# ANALYSIS OF THE LOCAL ROBUSTNESS OF STABILITY FOR MAPS

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**Abstract**

In this paper the problem of measuring the robustness of stability for a perturbed discrete time nonlinear system is studied. Various stability radii are introduced and their values for the nonlinear system and its linearization are compared and it is shown that they generically coincide. Some examples are presented showing that it is sometimes necessary to consider the nonlinear system directly, and not simply to rely on the information provided by the linearization.

## 1 Introduction

Robustness analysis of linear systems has played a prominent role in the theory of linear systems. In particular the state-state approach via stability radii has received considerable attention, see the survey [HP90]. In recent years there has been a great deal of work done on extending these results to more general perturbation classes, see for example the survey paper [PD93], and for recent results on stability radii with respect to real perturbations, see [QBR$^+$95]. To date, the problem of extending these results to nonlinear systems has received little attention, although see [CK95].

In this paper stability radii for discrete-time nonlinear systems are studied. The system is assumed to have an affine perturbation structure. The stability radius is then the norm of the smallest perturbation such that the system is no longer stable. In this fashion, radii may then be defined with respect to exponential, asymptotic and Lyapunov stability. Results on the relationships between the sizes of these stability radii are presented. It is seen that the nonlinear stability radii are contained in the interval defined by the radii of the linearized system, and that generically all radii are equal. This equality is seen to rely on the non-convexity of the set of Schur matrices. Examples are presented to illustrate this point.

The paper is organised as follows: In Section 2 preliminary definitions are presented, defining stability radii for the nonlinear system and its linearization. The relationships between these stability radii are studied in Section 3. In Section 4 examples are presented demonstrating systems where the stability radii differ. An important link between the nonlinear system and its linearization is provided by the center manifold theorem for discrete time systems. For a review of these results see [Car81] and the references contained therein.

## 2 Preliminaries

Consider the perturbed nonlinear discrete time system:

$$x^+ = f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \qquad (1)$$

where, $f_0(x^*) = x^*$, $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $f_i(x^*) = 0, i = 1, 2 \dots m$. A solution for a given initial condition $x_0$ is denoted by $\varphi(t; x_0, u)$, where $\varphi(0; x_0, u) = x_0$. When the system is unperturbed, we denote the solution $\varphi(t;x_0) = \varphi(t;x_0,0)$.

The following standard definitions of local (Lyapunov) stability, asymptotic and exponential stability are now presented for completeness.

**Definition 1** [Stability] *The system (1) is said to be stable (Lyapunov stable) about $x^*$ if $\forall \epsilon > 0$ there exists a neighborhood $U \subset \mathbb{R}^n$ of $x^*$ such that $\forall x_0 \in U$, $||\varphi(t;x_0) - x^*|| < \epsilon$.*

**Definition 2** [Asymptotic Stability] *The system (1) is said to be asymptotically stable at $x^*$ if it is stable about $x^*$ and also $\exists U : \forall x_0 \in U$, $\varphi(t;x_0) \to x^*$.*

**Definition 3** [Exponential Asymptotic Stability] *The system (1) is said to be exponentially asymptotically stable at $x^*$ if it is asymptotically stable at $x^*$ and additionally $\exists U : \forall x_0 \in U$, there exist $M, \alpha > 0$ such that $||\varphi(t;x_0) - x^*|| < Me^{-\alpha t}$, $\forall t > 0$.*

For nonlinear systems, we define stability radii, or measures of robustness, for exponential, asymptotic and Lyapunov stability directly from the definitions.

$$r_{ex}(f_0;(f_i)) := \inf\{\rho > 0 | \exists u \in \mathbb{R}^m, ||u|| \le \rho, \text{ s.t. (1) is not exp. stable}\} \quad (2)$$

$$r_{as}(f_0;(f_i)) := \inf\{\rho > 0 | \exists u \in \mathbb{R}^m, ||u|| \le \rho, \text{ s.t. (1) is not as. stable}\} \quad (3)$$

$$r_{st}(f_0;(f_i)) := \inf\{\rho > 0 | \exists u \in \mathbb{R}^m, ||u|| \le \rho, \text{ s.t. (1) is not stable}\} \quad (4)$$

Although the norm used in the definition could in principle be any norm on $\mathbb{R}^m$, we shall use the Euclidean norm in this paper.

