Computing time-varying stability radii via discounted optimal control

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Abstract

The problem of calculating the maximal Lyapunov exponent of a discrete inclusion (or equivalently its generalized spectral radius) is formulated as an average yield optimal control problem. It is shown that the maximal value of this problem can be approximated by the maximal value of discounted optimal control problems, where for irreducible inclusions the convergence is linear in the discount rate. This result is used to obtain convergence rates of an algorithm for the calculation of time-varying stability radii.

1 Introduction

The stability radius of the system

$$x(t+1) = (A + D\Delta E)x(t)\,, \quad t \in \mathbb{N}, \qquad (1)$$

where A represents the unperturbed system, D and E are given *structure matrices* of appropriate sizes and Δ is an unknown perturbation matrix is defined as the size of the smallest Δ (measured in some operator norm) for which system (1) becomes unstable. The problem of calculating stability radii for different perturbation classes has attracted the interest of several researchers. For an overview of the theory the reader is referred to the survey article [11].

A formula for the stability radius for a matrix subject to real time-invariant perturbations was obtained only recently in [15] for the particular case that the size of the perturbations is measured in the spectral norm. For this situation a feasible algorithm for the calculation of the real structured stability radius has been given in [16]. It is still an open problem how to calculate real stability radii with respect to other norms.

In this paper discrete inclusions of the form

$$x(t+1) \in \{Ax(t)\ ;\ A \in \mathcal{M}\}\,, \quad t \in \mathbb{N}, \qquad (2)$$

are studied, where $\mathcal{M}$ is a bounded set of real matrices. Stability and dynamics of such systems has been studied extensively in [1], [2], [10], [13] and [20]. In particular equality between the joint and generalized spectral radius and the largest Lyapunov exponent have been shown, three quantities that characterize exponential stability of (2). Methods for the calculation of the generalized spectral radius have been discussed in [1], [7] and [14]. These approaches have in common that they are based on the calculation of ever longer matrix products and evaluating norms and spectral radii of these products.

In our approach the maximal Lyapunov exponent is formulated as the optimal value of an optimal control problem on the $n-1$ dimensional sphere. The main idea is that the intrinsically hard problem of calculating the maximal Lyapunov exponent is solved by finding easier problems which approximate the original one. These are the so called discounted optimal control problems with low discount rates. It has been shown in [18], that in general it is not possible to approximate average yield optimal control problems by discounted ones. Here we pursue an approach which only yields convergence results for the maxima of the value functions, but which has the added advantage of supplying convergence rates in the discount rate. Also it has been shown that the procedure we present for the calculation of stability radii is reliable in the sense that the estimates for the stability radius are below the actual stability radius.

In the continuous-time case there has been substantial work on the Lyapunov spectrum of time-varying linear systems using an approach introduced by Colonius and Kliemann [6]. This has also led to an investigation of the real time-varying stability radius in [4], [5] where stability radii are examined under the assumption of further controllability properties.

In the following Section 2 we give a precise problem formulation. In Section 3 we briefly present existing algorithms for the calculation of generalized spectral radii. In the ensuing Section 4 the basic idea of our approach is presented. The main result for the approximation and an analysis of its complexity are given in Section 5. A numerical example is presented in Section 6.

2 Problem Formulation

Consider the stable time-invariant system

$$\begin{aligned} x(t+1) &= A_0 x(t)\,, \quad t \in \mathbb{N} \\ x(0) &= x_0 \quad \in \mathbb{R}^n\,, \end{aligned}$$

where $A_0 \in \mathbb{R}^{n\times n}$. Time-varying uncertainty may be modeled by a set $\mathcal{M} \subset \mathbb{R}^{n\times n}$, that contains A_0 and is bounded. To this end consider the discrete inclusion

$$\begin{aligned} x(t+1) &\in \{Ax(t)\,;\, A \in \mathcal{M}\}\,, \quad t \in \mathbb{N} \qquad (3) \\ x(0) &= x_0 \quad \in \mathbb{R}^n\,. \end{aligned}$$

A sequence $\{x(t)\}_{t\in\mathbb{N}}$ is called a solution of (3) with initial condition x_0 if $x(0) = x_0$ and for all $t \in \mathbb{N}$ there exists an $A(t) \in \mathcal{M}$ such that $x(t+1) = A(t)x(t)$. We denote the set of all finite products of length t by

$$\mathcal{S}_t := \{A(t-1)\dots A(0); A(s) \in \mathcal{M}\,, s = 0,\dots,t-1\}\,.$$

Exponential stability of the discrete inclusion (3) may now be defined as follows.

