Supply Network Engineering: An Approach to Robust Capacity Allocation for Stochastic Production Processes ⋆

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Abstract: Supply networks are complex dynamical systems. Already the perturbation of a single production process can put the timely satisfaction of customer demand at risk. In this paper we model supply networks as multiclass queueing networks and present an approach to robust capacity allocation with respect to perturbations of the production processes. To this end we consider the fluid model of a multiclass queueing network and use its stability radius to measure the robustness. The stability radius reflects the smallest perturbation that destabilizes the network. Based on results concerning this measure we set up an optimization problem for the capacity allocation.

Keywords: Capacity Management, Supply Network Stability, Integrated Logistics

1. INTRODUCTION

Supply networks are shaped by several factors (e.g. technological innovation, logistics services available world-wide) and have become global. These large-scale networks comprise external suppliers, distributed manufacturing plants, sales centers and transportation assets. In addition to the structural complexity, the underlying procurement, manufacturing and distribution processes are dynamic. Furthermore, the evolution of internal and external parameters that determines structural and dynamic properties of global supply networks is not certain. Hence the resulting behavior of the network is often complex and uncertain. Especially the long-term planning has to anticipate customer demand as well as the dynamics of the supply network that have a significant impact on the performance and sustainability of the network. A robust and sustainable supply network engineering is given by the capability of the network to cope with several possible future scenarios in an efficient manner. In this context not only locations and transportation links of the supply chain have to be properly chosen but also production capacities at each location for the processed products. Stochastic programming and robust optimization are two methods in order to set up a robust plan. Nevertheless the planning result strongly depends on the arbitrarily chosen deterministic future scenarios.

In this paper we present a new approach to robust capacity allocation for stochastic production processes in supply networks on the long term. To this end we assume that the production processes of the supply network can be modeled as a multiclass queueing network. Based on this starting point we consider the fluid approximation of the dynamic and uncertain multiclass queueing network. The deterministic fluid model facilitates stability and robustness analyses for such networks. In particular we use the stability radius to quantify the robustness. The stability radius is given by the smallest perturbation that destabilizes the network. During the process of capacity allocation we aim to maximize the robustness of a supply network in regard to perturbations of the production processes. Hence, we seek for a sustainable capacity allocation that enables the network to handle unexpected reduced production capacities (e.g. machine breakdown or strike) or an additional workload. Based on findings regarding the stability radius we set up a mathematical program that allows to find an optimal capacity allocation to the production processes.

The outline of the paper is as follows. In Section 2 a brief literature review of applied planning systems and methods, including supply network engineering, is given. Section 3 introduces the modeling concept of fluid networks. This modeling approach captures the essence of multiclass queueing networks on the long term. Stability and robustness analysis of fluid networks are discussed in Section 4. Based on the findings regarding the stability radius an optimization program is formulated in Section 5. The program aims to maximize the robustness of a modeled supply network by adjusting the allocated capacity to the production processes. This section is followed by the computational analysis in Section 6. At the end of the paper we present some conclusions and an outlook.

⋆ This work is part of the research project *Stability, robustness and approximation of dynamic large-scale networks - theory and applications in logistics networks*, funded by the Volkswagen Foundation.

2. ADVANCED PLANNING SYSTEMS AND METHODS

A sustainable creation of value is paramount aim of global supply networks and mainly determined by the network engineering. This goal is fostered by Advanced Planning Systems (APS) - applied to such networks. The underlying structure of APSs is illustrated by the Supply Chain Planning Matrix in Figure 1 (Rohde et al. (2000)).

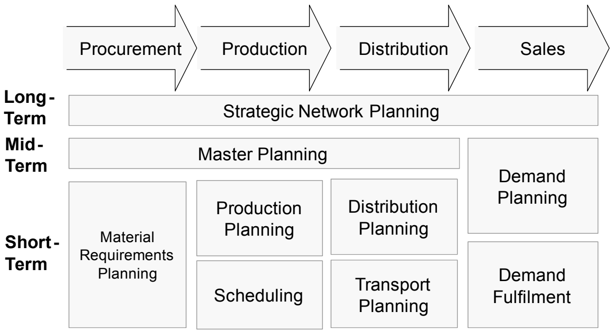

Fig. 1. Supply Chain Planning matrix.

