

Multichannel Small-Gain Theorems for Large Scale Networked Systems

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Abstract—We consider large scale interconnected systems where some of the interconnections are characterized by uncertain bounded delays. The delays may for instance be due to a communication network connecting some or all of the subsystems. Using a general formulation of comparison functions we derive a small-gain theorem for local practical stability of such large scale systems. The results build on some generalizations of recent results on the class of monotone aggregation functions and almost inversion of general small-gain operators.

I. INTRODUCTION

Since 1960 small-gain type theorems have proved to be a valuable tool for analyzing the stability properties of interconnected systems. With the introduction of the ISS framework by Sontag [1] nonlinear extensions of the linear small-gain theorem became possible. One of the first contributions dealing with the interconnection of two nonlinear ISS systems in a feedback manner was [2]. It has been generalized to deal with the interconnection of several systems in [3].

There exists many stability notions which are related to ISS (see e.g., [4]). One of them is the notion of input-to-output-practical-stability (IOpS) introduced in [2].

Andrew Teel derived Razumikhin-type theorems for functional differential equations (FDE) in [5] based on the ISS small-gain theorems from [2].

In [6] it is shown how to construct an ISS Lyapunov function for a large scale system from the ISS Lyapunov functions of the subsystems using small gain arguments.

The paper [7] uses so-called cyclic-small-gain arguments to construct Lyapunov function for an interconnected system. Two contributions dealing with the notion of ISS for time-delay systems can be found in [8] and [9].

Similar to [7], the paper [8] uses also the cyclic-small-gain argument, which is known to be equivalent to the *no joint increase condition* if the maximum formulation is used (see [10, Theorem 6.4]). Here we have to use a slightly stronger condition than the no joint increase condition (see Lemma 3.6).

Polushin *et al.* presented in [11] a small-gain theorem which guarantees the IOpS property of the interconnection of two IOpS systems. In this particular work the interconnection of the subsystems is over multiple channels, where each of those channels can be delayed. The delay could be introduced by a

communication network, where the information is possibly transmitted in packets as for instance when using TCP or UDP as a communication protocol. Using such protocols introduces delay or even loss of information i.e., packet loss. In [12] we relaxed the condition from [11] to a weaker small-gain condition. We show in [13] that this result can be generalized to the case of an arbitrary number of subsystems, where the interconnection can form an (almost) arbitrary topology. In this paper we continue our work on small-gain theorems from [13]. In [13] we investigate the notion of ISS in a maximum formulation. Sometimes it is more natural to consider the sum formulation of ISS or other norms. To be able to use different comparison functions we generalize the results of [13] to the class of so called *monotone aggregation functions* (see Definition 2.1).

In order to prove basically the same results as in [13], we have to change some of our technical lemmas. In particular, we have to change the lemma which describes the domain of our gain operator (Lemma 3.2) and the lemma which allows us the “inversion” (Lemma 3.4).

The paper is organized as follows. The problem setup as well as the notion of IOpS for FDEs is presented in Section II. The main contribution of this paper is presented in Section III. We will conclude our work with some remarks in Section IV.

II. PRELIMINARIES

A. Notations

In this section we introduce the class of systems under consideration and the stability notion we investigate. To this end we have to define some functional classes and their multi-dimensional extensions.

A continuous function $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is said to belong to class \mathcal{G} , if it is nondecreasing and satisfies $\gamma(0) = 0$. A function $\gamma \in \mathcal{G}$ is of class \mathcal{K} if it is strictly increasing. If γ is of class \mathcal{K} and unbounded, it is said to belong to \mathcal{K}_∞ .

A matrix $\Gamma = (\gamma_{ij})$, $\gamma_{ij} \in \mathcal{K}$ or $\gamma_{ij} = 0$ for $i, j = 1, \dots, n$ is said to belong to $\mathcal{G}^{n \times n}$. It defines an operator $\Gamma : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^{n \times n}$ by

$$(\Gamma(s))_i = (\gamma_{i1}(s_1), \dots, \gamma_{in}(s_n))_i, \quad s \in \mathbb{R}_+^n, \quad i = 1, \dots, n.$$

We write $a < b$, $a, b \in \mathbb{R}^k$ if and only if $a_i < b_i \forall i = 1, \dots, k$ ($\leq, >, \geq$ are defined analogously). Note that we are comparing vectors, therefore we also have to use the negations $\not\leq, \not<, \not>, \not\geq$. The inequality $a \not\leq b$ means that there must be at least one component i of a which is strictly less than the corresponding component of b i.e., $a \not\leq b \Leftrightarrow \exists i$ such that $a_i < b_i$. The other negations are defined in a similar manner. Furthermore we want to adopt a more general class of comparison functions. To this end we introduce the so

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called monotone aggregation functions, which incorporate the properties we need.