We assume that the functions $f_i$ are differentiable at $x^*$. The linearization of system (1) at $x^*$ is of particular interest, and is defined by

$$x^+ = \left(A_0 + \sum_{i=1}^{m} u_i A_i\right) x =: A(u)x \quad (5)$$

where $A_i := \left.\frac{\partial f_i}{\partial x}\right|_{x^*}, i = 0, 1, \ldots m$. Stability radii for the linearized system (5) are now defined as follows:

$$r_{\geq}(A_0;(A_i)) := \inf\{\rho > 0 \mid \exists u \in \mathbb{R}^n, ||u|| \le \rho, \text{ s.t. } r(A(u)) \geq 1\} \quad (6)$$

$$r_{>}(A_0;(A_i)) := \inf\{\rho > 0 \mid \exists u \in \mathbb{R}^n, ||u|| \le \rho, \text{ s.t. } r(A(u)) > 1\} \quad (7)$$

where $r(M)$ denotes the spectral radius of $M \in \mathbb{R}^{n\times n}$. $r_{\geq}(A_0;(A_i))$ measures the size of the smallest perturbation such that (5) is no longer exponentially stable, and corresponds to the stability radius usually studied for linear systems, and is motivated by the following result.

**Lemma 4** *Let $u \in \mathbb{R}^m$. Then the following statements are equivalent:*

*(i) (5) is asymptotically stable.*

*(ii) (5) is exponentially stable.*

*(iii) $r(A(u)) < 1$.*

*Furthermore (5) is stable iff $r(A(u)) \le 1$, and the geometric and algebraic multiplicity of the eigenvalues of $A(u)$ with norm 1 are equal.* □

**Proof.** See [Aga92] Theorem 5.5.1, Remark 5.5.3, and Corollary 5.4.2. ■

The radius $r_{>}(A_0;(A_i))$ is the infimum of the norms of the perturbations such that the system is exponentially unstable. To the best of our knowledge this is a new measurement of instability.

## 3 Results

In this section we examine the relationships between the stability radii defined in the first section. For the moment we will not consider the problem of exactly how to calculate these quantities.

The following inequalities are immediate from the definitions.

**Lemma 5**

$$r_{\geq}(A_0;(A_i)) \le r_{>}(A_0;(A_i)) \quad (8)$$

$$r_{ex}(f_0;(f_i)) \le r_{as}(f_0;(f_i)) \le r_{st}(f_0;(f_i)) \quad (9)$$

□

We now examine the relationship between the stability radii for a nonlinear system, and those of its linearization. The main ideas and intuition for the first result come from the center manifold theorem for discrete time systems.

**Lemma 6**

$$r_{\geq}(A_0;(A_i)) = r_{ex}(f_0;(f_i)) \quad (10)$$

□

**Proof.** Suppose $\gamma < r_{\geq}(A_0;(A_i))$. Then by definition, $\forall u \in \mathbb{R}^m$ such that $||u|| \leq \gamma$, $r(A(u)) < 1$. Thus by Corollary 5.6.3 of [Aga92] $\forall u \in \mathbb{R}^m$ such that $||u|| \leq \gamma$, (1) is exponentially stable, thus $\gamma < r_{ex}(f_0;(f_i))$ and $r_{\geq}(A_0;(A_i)) \leq r_{ex}(f_0;(f_i))$.