The matrix comprises modules for the planning tasks that are characterized by time horizon and involved business functions. The degree of detail increases and the planning horizon decreases by shifting from the long-term to the short-term. These consecutive modules allow to handle the complexity of an overall planning process (Fleischmann et al. (2004)). The modules are often based on mathematical programming formulations or heuristics that assume deterministic planning information (Scholl (2001)). However, it can be shown that such approaches fail to cope with a dynamic environment and the considerable uncertainty of the underlying planning information (Landeghem and van Vanmaele (2002)). Stochastic programming and robust optimization address uncertainty of relevant parameters (Mulvey et al. (1995)) and are used for robust planning. They are based on a set of deterministic future scenarios of relevant parameters (Helfrich and Cook (2002)) instead of point estimates. In this context a plan is considered robust as it remains close to a desired solution for each scenario. These approaches are applied to network design problems (Klibi et al. (2010)) and (Pan and Nagi (2010)). Nevertheless, the obtained planning results depend on the arbitrarily chosen scenarios.

In this paper we present an approach to supply network design that maximizes the robustness of a supply network in regard to perturbations of the production processes. This approach incorporates knowledge about the dynamic behavior of supply networks as well as a consideration of uncertainty. The dynamics of such complex networks can be modelled by multiclass queueing networks. A queueing network is said to be stable if the total number of products in the network remains bounded over all time. This means that the long-run input rate of the network equals the long-run output rate. For a precise definition of the stability of multiclass queueing networks see Bramson (2008). Dai (1995) and Bramson (2008) presented an approach to investigate the stability of queueing networks using the so-called fluid limit model. The fluid approximation model is a continuous deterministic analogue of the discrete stochastic model. The stability of a corresponding fluid limit model implies the stability of the original queueing network. In comparison to a queueing model the stability of a fluid model can be determined more easily.

Since the evolution of relevant external and internal parameters is not known prior we embed a measure for robustness of a fluid model with respect to perturbations of the process rates into our approach. The stability radius of a network quantifies the size of perturbations that are guaranteed not to destroy stability. Hence it allows to prepare a network precisely for unexpected events of a certain magnitude. First results and capabilities concerning this measure were presented in Scholz-Reiter et al. (2010). There the concept of the stability radius for dynamical systems (Hinrichsen and Pritchard (2005)) is adapted to the case of fluid network models.

3. DESCRIPTION OF THE FLUID MODEL

The following model description relies on (Bramson and Dai (2001)) and (Ye and Chen (2001)). The considered network consists of locations j with $j \in \mathcal{J} = \{1, 2, ..., J\}$ and different classes of products k with $k \in \mathcal{K} = \{1, 2, ..., K\}$. Every class of product is processed exclusively at one location. The mapping $s : \mathcal{K} \to \mathcal{J}$ determines which class of product is processed by which location and generates the so called *constituency matrix* C, with $c_{jk} = 1$ if $s(k) = j$ and $c_{jk} = 0$ else. For every location it is assumed that the set $C(j) = \{k \in \mathcal{K} : s(k) = j\}$ is nonempty. Further every class of product k has the exogenous arrival rate α_k and the process rate μ_k of products per time unit. After a product of class k has been processed at location $s(k)$ it either leaves the network or becomes a product of another class l, with $l \in \mathcal{K}$. The constant p_{lk} denotes the proportion of processed products of class l that become products of class k. Hence $1-\sum_{l=1}^{K} p_{lk}$ is the proportion that leaves the network. The corresponding $K \times K$ matrix P is referred to as the transition matrix. It is assumed that P has spectral radius strictly less than one, i.e. ultimately all products leave the network. The initial amount of products is represented through the K-dimensional vector $Q(0)$. The queue length of products k at time t is denoted by $Q_k(t)$ and the total amount of time in the interval $[0, t]$ that location $s(k)$ has devoted to processing products of class k is denoted by $T_k(t)$. The performance is described by the K-dimensional queue length process $\{Q(t) : t \geq 0\}$ and the K-dimensional allocation process $\{T(t) : t \geq 0\}$.