Definition 2.1: A map $\mu : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is called a *monotone aggregation function* (respectively MAF_n), if the following conditions are satisfied:

- 1) positive definiteness: $\mu(x) \geq 0$, $\forall x \in \mathbb{R}_+^n$ and $\mu(x) = 0$ if and only if $x = 0$
- 2) strict monotonicity: $\mu(x) < \mu(y)$ if $x < y$
- 3) unboundedness: $\mu(x) \rightarrow \infty$ if $|x| \rightarrow \infty$
- 4) continuity: μ is continuous

A $\mu \in \text{MAF}_n$ is said to have the subadditivity property, if $\mu(x+y) \leq \mu(x) + \mu(y)$, $\forall x, y \in \mathbb{R}_+^n$. Furthermore we call $\mu_1 \in \text{MAF}_n$, $\mu_2 \in \text{MAF}_{n+l}$ compatible, if

$$\mu_1(x) = \mu_2((x, 0)) = \mu_2((0, x)), \quad \forall x \in \mathbb{R}^n,$$

where in μ_2 we filled up the vector x with l zeros.

For instance, the maximum norm and the 1-norm as well as all p -norms are subadditive monotone aggregation functions and fulfill the compatibility condition.

The definition of monotone aggregation function raises the question whether there is a $\mu \in \text{MAF}_n$ which is not a norm. Considering $\mu(x) = \log(|x| + 1)$, where $|\cdot|$ can be any norm on \mathbb{R}^n gives a positive answer. It is easy to see that this μ fulfills the properties of Definition 2.1 and is subadditive but not a norm.

We define a multi-dimensional extension to the notion of monotone aggregation functions by $\mu : \mathbb{R}_+^{n \times n} \rightarrow \mathbb{R}_+^n$, $\mu = (\mu_1, \dots, \mu_n)^T$, $\mu_i \in \text{MAF}_n$, $\forall i = 1, \dots, n$. We denote such an operator by $\mu \in \text{MAF}_n^n$.

An operator $T : \mathbb{R}_+^n \rightarrow \mathbb{R}^n$ is called monotone, if the following implication holds: Given $x, y \in \mathbb{R}_+^n$ such that $x \leq y$ then it holds that $T(x) \leq T(y)$.

A given $\mu \in \text{MAF}_n^n$ together with $\Gamma \in \mathcal{G}^{n \times n}$ defines a continuous and monotone operator $\mu(\Gamma) : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ by

$$(\mu(\Gamma(s)))_i = \mu_i(\gamma_{i1}(s_1), \dots, \gamma_{in}(s_n)),$$

for $s \in \mathbb{R}_+^n$ and $i = 1, \dots, n$. The class of these matrix-induced operators has some nice properties. Most relevant is the fact that any finite composition of matrix-induced operators gives again a matrix-induced operator.

To shorten the notation we fix a $\mu \in \text{MAF}_k^m$, $\mu = (\mu_1, \dots, \mu_m)^T$, where k has to be chosen appropriately and assume that all other monotone aggregation functions are compatible with μ . Furthermore we assume that μ is subadditive. This justifies the simplification we will use throughout the paper by just using a single μ .

The IOpS notion mentioned in the introduction usually deals with ordinary differential equations. In this note we are interested in the interconnection of many systems, where the inputs of the subsystems are delayed. An appropriate mathematical object to model such a situation are the so called *functional differential equations* or FDEs. An FDE is a differential equation where the right hand side depends on a function rather than a single point in the state space for every time instance t . For a detailed introduction to FDEs see e.g., [14].

More precisely, we consider systems of the form:

$$\begin{aligned} \dot{x}(t) &= f(x_d, u_d^1, \dots, u_d^l, t) \\ y^1(t) &= h^1(x_d, u_d^1, \dots, u_d^l, t) \\ &\vdots \\ y^r(t) &= h^r(x_d, u_d^1, \dots, u_d^l, t), \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state, u_d^j , $j = 1, \dots, l$ are the inputs. The operators f and h are Lipschitz in x_d , uniformly continuous in u_d and Lebesgue measurable in t . The subscript d describes a retarded version of its variable in the following way $x_d(t) := \{(s, x(s+t)), s \in [-t_d(t), 0]\}$, $t_d : \mathbb{R} \mapsto \mathbb{R}_+$, $t_d(t_2) - t_d(t_1) \leq t_2 - t_1$ for all $t_1, t_2 \in \mathbb{R}$. Therefore x_d is a piece of trajectory starting at $s = t - t_d(t)$ and ending at $s = t$. Now define $\|x_d(t)\| := \sup_{s \in [t-t_d(t), t]} |x(s)|$, where $|\cdot|$ is an arbitrary norm on \mathbb{R}^n norm. In a similar manner we define $\|y_d\|$, $\|u_d\|$, and $\|w_d\|$.

To ease presentation we introduce $u_d^+ := (\|u_d^1\|, \dots, \|u_d^l\|)^T$ and $y_d^+ := (\|y_d^1\|, \dots, \|y_d^r\|)^T$.

Following [11], we use this ‘‘multichannel’’ formulation to model and analyze the effects of certain inputs on certain outputs. A second advantage of this approach is to have the possibility of different delays in every ‘‘channel’’ as we will see in the next section.

The ensuing definition is borrowed from [11].