Suppose now that $\gamma < r_{ex}(f_0;(f_i))$. Then $\forall u \in \mathbb{R}^m$ such that $||u|| \leq \gamma$ there exist $M_u, \beta_u > 0$ such that

$$||\varphi(t;x_0,u)|| \leq M_u e^{-\beta_u t}||x(0)|| \tag{11}$$

for $||x(0)||$ sufficiently small. Fix $u$ and suppose that $r(A(u)) \geq 1$. If $r(A(u)) > 1$, then the system is locally unstable, contradicting $\gamma < r_{ex}(f_0;(f_i))$. Now consider $r(A(u)) = 1$. Due to the center manifold theorem, we need only consider the behavior of that part of the system which corresponds to eigenvalues $\lambda$ of $A(u)$ for which $|\lambda| = 1$. Transform the system so that it may be written in the form (suppressing the dependence on $u$)

$$\begin{aligned} v^+ &= Sv + g(v,w) \\ w^+ &= Tw + h(v,w) \end{aligned} \tag{12}$$

where $T$ is Schur, and $S$ has eigenvalues on the unit circle. Then, by center manifold theory, there exists a function $c(v)$ such that the dynamics of the system (12) are governed by

$$z^+ = Sz + g(z,c(z)) \tag{13}$$

in the sense that $v(t) \to z(t)$ and $w(t) \to 0$ as $t \to \infty$. Now define $G(z) := g(z,c(z))$, which is the nonlinear part of the dominant dynamics of the system.

Now $\forall \epsilon > 0$, $\exists \delta > 0$ such that $||z|| < \delta \Rightarrow ||G(z)|| \leq \epsilon||z||$. Thus for $||z|| < \delta$

$$\begin{aligned} ||Sz + g(z,c(z))|| &= ||Sz + G(z)|| \\ \geq\ ||z|| - ||G(z)|| &\geq (1-\epsilon)||z|| \end{aligned}$$

Considering $||z(0)|| < \delta$ it follows that $||z(t)|| \geq (1-\epsilon)^t||z(0)||$. For $t$ large enough, this contradicts (11). We conclude that $r(A_0 + \sum u_i A_i) < 1$, and $\gamma < r_{\geq}(A_0;(A_i))$. It follows that $r_{ex}(f_0;(f_i)) \leq r_{\geq}(A_0;(A_i))$, and the proof is complete. ■

**Lemma 7**

$$r_{st}(f_0;(f_i)) \leq r_{>}(A_0;(A_i)) \tag{14}$$

□

**Proof.** If $\gamma > r_{>}(A_0;(A_i))$, then there exists a $u \in \mathbb{R}^m$ with $||u|| \leq \gamma$ such that $r(A_0 + \sum u_i A_i) > 1$. Thus the system is unstable, so $\gamma \geq r_{st}(f_0;(f_i))$ ■

When considering the possibility that

$$r_{\geq}(A_0;(A_i)) \neq r_{>}(A_0;(A_i))$$

it is important to note that the important characteristic of the equations is the movement of the roots of $\chi(A_0 + \sum u_i A_i)$ as the maximum allowable norm of $u$ increases. Clearly, if $r_{\geq}(A_0;(A_i)) < r_{>}(A_0;(A_i))$, then there must be a $\rho_1$ such that for some $u$ with $||u|| = \rho_1$ $r(A_0 + \sum u_i A_i) = 1$, but that for $||u|| = \rho_1 + \epsilon$, with $\epsilon > 0$ sufficiently small $r(A_0 + \sum u_i A_i) < 1$. This is only possible when the set of Schur matrices is non-convex. In order to prove a genericity result about this property the following concepts are useful.

Recall that a subset $X$ of $\mathbb{R}^n$ is called semi-algebraic if it is the finite union of sets of the form

$$\{x \in \mathbb{R}^n \ ; \ f_1(x) = 0 = \ldots = f_l(x), \\ g_1(x) > 0, \ldots, g_k(x) > 0\},$$

where the $f_i, g_j$ are all polynomials in $\mathbb{R}[X_1,\ldots,X_n]$. For semi-algebraic sets $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^l$ a function $f: X \to Y$ is called semi-algebraic if its graph $\{(x,y) \in \mathbb{R}^{n+l} \ ; \ y = f(x), x \in X\}$ is a semi-algebraic subset of $\mathbb{R}^{n+l}$. For an introduction to the theory of semi-algebraic sets we refer to [BCR87]. In particular we will use that the complement of a semi-algebraic set is semi-algebraic and that sets determined by a formula of first order in the language of real ordered fields is semi-algebraic. (See [BCR87] Définition 2.2.3 and Proposition 2.2.4).

