Growth conditions for the global stability of highspeed communication networks

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Abstract

This note considers the design of TCP-like congestion control protocols for deployment in highspeed communication networks. A basic problem in this area is to design congestion control strategies that probe more aggressively than standard TCP, but which coexist with each other and result in globally stable and equitable network behaviour. In this note we take a first step towards this goal, by formulating the TCP dynamics as a discrete linear system with nonlinear feedback gain. Under the assumption of player synchronisation, conditions for global network stability are derived in the form of growth bounds on the local nonlinear probing functions. Examples are given to illustrate the main features of our results.

I. Introduction

Traffic generated by the *Transmission Control Protocol* (TCP) accounts for 85% to 95% of all traffic in today's internet [1]. TCP, in congestion avoidance mode, is based primarily on the Chiu and Jain's [2] *Additive-Increase Multiplicative-Decrease* (AIMD) paradigm for decentralized allocation of a shared resource (e.g., bandwidth) among competing users. With some minor modifications, the AIMD algorithm has served the networking community well over the past two decades and it continues to provide the basic building block upon which today's internet communication is built.

Recently, in the context of designing high speed communication networks, several authors have suggested basic modifications to the AIMD algorithm; for example, see [3], [4], [5], [6], [7], [8]. One idea underlying these modifications is to replace the constant growth rate of window size in standard TCP with an increasing rate. The rationale for this change is that such protocols probe more aggressively for available bandwidth as network capacity increases [8], [3]. These algorithms, which we refer to as nonlinear AIMD (NAIMD), result in networks with different dynamic properties than those employing the basic (linear) AIMD; see [9], [10], [11], [12]. A basic question in the design of NAIMD networks is how to choose the probing action so that the resulting network exhibits desirable properties. Remarkably, despite increasing deployment of these algorithms (e.g., a high-speed TCP algorithm is implemented as part of the Linux operating system), little work has been carried out in this area and basic questions concerning the existence and nature of network equilibria have yet to be addressed. In this paper we extend results obtained in [13] to more general nonlinear settings.

Our objective in this note is to study basic convergence and stability properties of a class of NAIMD congestion control protocols. In this preliminary study, we restrict attention to deterministic networks in which all sources (players) are informed of network congestion simultaneously, and where all sources employ NAIMD protocols whose growth rates depend only on their most recent congestion window size. The study of synchronized networks is in general restrictive, however it is important for a number of reasons. Firstly, it has been observed by many researchers that source synchronisation is a feature of high-speed networks, and in the context of such networks, the assumption of synchronisation is not overly conservative. Secondly, the study of synchronised networks represents an important first step toward the

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study of more realistic network types in which sources are informed of congestion asynchronously.

Our basic finding in this paper is that NAIMD protocols are stable and lead to a reasonably equitable allocation of resources as long as the growth rate functions of all users in the network increase more slowly than linearly with window size. This linear growth rate is a sharp boundary between stable and unstable behavior; if two or more users employ protocols whose growth rate functions increase more rapidly than linearly, then there cannot be any stable equilibrium for the network.

The paper is organised as follows. In section II we introduce the basic model for NAIMD protocols. The main growth assumption on the nonlinearities which is needed for stability is presented. The main technical tool consists of a state space transformation which leads to transition maps, which are the product of a diagonal (state dependent) matrix and a row stochastic matrix. This leads to the main Lemma 2.1 which provides a Lyapunov type argument for the tranformed system. Its proof is deferred to the appendix. The main result is stated and proved in section III. It provides formulas for the fixed points of NAIMD networks and gives conditions under which these are globally asymptotically stable. We present some examples in section IV.

II. BACKGROUND

Throughout, we use notation $\mathbb{R}$ for the real numbers, $\mathbb{R}^n$ for the n-dimensional real Euclidean space and $\mathbb{R}^{n\times n}$ for the space of $n \times n$ matrices with real entries.

(a) The AIMD algorithm : The AIMD algorithm of Chiu and Jain [2] is a decentralised resource allocation algorithm. The algorithm consists of two operational modes. The first mode, a backoff mode, is invoked whenever a source (player) is informed that the network is congested. Users respond to this notification by down-scaling their utilization-rate in a multiplicative manner. This mode is called the *Multiplicative Decrease* (MD) phase. After reducing their utilisation rates, users probe for available capacity until congestion is reached again, at which point the mode of the next cycle is entered. The second mode of operation is called the *Additive Increase* (AI) phase.