Definition 2.2: A system of the form (1) is input-to-output-practical-stable (IOpS) at $t = t_0$ with $t_d(t) \geq 0$, $\beta \in \mathcal{K}_\infty^{r \times 1}$, IOpS gains $\Gamma \in \mathcal{G}^{r \times l}$, restrictions $\Delta_x \in \mathbb{R}_+$, $\Delta_u \in \mathbb{R}_+^l$ and offset $\delta \in \mathbb{R}_+$ if the conditions $\|x_d(t_0)\| \leq \Delta_x$ and $\sup_{t \geq t_0} u_d^+ \leq \Delta_u$, imply that the solution of (1) are well-defined for $t \geq t_0$ and the following inequalities hold:

$$\sup_{t \geq t_0} y^+ \leq \mu \left(\beta(\|x_d(t_0)\|), \Gamma(\sup_{t \geq t_0} u_d^+), \delta \right),$$

and

$$\limsup_{t \rightarrow \infty} y^+ \leq \mu \left(\Gamma(\limsup_{t \rightarrow \infty} u_d^+), \delta \right),$$

Remark 2.3: The first inequality in Definition 2.2 resembles a property which is called global stability, where the second can be regarded as an asymptotic gain property.

Our motivation for the name IOpS for FDEs defined in Definition 2.2 is the following well known fact. For the case of finite dimensional systems the ISS definition is equivalent to the global stability property together with the asymptotic gain property. Although it is not known yet, whether this equivalence is also true in the case of FDEs, we use the term IOpS for our definition nevertheless.

Definition 2.2 couples the practical notion in the sense of IOpS mentioned in [2] with the notion of ISS for FDEs mentioned in [5] and extends it to a semi global version. The term semi global refers to the restrictions Δ_x , Δ_u , while the offset δ reflects the practical notion of the IOpS definition. Setting $\Delta_x = \infty$, $\Delta_u = \infty$, $\delta = 0$, and restricting to ODEs recovers the usual IOS definition.

It has been shown in [15] that design methods for nonlinear sampled-data systems usually yield stability with finite restrictions and an offset bigger than zero.

B. Problem Setup

Consider n systems of FDEs Σ_i , $i = 1, 2, \dots, n$, $n \in \mathbb{N}$ of the form

$$\begin{aligned} \dot{x}_i &= f_i(x_{id}, u_{id}^1, \dots, u_{id}^{l_i}, w_{id}^1, \dots, w_{id}^{v_i}, t) \\ y_i^1 &= h_i^1(x_{id}, u_{id}^1, \dots, u_{id}^{l_i}, w_{id}^1, \dots, w_{id}^{v_i}, t) \\ &\vdots \\ y_i^{r_i} &= h_i^{r_i}(x_{id}, u_{id}^1, \dots, u_{id}^{l_i}, w_{id}^1, \dots, w_{id}^{v_i}, t). \end{aligned} \quad (2)$$

Here we distinguish between the controlled inputs u and the disturbance inputs w . The dimensions of the state spaces and the input spaces are as follows $x_i \in \mathbb{R}^{n_i}$, $u_i^j \in \mathbb{R}^{r_{ij}}$, $j = 1, \dots, l_i$ and $w_i^j \in \mathbb{R}^{q_{ij}}$, $j = 1, \dots, v_i$.

The following definition as well as Definition 2.7 are from [11].

Assumption 2.4: The systems Σ_i , $i = 1, 2, \dots, n$ are IOpS at $t = t_0$ with $t_{id}(t_0) \geq 0$, restrictions $\Delta_{x_i} \in \mathbb{R}$, $\Delta_{u_i} \in \mathbb{R}^{l_i}$, $\Delta_{w_i} \in \mathbb{R}^{v_i}$ and offsets $\delta_i \in \mathbb{R}^{r_i}$. More precisely, there exist $\beta_i \in \mathcal{K}^{r_i \times 1}$, $\Gamma_{iu} \in \mathcal{G}^{r_i \times l_i}$ and $\Gamma_{iw} \in \mathcal{G}^{r_i \times v_i}$, such that for each $i = 1, 2, \dots, n$ and each $t_0 \in \mathbb{R}$ the condition $\|x_{id}(t_0)\| \leq \Delta_{x_i}$, $\sup_{t \geq t_0} u_{id}^+ \leq \Delta_{u_i}$ and $\sup_{t \geq t_0} w_{id}^+ \leq \Delta_{w_i}$ imply that the corresponding solution of Σ_i is well-defined for all $t \geq t_0$ and the following inequalities hold

$$\sup_{t \geq t_0} y_i^+ \leq \mu_i \left(\beta_i(\|x_{id}(t_0)\|), \Gamma_{iu}(\sup_{t \geq t_0} u_{id}^+), \Gamma_{iw}(\sup_{t \geq t_0} w_{id}^+), \delta_i \right) \quad (3)$$

and

$$\limsup_{t \rightarrow \infty} y_i^+ \leq \mu_i \left(\Gamma_{iu}(\limsup_{t \rightarrow \infty} u_{id}^+), \Gamma_{iw}(\limsup_{t \rightarrow \infty} w_{id}^+), \delta_i \right). \quad (4)$$