**Lemma 8** *Let $n, m \in \mathbb{N}$ be fixed.*

*(i) The sets*

$$\begin{aligned} \mathcal{T}_{\geq} &:= \{(A_0,\ldots,A_m) \ ; \ r_{\geq}(A_0,\ldots,A_m) = \infty\} \\ \mathcal{T}_{>} &:= \{(A_0,\ldots,A_m) \ ; \ r_{>}(A_0,\ldots,A_m) = \infty\} \end{aligned}$$

*are semi-algebraic.*

*(ii) The functions*

$$\begin{aligned} r_{\geq} &: \mathbb{R}^{n\times n\times(m+1)} \setminus \mathcal{T}_{\geq} \to \mathbb{R} \\ r_{>} &: \mathbb{R}^{n\times n\times(m+1)} \setminus \mathcal{T}_{>} \to \mathbb{R} \end{aligned}$$

*are semi-algebraic.*

*(iii) $r_{\geq}$ is upper semicontinuous.*

*(iv) $r_{>}$ is lower semicontinuous.*

□

**Proof.**

*(i)* Note first that by the Schur-Cohn conditions the set of Schur-stable matrices $\mathcal{S}_n$ is semi-algebraic. The sets $\mathcal{T}_{\geq}, \mathcal{T}_{>}$ are the complements of

$$\{(A_0, \dots, A_m)\ ;\ \exists u \in \mathbb{R}^m : A(u) \in \complement\mathcal{S}_n\},$$

respectively

$$\{(A_0, \dots, A_m)\ ;\ \exists u \in \mathbb{R}^m : A(u) \in \complement\,\mathrm{cl}\,\mathcal{S}_n\}$$

and thus semi-algebraic.

*(ii)* Define the function

$$d_{A_1,\dots,A_m}(A_0, B) \quad := \quad \inf\{\|u\|\ ;\ u \in \mathbb{R}^m,\ B = A(u)\},$$

where $\inf \emptyset = \infty$. Note that $d_{\dots}(\cdot,\cdot)$ is a semi-algebraic function on $\mathbb{R}^{n\times n\times(m+2)}$ which is finite on the set given by the condition $B \in A_0 + \mathrm{span}\{A_1, \dots, A_m\}$. The graph of $r_{\geq}$ is given by

$$\Big\{(A_0, \dots, A_m, t)\ \Big|\ t \geq 0,\ \forall B \in (A_0 + \mathrm{span}\{A_1, \dots, A_m\})\backslash\mathcal{S}_n : t \leq d_{A_1,\dots,A_m}(A_0, B) \quad \text{and} \quad \forall \epsilon > 0,\ \exists B \in (A_0 + \mathrm{span}\{A_1, \dots, A_m\})\backslash\mathcal{S}_n : t + \epsilon > d_{A_1,\dots,A_m}(A_0, B)\Big\},$$

which defines a semi-algebraic set. The graph of $r_{>}$ has the same definition with the exception that $\mathcal{S}_n$ has to be replaced by $\mathrm{cl}\,\mathcal{S}_n$.

*(iii)* If $r_{\geq}(A_0; (A_i)) > c$ this implies that for all $u \in \mathbb{R}^m, \|u\| \leq c$ we have $r(A(u)) \leq \gamma < 1$ for a suitable constant $\gamma$. By continuity of the spectral radius there exists a neighborhood $V$ of $(A_0, \dots, A_m)$ such that $r(B(u)) \leq \gamma' < 1$ for all $(B_0, \dots, B_m) \in V$ and $u \in \mathbb{R}^m, \|u\| \leq c$. This implies $r_{\geq}(B_0; (B_i)) > c$ for all elements of $V$. (iv) can be proved using a similar argument. ■

