STABILITY INDICES OF INFINITE-DIMENSIONAL DISCRETE INCLUSIONS

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Abstract

For discrete inclusions in Banach spaces we study stability questions. The main result states that for discrete inclusions on a reflexive Banach space various characteristic exponents characterizing different concepts of stability coincide. Using this result it is shown that the convexification of an exponentially stable difference inclusions is exponentially stable. It is examined to what extent these results can be carried over to the time-varying case.

1 Introduction

In this work we are concerned with stability properties of time-varying discrete-time systems and of discrete inclusions. Several results that have been obtained for finite dimensional systems are extended to infinite dimensions. In particular we obtain results similar to those in [3], [1], [5].

The study of infinite-dimensional time-varying discrete-time systems has been carried out to a large extent in the articles [8], [12], [9], [10], [11] . In these papers it has also been noted that the study of discrete time systems on Banach space is an appropriate setting for the study of delay differential equations. Robustness of stability was studied for discrete time systems in infinite dimensions in [13]. As a discrete inclusion may be interpreted as a time invariant system with a specified region of uncertainty this paper extends and complements the results of the previous work on robust stability.

The motivation to study discrete inclusions in infinite dimensions comes from the consideration of robustness problems with respect to time-varying perturbations of infinite dimensional discrete time systems. Discrete inclusions are a very general way of describing time-varying uncertainty of a time-invariant system. Applications of the theory to delay differential equations are discussed in the forthcoming article [14].

Section 2 is devoted to the definition of various concepts of stability for time-invariant and time-varying systems. For exponential stability of a time-varying system it is not enough to require, that all trajectories decay exponentially. The fact as such has been discussed in the book on stability by Daleckii and Krein [2], and the ideas are easily transferred to the discrete-time case.

In Section 3 Lyapunov and Bohl exponents of time-varying systems are introduced. We will show that the Bohl exponent of an operator sequence $A(\cdot) \in \ell^\infty(\mathbb{N}, \mathcal{L}(X))$ can be represented as the logarithm of the spectral radius of an associated operator $\hat{A}$ on $\ell^2(\mathbb{N}, X)$. After this discussion of characteristic exponents we turn to the study of discrete inclusions in the following Section 4. For discrete inclusions on reflexive Banach spaces it is shown that the supremal Lyapunov exponent, the supremal Bohl exponent and a further "uniform" exponential growth rate coincide. In the final Section 5 time-varying discrete inclusions are considered and stability concepts are studied. These class of systems forms a generalization for time-varying systems as well as discrete inclusions and this section therefore encompasses all the results obtained up to that point.

2 Time-varying systems

Let X be a Banach space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. $\mathcal{L}(X)$ denotes the Banach algebra of bounded linear operators from X to X. The norm on X and the induced operator norm on $\mathcal{L}(X)$ are both denoted by $\|\cdot\|$.

A Banach space X is called reflexive if the range of the natural embedding of X into X^{**} is X^{**}. X is reflexive iff the unit ball in X is compact in the weak topology which is in turn equivalent to the

weak compactness of the unit ball in $\mathcal{L}(X)$ (see [4] Theorem V.4.7 and Exercise VI.9.6). Recall that a net $\{A_n\} \subset \mathcal{L}(X)$ converges weakly to A iff for all $x \in X$ and $f \in X^*$ it holds that $< A_n x, f >$ converges to $< Ax, f >$. Weak convergence is denoted by $\mathrm{w} - \lim_n A_n = A$. A set $\mathcal{M} \subset \mathcal{L}(X)$ is called weakly compact if it is compact with respect to the weak topology, i.e. open covers contain a finite subcover. The weak closure of a set V is denoted by $\mathrm{w} - \mathrm{cl}\, V$.
We consider time-varying linear discrete-time systems of the form

$$x(t+1) = A(t)x(t), \quad t \in \mathbb{N}, \tag{1}$$

where $A(\cdot) = (A(t))_{t\in\mathbb{N}} \in \mathcal{L}(X)^{\mathbb{N}}$ is a sequence of bounded linear operators on X. The evolution operator associated to this system is defined by

$$\Phi(t,t) = I_X,\ \Phi(t,s) = A(t-1)\cdots A(s),\ t > s.$$

Definition 2.1 [Stability] *System (1) is called*

(i) stable, *if for every $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that*

$$\|x_0\| < \delta \ \Rightarrow\ \|\Phi(t,t_0)x_0\| < \varepsilon \quad \text{for all } t \geq t_0\,,$$