The next step is to fix a policy that rules the order in which the arriving products are processed at each location. In this paper we use the head-of-the-line proportional processor sharing discipline (HLPPS). Under this discipline all nonempty product classes present at a location are produced simultaneously proportional to their queue length. Head-of-the-line means that a location processes only one product of each type at any time. The allocation rate $\dot{T}_k(t)$ for class k products is proportional to the total queue length at location $s(k)$ present at time t. That is,

$$\dot{T}_k(t) = \frac{Q_k(t)}{\sum_{l \in C(j)} Q_l(t)} \qquad \text{if} \qquad \sum_{l \in C(j)} Q_l(t) > 0.$$

Note that even when the queue length of class k products at location $s(k)$ is zero, $\dot{T}_k(t)$ may still be positive. Finally the idle time process $Y = \{Y(t) : t \geq 0\}$ is introduced, i.e. $Y_j(t)$ denotes the cumulative time that location $s(k)$ idles in the interval $[0, t]$. With $M = \text{diag}(\mu)$ the dynamics of the fluid network under HLPPS discipline can be summarized as follows

$$Q(t) = Q(0) + \alpha\, t - (I - P^T)MT(t) \geq 0, \quad (1)$$

$$W(t) = C\, M^{-1}\, Q(t), \quad (2)$$

$$Y(t) = et - C\, T(t), \quad (3)$$

$$0 = \dot{Y}_j(t)\, W_j(t) \text{ for all } t \geq 0 \text{ and for all } j \in \mathcal{J}, \quad (4)$$

$$\dot{T}_k(t) = \frac{Q_k(t)}{\sum_{l \in C(j)} Q_l(t)} \qquad \text{if} \qquad \sum_{l \in C(j)} Q_l(t) > 0. \quad (5)$$

From equation (4) it follows that the idle time for a product class k increases if and only if the workload $W_j(t)$ of location j is zero, i.e. the queues at location j are empty. Networks with this property are called work-conserving. Relation (1) is called the flow balance relation. Any pair $(Q(\cdot), T(\cdot))$ that satisfies (1)-(5) is called a solution of the HLPPS fluid network. The set of all feasible queue length processes is denoted as

$$\mathcal{Q} = \{Q(\cdot) : \exists\, T(\cdot) : (Q(\cdot), T(\cdot)) \text{ is a solution}\}.$$

We use the following notations. For $a \in \mathbb{R}^n_+$ define the weighted 1-norm with weight a by $||x||_a = \sum_{k=1}^{K} |a_k\, x_k|$. For the special case where $a = (1 \ldots 1)^T$ we write $|| \cdot ||_1$. The total amount of products in the network at time t is given by the sum of the queue length at time t for every class, i.e. by $||Q(t)||_1$.

Example 1. Throughout the paper we consider an example network of two locations that process three classes of products. A schematic illustration is given in Figure 2. The parameters for the test scenario are

$$\alpha = \begin{pmatrix} 0.15 \\ 0.15 \\ 0.10 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0.6 \\ 0.9 \\ 0.5 \end{pmatrix}, \quad P = \begin{pmatrix} 0.25 & 0.15 & 0.20 \\ 0.05 & 0.25 & 0.15 \\ 0.20 & 0.25 & 0.10 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

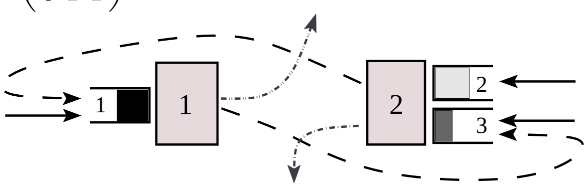

Fig. 2. Fluid network with two locations under HLPPS discipline.