**Proposition 9** *Given $n, m \in \mathbb{N}$, the set of matrices $A_0, A_1, \dots, A_m \in \mathbb{R}^{n\times n}$ for which*

$$r_{\geq}(A_0; (A_i)) < r_{>}(A_0; (A_i)) \tag{15}$$

*is a nowhere dense, semi-algebraic subset of $\mathbb{R}^{n\times n\times(m+1)}$.* □

**Proof.** By the previous lemma it follows that the set

$$\mathcal{B} := \{(A_0, \dots, A_m)\ ;\ r_{\geq}(A_0; (A_i)) < r_{>}(A_0; (A_i))\}$$

is semi-algebraic. Thus it suffices to show that $\mathcal{B}$ is nowhere dense in $\mathbb{R}^{n\times n\times(m+1)}$ and for this we need to show that in every neighborhood of any point in $\mathcal{B}$ there exists a point that does not belong to $\mathcal{B}$. We first prove the following intermediate claim: For any open set $V \subset \mathbb{R}^{n\times n\times(m+1)}$ it holds that

$$\inf_{(A_0,\dots,A_m)\in V} r_{>}(A_0; (A_i)) - r_{\geq}(A_0; (A_i)) = 0\,.$$

To see this define $d := \inf_{(A_0,\dots,A_m)\in V} r_{\geq}(A_0; (A_i)) \geq 0$. If the infimum is $\infty$ there is nothing to show. Fix $\epsilon > 0$ and choose $(B_0; (B_i)) \in V$ such that $r_{\geq}(B_0; (B_i)) - \epsilon > d$. Thus there exists $u, \|u\| \leq d + \epsilon$ such that $r(B(u)) \geq 1$. In any neighborhood of $(B_0; (B_i))$ there exists a point $(C_0; (C_i))$ such that $r(C(u)) > 1$ and hence $r_{>}(C_0; (C_i)) \leq d+\epsilon$ and so $r_{>}(C_0; (C_i)) - r_{\geq}(C_0; (C_i)) \leq d+\epsilon-d$. As $r_{>} - r_{\geq}$ is lower semicontinuous a standard argument shows that the infimum is actually attained on every open set $V$. This completes the proof. ■

To paraphrase the previous result is to say that *generically* it does not happen that $r_{\geq}(A_0; (A_i))$ and $r_{>}(A_0; (A_i))$ differ. This result extends to the case of nonlinear systems. We denote by $C^1(\mathbb{R}^n, \mathbb{R}^n, x^*)$ the set of continuous maps from $\mathbb{R}^n$ to itself, continuously differentiable at $x^* \in \mathbb{R}^n$ and satisfying $f(x^*) = x^*$. Furthermore $C^1(\mathbb{R}^n, \mathbb{R}^n, x^*, 0)$ denotes the set of continuous functions from $K^n$ to itself, continuously differentiable at $x^* \in \mathbb{R}^n$ and satisfying $f(x^*) = 0$. We endow these spaces with the topologies generated by the topology of uniform convergence on compact sets and the topology of convergence of the Jacobians at $x^*$.

**Corollary 10** *Given $n, m \in \mathbb{N}$, the set of functions $\{f_0, f_1, \dots, f_m\} \in C^1(\mathbb{R}^n, \mathbb{R}^n, x^*) \times C^1(\mathbb{R}^n, \mathbb{R}^n, x^*, 0)^m$ for which*

$$r_{ex}(f_0; (f_i)) < r_{st}(f_0; (f_i)) \tag{16}$$

*is a thin subset of $C^1(\mathbb{R}^n, \mathbb{R}^n, x^*) \times C^1(\mathbb{R}^n, \mathbb{R}^n, x^*, 0)^m$ endowed with the product topology.* □