(ii) asymptotically stable, *if it is stable and for every $x_0 \in X$, $t_0 \in \mathbb{N}$ $\lim_{t\to\infty} \Phi(t,t_0)x_0 = 0$.*

(iii) exponentially stable, *if there are $c, \beta > 0$ such that $\|\Phi(t,s)\|_{\mathcal{L}(X)} \leq c\, e^{-\beta(t-s)}$, $s,t \in \mathbb{N}$, $t \geq s$ holds.*

An immediate consequence of the definition is that if (1) is exponentially stable, then $A(\cdot) \in \ell^\infty(\mathbb{N}, \mathcal{L}(X))$. For time-varying systems asymptotic and exponential stability are not equivalent and exponential stability is not characterized by the spectrum of the transition matrices $A(t)$ even in the finite dimensional case, see e.g. [7] Chapter 4.4. An example in [2] shows that for exponential stability it is not sufficient that all trajectories of a system of the form (1) decay exponentially.

3 Characteristic Exponents

To characterize exponential stability the concepts of Lyapunov and Bohl exponents have been introduced. The largest exponential growth rate of system (1) is given by the discrete time version of the (upper) *Bohl* exponent [2] (named *generalized spectral radius* in [10]). In the following definition we do not assume (1) to be exponentially stable and let $(A(t))_{t\in\mathbb{N}}$ be an arbitrary sequence in $\mathcal{L}(X)$.
Definition 3.2 [Bohl exponent] *Given a sequence $(A(t))_{t\in\mathbb{N}}$ in $\mathcal{L}(X)$ the* (upper) Bohl exponent *of the system (1) is*

$$\beta(A(\cdot)) = \inf\{\beta \in \mathbb{R}; \exists c_\beta \geq 1 : t \geq s \geq 0 \Rightarrow \|\Phi(t,s)\| \leq c_\beta e^{\beta(t-s)}\}\,,$$

where $\inf \emptyset = \infty$.
$\beta(A(\cdot))$ may be infinite, but it is easy to see that $\beta(A(\cdot)) < \infty$ if and only if $(A(t))_{t\in\mathbb{N}}$ is bounded.
In contrast Lyapunov exponents focus on the exponential growth rates of trajectories.
Definition 3.3 [Lyapunov exponent] *Given a sequence $(A(t))_{t\in\mathbb{N}}$ in $\mathcal{L}(X)$ and an initial condition $(t_0, x_0) \in \mathbb{N} \times (X \setminus \{0\})$ the* Lyapunov exponent *corresponding to (t_0, x_0) is defined by*

$$\lambda(t_0,x_0) = \inf\{\lambda \in \mathbb{R}; \exists c_\lambda \geq 1 : t \geq t_0 \Rightarrow \|\Phi(t,t_0)x_0\| \leq c_\lambda e^{\lambda(t-t_0)}\|x_0\|\}. \tag{2}$$

Furthermore we define the supremal Lyapunov exponent by

$$\kappa(A(\cdot)) := \sup\{\lambda(t_0,x_0); (t_0,x_0) \in \mathbb{N} \times (X \setminus \{0\})\}.$$

The following statement is immediate from the definition.
Lemma 3.4 *Let $(A(t))_{t\in\mathbb{N}}$ in $\mathcal{L}(X)$ and an initial condition $(t_0, x_0) \in \mathbb{N} \times (X \setminus \{0\})$. Then*

(i) $\beta(A(\cdot)) < 0$ *iff (1) is exponentially stable.*

(ii) $\lambda(t_0,x_0) < 0$ *iff $\Phi(t,t_0)x_0$ goes to zero exponentially fast.*

(iii) $\beta(A(\cdot)) \geq \kappa(A(\cdot))$

□

It should be noted that Lyapunov and Bohl exponents do not characterize asymptotic stability. Also, in general the inequality in Lemma 3.4 (iii) may be strict.
Example 3.5 Let $X = \mathbb{R}$ and consider the sequence $A(\cdot)$ given by

$$A(\cdot) = (\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, \ldots)\,.$$