(b) A model of NAIMD : The evolution of the network states over one cycle of operation may be conveniently written as follows. Let $w(k) = [w_1(k), \ldots, w_n(k)]^T$ denote the share of network capacity allocated to each user at the time of the k^{th} congestion event $t(k)$. The capacity constraint requires that $\sum_{i=1}^n w_i(k) = C$, with C as the total capacity of the resource available to the entire system. After the next cycle of multiplicative decrease and additive increase, the utilization-rate of player i becomes

$$w_i(k+1) = \beta_i w_i(k) + a_i(w_i(k))\,(t(k+1) - t(k)), \tag{1}$$

where β_i is a constant in the open interval $(0, 1)$, and $a_i(\cdot)$ is the growth rate function of user i. It follows that the time between congestion events $t(k+1) - t(k)$ is a function of $w(k)$, that is $t(k+1) - t(k) = T(w(k))$ where

$$T(w(k)) = \frac{C - \sum_{j=1}^n \beta_j w_j(k)}{\sum_{j=1}^n a_j(w_j(k))} = \frac{\sum_{j=1}^n (1-\beta_j) w_j(k)}{\sum_{j=1}^n a_j(w_j(k))}\,. \tag{2}$$

Combining (1) and (2) we obtain the discrete time dynamical systems

$$w_i(k+1) = F_i(w_i(k)) := \beta_i w_i(k) + a_i(w_i(k)) \frac{\sum_{j=1}^n (1-\beta_j) w_j(k)}{\sum_{j=1}^n a_j(w_j(k))}\,, \quad i = 1, \ldots, n \tag{3}$$

with initial condition $w(0) = \begin{bmatrix} w_1(0) & \ldots & w_n(0) \end{bmatrix}^T$ satisfying $\sum_{i=1}^n w_i(k) = C$ and $w_i(0) \geq 0, i = 1, \ldots, n$. We note that regardless of initial condition we have $w_i(k) > 0$ for all $k \geq 1$, so that in what follows we will always assume without loss of generality that $w_i(k) > 0$ for all $k \geq 0$.

(c) Growth assumptions: The key assumptions concern the growth rate functions $a_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$. We assume that the functions $a_i, i = 1, \dots, n$ are continuous, nondecreasing and satisfy $a_i(x) > 0$ for all $x > 0$. In addition we will assume that the functions $f_i : (0, \infty) \to (0, \infty)$ defined by

$$f_i(x) := \frac{x}{a_i(x)}\,, \quad i = 1, \dots, n \tag{4}$$

are strictly increasing. E.g. if $a_i(x) = cx^p$, this implies that $p \in (0, 1)$. In the case where a_i is differentiable, the non-decreasing and growth properties can be expressed compactly as the inequalities

$$0 \le \frac{d}{d\log x} \log a_i(x) = \frac{a_i'(x)}{a_i(x)}\, x < 1\,, \quad \forall\, x \ge 0\,. \tag{5}$$

(d) Matrix formulation: We define a new set of network states $z = [z_1, \dots, z_n]^T$ as follows:

$$z_i := \Phi_i(w_i) := (1 - \beta_i)\frac{w_i}{a_i(w_i)} = (1 - \beta_i) f_i(w_i)\,. \tag{6}$$

By assumption on the f_i the transformations Φ_i are bijections of $(0, \infty)$ on their range and so the new variables are well-defined. In terms of these variables the network dynamics (1) becomes

$$\begin{aligned} z_i(k+1) &= \frac{a_i(w_i(k))}{a_i(w_i(k+1))}\Big(\beta_i z_i(k) + (1-\beta_i)\, T(w(k))\Big) \\ &= \frac{a_i(w_i(k))}{a_i(w_i(k+1))}\Big(\beta_i z_i(k) + (1-\beta_i)\, \sum_{j=1}^n z_j(k) p_j(w(k))\Big) \qquad (7) \\ &= \frac{a_i(\Phi_i^{-1} z_i(k))}{a_i(F_i(\Phi_i^{-1}(z_i(k)))}\Big(\beta_i z_i(k) + (1-\beta_i)\, \sum_{j=1}^n z_j(k) p_j(\Phi^{-1}(z(k)))\Big) \qquad (8) \end{aligned}$$