**Proof.** Note that the situation of (16) is only possible if for the linearized system (15) holds. It is thus sufficient to show that the preimage of a nowhere dense semi-algebraic set under the continuous, linear map

$$\{f_0, f_1, \dots, f_m\} \mapsto \{\frac{\partial f_0}{\partial x}(x^*), \dots, \frac{\partial f_m}{\partial x}(x^*)\}$$

is a thin subset. This, however, is clear by definition of the topology. ■

**Remark 11** Due to the fact that for 1 dimensional systems the set of Schur matrices is convex, it is immediate that for $n = 1$

$$r_{ex}(f_0;(f_i)) = r_{as}(f_0;(f_i)) = r_{st}(f_0;(f_i)) . \qquad (17)$$

# 4 Examples

In this section we present examples demonstrating when the inequalities proven in the previous section are strict inequalities.

**Example 12**

To demonstrate what may happen in a situation where the inequalities (15) and (16) hold, consider the following system

$$x^+ = A_0 x + u A_1 x \qquad (18)$$

where

$$A_0 = \begin{bmatrix} 7/22 & -7/8 & -37/264 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -13/22 & -1/8 & -35/264 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we use the standard norm, $||u|| = |u|$.

Due to the special form of the matrices, the characteristic equation of the system (18) may be seen to be

$$\chi(A_0+uA_1) = s^3 - \frac{7\,s^2}{22} + \frac{7\,s}{8} + \frac{37}{264} + \left(\frac{13\,s^2}{22} + \frac{s}{8} + \frac{35}{264}\right) u \qquad (19)$$

Note that $A_0 + uA_1$ is Schur for $u \in (-2, 271/79)\backslash\{1\}$[1]. The matrix $A_0 + A_1$ has eigenvalues $i, -i, -3/11$, and the martix $A_0 - 2A_1$ has eigenvalues $1, (1+i)/4, (1-i)/4$. See Fig. 1 for a root locus diagram for the system.

It is thus clear that the system (18) has the following stability radii: $r_{\geq}(A_0,(A_1)) = 1$, and $r_{>}(A_0,(A_1)) = 2$. Furthermore, by virtue of Lemma 4, system (18) is an example where

$$r_{\geq}(A_0,(A_1)) = r_{ex}(A_0,(A_1)) = r_{as}(A_0,(A_1))$$

$$< r_{st}(A_0,(A_1)) = r_{>}(A_0,(A_1)) .$$

[1] $271/79 \approx 3.43$

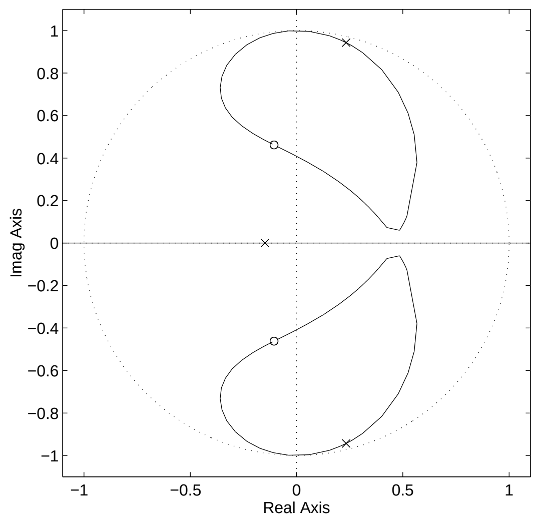


Figure 1: Root loci for $\chi(A_0 + uA_1)$.

If we consider the system in $\mathbb{R}^6$ given by

$$\begin{aligned} x^+ &= \left(\begin{bmatrix} A_0 & I \\ 0 & A_0 \end{bmatrix} + u \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix}\right) x \\ &=: B_0 x + u B_1 x \end{aligned}$$

then of course the values of $r_{\geq}, r_{>}$ are the same as for system (18), but again by virtue of Lemma 4 we have an example where

$$r_{\geq}(B_0,(B_1)) = r_{ex}(B_0,(B_1)) = r_{as}(B_0,(B_1))$$

$$= r_{st}(B_0,(B_1)) < r_{>}(B_0,(B_1)) .$$

□

A difference between the exponential and the asymptotic stability radius is a truely nonlinear phenomenon. To exhibit this we study a nonlinear system which has (18) as its linearization.