If we introduce the following notation

$$\begin{aligned} B(x_d^+(t_0)) &= (\beta_1(\|x_{1d}(t_0)\|)^T, \dots, \beta_n(\|x_{nd}(t_0)\|)^T)^T, \\ \Gamma_U &= \text{diag}\{\Gamma_{1u}, \dots, \Gamma_{nu}\}, \quad \Gamma_W = \text{diag}\{\Gamma_{1w}, \dots, \Gamma_{nw}\} \\ \delta_{off} &= (\delta_1^T, \dots, \delta_n^T)^T, \quad u_d^+ = ((u_{1d}^+)^T, \dots, (u_{nd}^+)^T)^T \text{ and} \\ w_d^+ &= ((w_{1d}^+)^T, \dots, (w_{nd}^+)^T)^T \text{ we can rewrite (3) and (4) to} \\ &\text{obtain} \end{aligned}$$

$$\sup_{t \geq t_0} y^+ \leq \mu \left(B(x_d^+(t_0)), \Gamma_U(\sup_{t \geq t_0} u_d^+), \Gamma_W(\sup_{t \geq t_0} w_d^+), \delta_{off} \right), \quad (5)$$

and, respectively,

$$\limsup_{t \rightarrow \infty} y^+ \leq \mu \left(\Gamma_U(\limsup_{t \rightarrow \infty} u_d^+), \Gamma_W(\limsup_{t \rightarrow \infty} w_d^+), \delta_{off} \right). \quad (6)$$

Before we can describe the interconnection of the n subsystems, we have to introduce a delayed versions of y_i^+ . To this end define

$$\hat{y}_i^+(t) = (|y_i^1(t - \tau_i^1(t))|, \dots, |y_i^{r_i}(t - \tau_i^{r_i}(t))|)^T, \quad i = 1, \dots, n,$$

where $\tau_i^j : \mathbb{R} \rightarrow \mathbb{R}_+$, $i = 1 \dots n$, $j = 1, \dots, r_i$ are Lebesgue measurable functions. They describe the delay of the j -th component of the output of the i -th subsystem. Again, to shorten the notation we introduce

$$\hat{y}^+(t) = (\hat{y}_1^+(t)^T, \dots, \hat{y}_n^+(t)^T)^T.$$

Assumption 2.5: The interconnection of the n subsystems is described by

$$u_d^+(t) \equiv 0, \quad \forall t < T_0 \quad (7)$$

$$u_d^+(t) \leq \mu(\Psi(\hat{y}^+(t))), \quad \forall t \geq T_0 \quad (8)$$

where the operator Ψ is of the form

$$\Psi = \begin{pmatrix} 0 & \Psi_{12} & \cdots & \Psi_{1n} \\ \Psi_{21} & 0 & \cdots & \Psi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{n1} & \Psi_{n2} & \cdots & 0 \end{pmatrix},$$

$\Psi_{ij} : \mathbb{R}_+^{r_j} \rightarrow \mathbb{R}_+^{l_i}$ is a continuous and monotone operator for all $i, j = 1, \dots, n$.

Remark 2.6: Assumption 2.5 states that there exists a $T_0 \in \mathbb{R}$ which is the first time instance a connection has been established. Before that time the input is constantly 0. After T_0 the operator Ψ_{ij} describes how the output of the j -th subsystem influences the input of the i -th subsystem. To ensure that communication between the subsystems happens at least sometime, we have to make the following assumption on the delays.

Assumption 2.7: There exists $\tau_* > 0$ and a piecewise continuous function $\tau^* : \mathbb{R} \mapsto \mathbb{R}_+$ with $\tau^*(t_2) - \tau^*(t_1) \leq t_2 - t_1$ for all $t_2 \geq t_1$ such that

$$\tau_* \leq \min_{i=1, \dots, n} \{ \tau_i^j(t) \} \leq \max_{i=1, \dots, n} \{ \tau_i^j(t) \} \leq \tau^*(t), \quad (9)$$

and

$$t - \max_{i=1, \dots, n} \{ \tau_i^j(t) \} \rightarrow \infty \text{ as } t \rightarrow \infty \quad (10)$$

for all $t \geq 0$.

Remark 2.8: The inequalities (9) say that the delays should be bounded from above by $\tau^*(t)$ and from below by τ_* . Because of the propagation delay of any physical system the existence of a lower bound τ_* is guaranteed.

Basically, (10) states that the delay should not grow faster than the time itself. In the literature an assumption of the kind of $\tau^*(t) < 1$ can be found to ensure the property (10). To account for possible information losses we have to adopt the more general Assumption 2.7.

In [11] a methodology to satisfy Assumption 2.7 either by timestamping or by sequence numbering can be found. Timestamping refers to an approach where in a packet based transmission (e.g., TCP) every packet is marked with the current time, while sequence numbering maps an uniquely defined number to every packet.