where $p_i(w) = \Big(\sum_{j=1}^n a_j(w_j)\Big)^{-1} a_i(w_i)$. Gathering equation (7) for all i we get the vector equation

$$z(k+1) = R(z(k))z(k) \tag{9}$$

where $R(z) \in \mathbb{R}^{n \times n}$ is defined by

$$R(z)_{ij} = \frac{a_i(w_i)}{a_i(F_i(w_i))}\Big(\delta_{ij}\beta_i + (1-\beta_i)\, p_j(w)\Big)\,, \tag{10}$$

with $z_i = \Phi_i(w_i)$ and where δ_{ij} denotes the standard Kronecker symbol. The matrix $R(z)$ may be factorized as

$$R(z) = D(z)\tilde{R}(z)\,,$$

with a diagonal part $D(z) = \operatorname{diag}\left(\frac{a_1(w_1)}{a_1(F_1(w_1))}, \dots, \frac{a_n(w_n)}{a_n(F_n(w_n))}\right)$ and where the second part of this expression with entries $\tilde{R}(z)_{ij} = \Big(\delta_{ij}\beta_i + (1-\beta_i)\, p_j(w(k))\Big)$ defines a positive, row stochastic matrix. As is well-known, positive, row stochastic matrices define contractions on the positive orthant $\mathbb{R}^n_+$ [14, Theorem 1.1]. This has been called the averaging effect of stochastic matrices. The following lemma shows that $R(z(k))$ asymptotically shares this property in the limit $k \to \infty$. We write $z_{\max}$ and $z_{\min}$ to denote the largest and smallest components of the vector z.

Lemma 2.1: Consider the dynamic system defined in Equation (3), and suppose that all growth functions $a_i(x)$ satisfy condition (4). Then

(i) for all $k \ge 0$

$$z_{\min}(k) \le z_{\min}(k+1) \le z_{\max}(k+1) \le z_{\max}(k)\,.$$

(ii) there is $r < 1$ and a sequence $\{\eta_k\}_{k\in\mathbb{N}}$ converging to zero, such that for all $k \geq 0$

$$z_{\max}(k+1) - z_{\min}(k+1) \leq r\Big(z_{\max}(k) - z_{\min}(k)\Big) + \eta_k\,. \tag{11}$$

Lemma 2.1 is proved in the appendix. We will use it in the proof of our main result to show that the nonlinear system (9) behaves essentially as a linear system with row stochastic matrices, and that the solution $z(k)$ is driven to the fixed point $z = [1, \ldots, 1]^T$. This will imply in turn that the network dynamics (3) converges to a stable solution.

III. Main result

Our principal goal in this paper is to derive conditions for network stability in terms of the network rate functions $a_i(\cdot)$ and the network capacity C. Referring to (1), a *fixed point* of the network is defined to be a vector $w^* \in \operatorname{int}\mathbb{R}^n_+$ satisfying $w_i^* = \beta_i w_i^* + a_i(w_i^*)T(w^*)$ and $\sum w_i^* = C$, where

$$\operatorname{int}\mathbb{R}^n_+ = \{w \in \mathbb{R}^n \,:\, w_i > 0 \text{ for } i = 1, \ldots, n\}\,. \tag{12}$$

Theorem 3.1: Consider the network of NAIMD sources defined in the preamble. Then the following properties hold.

(i) If all $a_i(\cdot)$ satisfy the growth condition (4), then for all $C > 0$ the network has a unique fixed point which is globally asymptotically stable with respect to $\mathbb{R}^n_+$.

(ii) Recall the definition (4), and suppose that a_i is differentiable for all i. If at least two of the $f_i(\cdot)$ are strictly decreasing, then any fixed point is unstable.

Remark 3.1: Theorem 3.1 gives conditions for acceptable network behaviour. Provided that all of the $a_i(\cdot)$ satisfy the growth condition (4), the network has a unique globally asymptotically stable fixed point. If at least two sources do not satisfy these bounds, and produce functions f_i that are strictly decreasing, then the network is unstable. In the case where exactly one of the $f_i(x)$ is decreasing, and all others are increasing there is a locally stable fixed point in $\tilde{\Sigma}$, assuming the capacity C is large enough. In this case the network may or may not be stable.