**Example 13** Consider the system

$$x^+ = A_0 x + u_1 A_1 x - x^T x (A_0 + A_1) x + u_2 x^T x (A_0 + A_1) x \qquad (20)$$

with the Euclidean norm $||u|| = \left(u_1^2 + u_2^2\right)^{\frac{1}{2}}$. Note that the linearization of (20) is (18), thus exponential stability or instability of the system will be determined by the parameter $u_1$. Thus $r_{ex}(f_0;(f_i)) = 1 = r_{\geq}(f_0;(f_i))$.

Consider now that $u_1 = 1$, so that the linearization is stable, but not asymptotically stable. We examine the stability of the system using the center manifold theorem and a Lyapunov argument. First note that $(A_0 + A_1)$ is similar to the matrix

$$B := \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -3/11 \end{bmatrix}.$$

Let $S$ be the transformation matrix. In this basis, and using $u_1 = 1$ the system(20) has the special form:

$$x^+ \;=\; (1 - (1+u_2)x^T S^T S x)Bx\,.$$

Due to the center manifold theorem, we need only consider what happens on the center manifold to determine stability of the system. In this case the subspace $(x_1, x_2, 0)$ is a center manifold for the system. On this subspace (21) takes the form:

$$\begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} \;=\; \left(1 - (1+u_2)(2x_1^2 + x_2^2)\right)\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}.$$

Consider the function $V(x) = x_1^2 + x_2^2$. Then

$$\begin{aligned} \Delta V(x) &:= V(x^+) - V(x) \\ &= (1+u_2)(-2 + (2x_1^2 + x_2^2)(1+u_2)) * \\ &\qquad (x_1^2 + x_2^2)(2x_1^2 + x_2^2)\,. \end{aligned}$$

Now, $\forall (x_1, x_2) \neq 0$ the factor $(x_1^2+x_2^2)(2x_1^2+x_2^2)$ is strictly positive, and $\forall u$ there exists a neighborhood of the origin such that $-2 + (2x_1^2 + x_2^2)(1+u_2) < 0$. Within this neighborhood the sign of $V(x)$ is thus purely determined by the sign of $(1+u_2)$. Thus, for $u_2 > -1$ the origin is asymptotically stable, for $u_2 = -1$, the origin is stable, and for $u_2 < -1$ the origin is unstable.

It is thus clear that

$$1 = r_{ex}(f_0; (f_i)) < r_{as}(f_0; (f_i)) = r_{st}(f_0; (f_i)) = \sqrt{2}\,.$$

□

Note that using combinations of Examples 12 and 13 it is easy to construct a higher dimensional example where indeed

$$r_{\geq} = r_{ex} < r_{as} < r_{st} < r_{>}\,.$$

## 5 Conclusions

In this paper we have shown that for a nonlinear system (1) and its linearization (5),the stability radii are related in the following way:

$$\begin{aligned} r_{\geq}(A_0; (A_i)) = r_{ex}(f_0; (f_i)) &\leq r_{as}(f_0; (f_i)) \\ &\leq r_{st}(f_0; (f_i)) \leq r_{>}(A_0; (A_i))\,, \end{aligned}$$

where $A_i = \left.\frac{\partial f_i}{\partial x}\right|_{x=0}$. Examples have been presented to show that the inequalities may not be replaced by equalities, however for systems of dimension 1 or 2, and generically for systems of higher dimension

$$\begin{aligned} r_{\geq}(A_0; (A_i)) = r_{ex}(f_0; (f_i)) &= r_{as}(f_0; (f_i)) \\ &= r_{st}(f_0; (f_i)) = r_{>}(A_0; (A_i))\,. \end{aligned}$$

In this paper we only consider real, time invariant perturbations of the system. In further work we intend extending these results to include complex or time-varying perturbations.

It is expected that all the results presented in this paper may be proven for continuous time systems. This will be the subject of another study.

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