Definition 2.1 *The discrete inclusion (3) is called* exponentially stable, *if there exists constants $c \geq 1, \beta < 0$ such that*

$$\|S_t\| \leq ce^{\beta t}\,, \quad \textit{for all } t \geq 0\,, \quad S_t \in \mathcal{S}_t\,. \qquad (4)$$

Two quantities which have been studied in [2] in order to analyze exponential stability of discrete inclusions are the joint and the generalized spectral radius. We depart slightly from the conventions in this area as we take logarithms of all quantities, as is the custom if Lyapunov exponents are considered. Let $\mathcal{M} \subset \mathbb{R}^{n\times n}$ be fixed, let $r(\cdot)$ denote the spectral radius and $\|\cdot\|$ be some operator norm on $\mathbb{R}^{n\times n}$ and define:

$$\overline{\rho}_t(\mathcal{M}) := \sup\{\frac{1}{t}\log r(S_t)\,;\, S_t \in \mathcal{S}_t\}\,,$$

$$\hat{\rho}_t(\mathcal{M}) := \sup\{\frac{1}{t}\log \|S_t\|\,;\, S_t \in \mathcal{S}_t\}\,.$$

Theorem 4 in [2] states that for bounded $\mathcal{M}$ the following equality holds

$$\rho(\mathcal{M}) := \lim_{t\to\infty} \hat{\rho}_t(\mathcal{M}) = \limsup_{t\to\infty} \overline{\rho}_t(\mathcal{M})\,. \qquad (5)$$

It is easy to see that (3) is exponentially stable iff $\rho(\mathcal{M}) < 0$. Furthermore we have for all $t \geq 1$

$$\overline{\rho}_t(\mathcal{M}) \leq \rho(\mathcal{M}) \leq \hat{\rho}_t(\mathcal{M})\,. \qquad (6)$$

Recall that $\mathcal{M} \subset \mathbb{R}^{n\times n}$ is called *irreducible* if only the trivial subspaces $\{0\}$ and $\mathbb{R}^n$ are invariant under all matrices $A \in \mathcal{M}$. Otherwise $\mathcal{M}$ is called *reducible.* The following lemma is shown in [19]. It provides the basic fact needed for our convergence analysis.

Lemma 2.2 *Assume that $\mathcal{M} \subset \mathbb{R}^{n\times n}$ is bounded.*

(i) If $\mathcal{M}$ is irreducible, then there exists a constant $M > 0$ such that for all $t \geq 1$

$$|\hat{\rho}_t(\mathcal{M}) - \rho(\mathcal{M})| < Mt^{-1}\,.$$

(ii) If $\mathcal{M}$ is reducible then there exists an $M > 0$ such that for all $t \geq 1$

$$|\hat{\rho}_t(\mathcal{M}) - \rho(\mathcal{M})| < M\frac{1+\log t}{t}\,.$$

The concept of the stability radius may now be formulated as follows, see also [4]. Assume we are given an increasing family $\mathcal{U} := \{\mathcal{M}_\gamma;\ \gamma \geq 0\}$ of bounded subsets of $\mathbb{R}^{n\times n}$, i.e. $\mathcal{M}_{\gamma_1} \subseteq \mathcal{M}_{\gamma_2}$ if $\gamma_1 \leq \gamma_2$. The set $\mathcal{M}_\gamma$ is the set of admissible perturbations at the *perturbation intensity* γ. We will assume that $\gamma = 0$ represents the case, when no perturbations are present i.e. $\mathcal{M}_0 = \{A_0\}$. Given the family $\mathcal{U}$ the problem is then to find the smallest γ such the discrete inclusion given by $\mathcal{M}_\gamma$ is not exponentially stable.

