

Capability and limitation of max- and sum-type construction of Lyapunov functions for networks of iISS systems [★]

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Abstract

This paper addresses the problem of verifying stability of networks whose subsystems admit dissipation inequalities of integral input-to-state stability (iISS). We focus on two ways of constructing a Lyapunov function satisfying a dissipation inequality of a given network. Their difference from one another is elucidated from the viewpoint of formulation, relation, fundamental limitation and capability. One is referred to as the max-type construction resulting in a Lipschitz continuous Lyapunov function. The other is the sum-type construction resulting in a continuously differentiable Lyapunov function. This paper presents geometrical conditions under which the Lyapunov construction is possible for a network comprising $n \geq 2$ subsystems. Although the sum-type construction for general $n > 2$ has not yet been reduced to a readily computable condition, we obtain a simple condition of iISS small gain in the case of $n = 2$. It is demonstrated that the max-type construction fails to offer a Lyapunov function if the network contains subsystems which are not input-to-state stable (ISS).

Key words: Nonlinear systems; Interconnected systems; Lyapunov function; Integral Input-to-state stability; Dissipation inequalities.

1 Introduction

In order to verify stability of an interconnected system, the notion of input-to-state stability (ISS) is useful for dealing with the subsystems which do not admit a finite linear gain [23]. For example, the ISS small-gain theorem is available for establishing the ISS property of interconnection of two ISS subsystems [16,26]. The notion of integral input-to-state stability (iISS) has been also developed to characterize nonlinear systems which are not finite in the sense of ISS [2]. For the interconnection of two subsystems, the philosophy of the ISS small-gain

theorem has been extended to the iISS case [11,14]. On the other hand, many practical systems such as logistic systems, biological systems, communication networks and power networks consist of more than two subsystems and have complex interconnection structures. To address such large-scale systems of ever-increasing importance, the ISS small-gain theorem has been extended to the case of general networks recently [8,17].

The ISS small-gain theorem was originally given in terms of bounds for trajectories. Having Lyapunov functions is sometimes advantageous in analysis and design of nonlinear systems. A Lyapunov formulation of the ISS small-gain theorem was given in [15] for the first time, and extended to the general networks in [7,9,18]. The ISS Lyapunov functions constructed there are defined as the maximum among ISS Lyapunov functions of the subsystems, which directly yield Lipschitz continuous Lyapunov functions of the networks¹. In contrast, the iISS small-gain theorem developed in [11,14] is proved by

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¹ Historically, the max-type and the sum-type construction incorporates the idea of vector and scalar Lyapunov functions, respectively[22,19].

using the sum of iISS Lyapunov functions of two subsystems, which directly results in continuously differentiable Lyapunov functions. With the aim of obtaining continuously differentiable Lyapunov functions for general networks of ISS subsystems, an attempt has been made in [5] and a max-type Lyapunov function yielding a *dissipative inequality* of the network have been derived from the ISS subsystems defined in the *dissipative form* although the constructed Lyapunov function is only Lipschitz continuous. Note that the max-type construction was originally derived in the so-called *implication form* [15,7,9,18]. The dissipation form has the advantage that it unifies the definition of ISS and iISS systems, while the implication form is invalid for iISS systems which are not ISS. An attempt to tackle iISS networks was made in [20]. These investigations show that a new scheme is required for establishing the stability of networks involving non-ISS subsystems.

The purpose of this paper is to deal with subsystems described by dissipative inequalities covering the iISS property, and to elucidate capabilities, limitations and relations of the two constructions of Lyapunov functions for general networks. This paper shows that the max-type construction yields a dissipation inequality of the general network consisting of general n subsystems if a matrix-like small-gain condition holds without any assumption on the interaction with external disturbance. From the sum-type construction, this paper also derives a sufficient condition for the stability of the network. Although the condition has not yet been expressed in a computationally convenient form for general n , it can be reduced to a small-gain condition in the case of two subsystems. Moreover, this paper proves that the max-type construction can only deal with ISS subsystems, while the sum-type construction can handle non-ISS as well as ISS subsystems. This paper gives geometrical insights into the capabilities and limitations of the two constructions. In order to avoid confusion, it is made clear here that the focus of this paper is on how to *compose a Lyapunov function the entire network*, which is independent of another interesting issue of how to formulate interaction between individual subsystems such as sum and maximum [9,6]².