Remark 3.2: Networks of standard TCP flows satisfy the conditions of Theorem 3.1. Further, our result shows that any mix of standard TCP flows and flows satisfying the growth conditions will result in a globally stable network.

Remark 3.3: Network flows that employ MIMD (multiplicative increase, multiplicative decrease) do not satisfy the growth conditions. It has been shown that such networks either have multiple stable equilibria or are unstable [12]. Scalable TCP is an example of one such protocol.

Proof of Part (i) of Theorem: Here we present the proof of (i) in two parts. In Part A. we prove the existence of unique fixed points for (1), and in Part (B) we prove the global asymptotic stability of these solutions.

Part A. Fixed point: From the fixed point equations

$$w_i^* = \beta_i w_i^* + T(w^*)a_i(w_i^*)\,, \quad i = 1, \ldots, n \tag{13}$$

we obtain the condition

$$f_i(w_i^*) = \frac{T(w^*)}{1-\beta_i}\,, \quad i = 1, \ldots, n\,. \tag{14}$$

In addition to (14) the fixed point has to satisfy the condition $\sum_{i=1}^n w_i^* = C$. Thus the constant $T^* := T(w^*)$ has to satisfy

$$C = \sum_{i=1}^n w_i^* = \sum_{i=1}^n f_i^{-1}\left(\frac{T^*}{1-\beta_i}\right). \tag{15}$$

Now by assumption the functions f_i are all increasing and continuous. Hence the map

$$T \mapsto \sum_{i=1}^{n} f_i^{-1}\left(\frac{T}{1-\beta_i}\right)$$

is continuous with range $(0,\infty)$, so for all $C>0$ there is a unique T^* satisfying (15). This together with (14) determines a unique fixed point w^* in $\tilde{\Sigma}$.

Part B. Global stability: Lemma 2.1 will be proved in the Appendix; we use it now to complete the proof of Theorem 3.1.From (ii) of Lemma 2.1 we deduce that given any $\epsilon>0$, there are integers $K, L>0$ so that

$$\eta_k < (1-r)\frac{\epsilon}{2} \text{ for all } k \geq K, \qquad r^L\Big(z_{\max}(K) - z_{\min}(K)\Big) < \frac{\epsilon}{2} \tag{16}$$

Then for all $k \geq K+L$ we get

$$\begin{aligned} z_{\max}(k+1) - z_{\min}(k+1) &\leq r^{k+1-K}\Big(z_{\max}(K) - z_{\min}(K)\Big) + \sum_{j=0}^{k-K} r^j \eta_{k-j} \\ &\leq r^L(z_{\max}(K) - z_{\min}(K)) + (1-r)\frac{\epsilon}{2}\sum_{j=0}^{k-K} r^j \\ &< \epsilon \end{aligned} \tag{17}$$

Since this holds for any $\epsilon>0$, it follows that $\lim_{k\to\infty}(z_{\max}(k+1) - z_{\min}(k+1)) = 0$. Together with (i) of Lemma 2.1 this implies that $z(k)$ converges to the fixed point $(1,\ldots,1)^T$, and hence $w(k)$ converges to w^*.

Proof of Part (ii) of Theorem: Suppose there is a fixed point $w^* = [w_1^*,\ldots,w_n^*]^T$ in $\operatorname{int}\mathbb{R}_+^n$. The evolution equation (3) can be written as $w(k+1) = F(w(k))$ where $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ is differentiable under our assumptions. We will compute the Jacobian of this map at the fixed point.