4. STABILITY AND ROBUSTNESS ANALYSIS

This section provides the definition of stability for fluid networks. Further, relevant variables as well as a necessary and sufficient condition for the stability of fluid networks under HLPPS discipline are introduced. In addition, in this section the stability radius is defined and a theoretical framework for the computation of the stability radius is established.

Definition 1. A fluid network $\mathcal{Q}$ is said to be stable, if there exists a finite time $\tau \geq 0$ such that $Q(\tau + \cdot) \equiv 0$ for any $Q(\cdot) \in \mathcal{Q}$ with $||Q(0)||_1 = 1$.

Since fluid networks contain reentrant flows, the so-called effective arrival rate for class k products, denoted λ_k, is given by

$$\lambda_k = \alpha_k + \sum_{l=1}^{K} \lambda_l\, p_{lk}. \quad (6)$$

As the spectral radius of P is assumed to be strictly less than one, (6) can be written as $\lambda = (I - P^T)^{-1}\, \alpha$. The nominal workload of location j is given by

$$\rho_j = \sum_{k \in C(j)} \frac{\lambda_k}{\mu_k}. \quad (7)$$

In vector form (7) can be written as $\rho = CM^{-1}\lambda$ and by using the expression for λ the nominal workload can be expressed as

$$\rho = C\, M^{-1}\, (I - P^T)^{-1}\, \alpha. \quad (8)$$

Sometimes ρ is also referred to as the traffic intensity. Clearly, a necessary condition for the stability of a fluid network is that the nominal workload for every location j is strictly less than one. By using the J-dimensional vector $e = (1, ..., 1)^T$ this is

$$\rho < e. \quad (9)$$

Here the inequality $<$ has to be understood componentwise. Condition (9) is necessary for fluid networks under any service discipline. However, a sufficient condition depends on the service discipline, i.e. fluid networks may be stable under some discipline but not under another, see Kumar and Seidman (1990). The following theorem states that condition (9) is also sufficient for HLPPS fluid networks (Bramson (2008)).

Theorem 1. A HLPPS fluid network is stable if and only if $\rho < e$.

The nominal workload for the test scenario in Example 1 is $\rho = (0.472\ 0.829)^T$, which implies that the network is stable. In this paper we aim to measure the robustness of a given fluid network $\mathcal{Q}$ with respect to perturbations of the process rates μ. Consequently, we are interested in the the smallest process rates μ such that the fluid network keeps the property of stability. To this end, we perturb the process rate by subtraction of a vector $\delta \in \mathbb{R}^K_+$ and consider the fluid network $\mathcal{Q}(\delta) = (\alpha, \mu - \delta, P, C)$. So we are interested in the smallest perturbation that destabilizes the network. Thus according to Scholz-Reiter et al. (2010) we define the following.

Definition 2. Let $a \in \mathbb{R}^K_+$ be fixed. The a-weighted stability radius of the fluid network $\mathcal{Q}$ with respect to perturbations of the production rate is defined by

$$r(\mathcal{Q}) = \inf\{\ ||\delta||_a : \mathcal{Q}(\delta) \text{ is not stable }\}. \quad (10)$$

Note that the stability radius is a property of a given system. Consequently different values of α, μ, P or C lead to different stability radii. Since the condition (9) is necessary and sufficient for the stability for HLPPS fluid networks the stability radius can equivalently be described by the nominal workload condition (9). To be precise, let $M(\delta) = \text{diag}(\mu - \delta)$ reflect the perturbed process rates and let

$$\rho(\delta) = C\, M(\delta)^{-1}\, (I - P^T)^{-1}\, \alpha \quad (11)$$

denote the perturbed nominal workload. Then the stability radius can be expressed as

$$r(\mathcal{Q}) = \min\{\; ||\delta||_a \;:\; \rho(\delta) \not< 1 \;\}\,. \tag{12}$$