It is easy to see that $\beta(A(\cdot)) = 0$ while $\kappa(A(\cdot)) = -\frac{1}{2}\log 2$. □

Both Bohl- and Lyapunov-exponents have asymptotic expressions that are easily shown from the definition. For the Bohl exponent see [10].
Proposition 3.6 *Let $A(\cdot) \in \mathcal{L}(X)^{\mathbb{N}}$, then*

(i) If $\beta(A(\cdot)) < \infty$ *then*

$$\beta(A(\cdot)) = \limsup_{s,t-s\to\infty} \frac{1}{t-s}\log\|\Phi(t,s)\|\,.$$

(ii) For every $(t_0, x_0) \in \mathbb{N} \times X$

$$\lambda(t_0, x_0) = \limsup_{t\to\infty} \frac{1}{t - t_0} \log \|\Phi(t, t_0)x_0\|.$$

$\square$

In [10] Bohl exponents have been named generalized spectral radius, as properties of the spectral radius for time-invariant systems coincide with properties of the Bohl exponent for time-varying systems. Let us now explain why the relation between these concepts is such a close one. The proofs of the following two propositions are omitted due to space limitations. Let $\hat{X} = \ell^2(\mathbb{N}, X)$ be the space of all sequences $(x_i)_{i\in\mathbb{N}} \subset X$ satisfying $\sum_{i\in\mathbb{N}} \|x_i\|^2 < \infty$. $\hat{X}$ is a Banach space if it is endowed with the norm $\|(x_i)_{i\in\mathbb{N}}\|_{\hat{X}} = (\sum_{i\in\mathbb{N}} \|x_i\|^2)^{\frac{1}{2}}$. On $\hat{X}$ we define the operator $\hat{A}$ by defining for $x = (x_0, x_1, \dots) \in \hat{X}$

$$\hat{A}(x) := (0, A(0)x_0, A(1)x_1, \ldots). \tag{3}$$

From the definition it is immediate that $\hat{A} \in \mathcal{L}(\hat{X})$ iff $A(\cdot) \in \ell^\infty(\mathbb{N}, \mathcal{L}(X))$. The following relation holds between the spectral radius of $\hat{A}$ and the Bohl exponent of $A(\cdot)$.

Proposition 3.7 *Let $A(\cdot) \in \ell^\infty(\mathbb{N}, \mathcal{L}(X))$ and $\hat{A}$ be defined by (3), then*

$$e^{\beta(A(\cdot))} = r(\hat{A}), \tag{4}$$

where we use the convention $e^{-\infty} = 0$. $\square$

As the Bohl exponent can be expressed as the spectral radius of the operator $\hat{A} \in \mathcal{L}(\ell^2(\mathbb{N}, X))$ we have obtained a new way to prove the following results that first appear in [10] and [12].

Proposition 3.8 *Let $(A(t))_{t\in\mathbb{N}} \in l^\infty(\mathbb{N}; \mathcal{L}(X))$. Then*

(i) The function $\beta : (l^\infty(\mathbb{N}; \mathcal{L}(X)), \|\cdot\|_\infty) \to \mathbb{R} \cup \{-\infty\}$ is upper semi-continuous.

(ii) If $A(t) \equiv A \in \mathcal{L}(X)$ is constant in $t \in \mathbb{N}$ then

$$e^{\beta(A)} = \exp(\lim_{t\to\infty} \frac{1}{t} \log \|A^t\|) = r(A)$$

is the spectral radius of A.

(iii) The following statements are equivalent:

(a) (1) is exponentially stable.

(b) $\beta(A(\cdot)) < 0$.

(c) $r(\hat{A}) < 1$.

(d) $\exists\gamma > 1 \ \forall s \in \mathbb{N} \ \forall x_0 \in X$: $\sum\limits_{t=s}^{\infty} \|\Phi(t, s)x_0\|^2 \le \gamma^2 \|x_0\|^2$.