For $i = 1,\ldots,n$ define

$$\kappa_i = w_i^* \frac{a_i'(w_i^*)}{a_i(w_i^*)} = \left(\frac{d}{d\,\log x}\log a_i(x)\right)_{x=w_i^*} \tag{18}$$

Let $\gamma_i = \beta_i + (1-\beta_i)\kappa_i$, $\gamma = [\gamma_1,\ldots,\gamma_n]^T$ and $D_\gamma = \operatorname{diag}(\gamma_1,\ldots,\gamma_n)$ be the diagonal matrix whose (i,i) entry is γ_i. Finally let $p = [p_1(w^*),\ldots,p_n(w^*)]^T$ where $p_i(w)$ is defined after (7). Then a straightforward calculation shows that the Jacobian of F at w^* is

$$JF(w^*) = D_\gamma - p\,\gamma^T \tag{19}$$

By assumption and without loss of generality it holds for the indices $n-1, n$ that f_{n-1} and f_n are both strictly decreasing, and hence (18) implies that $\kappa_{n-1}, \kappa_n > 1$. This means that also $\gamma_{n-1}, \gamma_n > 1$. We will show by an interlacing argument, that the Jacobian $JF(w^*)$ has an eigenvalue which is bigger than one.

To this end we will assume that the values γ_i are ordered $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n$. Consider the eigenvalue equation

$$(D_\gamma - p\gamma^T)x = \lambda x$$

If $\gamma^T x = 0$, this implies that x is an eigenvector of D_γ, so that $\lambda = \gamma_i$ for some i. Assume now that $\gamma^T x \neq 0$, so that we may assume it is equal to 1. Then the eigenvalue equation reads in componentwise form

$$\gamma_i x_i - p_i = \lambda x_i\,, \quad i = 1,\ldots,n\,. \tag{20}$$

From this we see that $\lambda = \gamma_i$ for some i if and only if $p_i = 0, i = 1, \ldots, n$, which contradicts $p > 0$. Thus the assumption $\gamma^T x \neq 0$ implies, that λ is different from the γ_i and hence (20) is equivalent to

$$x_i = \frac{p_i}{\gamma_i - \lambda}\,, \quad i = 1, \ldots, n\,.$$

As we have the condition $\gamma^T x = 1$ we obtain

$$1 = \sum \gamma_i x_i = \sum \frac{\gamma_i p_i}{\gamma_i - \lambda} =: q(\lambda)\,.$$

By these consideration we see, that λ is an eigenvalue different from the γ_i, if and only if $q(\lambda) = 1$.

Clearly, the rational function q has poles in $\gamma_1, \ldots, \gamma_n$. Note that for $\eta \searrow \gamma_i$ we have $q(\eta) \to -\infty$ and for $\eta \nearrow \gamma_i$ if holds that $q(\eta) \to \infty$. Thus q maps the interval (γ_i, γ_{i+1}) to $\mathbb{R}$ and consequently, there is a $\gamma_i < \lambda < \gamma_{i+1}$ with $q(\lambda) = 1$, so that λ is an eigenvalue of $Jf(w^*)$. Thus in total every interval (γ_i, γ_{i+1}) contains an eigenvalue, if we set $(\gamma_i, \gamma_{i+1}) = \{\gamma_{i+1}\}$ for the degenerate case $\gamma_i = \gamma_{i+1}$, in which case, clearly γ_i is an eigenvalue of $D_\gamma - p\gamma^T$.

As we have assumed $\gamma_{n-1}, \gamma_n > 1$, this shows that $JF(w^*)$ has an eigenvalue bigger than one. Therefore the fixed point w^* is unstable. ■

IV. Examples

We now give a number of examples to illustrate the main features of Theorem 3.1.

Example 1: Consider a network of three users that compete for bandwidth according to:

$$w_i(k+1) = \beta_i w_i(k) + w_i(k)^{p_i} T(k), \tag{21}$$

with $\beta_i = \{0.9, 0.7, 0.6\}$, $p_i = \{0.5, 0.7, 0.1\}$ and $C = 10$. The fixed point of the network satisfies:

$$w_i^* = \left(\frac{T^*}{1 - \beta_i}\right)^{\frac{1}{1-p_i}}, \tag{22}$$

with $w_1^* + w_2^* + w_3^* = C = 10$. In this case it is easy to determine numerically that $T^* \approx 0.289962$ and so $w_1^* \approx 8.4077, w_2^* \approx 0.8928, w_3^* \approx 0.6995$. Since the $f_i(w(k)) = w(k)^{1-p_i}$ are all strictly increasing, Theorem 4.1 asserts that the fixed point of the network is unique and globally asymptotically stable. Convergence of the network to a unique fixed point can be clearly seen from Figure 1. Note that in this figure the evolution of the complete network is modelled, whereas the discrete model only considers the times at which the constraint is satisfied. Thus necessarily there are oscillations in the figure. The point is that the peak values of the flows converge.