Here $\rho(\delta) \not< 1$ means that there is at least one component of $\rho(\delta)$ that is greater or equal to one. This reflects the fact that for some location j the nominal workload is at least one and thus the network is unstable. To give a geometric interpretation of Definition 2 and the equivalent representation (12) of the stability radius we focus on location $\mathcal{S}_2$ from Example 1. That is, we consider one location that processes two classes of products. For given

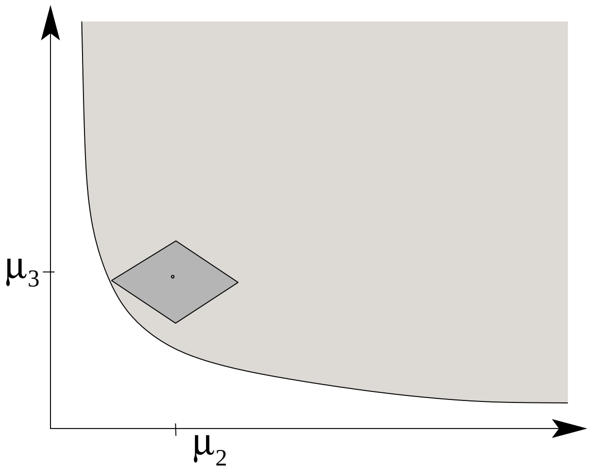

Fig. 3. Illustration of the stability radius for one station processing two product classes.

effective arrival rates λ_2, λ_3 the light grey domain in Figure 3 represents the set of process rates that satisfy condition (9), i.e.

$$\{(\mu_2, \mu_3) \in \mathbb{R}^2_+ \;:\; \frac{\lambda_2}{\mu_2} + \frac{\lambda_3}{\mu_3} < 1\}.$$

By Theorem 1, for any process rates in the interior of the light gray domain the network is stable, while for process rates on the boundary the network is unstable. So $(\mu_2, \mu_3) = (0.9, 0.5)$ of the example network is an interior point of the light gray domain. Let $B_{||\cdot||_a}(x, r)$ denote the neighborhood around $x \in \mathbb{R}^K$ of radius r, where r is measured with respect to the norm $||\cdot||_a$. From this perspective the stability radius can be described as the radius of largest neighborhood around μ that is completely contained in the interior of the light gray domain such that at least one edge of $B_{||\cdot||_a}(\mu, r)$ intersects the boundary of the light grey domain. In Figure 3 the dark grey domain illustrates the neighborhood $B_{||\cdot||_a}(\mu, r)$. Based on this, the stability radius can be calculated by the following optimization problem

$$\begin{aligned} \min \quad & \sum_{k=1}^{K} a_k\, \delta_k \\ \text{such that} \quad & C\, M(\delta)^{-1}\, \lambda \not< e \\ & 0 \le \delta \le \mu. \end{aligned} \tag{13}$$

The first constraint can also be expressed as

$$\max_{j \in \mathcal{J}} \rho_j(\delta) \ge 1. \tag{14}$$

To compute the stability radius we split the problem into J subproblems as follows. We solve the optimization problem (13) for each location $j \in \mathcal{J}$ individually and the smallest solution in magnitude represents the solution to (13). In the sequel we describe how to solve the optimization problem for a single location $j \in \mathcal{J}$. That is, for each location $j \in \mathcal{J}$ the smallest perturbation $r_j(\mathcal{Q})$ that leads to instability can be described by an optimization problem of the form

$$\begin{aligned} \min \quad & \sum_{k \in C(j)} a_k\, \delta_k \\ \text{such that} \quad & \sum_{k \in C(j)} \frac{\lambda_k}{\mu_k - \delta_k} \ge 1 \\ & 0 \le \delta_k \le \mu_k \qquad k \in C(j). \end{aligned} \tag{15}$$

Consequently, the stability radius of the whole fluid network $\mathcal{Q}$ is given by

$$r(\mathcal{Q}) = \min_{j \in \mathcal{J}} r_j(\mathcal{Q}). \tag{16}$$

In the following we want to analyze the solution of the optimization problem (15) for a single location. We recall some concepts from Rockafellar (1970). For a set $A \subset \mathbb{R}^n$ the boundary is denoted by ∂A. A set $A \subset \mathbb{R}^n$ is called convex if $(1-c)x + cy \in A$ whenever $x, y \in A$ and $c \in (0,1)$. A point x of a convex set A is called an extreme point if there is no way to express x as a convex combination $(1-c)x + cy$ such that $x, y \in A$ are distinct and $c \in (0,1)$. The set of all extreme points of A is denoted by $\text{ext}(A)$. The convex hull of a set A, denoted $\text{conv}(A)$, is the smallest convex set that contains A.