Definition 2.3 (Stability radius) *For an increasing family $\mathcal{U}$, such that $\{A_0\} = \mathcal{M}_0$ we define the* time-varying stability radius *of A_0 by*

$$r_{tv}(A_0, \mathcal{U}) := \inf\{\gamma;\ \rho(\mathcal{M}_\gamma) \geq 0\}\,. \qquad (7)$$

By the results of Barabanov [1] and Gurvits [10] it is known that if the set of matrices $\mathcal{M}$ is exponentially stable and bounded then the same holds true for the closure of the convex hull $\mathrm{cl\,conv}(\mathcal{M})$. Thus all considerations can be restricted to affine perturbations and compact $\mathcal{M}$. The following general assumption will be made throughout the remainder of this paper.

Assumption 2.4 *We assume that the set $\mathcal{U} = \{\mathcal{M}_\gamma\,;\, \gamma \geq 0\}$ satisfies:*

(i) $\mathcal{M}_0 = \{A_0\}$.

(ii) The family $\mathcal{U}$ is increasing in γ.

(iii) For all $\gamma \geq 0$ the set $\mathcal{M}_\gamma$ is compact and convex.

(iv) For all $\gamma > 0$ the set $V_\gamma := \{x \in \mathbb{R}^n\,;\, Ax = 0, \forall A \in \mathcal{M}\} = \{0\}$.

Note that Assumption (iv) is without loss of generality for robustness analysis as V_γ is a linear subspace and we can study exponential stability on the quotient space $\mathbb{R}^n/V_\gamma$ if necessary.

3 Existing Algorithms

It is known that the calculation of the generalized spectral radius is NP-hard in the dimension of the matrices involved [17]. For finite sets of matrices $\mathcal{M} = \{A_1, \dots, A_m\}$ there are algorithms available for the calculation of upper and lower bounds. As the top Lyapunov exponents does not change under convexification of $\mathcal{M}$ these algorithms are applicable to the special case of system (3) and Assumption 2.4, when $\mathcal{M}_\gamma = A_0 + \gamma\mathcal{M}$ and $\mathcal{M}$ is a polygon.
For this case an approximation algorithm is presented in [7], which uses (6) to obtain upper and lower bounds. It converges to a value within a predefined error bound of the generalized spectral radius. The idea is that the algorithm evaluates the norm and the spectral radius of matrix products of length t of the form $A(t-1)\dots A(0)$ only in the case that all intermediate products $A(s)\dots A(0)$ have a norm bigger than the previously obtained lower bound, thus reducing the number of matrix products to be computed. In [14] the calculation of the lower bound $\bar{\rho}_t(\mathcal{M})$ in (6) is further simplified. In fact, it is shown that it can be performed involving no more than m^t/t matrix products of length t. The spectral radius of each of these matrices has then to be evaluated. At the moment it is, however, not known how quickly $\bar{\rho}_t(\mathcal{M})$ converges to $\rho(\mathcal{M})$ also there are no convergence rates available for Gripenberg's algorithm.

4 Infinite horizon optimal control

In this section we aim to show how to formulate the generalized spectral radius problem as an infinite horizon optimal control problem. This leads to an algorithm that does not involve matrix multiplications. In order to do this we introduce exponential growth rates of trajectories, following an approach introduced in [4].

Definition 4.1 *Given a sequence* $\mathbf{A} \in \mathcal{M}^{\mathbb{N}}$ *and an initial condition* $x_0 \in \mathbb{R}^n \setminus \{0\}$ *the* Lyapunov exponent *corresponding to* $(x_0, \mathbf{A})$ *is defined by*

$$\lambda(x_0, \mathbf{A}) = \limsup_{t\to\infty} \frac{1}{t} \log \|\mathbf{A}(t-1)\dots\mathbf{A}(0)x_0\|\,,$$

where we use the convention $\log 0 = -\infty$.