$\square$

4 Discrete Inclusions

Let X be a Banach space and $\mathcal{M} \subset \mathcal{L}(X)$ be a bounded set. We consider the discrete inclusion

$$x(t + 1) \in \{Ax(t) \ ; \ A \in \mathcal{M}\} \quad t \in \mathbb{N}. \tag{5}$$

A sequence $\{x(t)\}_{t\in\mathbb{N}}$ is called solution of (5) with initial condition $x_0 \in X$ if $x(0) = x_0$ and for all $t \in \mathbb{N}$ there exists an $A(t) \in \mathcal{M}$ such that $x(t + 1) = A(t)x(t)$.

Two concepts as regards characteristic exponents of (5) are immediate.

$$\bar{\kappa}(\mathcal{M}) := \sup\{\kappa(A(\cdot)) \ ; \ A(\cdot) \in \mathcal{M}^{\mathbb{N}}\}, \tag{6}$$

$$\bar{\beta}(\mathcal{M}) := \sup\{\beta(A(\cdot)) \ ; \ A(\cdot) \in \mathcal{M}^{\mathbb{N}}\}. \tag{7}$$

A further quantity that will be of interest is given by the uniform exponential growth rate

$$\bar{\delta}(\mathcal{M}) := \limsup_{t\to\infty} \frac{1}{t} \log \sup_{A(\cdot)\in\mathcal{M}^{\mathbb{N}}} \|\Phi_{A(\cdot)}(t, 0)\|. \tag{8}$$

Corresponding to these definitions we introduce the following concepts of stability:

Definition 4.9 [Stability] *Let X be a Banach space and $\mathcal{M} \subset \mathcal{L}(X)$ be a bounded set. The discrete time inclusion of the form (5) given by $\mathcal{M}$*

(i) has exponentially decaying trajectories *if $\bar{\kappa}(\mathcal{M}) < 0$.*

(ii) is called exponentially stable *if $\bar{\beta}(\mathcal{M}) < 0$.*

(iii) is called uniformly exponentially stable *if $\bar{\delta}(\mathcal{M}) < 0$.*

Let us note that every bounded $\mathcal{M}$ satisfies:

$$\bar{\kappa}(\mathcal{M}) \le \bar{\beta}(\mathcal{M}) \le \bar{\delta}(\mathcal{M}), \tag{9}$$

It is our goal to show that the three quantities in (9) are in fact equal for discrete inclusions on reflexive Banach spaces. In order to do this we need the following proposition.

Proposition 4.10 *Let X be a reflexive Banach space and let $\mathcal{M} \subset \mathcal{L}(X)$ be weakly compact. Then for every sequence $\{x_k(t)\}$ of solutions of (5) with bounded initial condition, i.e. $\|x_k(0)\| \le c$ for all $k \in \mathbb{N}$, there exists a subsequence $\{x_{k_l}(t)\}$ of solutions and a solution $\{x(t)\}$ of (5) satisfying*

$$\mathrm{w} - \lim_{l\to\infty} x_{k_l}(t) = x(t), \quad \textit{for all } t \in \mathbb{N}. \tag{10}$$

$\square$

Proof. As every closed ball $B(0, r)$ is weakly compact in X, we may choose a subsequence $\{x_{k_l^0}(t)\}$ of the original sequence such that $\mathrm{w} - \lim_{l\to\infty} x_{k_l^0}(0) =$

x_0 for an appropriate $x_0 \in X$. Now choose a subsequence $\{x_{k_l^1}(t)\}$ of the sequence $\{x_{k_l^0}(t)\}$ such that $\text{w}-\lim_{l\to\infty} A_{k_l^1}(0) = A(0)$ for some appropriate $A(0) \in \mathcal{M}$. Continue this process inductively and consider the diagonal sequence $\{x_{k_l}(t)\} := \{x_{k_l^l}(t)\}$ and the solution $\{x(t)\}$ of the discrete inclusion (5) determined by the initial condition x_0 and the operator sequence $(A(0), A(1), \dots) \in \mathcal{M}^{\mathbb{N}}$.
We will prove (10) by induction. For $t = 0$ the assertion is clear by construction, so consider the case $t+1$, then for any $f \in X^*$ it holds that

$$
\begin{aligned}
&< x(t+1) - x_{k_l}(t+1), f >= \\
&< A(t)x(t) - A(t)x_{k_l}(t), f > + \\
&< A(t)x_{k_l}(t) - A_{k_l}(t)x_{k_l}(t), f > .
\end{aligned}
$$

The first term of the right hand side converges to zero by the induction hypothesis, while the second term converges to zero by the weak convergence of the $A_{k_l}(t)$ to $A(t)$. ■

Using the preceding fact we can show the following proposition on the boundedness of the solutions of an asymptotically stable discrete inclusion partly using ideas that were also used in [3] for the finite dimensional case.