Proposition 2. Given a closed and convex set $B \subset \mathbb{R}^n$ and let $A \subset B$ be convex and compact such that $\partial A \cap \partial B \neq \emptyset$. Then $\text{ext}(A) \cap \partial B \neq \emptyset$.

Proof. The assertion is shown by contradiction. So assume that $\text{ext}(A) \cap \partial B = \emptyset$. This implies that $\text{ext}(A) \subset \text{int}(B)$. Further, by Minkowski's Theorem ((Borwein and Lewis, 2006, Theorem 4.1.8)) it holds that $A = \text{conv}(\text{ext}(A)) \subset \text{int}(B)$, which is a contradiction to $\partial A \cap \partial B \neq \emptyset$. $\square$

To analyze the solution of the optimization problem (15) we first regard the following situation. For given λ_1, λ_2 and $c > 0$ consider the set

$$M = \{x = (x_1, x_2) \in \mathbb{R}^2_+ \;:\; \frac{\lambda_1}{x_1} + \frac{\lambda_2}{x_2} \le c\}.$$

Note that M is closed and convex and the boundary ∂M equals the set of extreme points $\text{ext}(M)$. The minimal distance r from $x = (x_1, x_2) \in M$ to the boundary ∂M can be described by

$$r = \max\{\delta \;:\; B_{||\cdot||_a}(x, \delta) \subset M\}$$

Further, it holds for every $\varepsilon > 0$ that $B_{||\cdot||_a}(x, r+\varepsilon) \not\subset M$ and this implies that $\partial M \cap \partial B_{||\cdot||_a}(x, r+\varepsilon) \neq \emptyset$. By Proposition 2 it holds that

$$\partial M \cap \text{ext}\left(B_{||\cdot||_a}(x, r+\varepsilon)\right) \neq \emptyset.$$

This is used to analyze the optimization problem

$$\begin{aligned} \min \quad & a_1\, \delta_1 + a_2\, \delta_2 \\ \text{such that} \quad & \frac{\lambda_1}{x_1 - \delta_1} + \frac{\lambda_2}{x_2 - \delta_2} = c \\ & 0 \le \delta_k \le x_k \qquad k = 1, 2. \end{aligned} \tag{17}$$

Consequently, the application of Proposition 2 implies

Proposition 3. If r is a solution to (17) it holds that

$$\frac{\lambda_1}{x_1 - r} + \frac{\lambda_2}{x_2} = c \quad \text{or} \quad \frac{\lambda_1}{x_1} + \frac{\lambda_2}{x_2 - r} = c.$$

The next step is to generalize the above proposition to the setting

$$\begin{aligned} \min \quad & a_1\,\delta_1 + \ldots + a_n\,\delta_n \\ \text{such that} \quad & \frac{\lambda_1}{x_1-\delta_1} + \ldots + \frac{\lambda_n}{x_n-\delta_n} = c \\ & 0 \le \delta_k \le x_k \qquad k = 1,2,\ldots,n. \end{aligned} \tag{18}$$

Lemma 4. If r is a solution to (18) it holds that

$$\delta_1 = r \quad \text{or} \ldots \text{or} \quad \delta_n = r.$$

Proof. Assume that $(r_1\,r_2\,\ldots\,r_n)^T$ is a solution to (18), i.e. $r = ||(r_1\,r_2\,\ldots\,r_n)^T||$ and

$$\frac{\lambda_1}{x_1-r_1} + \frac{\lambda_2}{x_2-r_2} + \ldots + \frac{\lambda_n}{x_n-r_n} = c.$$