From (6) it is easy to see that

$$\rho(\mathcal{M}) = \sup\{\lambda(x_0,\mathbf{A}); x_0 \in \mathbb{R}^n \setminus \{0\}, \mathbf{A} \in \mathcal{M}^{\mathbb{N}}\}\,,$$

which is the quantity studied in [1]. Note that in order to characterize exponential stability of time-varying systems it is not sufficient to consider Lyapunov exponents, but rather Bohl exponents have to be introduced. However, it follows from (5) that for discrete inclusions determined by a bounded set of matrices maximal Lyapunov and maximal Bohl exponent coincide.

We study now the projection of the discrete inclusion onto the sphere. In our discrete–time system we do not exclude the possibility that the origin may be reached from non–zero states. Denote $\mathcal{M}(x) := \{A \in \mathcal{M}\ ;\ Ax \neq 0\}$ and $\mathcal{M}^{\mathbb{N}}(x) := \{\mathbf{A} \in \mathcal{M}^{\mathbb{N}}\ ;\ \mathbf{A}(t)\dots\mathbf{A}(0)x \neq 0, \forall t \in \mathbb{N}\}$. By Assumption 2.4 (iv) it holds for all $x \neq 0$ that $\mathcal{M}(x) \neq \emptyset, \mathcal{M}^{\mathbb{N}}(x) \neq \emptyset$. With this notation the projected inclusion corresponding to (3) is given by

$$\begin{aligned} \xi(t+1) &\in \left\{ \frac{A\xi(t)}{\|A\xi(t)\|}\ ;\ A \in \mathcal{M}(\xi(t)) \right\},\ t \in \mathbb{N} \\ \xi(0) &= \xi_0 \in \mathbb{S}^{n-1}. \end{aligned} \tag{8}$$

We denote the solution of (8) corresponding to an initial value $\xi_0 \in \mathbb{S}^{n-1}$ and a control sequence $\mathbf{A} \in \mathcal{M}^{\mathbb{N}}(\xi_0)$ by $\xi(\cdot;\xi_0,\mathbf{A})$. In order to obtain the Lyapunov exponent $\lambda(x_0,\mathbf{A})$ from the trajectory $\xi(\cdot; x_0/\|x_0\|, \mathbf{A})$ of the projected system define for $\xi \in \mathbb{S}^{n-1}$, $A \in \mathcal{M}(\xi)$

$$q(\xi, A) := \log\|A\xi\|\,. \tag{9}$$

A straightforward calculation yields the following expression for Lyapunov exponents.

Lemma 4.2 *For* $\xi_0 \in \mathbb{S}^{n-1}$, $\mathbf{A} \in \mathcal{M}^{\mathbb{N}}(\xi_0)$ *it holds that*

$$\lambda(\xi_0, \mathbf{A}) = \limsup_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t-1} q(\xi(s;\xi_0,\mathbf{A}), \mathbf{A}(s))\,,$$

and otherwise $\lambda(\xi_0,\mathbf{A}) = -\infty$.

Thus Lyapunov exponents may interpreted as the average yield along a trajectory on the sphere. Analogously, we introduce for $\delta > 0$ and $\mathbf{A} \in \mathcal{M}^{\mathbb{N}}(\xi)$ the δ-discounted yield

$$J_\delta(\xi,\mathbf{A}) := \sum_{t=0}^{\infty} e^{-\delta t} q(\xi(t;\xi,\mathbf{A}),\mathbf{A}(t)) \tag{10}$$

Otherwise let $J_\delta(\xi,\mathbf{A}) = -\infty$. The associated value functions are given by

$$V_\delta(\xi) := \sup_{\mathbf{A}\in\mathcal{M}^{\mathbb{N}}} J_\delta(\xi,\mathbf{A})\,,\ V_0(\xi) := \sup_{\mathbf{A}\in\mathcal{M}^{\mathbb{N}}} \lambda(\xi,\mathbf{A})\,.$$