Theorem 4.11

Let X be a reflexive Banach space and let $\mathcal{M} \subset \mathcal{L}(X)$ be weakly compact. If for all $A(\cdot) \in \mathcal{M}^{\mathbb{N}}$ it holds that

$$\lim_{t\to\infty} \Phi_{A(\cdot)}(t,0)x_0 = 0\,,$$

then there exists a constant $c_{\mathcal{M}} \in \mathbb{R}$ such that

$$\sup\{\|\Phi_{A(\cdot)}(t,0)\|\ ;\ t \in \mathbb{N}, A(\cdot) \in \mathcal{M}^{\mathbb{N}}\} < c_{\mathcal{M}}\,. \quad (11)$$

□

Proof. In order to show (11) we have to show that

$$\sup\{\|\Phi_{A(\cdot)}(t,0)x_0\|\ ;\ t \in \mathbb{N}, A(\cdot) \in \mathcal{M}^{\mathbb{N}}, \|x_0\| \le 1\}$$

is finite. By the principle of uniform boundedness (see, [6], Theorem III.1.27) it is sufficient that for each $f \in X^*$ there exists a constant c_f satisfying

$$
\begin{aligned}
&\sup\{| < \Phi_{A(\cdot)}(t,0)x_0, f > |\ ; \\
&t \in \mathbb{N}, A(\cdot) \in \mathcal{M}^{\mathbb{N}}, \|x_0\| \le 1\} < c_f\,. \qquad (12)
\end{aligned}
$$

Assume there exists an $f \in X^*$ for which (12) does not hold, i.e. there exist sequences $\{x_n\}_{n\in\mathbb{N}}, \{t_n\}_{n\in\mathbb{N}}$ and $\{A_n(\cdot)\}_{n\in\mathbb{N}}$ such that

$$| < \Phi_{A_n(\cdot)}(t_n,0)x_n, f > | \ge n+1\,. \quad (13)$$

Let us assume that the sequences have been chosen in such a way as to guarantee that for all solutions $\{x(t)\}$ of (5) with initial condition $x_0, \|x_0\| \le 1$ it holds that

$$| < x(t), f > | < n+1 \quad \text{for } t = 0,1,\dots,t_n-1\,. \quad (14)$$

As $\mathcal{M}$ is bounded it follows immediately that

$$t_1 \le t_2 \le t_3 \dots\,, \quad t_n \to \infty\,.$$

We claim that for every n it holds that

$$| < x_n(t), f > | > 1\,, \quad t = 1,\dots,t_n\,, \quad (15)$$

for otherwise consider a $1 \le t' \le t_n$ with

$$| < x_n(t'), f > | \le 1\,.$$

The sequence $z(t) = x_n(t+t')$ is a solution of (5) with initial condition $x_0 = x_n(t'), \|x_0\| \le 1$, by (14) it follows that

$$n+1 > | < z(t_n - t'), f > | = | < x_n(t_n), f > |\,,$$

in contradiction to (13). Now using Lemma 4.10 we may choose a subsequence $\{x_{n_j}(t)\}$ of $\{x_n(t)\}$ such that

$$\text{w}-\lim_{j\to\infty} x_{n_j}(t) = x(t)\,,$$

for some solution $\{x(t)\}$ of (5). By assumption there exists a $T \in \mathbb{N}$ such that $\|x(t)\| < 1/2\|f\|$ for all $t \ge T$ and hence

$$| < x(t), f > | < \frac{1}{2}\,, \quad \text{for all } t \ge T\,.$$

Now for all j big enough it holds that

$$| < x_{n_j}(T), f > | > 1\,,$$

by (15) and thus $x(T)$ is not the weak limit of the $x_{n_j}(T)$. This contradicts our construction, which completes the proof. ■