Example 2: Consider again the network in Example 1 with $p_i = \{1.5, 1.7, 1.1\}$ and again $C = 10$. Since the $f_i(w(k)) = w(k)^{1-p_i}$ are all strictly decreasing, Theorem 4.1 asserts that the network is unstable, and the simulation shows that some of the network sources end up with no share of the resource.

V. Conclusions

In this paper we have studied stability properties of heterogeneous networks in which different users implement different versions of nonlinear AIMD algorithms and with different levels of aggressiveness. Conditions for global stability are given in the form of growth conditions on the network growth functions a_i. It has been shown that if the level of aggressiveness a_i is linear or faster, then this results in an unstable situation. This applies in particular to Scalable TCP, in which users set their aggressiveness function to be linear. On the other hand minor modifications of this, in which users are allowed to set their aggressiveness to be slightly sublinear result in a unique exponentially stable fixed point. The rate of convergence to this fixed point, however, deteriorates as the behavior approaches linear.

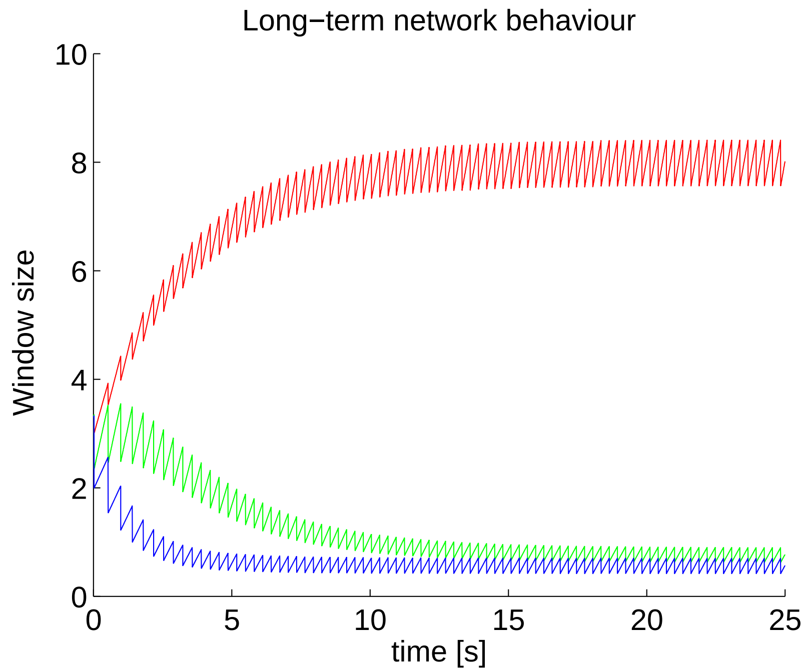

Fig. 1. Example 1 - A network with unique g.a.s. fixed point. Note that the oscillations in the figure do not violate the existence of a fixed point. The model (3) models the peak window sizes and the evolution of the peaks over successive congestion events.

Acknowledgements

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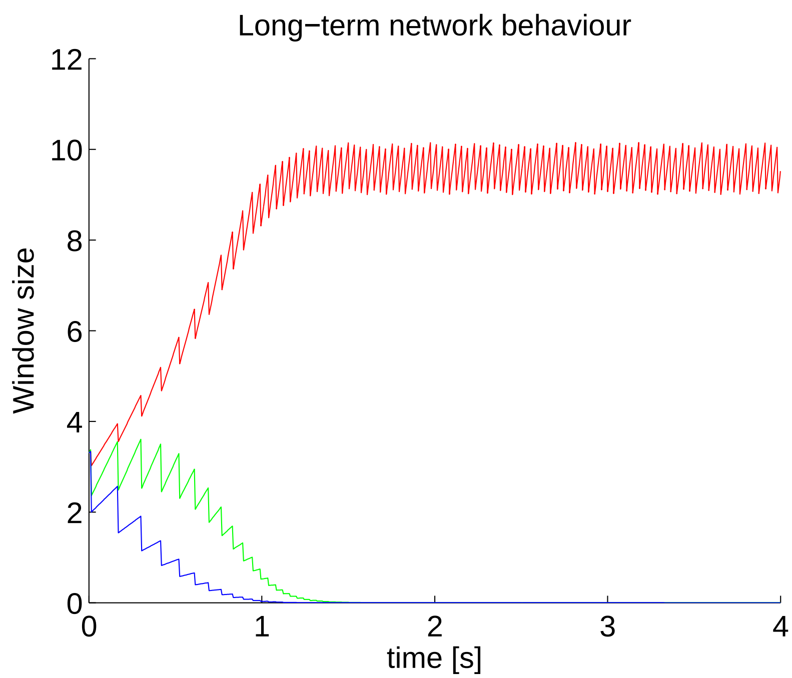