The statement is shown by induction. For $n = 2$ the claim is valid by Proposition 3. In the inductive step we a Assume that the claim is true n and consider the case for $n+1$. Then it holds that

$$\frac{\lambda_1}{x_1-r_1} + \frac{\lambda_2}{x_2-r_2} + \ldots + \frac{\lambda_{n+1}}{x_{n+1}-r_{n+1}} = c.$$

By hypothesis it holds that $r_1 = r$ or...or $r_n = r$. Without loss of generality, let $r_1 = r$. Hence

$$\frac{\lambda_1}{x_1-r_1} + \frac{\lambda_{n+1}}{x_{n+1}-r_{n+1}} = \eta - \frac{\lambda_2}{x_2} - \ldots - \frac{\lambda_n}{x_n}.$$

So by applying Proposition 3 it follows that $r_1 = r$ or $r_{n+1} = r$, which shows the assertion. □

Remark 1. The significance of the previous results lies in the fact that the stability radius can be computed by solving an optimization problem and its solution can be obtained in the following way: Consider every location separately. Then perturb the process rate of exactly one product class and solve the corresponding optimization problem. Finally take the minimum of all results.

5. OPTIMIZATION MODEL FOR THE CAPACITY ALLOCATION

In this section we formulate an optimization model to maximize the robustness of a supply network. This is based on the optimization problem (13). For given α, P and C the capacity allocation to process rates μ is flexible within given bounds and allows to maximize the stability radius with the weight $a = e$.

Sets

$\mathcal{K}$	Product classes $(k,l,n \in \mathcal{K})$
$\mathcal{J}$	Locations $(j \in \mathcal{J})$
$C(j,k)$	Set, that determines which product class k is processed at which location j

Parameters

α_k	External arrival rate of class k products
z_j	Maximal process rate of location j that is available for processing all assigned product classes
$C_{j,k}$	Constituency matrix, that determines which product class k is processed by a certain location j
$P_{l,k}$	Routing matrix that determines the proportion of product class l which becomes product class k after being processed
$I_{l,k}$	Identity matrix of product classes
R	Inverse matrix of $(I-P^T)$
L	Large scalar (big M)

Variables

μ_k	Process rate of class k products at the assigned location
Δ	Perturbation of the process rate of each product class in the case that the other product classes are not disturbed (measure for the stability radius)
$\rho_{j,k}$	Nominal workload of location j in the case that the process rate μ_k of product class k is perturbed
$CM_{j,k,n}$	Auxiliary matrix CM^{-1}
$A_{j,k,n}$	Auxiliary matrix $CM^{-1}(I-P^T)^{-1}$

Binary variables

$X_{j,k}$	Binary variable denoting that location j has nominal workload $\rho_{j,k} = 1$ if the process rate μ_k of product class k is perturbed

Mathematical model

In (19) and (20) two auxiliary matrices are calculated. Equation (19) implements the idea of Remark 1. Hence, the matrix $CM_{j,k}$ is created n times, each time the process rate μ_k is disturbed if and only if $k = n$.

$$CM_{j,k,n} = C_{j,k}(\mu_k - \Delta\cdot\delta_{k,n})^{-1} \quad (j \in \mathcal{J}; k,n \in \mathcal{K}) \tag{19}$$

Here we use the notation of the Kronecker delta

$$\delta_{n,k} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{else }. \end{cases}$$

$A_{j,k,n}$ describes the relation between the arrival rate α and the nominal workload ρ, see (8).

$$A_{j,k,n} = \sum_l CM_{j,l,n} R_{l,k} \qquad (j \in \mathcal{J}; k,n \in \mathcal{K}) \tag{20}$$

$\rho_{j,n}$ is the nominal workload of location j if the process rate of product class n is disturbed.

$$\rho_{j,n} = \sum_k A_{j,k,n}\alpha_k \qquad (j \in \mathcal{J}; n \in \mathcal{K}) \tag{21}$$