Remark 4.12 In the finite dimensional theory Theorem 4.11 can be used to show that an asymptotically stable discrete inclusion is in fact exponentially stable [3]. Note that this is false in infinite dimensions as even for time-invariant systems asymptotic stability does not imply exponential stability, see [8]. The reason for this appears to be that the boundary of the spectrum of an operator A need not contain elements of the point spectrum, so that the existence of $\lambda \in \sigma(A), |\lambda| = 1$ does not guarantee the existence of a trajectory with invariant norm.
Using Theorem 4.11 we obtain the following result on growth bounds of discrete inclusions:

Theorem 4.13

Let X be a reflexive Banach space and assume that $\mathcal{M} \subset \mathcal{L}(X)$ is weakly compact, then

$$\bar{\kappa}(\mathcal{M}) = \bar{\beta}(\mathcal{M}) = \bar{\delta}(\mathcal{M})\,. \quad (16)$$

I.e. a discrete inclusion given by a weakly compact $\mathcal{M}$ has exponentially decaying trajectories iff it is exponentially stable iff it is uniformly exponentially stable. □

Proof. By (9) it remains to show that $\bar{\delta}(\mathcal{M}) \leq \bar{\kappa}(\mathcal{M})$. Assume without loss of generality that $\bar{\kappa}(\mathcal{M}) < 0 < \bar{\delta}(\mathcal{M})$. By the definition of the Lyapunov exponents this implies that all solution of (5) converge to zero and hence by Theorem 4.11 there exists a constant $c_{\mathcal{M}}$ such that

$$\sup\{\|\Phi_{A(\cdot)}(t,0)\| \; ; \; t \in \mathbb{N}, A(\cdot) \in \mathcal{M}^{\mathbb{N}}\} < c_{\mathcal{M}} .$$

Hence $\bar{\delta}(\mathcal{M}) \leq \limsup_{t\to\infty} 1/t \log c_{\mathcal{M}} = 0$, a contradiction. ■

Corollary 4.14 *Let X be a reflexive Banach space and let $\mathcal{M} \subset \mathcal{L}(X)$ be weakly compact. For every $\varepsilon > 0$ there exists a constant $c_{\mathcal{M},\varepsilon}$ such that*

$$\|\Phi_{A(\cdot)}(t,0)\| \leq c_{\mathcal{M},\varepsilon} e^{(\bar{\beta}(\mathcal{M})+\varepsilon)t} , \tag{17}$$

for all $A(\cdot) \in \mathcal{M}^{\mathbb{N}}$ and all $t \in \mathbb{N}$. □

Corollary 4.15 *Let X be a reflexive Banach space and let $\mathcal{M} \subset \mathcal{L}(X)$ be weakly compact. If the discrete inclusion (5) given by $\mathcal{M}$ has a positive uniform growth rate, then there exists a trajectory with positive exponential growth rate. In particular there exists an unbounded trajectory.* □

Let us note that Theorem 4.13 also has implications on balanced convex sets. For $\mathcal{M} \subset \mathcal{L}(X)$ we denoted the balanced convexification by

$$\operatorname{bco} \mathcal{M} := \operatorname{co} \bigcup_{a \in \mathbb{K}, |a|=1} a\mathcal{M} .$$

Corollary 4.16 *Let X be a reflexive Banach space and assume that $\mathcal{M} \subset \mathcal{L}(X)$ is weakly compact, then*

$$\bar{\beta}(\mathcal{M}) = \bar{\beta}(\operatorname{co} \mathcal{M}) , \tag{18}$$

and

$$\bar{\beta}(\mathcal{M}) = \bar{\beta}(\operatorname{bco} \mathcal{M}) . \tag{19}$$