Remark 4.3 (i) The stability radius can now be formulated in terms of value functions in the following sense. Let $A_0 \in \mathbb{R}^{n\times n}$ and let $\mathcal{U} \subset \mathbb{R}^{n\times n}$ satisfy Assumption 2.4 then

$$r_{tv}(A_0,\mathcal{U}) = \inf\{\gamma \geq 0; \sup_{\xi\in\mathbb{S}^{n-1}} V_{0,\gamma}(\xi) \geq 0\}\,,$$

where $V_{0,\gamma}$ denotes the value function of the average yield problem corresponding to $\mathcal{M}_\gamma$.
(ii) Note that for every $\mathbf{A} \in \mathcal{M}^{\mathbb{N}}(\xi)$ the expression for $J_\delta(\xi,\mathbf{A})$ is well defined. In fact, it may be shown that the infinite sum is either absolutely convergent, or the partial sums tend to $-\infty$.

The discounted optimal control problem is far easier to analyze, which is why one tries to obtain a relation between it and the average yield problem. In [3] Chapter V it is shown that V_δ satisfies the Hamilton-Jacobi-Bellman partial difference equation. A considerable amount of effort has been spent in recent years on the numerical solutions of such equations. It remains to analyze the relation between V_δ and V_0, so that we can make use of the results of these efforts.

5 Convergence analysis

Let us first examine properties of the different values along periodic trajectories. This may then be employed in the analysis of trajectories evolving in eigenspaces. The following results are shown in [19].

Proposition 5.1 *Let $\xi_0 \in \mathbb{S}^{n-1}$, $\mathbf{A} \in \mathcal{M}^{\mathbb{N}}(\xi_0)$ be such that there exists a $p \geq 1$ satisfying*

(i) $\mathbf{A}(t+p) = \mathbf{A}(t)$ for all $t \in \mathbb{N}$,

(ii) $\xi(t+p;\xi_0,\mathbf{A}) = \xi(t;\xi_0,\mathbf{A})$ for all $t \in \mathbb{N}$.

Then the following statements hold:

(i) $\lambda(\xi_0,\mathbf{A}) = \frac{1}{p}\sum_{t=0}^{p-1} q(\xi(t;\xi_0,\mathbf{A}),\mathbf{A}(t))$.

(ii) $$\max_{0\leq t\leq p-1} \delta J_\delta(\xi(t;\xi_0,\mathbf{A}),\mathbf{A}(t+\cdot)) \geq \frac{\delta}{1-e^{-\delta}}\lambda(\xi_0,\mathbf{A}).$$

(iii) If $\lambda(\xi_0,\mathbf{A}) \geq 0$ then for all $\delta > 0$ it holds that

$$\max_{0\leq t\leq p-1} \delta J_\delta(\xi(t;\xi_0,\mathbf{A}),\mathbf{A}(t+\cdot)) \geq \lambda(\xi_0,\mathbf{A}).$$

The preceding proposition may be used to analyze the behavior in eigenspaces given by periodic sequences $\mathbf{A}$. For an eigenvalue $\mu \in \mathbb{R}$ of $S_t \in \mathcal{S}_t$ let $E(\mu)$ denote the corresponding eigenspace, or if $\mu \notin \mathbb{R}$, let $E(\mu)$ denote the real part of the sum of the eigenspaces corresponding to $\mu, \bar{\mu}$.

Corollary 5.2 *Let $S_t \in \mathcal{S}_t$ and $\xi_0 \in E(\mu), \|\xi_0\| = 1$ for some $\mu \in \sigma(S_t)$. Let $\mathbf{A} \in \mathcal{M}^{\mathbb{N}}$ be a t-periodic sequence satisfying $S_t = \mathbf{A}(t-1)\dots\mathbf{A}(0)$. then*

(i) $$\max_{0\leq s\leq t-1} \delta J_\delta(\xi(s;\xi_0,\mathbf{A}),\mathbf{A}(s+\cdot)) \geq \frac{\delta}{1-e^{-\delta}}\lambda(\xi_0,\mathbf{A}) = \frac{\delta}{1-e^{-\delta}}\frac{1}{t}\log|\mu|.$$