III. MAIN RESULT

We find it convenient to define

$$\Gamma = \Gamma_U \circ \Psi \text{ and } \Gamma_\mu = \mu \circ \Gamma. \quad (11)$$

The operator Γ resembles the matrix multiplication of Ψ and Γ_U . Hence Γ has the following form:

$$\Gamma = \begin{pmatrix} 0 & \Gamma_{1u} \circ \Psi_{12} & \cdots & \Gamma_{1u} \circ \Psi_{1n} \\ \Gamma_{2u} \circ \Psi_{21} & 0 & \cdots & \Gamma_{2u} \circ \Psi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{nu} \circ \Psi_{n1} & \Gamma_{nu} \circ \Psi_{n2} & \cdots & 0 \end{pmatrix}.$$

Let $m = \sum_{i=1}^n r_i$, clearly, Γ_μ is a continuous and monotone operator from \mathbb{R}_+^m to \mathbb{R}_+^m .

Small-gain type theorems in the spirit of [3] compare some operator with the identity. Because of the restrictions and the offset of Assumption 2.4 we need a slightly stronger small-gain condition.

Definition 3.1: Let Γ_μ be defined as in (11). We say that Γ_μ fullfills the *local small-gain* condition if there exists $\delta, \Delta \in \mathbb{R}_+^m$, $\delta < \Delta$, such that

$$\limsup_{k \rightarrow \infty} \Gamma_\mu^k(\Delta) < \delta. \quad (12)$$

Before we can prove the main technical lemma, we have to exploit some topological properties of our *local small-gain condition*.

Lemma 3.2: Let Γ_μ be defined as in (11) and assume that Γ_μ fullfills the local small-gain condition (12), then

$$\limsup_{k \rightarrow \infty} \Gamma_\mu^k(s) < \delta \quad \forall s \in \Omega, \text{ where}$$

$$\Omega := \bigcup_k \{s \in \mathbb{R}_+^m \mid s \leq \Gamma_\mu^k(\Delta)\}.$$

Proof: Let $s \leq \Delta$. From the monotonicity of Γ_μ we deduce

$$\Gamma_\mu^k(s) \leq \Gamma_\mu^k(\Delta), \quad \forall k \in \mathbb{N}.$$

And hence

$$\limsup_{k \rightarrow \infty} \Gamma_\mu^k(s) \leq \limsup_{k \rightarrow \infty} \Gamma_\mu^k(\Delta) < \delta,$$

where the last inequality follows from (12). Repeating the arguments for $s \leq \Gamma_\mu^l(\Delta)$ with l arbitrary yields

$$\limsup_{k \rightarrow \infty} \Gamma_\mu^k(s) \leq \limsup_{k \rightarrow \infty} \Gamma_\mu^{k+l}(\Delta) < \delta,$$

and the proof is finished. \blacksquare

Remark 3.3: In contrast to [13] where we have used the maximum formulation for the IOpS definition, in this paper we end up with a different Ω as depicted in Figure 1. Using the maximum formulation would lead to $\Omega = \{s \in \mathbb{R}_+^m \mid s \leq \max_k \{\Gamma_\mu^k(\Delta)\}\}$. The reason for the simpler Ω in the maximum case is that the maximum commutes with monotone operators.

For a more extensive study of the properties of such monotone operators and their induced discrete dynamical systems see [10].

The next lemma is the main technical tool for the proof of our main theorem. It is a generalization of the corresponding

lemma from [13]. Some of the ideas of the proof can already be found in [16] respectively [3].

Lemma 3.4: Let the premise of Lemma 3.6 hold. Then for all $a, b \in \Omega$,

$$a \leq \mu(\Gamma(a), b) \quad (13)$$

implies

$$a \leq \max\{\Gamma_d(\mu(b)), \delta\}, \quad (14)$$

where $\Gamma_d = \text{diag}(\gamma_1, \dots, \gamma_m) \in \mathcal{G}^{m \times m}$.

Proof: First we have to rewrite (13) to

$$a \leq \mu((\Gamma(a), 0) + (0, b)).$$

Using the subadditivity we end up with

$$a \leq \mu(\Gamma(a)) + \mu(b). \quad (15)$$

If we apply Γ_μ on both sides of the inequality, use the subadditivity, and insert the result in (15) we obtain

$$a \leq \Gamma_\mu^2(a) + \Gamma_\mu(\mu(b)) + \mu(b)$$

Repeating the same steps $N - 1$ times yields

$$a \leq \Gamma_\mu^N(a) + \sum_{k=0}^{N-1} \Gamma_\mu^k(\mu(b)).$$

Because of (12) we can choose N large enough to get

$$a \leq \delta + \sum_{k=0}^{N-1} \Gamma_\mu^k(\mu(b)).$$

Since the concatenation and the sum of the matrix induced monotone operators in the right hand side of the previous expression is again a matrix induced monotone operator, this can easily be rewritten to get (14). \blacksquare

The next observation will be needed later on.

Remark 3.5: From the construction of Γ_d in Lemma 3.4 it is easy to see that the following holds.

$$\mu(b) \leq \Gamma_d(\mu(b)). \quad (16)$$

Usually, small-gain type conditions compare some operator with the identity. As we will see in the next lemma, condition (12) can also be interpreted in this manner.