Fig. 2. Example 2 - A network of flows all of which violate the growth condition.

APPENDIX

Proof of Lemma 2.1 :

Part (i): First we show that $z_{\max}(k)$ is decreasing. It follows from the basic evolution equations (1) and (2) that

$$\begin{aligned} w_i(k+1) &= \beta_i w_i(k) + a_i(w_i(k)) \frac{\sum_{j=1}^n (1-\beta_j) w_j(k)}{\sum_{j=1}^n a_j(w_j(k))} \\ &= \beta_i w_i(k) + a_i(w_i(k)) \frac{\sum_{j=1}^n z_j(k) a_j(w_j(k))}{\sum_{j=1}^n a_j(w_j(k))} \\ &\leq \beta_i w_i(k) + a_i(w_i(k)) z_{\max}(k) \\ &= \beta_i \frac{z_i(k)}{1-\beta_i} a_i(w_i(k)) + a_i(w_i(k)) z_{\max}(k) \\ &\leq a_i(w_i(k)) z_{\max}(k) \left(1 + \frac{\beta_i}{1-\beta_i}\right) . \end{aligned}$$

Thus we obtain the estimate

$$w_i(k+1) \leq \frac{a_i(w_i(k))}{1-\beta_i} z_{\max}(k) . \tag{23}$$

By assumption f_i is an increasing function, hence applying f_i to (23) gives

$$\begin{aligned} z_i(k+1) &\leq (1-\beta_i) f_i\left(\frac{a_i(w_i(k))}{1-\beta_i} z_{\max}(k) \right) \\ &= \frac{a_i(w_i(k)) z_{\max}(k)}{a_i\left(\frac{a_i(w_i(k))}{1-\beta_i} z_{\max}(k) \right)} = \frac{a_i(w_i(k)) z_{\max}(k)}{a_i\left(\frac{w_i(k)}{z_i(k)} z_{\max}(k) \right)} . \end{aligned} \tag{24}$$

Since a_i is non-decreasing and $z_i(k) \le z_{\max}(k)$, we have $a_i\left(\frac{w_i(k)}{z_i(k)}\, z_{\max}(k)\right) \ge a_i(w_i(k))$ and from (24) it then follows that $z_i(k+1) \le z_{\max}(k)$. Maximizing this inequality over the index i shows that $z_{\max}(k+1) \le z_{\max}(k)$ which means that the sequence $\{z_{\max}(k)\}$ is decreasing. A similar argument shows that $\{z_{\min}(k)\}$ is increasing.

Remark 5.1: Note that the decreasing sequence $\{z_{\max}(k)\}$ is bounded below by zero, and therefore it converges as $k \to \infty$; similarly $\{z_{\min}(k)\}$ is increasing and hence also converges.

Remark 5.2: The fact that $\{z_{\max}(k)\}$ is decreasing means in particular that $z_i(k)$ is uniformly bounded from above for all i and all k. Because f_i are increasing and continuous this also means that $w_i(k)$ is uniformly bounded for all i, k. Similarly the fact that $\{z_{\min}(k)\}$ is increasing means that $w_i(k)$ is uniformly bounded away from zero for all i and k.