(ii) If $|\mu| \geq 1$ then

$$\max_{0\leq s\leq t-1} \delta J_\delta(\xi(s;\xi_0,\mathbf{A}),\mathbf{A}(s+\cdot)) \geq \lambda(\xi_0,\mathbf{A}),$$

Theorem 5.3 *Let $\mathcal{M} \subset \mathbb{R}^{n\times n}$ be bounded.*

(i) If $\mathcal{M}$ is irreducible then there exists a constant $M > 0$ such that for all $\delta > 0$

$$|\max_{\xi\in\mathbb{S}^{n-1}} \delta V_\delta(\xi) - \rho(\mathcal{M})| \leq M\delta.$$

(ii) If $\mathcal{M}$ is reducible then for any $\bar{\delta} > 0$ there exists a constant $M > 0$ such that for all $\bar{\delta} > \delta > 0$

$$|\max_{\xi\in\mathbb{S}^{n-1}} \delta V_\delta(\xi) - \rho(\mathcal{M})| \leq M\delta(1+\log(\delta^{-1})).$$

(iii) If $\rho(\mathcal{M}) \geq 0$ then

$$\lim_{\delta\to 0}\max_{\xi\in\mathbb{S}^{n-1}} \delta V_\delta(\xi) = \inf_{\delta>0}\max_{\xi\in\mathbb{S}^{n-1}} \delta V_\delta(\xi) = \rho(\mathcal{M}).$$

In [9] an adaptive grid scheme for the calculation of approximations $\tilde{V}_\delta$ to V_δ are presented. For grids of constant node distance k it holds that the deviation satisfies $\|\tilde{V}_\delta - V_\delta\|_\infty < Ck^{D/2\delta}$, for suitable constants C, D. As the number of nodes in $\mathbb{S}^{n-1}$ is of the order k^{-n+1} we see that obtaining a good approximation of $\rho(\mathcal{M})$ using small values of $\delta > 0$ entails exponential effort in the dimension of the problem.
Let us also note the consequences of the previous theorem for the approximate calculation of time-varying stability radii. By definition the time-varying stability radius is the infimum of the set $\{\gamma\,;\ \rho(\mathcal{M}_\gamma) \geq 0\}$. Let us assume that the sets $\mathcal{M}_\gamma$ are compact and the map $\gamma \mapsto \mathcal{M}_\gamma$ is continuous with respect to the Hausdorff topology. Then the map

$$g : \gamma \mapsto \rho(\mathcal{M}_\gamma)$$

is also continuous (see [1]), $g(r_{tv}(A_0,\mathcal{U})) = 0$ and g is clearly monotone by Assumption 2.4 (ii). Define

$$c(\mathcal{U}) := \sup\left\{c \in \mathbb{R}\,;\ \overline{\lim_{h\downarrow 0}}\frac{g(r_{tv}(A_0,\mathcal{U})-h)}{h} \leq -c\right\}.$$

The number $c(\mathcal{U})$ may be interpreted as the supremum of the gradients of those linear functions that have their zero in $r_{tv}(A_0,\mathcal{U})$ and are larger than g on some interval of the form $[a, r_{tv}(A_0,\mathcal{U})]$, where $a < r_{tv}(A_0,\mathcal{U})$. It is nonnegative by the monotonicity of g. The following theorem is the main result for our approximation procedure, we give a brief outline of the proof.

Theorem 5.4 *Let $A_0 \in \mathbb{R}^{n\times n}$. Let $\mathcal{U}$ satisfy Assumption 2.4, then the following properties hold.*

(i) For all $\delta > 0$ it holds that

$$r_{tv}(A_0,\mathcal{U}) \geq r_\delta(A_0,\mathcal{U}) := \inf_{\gamma>0}\{\gamma;\ \max_{\xi\in\mathbb{S}^{n-1}} \delta V_{\delta,\gamma}(\xi) \geq 0\}.$$

(ii) $r_{tv}(A_0,\mathcal{U}) = \lim_{\delta\to 0} r_\delta(A_0,\mathcal{U})$.