Lemma 3.6: Let the premise of Lemma 3.2 hold. then

$$\Gamma_\mu(s) \not\leq s \quad \forall s \in \Omega, s \geq \delta \quad (17)$$

where Ω comes from Lemma 3.2

Proof: We will prove this by contradiction. So assume there exists $s \in \Omega$, $s \geq \delta$ such that $\Gamma_\mu(s) \geq s$. From the monotonicity of Γ_μ it follows readily that $\Gamma_\mu^k(s) \geq s \geq \delta$ for all k . Realizing that this contradicts (12) finishes the proof. \blacksquare

Remark 3.7: The inequality (17) is known as the *no joint increase condition*.

Remark 3.8: From the following example it can be seen that condition (12) is indeed stronger than $\Gamma \not\leq \text{id}$. Consider the following operator.

$$T = \begin{pmatrix} 2 \text{id} & 0 \\ 0 & \frac{1}{2} \text{id} \end{pmatrix}$$

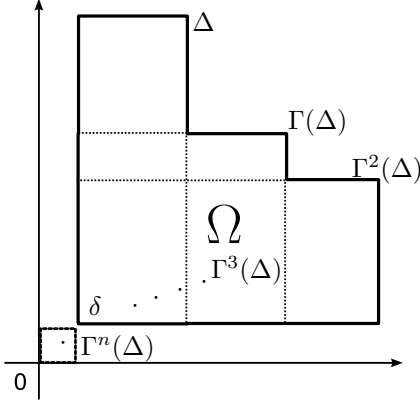


Fig. 1. Sketch of the evolution of Γ_μ in two dimensions. The black bounded region is Ω , where $\Gamma_\mu \not\leq \text{id}$ holds. By condition (12) the iterates of Γ_μ will end in the smaller box on the left.

It is easy to verify, that $T \in \mathcal{G}^{2 \times 2}$ and $T(s) \not\leq s \forall s \in [\delta, \nu]$, $\delta > 0$ and arbitrary $\nu > \delta$. On the other hand $T^k(s)$, $k \rightarrow \infty$ is unbounded, contradicting (12).

From the last example we see that we have to exclude the possibility of unbounded growth to achieve a property like (12). The next lemma shows that a descent in one single point is needed to ensure that property.

Lemma 3.9: Let Γ_μ be defined as in (11). If there exists an $a \in \mathbb{R}_+^m$ and some $k \in \mathbb{N}$ such that $\Gamma_\mu^k(a) < a$, then there exists $b \leq a$ such that

$$\limsup_{k \rightarrow \infty} \Gamma_\mu^k(a) < b.$$

Proof: Choose an $l > k$. Then the following holds

$$0 \leq \Gamma_\mu^l(a) \leq \dots \leq \Gamma_\mu^{k+1}(a) \leq \Gamma_\mu^k(a) < a.$$

This is a bounded and monoton sequence. Hence

$$\lim_{r \rightarrow \infty} \Gamma_\mu^r(a) = b$$

exists. The proof is finalized by noting that $b \leq a$, which follows from $\Gamma_\mu^k(a) < a$. ■

Remark 3.10: The example from Remark 3.8 suggests that $\delta = 0$ plays a special role. Indeed, setting $\delta = 0$ would allow us to use existing results to prove our main result. If we would change condition (12) to

$$\limsup_{k \rightarrow \infty} \Gamma_\mu^k(\Delta) = 0$$

we could deduce with the help of Lemma 3.6 that

$$\Gamma_\mu(s) \not\leq s \forall s \in \Omega.$$

And hence there exists some $w \in \Omega$ such that

$$\Gamma_\mu(s) \not\leq s \forall s \in [0, w].$$

Now we could replace Lemma 3.4 with [17, Proposition 4.4].

Before we state the main contribution of this paper we introduce $\Delta_x = (\Delta_{x1}, \dots, \Delta_{xn})^T$, $\Delta_u = (\Delta_{u1}^T, \dots, \Delta_{un}^T)^T$ and $\Delta_w = (\Delta_{w1}^T, \dots, \Delta_{wn}^T)^T$.

Theorem 3.11: Suppose the system (2), satisfies Assumptions 2.4, 2.5 and 2.7 and that there exist $\delta, \Delta \in \mathbb{R}_+^m$, $0 \leq \delta < \Delta$, such that the following conditions hold:

$$\limsup_{k \rightarrow \infty} \Gamma_\mu^k(\Delta) < \delta, \quad (18)$$

$\Delta^* \in \Omega$, where

$$\Delta^* := \max \left\{ \Gamma_d(\mu(B(\Delta_x), \Gamma_W(\Delta_w), \delta_{off})), \delta \right\}, \quad (19)$$

and

$$\mu(\Psi(\Delta^*)) \leq \Delta_u, \quad (20)$$

then system (2) is IOpS at $t = T_0$ in the sense of Definition 2.2 with

$$t_d(T_0) = \max_{i=1, \dots, n} \{t_{id}(T_0)\} + \tau^*(T_0) + \tau^*(T_0 - \tau^*(T_0)). \quad (21)$$