Equation (22) enforces that the nominal workload of location j is greater or equal to one, if and only if exactly the process rate of product class n is disturbed.

$$\rho_{j,n} \ge 1 - (1 - X_{j,n})L \qquad (j \in \mathcal{J}; n \in \mathcal{K}) \tag{22}$$

Equation (23) ensures that the nominal workload does not exceed 1.

$$\rho_{j,n} \le 1 \qquad (j \in \mathcal{J}; n \in \mathcal{K}) \tag{23}$$

Condition (24) guarantees that the nominal workload of exactly one location equals one for a perturbation with exactly one strictly positive component

$$\sum_j \sum_n X_{j,n} = 1. \tag{24}$$

The total capacity of each location j is bounded by z_j.

$$\sum_{k \in C(j,k)} \mu_k \le z_j \qquad (j \in \mathcal{J}) \tag{25}$$

The objective function (26) maximizes the perturbation Δ that can be subtracted from the processing rates before the whole network becomes unstable. The stability radius is thus the solution of the problem:

$$\max \Delta\,. \tag{26}$$

6. COMPUTATIONAL ANALYSIS

We implemented the optimization model from Section 5 in GAMS 22 with the solver DICOPT and applied the introduced test case. The maximal process rate for the locations is $z = (1\ 1.4)^T$ and $L = 1000$. In the case that the production capacity allocation is pre-given ($\mu = (0.6\,0.9\,0.5)^T$) the stability radius of the test case is $\Delta = 0.135$. Table 1 shows the nominal workload $\rho_{j,k}$ of each location j in regard to the disturbed process rate of product class k. Column one shows for instance the workload of the two locations if only process rate of product class 1 is disturbed by Δ. Since the stability radius reflects the smallest perturbation that leads to instability Table 1 shows that the network becomes unstable at location 2 if a perturbation of magnitude $\Delta = 0.135$ is added to the process rate of product class 3.

$\rho_{j,k}$	$k = 1$	$k = 2$	$k = 3$
$j = 1$	0.610	0.472	0.472
$j = 2$	0.830	0.895	1.000

Table 1.

The allocation of available production capacity to the different product classes at a certain production location is performed by the mathematical program of Section 5. In the following we set the maximal production capacity of location 2 to 1.4, which equals the sum of the required production capacities of product class 2 and 3 in the fixed case. Furthermore we set the available production capacity at location 1 to 1. The obtained capacity allocation by the program is $\mu_1 = 0.519, \mu_2 = 0.752$ and $\mu_3 = 0.648$. Moreover the stability radius takes a value of 0.236. The workload of each of the production locations for the individually disturbed process rates of the product classes are given by Table 2.

$\rho_{j,k}$	$k = 1$	$k = 2$	$k = 3$
$j = 1$	1.000	0.546	0.546
$j = 2$	0.798	1.000	1.000

Table 2.

Table 2 shows that the network becomes unstable if a perturbation of magnitude 0.236 is added either to the process rate of product class 1, 2 or 3. In this context it is remarkable that the stability radius can be increase by 75% without adding additional production capacity. Moreover, the total required capacity can be reduced by 4% with an advanced capacity allocation. These promising results demonstrate the capabilities of our approach for a robust capacity allocation.

7. CONCLUSIONS AND OUTLOOK

In this paper we have introduced a new approach to robust capacity allocation at locations for sustainable supply network engineering. In particular, we focused on the question of measuring and maximizing the robustness of a supply network. To this end we introduced a fluid network model that provides a sufficient condition for the stability of the corresponding multiclass queueing network. We defined the stability radius of the fluid network as a measure for the robustness of the supply network. Using the fact that the stability of a fluid network under proportional processor sharing discipline is equivalent to a nominal workload strictly less than one, we formulated an optimization scheme to maximize the stability radius. In particular, the program chooses the process rates subject to pre-given bounds. In the future combined sources of perturbations (e.g. perturbation of arrival and process rates) have to be considered as well as other service disciplines. Furthermore the approach might be embedded in other design and allocation problems.

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