(iii) If $c(\mathcal{U}) > 0$ *then there exist* $\bar\delta > 0$ *and a constant* $M > 0$ *such that for all* $0 < \delta < \bar\delta$

$$r_{tv}(A_0,\mathcal{U}) - r_\delta(A_0,\mathcal{U}) \le M\delta(1+\log(\delta^{-1}))\,.$$

If, furthermore, $\mathcal{M}_\gamma$ *is irreducible for all* $\gamma > 0$ *then* M *may be chosen such that for all* $0 < \delta < \bar\delta$

$$r_{tv}(A_0,\mathcal{U}) - r_\delta(A_0,\mathcal{U}) \le M\delta\,.$$

Proof. (i) and (ii) follow using Theorem 5.3.
To prove (iii) choose $\varepsilon > 0$ small enough such that $c(\mathcal{U}) - \varepsilon > 0$. Then there exists an $\eta > 0$ such that

$$g(\gamma) < (c(\mathcal{U}) - \varepsilon)(\gamma - r_{tv}(A_0,\mathcal{U}))$$

for all

$$\gamma \in [r_{tv}(A_0,\mathcal{U}) - \eta, r_{tv}(A_0,\mathcal{U})].$$

Choose $\delta' > 0$ then by Theorem 5.3 (ii) for every $\gamma \in [r_{tv}(A_0,\mathcal{U}) - \eta, r_{tv}(A_0,\mathcal{U})]$ there exists an $M_\gamma > 0$ such that for all $0 < \delta < \delta'$ we have

$$\max_{\xi\in\mathbb{S}^{n-1}} \delta V_{\delta,\gamma}(\xi) \le$$

$$\frac{(c(\mathcal{U}) - \varepsilon)}{(\gamma - r_{tv}(A_0,\mathcal{U}))} + M_\gamma\delta(1+\log(\delta^{-1}))\,.$$

It follows from Theorem 2 in [1] and the construction of the constant M in Lemma 2.2 (ii) that $M := \sup\{M_\gamma\ ;\ \gamma \in [r_{tv}(A_0,\mathcal{U}) - \eta, r_{tv}(A_0,\mathcal{U})]\}$ exists. Denote the zero of $(c(\mathcal{U}) - \varepsilon)(\gamma - r_{tv}(A_0,\mathcal{U})) + M\delta(1+\log(\delta^{-1}))$ by

$$\tilde r_\delta := r_{tv}(A_0,\mathcal{U}) - \frac{M}{c(\mathcal{U}) - \varepsilon}\delta(1+\log(\delta^{-1}))\,.$$

Then for all $0 < \delta < \delta'$ small enough so that $M\delta(1+\log(\delta^{-1}))(c(\mathcal{U}) - \varepsilon)^{-1} < \eta$ we obtain

$$r_{tv}(A_0,\mathcal{U}) - r_\delta(A_0,\mathcal{U}) \le$$

$$\frac{M}{c(\mathcal{U}) - \varepsilon}\delta(1+\log(\delta^{-1}))\,.$$

The claim for the irreducible case follows the same way by replacing $\delta(1+\log(\delta^{-1}))$ by δ and using Theorem 5.3 (i). ∎

6 Numerical Examples

At the heart of the calculation of time-varying stability radii is the algorithm for the solution of the discrete Hamilton-Jacobi-Bellman equation, as described in [8]. Using these existing algorithms $\max_{\xi\in\mathbb{S}^{n-1}} \delta V_\delta(\xi)$ may be calculated and a bisection algorithm may then be applied to obtain $r_\delta(A,\mathcal{U})$ as an approximation of the

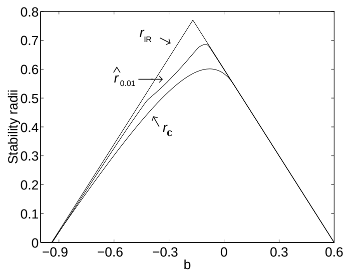

Figure 1: Comparison of the real structured, the complex structured and the time-varying stability radius.

