(t) = \begin{cases}\Delta(t) & t = lp + k\\ 0 & else\end{cases}, \quad k = 0, \ldots, p-1, \quad l, t \in \mathbb{N}.$$

As also $\|\Delta(\cdot)\|_\infty = \|\hat{\Delta}\|_\infty$ we obtain

$$r^1_{\mathbb{C}}(A; (D, E)) \geq r^1_{\mathbb{C}}(\hat{A}; (\hat{D}_k, \hat{E}_k)). \tag{6.6}$$

On the other hand if $\hat{\Delta}$ destabilizes (6.5) then

$$\Delta(t) = \hat{\Delta}_{k(t)}(t), \quad t \in \mathbb{N}$$

where $k(t+1) = (k(t)+1) \bmod p$, $0 \leq k(0) \leq p-1$ destabilizes (6.2) for some $0 \leq k(0) \leq p-1$, because these sequences are just the perturbations corresponding to one of the p trajectories of (6.1) in the trajectory of (6.4). So that

$$r^1_{\mathbb{C}}(A; (D, E)) = r^1_{\mathbb{C}}(\hat{A}; (\hat{D}_k, \hat{E}_k)). \tag{6.7}$$

In the periodic case it is natural to assume the disturbance operators $\Delta(\cdot)$ to be periodic as well. Then the system (6.5) turns out to be time invariant, and we obtain

Proposition 6.1 *Assume (1.1) is a periodic system, where $A(\cdot)$, $D(\cdot)$, $E(\cdot)$ are all of period p. Then the stability radius with respect to periodic perturbations satisfies*

$$r^1_{\mathbb{C},p}(A; (D, E)) = r^1_{\mathbb{C},c}(\hat{A}; (\hat{D}_k, \hat{E}_k)) = \left[\sup_{\omega \in [0, 2\pi]} \mu(G(e^{i\omega}))\right]^{-1} \tag{6.8}$$

where $G(s)$ is defined by

$$G(s) = \hat{E}(sI - \hat{A})^{-1}\hat{D} =$$

$$\begin{bmatrix} \hat{E}_0(sI-\hat{A})^{-1}\hat{D}_0 & \hat{E}_0(sI-\hat{A})^{-1}\hat{D}_1 & \cdots & \hat{E}_0(sI-\hat{A})^{-1}\hat{D}_{p-1} \\ \hat{E}_1(sI-\hat{A})^{-1}\hat{D}_0 & \hat{E}_1(sI-\hat{A})^{-1}\hat{D}_1 & \cdots & \hat{E}_1(sI-\hat{A})^{-1}\hat{D}_{p-1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \hat{E}_{p-1}(sI-\hat{A})^{-1}\hat{D}_0 & \hat{E}_{p-1}(sI-\hat{A})^{-1}\hat{D}_1 & \cdots & \hat{E}_{p-1}(sI-\hat{A})^{-1}\hat{D}_{p-1} \end{bmatrix}.$$

Proof: The right equation in (6.8) follows from Theorem 4.3. The left one by construction.

□

The above Poposition shows that robust stability analysis of a periodic system with single structured perturbations is equivalent to robust stability analysis of a time-invariant system with multi-structured time-invariant perturbations.

We will now express $G(s) = \hat{E}(sI - \hat{A})^{-1}\hat{D}$ in terms of the original data. Note that for $i, j = 0, \ldots, p-1$

$$\hat{E}_j(sI - \hat{A})^{-1}\hat{D}_i = E(j)B_{i+1,j}(s)D(i)$$

where $B_{i,j}(s)$ denotes the i,j-th block of $(sI-\hat{A})^{-1}$ corresponding to the decomposition $\hat{X} = X^p$, and the indices are to be read modulo p. If $\Phi(t,s)$ denotes the evolution operator (1.3) of (6.1), it can be verified by direct computation that

$$B_{i,j}(s) = \begin{cases} s^{p-1}(s^p - \Phi(i+p,i))^{-1} & , \quad i = j \\ s^{j-i-1}(s^p - \Phi(i+p,i))^{-1}\Phi(i+p,j) & , \quad i < j \\ s^{p+j-i-1}(s^p - \Phi(i+p,i))^{-1}\Phi(i,j) & , \quad i > j \end{cases} \tag{6.9}$$

for $i, j = 0, \ldots, p-1; s \in \mathbb{C}\backslash\mathbb{D}$. In this notation we make implicitly use of the fact, that $A(t+p) = A(t), t \in \mathbb{N}$. Thus we obtain the

Corollary 6.2 *With the assumptions of Proposition 6.1 it holds*

$$r^1_{\mathbb{C},p}(A;(D,E)) = \left[\sup_{\omega\in[0,2\pi]} \mu(G(e^{i\omega}))\right]^{-1} \tag{6.10}$$

where $G(s)$ is defined by

$$G(s) = (E(j)B_{i+1,j}(s)D(i))_{i,j=0,\ldots,p-1}$$

and $B_{i,j}$ are defined by (6.9).

In view of Corollary 6.2 it is natural to ask if $r^1_{\mathbb{C},p}(A;(D,E)) = r^1_{\mathbb{C}}(A;(D,E))$ or if the stability radius with respect to p-periodic perturbations is strictly larger, in general, than the robustness of the system with respect to arbitrary time-varying perturbations. For time-invariant systems with *single* perturbation structure in Hilbert spaces we have $r^1_{\mathbb{C},c}(A;(D,E)) = r^1_{\mathbb{C}}(A;(D,E))$. If the same equality held in Hilbert spaces for $(\hat{A},(\hat{D}_k,\hat{E}_k))$ Proposition 6.1 would imply $r^1_{\mathbb{C},p}(A;(D,E)) = r^1_{\mathbb{C}}(A;(D,E))$.

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