□

Proof. In view of the preceding Theorem 4.13 for (18) it is sufficient to show that $\bar{\delta}(\mathcal{M}) = \bar{\delta}(\operatorname{co} \mathcal{M})$ as it holds that $\bar{\beta}(\mathcal{M}) \leq \bar{\beta}(\operatorname{co} \mathcal{M}) \leq \bar{\delta}(\operatorname{co} \mathcal{M})$. To this end it is sufficient to note that for each $t \in \mathbb{N}$ it holds that

$$\sup_{A(\cdot) \in \mathcal{M}^{\mathbb{N}}} \|\Phi_{A(\cdot)}(t,0)\| \geq \sup_{A(\cdot) \in \operatorname{co} \mathcal{M}^{\mathbb{N}}} \|\Phi_{A(\cdot)}(t,0)\| ,$$

due to the convexity of the norm. Now (19) follows from (18) as the matrix products considered in (19) are up to a constant of modulus one the same as in (18). ■

Corollary 4.17 *Let X be a reflexive Banach space and assume that $\mathcal{M} \subset \mathcal{L}(X)$ is weakly compact, then*

$$\bar{\beta}(\mathcal{M}) = \bar{\beta}(\mathrm{w} - \mathrm{cl}\, \operatorname{bco} \mathcal{M}) . \tag{20}$$

□

Proof. As in the proof of the preceding corollary we have to show that $\bar{\delta}(\mathcal{M}) = \bar{\delta}(\mathrm{w} - \mathrm{cl}\, \operatorname{co} \mathcal{M})$ and for this it is sufficient to see that $\bar{\delta}(\operatorname{co} \mathcal{M}) = \bar{\delta}(\mathrm{w} - \mathrm{cl}\, \operatorname{co} \mathcal{M})$. As for convex subsets of $\mathcal{L}(X)$ weak closure and strong closure coincide (see [4], Corollary VI.1.5) the assertion follows after noting that for a strong limit A of an operator sequence $\{A_n\}$ it holds that $\|A\| \leq \limsup_{n\to\infty} \|A_n\|$. ■

5 Time-varying Discrete Inclusions

Let X be a Banach space and let for every $t \in \mathbb{N}$ the set $\mathcal{M}(t) \subset \mathcal{L}(X)$ be bounded. We consider the time-varying discrete inclusion

$$x(t+1) \in \{Ax(t) \; ; \; A \in \mathcal{M}(t)\} \quad t \in \mathbb{N} . \tag{21}$$

A sequence $\{x(t)\}_{t\in\mathbb{N}}$ is called solution of (21) with initial condition $x_0 \in X$ if $x(0) = x_0$ and for all $t \in \mathbb{N}$ there exists an $A(t) \in \mathcal{M}(t)$ such that $x(t+1) = A(t)x(t)$. Also we say that a sequence $A(\cdot) \in \mathcal{M}(\cdot)$ if $A(t) \in \mathcal{M}(t)$ for all $t \in \mathbb{N}$. We say that $\mathcal{M}(\cdot)$ is uniformly bounded if there exists a constant c such that $\sup\{\|A\| \; ; \; A \in \mathcal{M}(t), t \in \mathbb{N}\} < c$. We introduce the characteristic exponents

$$\bar{\kappa}(\mathcal{M}(\cdot)) := \sup\{\kappa(A(\cdot)) \; ; \; A(\cdot) \in \mathcal{M}(\cdot)\} , \tag{22}$$

$$\bar{\beta}(\mathcal{M}(\cdot)) := \sup\{\beta(A(\cdot)) \; ; \; A(\cdot) \in \mathcal{M}(\cdot)\} . \tag{23}$$

$$\bar{\delta}(\mathcal{M}(\cdot)) :=$$

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{A(\cdot) \in \mathcal{M}(\cdot), s \in \mathbb{N}} \|\Phi_{A(\cdot)}(t+s,s)\| , \tag{24}$$

and define the concepts of exponentially decaying trajectories, exponentially stable and uniformly exponentially stable as in Definition 4.9.
Let us note that for $\mathcal{M}(\cdot)$ uniformly bounded we have

$$\bar{\kappa}(\mathcal{M}(\cdot)) \leq \bar{\beta}(\mathcal{M}(\cdot)) \leq \bar{\delta}(\mathcal{M}(\cdot)) , \tag{25}$$

where the first inequality may be strict as time-varying systems are a special case of time-varying discrete inclusions where each $\mathcal{M}(t)$ is a singleton set. Also for time-varying systems it is clear that both $\bar{\beta}$ and $\bar{\delta}$ reduce to the Bohl exponent and are therefore equal. This extends to the general case. A proof of the following theorem can be found in [14]. Here we will only give a brief sketch.