Example 6.1

stability radius $r_{tv}(A,\mathcal{U})$. To make a clear distinction between the value $r_\delta(A,\mathcal{U})$, which is theoretically defined and the values that are the result of the numerical algorithm, we denote the latter by $\hat r_\delta(A,\mathcal{U})$.

To consider a three-dimensional example let

$$\begin{aligned} A(u) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{10} & b & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[0\ 1\ 0] \quad (11)\\ &=: A_0(b) + DuE\,. \quad (12)\end{aligned}$$

and $U_1 := \{u \in \mathbb{R}, |u| \le 1\}$. The parameter b is assume to be known. In this example it is used simply to be able to present different features of the time-varying stability radius. It is easy to see that for all $\gamma > 0$ and all $b \in \mathbb{R}$ the set $\mathcal{M}_\gamma(b) = A_0(b) + \gamma U_1$ is irreducible. With these data we obtain an uncertainty model of feedback type and the results presented in [11] may be compared with the results on the time-varying stability radius. The real stability radius may be obtained by the following simple calculation. The characteristic equation of $A_0(b)$ is given by

$$P_b(\lambda) = \lambda^3 + \frac{1}{2}\lambda^2 - b\lambda - \frac{1}{10} = 0\,, \quad (13)$$

and it is straightforward to see that $A_0(b)$ is stable iff $-0.94 < b < 0.6$ and for these values of b the real time-invariant stability radius satisfies

$$r_\mathbb{R}(A_0(b); D, E) = \min\{|-0.94 - b|, |0.6 - b|\}\,. \quad (14)$$

The complex stability radius $r_\mathbb{C}(A_0(b); D, E)$ can be calculated using the `MATLAB` routine Stabrad-Bruinsma, by L. Schwiedernoch. From [12] it follows that in this case

$$r_\mathbb{R} \ge r_{tv} \ge r_\mathbb{C}\,. \quad (15)$$

We calculated an approximation of the Lyapunov-stability radius with discount rate $\delta = 0.01$. The three different stability radii are shown in Figure 1.

In the interval $[0.1, 0.6]$ it holds that $r_\mathbb{R}(A_0(b); D, E) = r_\mathbb{C}(A_0(b); D, E)$, so that the reliability of the algorithm may be tested by comparing the results for

$r_{\mathbb{R}}(A(b);D,E)$ and $\hat{r}_{0.01}(A(b);D,E)$. The difference is shown in Figure 2.

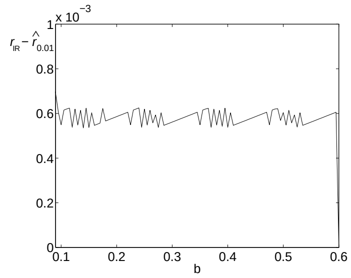

Figure 2: Error in the calculation of the time-varying stability radius.

The value $b = -0.165$ has been chosen to study the convergence of r_δ. To this end $\hat{r}_\delta(A_0(-0.165),\mathcal{U})$ was calculated for δ from 10^{-4} to $2 \cdot 10^{-2}$. The result is shown in Figure 3. The rate of convergence is of order 1 in δ as predicted by the theory. In fact, using the MATLAB polyfit function we obtain that

$$r_\delta(A_0(-0.165);D,E) \approx -0.6212\,\delta + 0.6629\,,$$

where the least squares error is $6.7174 \cdot 10^{-04}$. The difference between the linear and the second order fit in $\delta = 0$ is $1.7686 \cdot 10^{-05}$ showing that a linear fit is adequate.

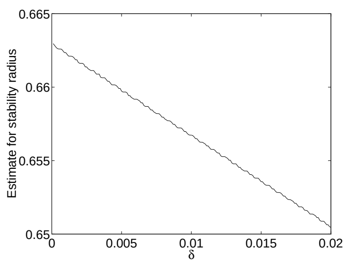

Figure 3: $\hat{r}_\delta$ for small discount rates, $b = -0.165$.

Similar calculations can be performed for the values of b considered in Figure 2, suggesting that the error displayed is mainly due to the discount rates and only to a small extent due to the discretization of the state space necessary for the solution of the HJB equation.

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