More precisely, the conditions $x_d^+(T_0) \leq \Delta_x$, $\sup_{t \geq T_0} w_d^+ \leq \Delta_w$ imply that the following inequalities hold

$$\sup_{t \geq T_0} y^+ \leq \max \left\{ \Gamma_d \left(\mu(B(x_d^+(T_0)), \Gamma_W(\sup_{t \geq T_0} w_d^+), \delta_{off})), \delta \right), \right\}, \quad (22)$$

and

$$\limsup_{t \rightarrow \infty} y_d^+ \leq \max \left\{ \Gamma_d \left(\mu(\Gamma_w(\limsup_{t \rightarrow \infty} w_d^+), \delta_{off})), \delta \right). \right\} \quad (23)$$

Proof: Consider system (2) and assume

$$x_d^+(T_0) \leq \Delta_x \quad \text{and} \quad \sup_{t \in [T_0, \infty)} w_d^+ \leq \Delta_w. \quad (24)$$

In order to achieve the IOpS property of the overall system it remains to show that the restrictions on the inputs are not violated. To this end we will first show that the inputs will be bounded for all positive times. This allows us to conclude the existence of solutions for the overall system. Then the claim will be concluded by an application of Lemma 3.4. Assumption 2.4 together with (5), (24) as well as causality arguments imply that

$$y_d^+(T_0) \leq \mu(B(\Delta_x), \Gamma_W(\Delta_w), \delta_{off}).$$

With the help of (7), (8) and Assumption 2.7 we can deduce

$$\begin{aligned} \sup_{t \in [T_0 - t_d(T_0), T_0 + \tau_*]} u^+ &\leq \mu(\Psi(\mu(B(\Delta_x), \Gamma_W(\Delta_w), \delta_{off}))) \\ &\leq \mu(\Psi(\Delta^*)), \end{aligned}$$

where the last inequality follows from (16) and (19). From the last inequality together with (20) we see that the restrictions on the inputs are satisfied for $t \in [T_0 - t_d(T_0), T_0 + \tau_*]$. Hence there exists $T_{\max} > T_0 + \tau_*$ such that the solutions of (2) are well-defined for all $t \in [T_0, T_{\max}]$. Now we want to show that

$$\sup_{t \in [T_0, T_{\max}]} y_d^+ \leq \Delta^*. \quad (25)$$

We will prove (25) by contradiction. To this end assume $T_1 \in [T_0, T_{\max} - \tau_*)$ is the first time that

$$\sup_{t \in [T_0, T_1]} y_d^+ \leq \Delta^* \text{ and } \sup_{t \in [T_0, T_1 + \tau_*)} y_d^+ \not\leq \Delta^*. \quad (26)$$

Combining (5), (21), (24) with (8) and Assumption 2.7, we obtain

$$\sup_{t \in [T_0, T_1 + \tau_*)} y_d^+ \leq \mu \left(B(\Delta_x), \Gamma_W(\Delta_w), \Gamma \left(\sup_{t \in [T_0, T_1]} y_d^+ \right), \delta_{off} \right).$$

By monotonicity we get

$$\sup_{t \in [T_0, T_1 + \tau_*)} y_d^+ \leq \mu \left(B(\Delta_x), \Gamma_W(\Delta_w), \Gamma \left(\sup_{t \in [T_0, T_1 + \tau_*)} y_d^+ \right), \delta_{off} \right). \quad (27)$$

Since T_1 is the first time that (26) holds,

$$\sup_{t \in [T_0, T_1 + \tau_*)} y_d^+ \leq \Delta^*.$$

Because of the continuity of Γ resp. Γ_μ the small-gain condition still holds on the closed interval $t \in [T_0, T_1 + \tau_*]$, therefore we can use Lemma 3.4 to get

$$\sup_{t \in [T_0, T_1 + \tau_*)} y_d^+ \leq \max \left\{ \Gamma_d \left(\mu \left(B(\Delta_x), \Gamma_W(\Delta_w), \delta_{off} \right) \right), \delta \right\} \leq \Delta^*,$$

which contradicts the second inequality in (26). This contradiction proves (25). Next we want to show that $T_{\max} = \infty$. Again we will prove this by contradiction. Due to the IOPs assumption on the subsystems and (24) $T_{\max} < \infty$ implies

$$\sup_{t \in [T_0, T_{\max})} u^+ \not\leq \Delta_u. \quad (28)$$

From (8) and (20) we can see that (28) implies

$$\mu \left(\Psi \left(\sup_{t \in [T_0, T_{\max})} \hat{y}^+ \right) \right) \not\leq \mu \left(\Psi(\Delta^*) \right).$$

Because of the monotonicity of $\mu \circ \Psi$ and the fact that $\sup \hat{y}^+ \leq \sup y_d^+$ we get

$$\sup_{t \in [T_0, T_{\max})} y_d^+ \not\leq \Delta^*,$$

which contradicts (25), hence $T_{\max} = \infty$.