Part (ii): The matrix $R(z(k)) = D(z(k))\tilde{R}(z(k))$ is the product of a row stochastic matrix $\tilde{R}(k)$ followed by a diagonal matrix, where

$$\tilde{R}(z(k))_{ij} = \delta_{ij}\beta_i + (1-\beta_i)\, p_j(w(k)) \tag{25}$$

Remark 5.2 above means that there is $\delta > 0$ such that $p_i(w(k)) > \delta$ for $i = 1, \dots, n$ and all $k \ge 0$, and hence the entries of $\tilde{R}(z(k))$ are positive and uniformly bounded away from zero. If we define

$$\tilde{z}(k) = \tilde{R}(z(k))z(k) \tag{26}$$

then by [14, Theorem 1.1] there is $r < 1$ (uniform in k) such that

$$\tilde{z}_{\max}(k) - \tilde{z}_{\min}(k) \le r(z_{\max}(k) - z_{\min}(k)) \tag{27}$$

We will show below that

$$\limsup_{k\to\infty} \Big(z_{\max}(k+1) - \tilde{z}_{\max}(k)\Big) \le 0, \quad \liminf_{k\to\infty} \Big(z_{\min}(k+1) - \tilde{z}_{\min}(k)\Big) \ge 0 \tag{28}$$

Combining (27) and (28) immediately yields the stated result (11).

We now show that the first inequality in (28) holds; the second inequality follows by an identical argument involving $z_{\min}$ in place of $z_{\max}$. For each $k \ge 0$ let i_k be the index such that $z_{\max}(k) = z_{i_k}(k)$. We define two sequences s_k, r_k corresponding to the diagonal factor $D(z(k))$ of the matrix $R(z(k))$:

$$s_k = a_{i_{k+1}}(w_{i_{k+1}}(k)), \quad r_k = a_{i_{k+1}}(w_{i_{k+1}}(k+1)) \tag{29}$$

It follows that

$$z_{\max}(k+1) = \frac{s_k}{r_k}\tilde{z}_{i_{k+1}}(k) \le \frac{s_k}{r_k}\tilde{z}_{\max}(k) \tag{30}$$

and so the first part of (28) will follow from

$$\lim_{k\to\infty} \frac{s_k}{r_k} = 1 \tag{31}$$

We now show that (31) holds. Note that from (7) it follows that for all j we have $z_j(k+1) \le \frac{a_j(w_j(k))}{a_j(w_j(k+1))}\, z_{\max}(k)$, so choosing $j = i_{k+1}$ gives the lower bound

$$\frac{s_k}{r_k} \ge \frac{z_{\max}(k+1)}{z_{\max}(k)} \ge 1\,. \tag{32}$$

There is a similar upper bound: for any index j,

$$\frac{a_j(w_j(k))}{a_j(w_j(k+1))} = \frac{h_j(z_j(k))}{h_j(z_j(k+1))} \tag{33}$$

where $h_j(z_j) = a_j(f_j^{-1}(\frac{z_j}{1-\beta_j}))$. Choosing $j = i_{k+1}$, and using the fact that a_j, f_j and hence h_j are non-decreasing, we deduce that

$$\frac{s_k}{r_k} = \frac{h_{i_{k+1}}(z_{i_{k+1}}(k))}{h_{i_{k+1}}(z_{\max}(k+1))} \leq \frac{h_{i_{k+1}}(z_{\max}(k))}{h_{i_{k+1}}(z_{\max}(k+1))} \tag{34}$$

From Remark 5.2 above we know that there is a closed interval in $(0, \infty)$ containing $z_{\max}(k)$ for all k. For any fixed index j, uniform continuity of h_j on this interval and the convergence noted in Remark 5.1 imply that $\frac{h_j(z_{\max}(k))}{h_j(z_{\max}(k+1))} \to 1$. Since this holds for every index $j = 1, \ldots, n$, convergence also holds for any sequence of indices $\{j_k\}$, that is $\frac{h_{j_k}(z_{\max}(k))}{h_{j_k}(z_{\max}(k+1))} \to 1$. Choosing the sequence $j_k = i_{k+1}$ then shows that

$$\lim_{k\to\infty} \frac{h_{i_{k+1}}(z_{\max}(k))}{h_{i_{k+1}}(z_{\max}(k+1))} = 1 \tag{35}$$

Hence the right side of (34) converges to 1 as $k \to \infty$, as does the right side of (32). Together these imply the convergence (31) as required. ■