Theorem 5.18

Let X be a reflexive Banach space and let $\mathcal{M}(\cdot)$ be uniformly bounded. Assume furthermore that for each $t \in \mathbb{N}$ the set $\mathcal{M}(t)$ is weakly compact, then

$$\bar{\beta}(\mathcal{M}(\cdot)) = \bar{\delta}(\mathcal{M}(\cdot)) . \tag{26}$$

Thus a time-varying discrete inclusion is exponentially stable iff it is uniformly exponentially stable. □

Idea of proof: Recall the definition of the space $\hat{X}$ from (3) and introduce the set $\hat{\mathcal{M}} := \{\hat{A} \; ; \; A(\cdot) \in \mathcal{M}(\cdot)\}$. For the discrete inclusion on $\hat{X}$ given by $\hat{\mathcal{M}}$ we can apply Theorem 4.13: it is quite straightforward to see that $\hat{X}$ is reflexive and that $\hat{\mathcal{M}}$ is weakly compact. Thus we have that

$$\bar{\kappa}(\hat{\mathcal{M}}) = \bar{\beta}(\hat{\mathcal{M}}) = \bar{\delta}(\hat{\mathcal{M}}) .$$

Now $\bar{\beta}(\hat{\mathcal{M}}) \geq \sup_{\hat{A} \in \hat{\mathcal{M}}} \log r(\hat{A}) = \bar{\beta}(\mathcal{M}(\cdot))$ and on the other hand $\bar{\delta}(\mathcal{M}(\cdot))$ is equal to

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup_{A(\cdot) \in \mathcal{M}(\cdot), s \in \mathbb{N}} \|\Phi_{A(\cdot)}(t+s,s)\| =$$

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup_{\hat{A}(\cdot) \in \hat{\mathcal{M}}} \|\Phi_{\hat{A}(\cdot)}(t,0)\| = \bar{\delta}(\hat{\mathcal{M}}) .$$

It therefore remains to show that $\sup_{\hat{A} \in \hat{\mathcal{M}}} \log r(\hat{A}) = \bar{\kappa}(\hat{\mathcal{M}})$. This is omitted here. ■

Corollary 5.19 *Let X be a reflexive Banach space and let $\mathcal{M}(\cdot)$ be uniformly bounded. Assume furthermore that for each $t \in \mathbb{N}$ the set $\mathcal{M}(t)$ is weakly compact, then for every $\varepsilon > 0$ there exists a constant $c_{\mathcal{M}(\cdot),\varepsilon}$ such that*

$$\|\Phi_{A(\cdot)}(t+s,s)\| \leq c_{\mathcal{M}(\cdot),\varepsilon} e^{(\bar{\beta}(\mathcal{M}(\cdot))+\varepsilon)t} .$$

□

Note that an equivalent statement to Corollary 4.15 is false, as $\mathcal{M}(t) = \{0\}$ may occur for infinitely many t while a positive uniform exponential growth rate exists.

Corollary 5.20 *Let X be a reflexive Banach space and let $\mathcal{M}(\cdot)$ be uniformly bounded. Assume furthermore that for each $t \in \mathbb{N}$ the set $\mathcal{M}(t)$ is weakly compact. Then for the time-varying discrete inclusion $\mathcal{N}(\cdot)$ given by*

$$\mathcal{N}(t) = \mathrm{w} - \mathrm{cl}\,\mathrm{bco}\,\mathcal{M}(t) ,$$

it holds that

$$\bar{\beta}(\mathcal{N}(\cdot)) = \bar{\beta}(\mathcal{M}(\cdot)) .$$

□

6 Conclusion

For a large class of discrete inclusions it has been shown that different concepts of stability resulting from the consideration of trajectories, evolution operators or supremal norms coincide. For time-varying inclusions the latter two indices are the same while simple examples show that here exponential decay of trajectories does not guarantee exponential stability. The results also show that the set of subsets of $\mathcal{L}(X)$ that yield stable discrete inclusions is stable under convexification and weak closure.

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