Summarizing, the restrictions on the inputs hold for all $t \in [T_0, \infty)$. Hence we can use (5) to get

$$\sup_{t \geq T_0} y_d^+ \leq \mu \left(B(x_d^+(T_0)), \Gamma_W \left(\sup_{t \geq T_0} w_d^+ \right), \Gamma \left(\sup_{t \geq T_0} y_d^+ \right), \delta_{off} \right).$$

Using Lemma 3.4 we conclude

$$\sup_{t \geq T_0} y_d^+ \leq \max \left\{ \Gamma_d \left(\mu \left(B(x_d^+(T_0)), \Gamma_W \left(\sup_{t \geq T_0} w_d^+ \right), \delta_{off} \right) \right), \delta \right\},$$

which is exactly inequality (22). Similarly we can use (6) together with Lemma 3.4 to get

$$\limsup_{t \rightarrow \infty} y_d^+ \leq \max \left\{ \Gamma_d \left(\mu \left(\Gamma_W \left(\limsup_{t \rightarrow \infty} w_d^+(t) \right), \delta_{off} \right) \right), \delta \right\}.$$

Realizing that this is just (23) finishes the proof. \blacksquare

IV. CONCLUSION

In this paper we continued our work on small-gain theorems which uses the notion of IOPs for FDEs to ensure that the interconnection of an arbitrary number of subsystems is again IOPs. In particular we considered the case where the communication is over delayed, possible lossy communication channels. The contribution of this paper is the generalization of our work in [13] to the case of a more general class of comparison functions.

REFERENCES

- [1] E. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Autom. Control*, vol. 34, no. 4, pp. 435–443, 1989.
- [2] Z. P. Jiang, A. R. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Math Control Signal Syst*, vol. 7, no. 2, pp. 95–120, 1994.
- [3] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth, "An ISS small gain theorem for general networks," *Math Control Signal Syst*, vol. 19, no. 2, pp. 93–122, 2007.
- [4] E. D. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE Trans. Autom. Control*, vol. 41, no. 9, pp. 1283–1294, 1996.
- [5] A. R. Teel, "Connections between Razumikhin-type theorems and the ISS nonlinear small-gain theorem," *IEEE Trans. Autom. Control*, vol. 43, no. 7, pp. 960–964, 1998.
- [6] S. Dashkovskiy, B. Rüffer, and F. Wirth, "Small gain theorems for large scale systems and construction of ISS Lyapunov functions," *SIAM Journal on Control and Optimization*, vol. 48, no. 6, pp. 4089–4118, 2010.
- [7] L. Tengfei, D. Hill, and Z.-P. Jiang, "Lyapunov formulation of ISS small-gain in dynamical networks," in *Proc. 48th Conf. on Decision and Control 2009, held jointly with the 28th Chinese Control Conf., Shanghai, China, 2009*, pp. 4204–4209.
- [8] S. Tiwari, Y. Wang, and Z.-P. Jiang, "A nonlinear small-gain theorem for large-scale time delay systems," in *Proc. 48th Conf. on Decision and Control 2009, held jointly with the 28th Chinese Control Conf., Shanghai, China, 2009*, pp. 7204 – 7209.
- [9] I. Karafyllis and Z. Jiang, "A vector Small-Gain theorem for general nonlinear control systems," in *Proc. 48th Conf. on Decision and Control 2009, held jointly with the 28th Chinese Control Conf., Shanghai, China, 2009*, pp. 7996 – 8001.
- [10] B. Rüffer, "Monotone inequalities, dynamical systems, and paths in the positive orthant of Euclidean n -space," *Positivity*, vol. 14, no. 2, pp. 257–283, 2010.
- [11] I. G. Polushin, H. J. Marquez, A. Tayebi, and P. X. Liu, "A multichannel IOS small gain theorem for systems with multiple time-varying communication delays," *IEEE Trans. Autom. Control*, vol. 54, no. 2, pp. 404–409, 2009.
- [12] B. Rüffer, R. Sailer, and F. Wirth, "Comments on a multichannel IOS small gain theorem for systems with multiple time-varying communication delays," *IEEE Trans. Autom. Control*, vol. 55, no. 7, pp. 1722–1725, 2010.
- [13] R. Sailer and F. Wirth, "A multichannel IOPs small-gain theorem for large scale systems," in *Proc. of the 19th International Symposium on Mathematical Theory of Networks and Systems (MTNS), Budapest, Hungary, 2010*, pp. 1477–1482.
- [14] J. K. Hale, *Theory of Functional Differential Equations*. New York, NY: Springer-Verlag, 1977.
- [15] D. Nešić and A. Teel, "Sampled-data control of nonlinear systems: an overview of recent results," *Perspectives in robust control*, pp. 221–239, 2001.
- [16] B. Rüffer, "Monotone dynamical systems, graphs, and stability of large-scale interconnected systems," Ph.D. dissertation, Universität Bremen, 2007. [Online]. Available: <http://nbn-resolving.de/urn:nbn:de:gbv:46-diss000109058>
- [17] S. Dashkovskiy and B. Rüffer, "Local ISS of large-scale interconnections and estimates for stability regions," *Systems & Control Letters*, vol. 59, pp. 241–247, 2010.