We use the following notation. The symbol $|\cdot|$ stands for the Euclidean norm. A continuous function $\omega : \mathbb{R}_+ := [0, \infty) \rightarrow \mathbb{R}_+$ is said to be positive definite and denoted by $\omega \in \mathcal{P}$ if it satisfies $\omega(0) = 0$ and $\omega(s) > 0$ holds for all $s > 0$. A function is of class \mathcal{K} if it belongs to \mathcal{P} and is strictly increasing; of class \mathcal{K}_∞ if it is of class \mathcal{K} and is unbounded. The symbol \mathbf{Id} denotes the identity map. The symbols \vee and \wedge denote logical sum and logical product, respectively. Negation is \neg . For $f, g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, we use the simple notation $\lim f(s) = \lim g(s)$ to describe $\{\lim f(s) = \infty \wedge \lim g(s) = \infty\} \vee \{\infty > \lim f(s) =$

² For example, one may want to use maximization instead of the sum on the right side of (1)

$\lim g(s)\}$. Note that the ∞ case is included. In a similar manner, $\lim f(s) \geq \lim g(s)$ denotes $\{\lim f(s) = \infty \vee \infty > \lim f(s) \geq \lim g(s)\}$. For vectors $a, b \in \mathbb{R}^n$ the relation $a \geq b$ is defined by $a_i \geq b_i$ for all $i = 1, \dots, n$. The relations $>, \leq, <$ for vectors are defined in the same manner. The negation of $a \geq b$ is denoted by $a \not\geq b$ and this means that there exists an $i \in \{1, \dots, n\}$ such that $a_i < b_i$. For a function of time t , a dot over its symbol stands for d/dt . A preliminary version of the material in this paper was presented at the 48th IEEE Conference on Decision and Control, December, 2009, Shanghai, China.

2 Problem statement

Consider a network Σ whose state vector $x(t) = [x_1(t)^T, x_2(t)^T, \dots, x_n(t)^T]^T \in \mathbb{R}^N$ is governed by $\dot{x} = f(x, r)$ and admits the existence of a positive definite and radially unbounded \mathbb{R}_+ -valued \mathcal{C}^1 function $V_i(x_i)$ satisfying

$$\dot{V}_i(x_i) \leq -\alpha_i(V_i(x_i)) + \sum_{j \neq i} \gamma_{ij}(V_j(x_j)) + \kappa_i(|r|) \quad (1)$$

along the trajectories $x_i(t) \in \mathbb{R}^{N_i}$ for each $i = 1, 2, \dots, n$. The vector $r(t) \in \mathbb{R}^M$ denotes an exogenous signal. The property (1) is usually called a dissipation inequality of a subsystem Σ_i . It is assumed that $\alpha_i \in \mathcal{K}$, $\gamma_{ij} \in \mathcal{K} \cup \{0\}$ and $\kappa_i \in \mathcal{K} \cup \{0\}$ hold. This assumption means that each subsystem Σ_i defined with the state x_i and the inputs $x_j, j \neq i$, and r is integral input-to-state stable (iISS), and that V_i is an iISS Lyapunov function for the individual subsystem Σ_i for each $i = 1, 2, \dots, n$. We borrow the notions of ISS and iISS properties from the references [23,25,2]. Under a stronger assumption $\alpha_i \in \mathcal{K}_\infty$, the system Σ_i is guaranteed to be input-to-state stable (ISS), and the function V_i is entitled to be a (dissipative) ISS Lyapunov function. The original definition of iISS and ISS is given in terms of trajectories, which is equivalent to the existence of \mathcal{C}^1 iISS and ISS Lyapunov functions, respectively [2,25]. By definition, an ISS system is always iISS. The converse does not hold.

Remark 1 *The function V_i satisfying (1) is an iISS Lyapunov function even when $\alpha_i \in \mathcal{P}$ [2]. Nevertheless, to allow for feedback loops in the network Σ , this paper assumes $\alpha_i \in \mathcal{K}$ which is a strict subset of \mathcal{P} . It is proved in [13] that a feedback interconnection of iISS systems defined with the dissipation inequalities (1) is guaranteed to be iISS only if for each i the function $\hat{\alpha}_{ii}$ can be bounded from below by a class \mathcal{K} function.*

The objective of this paper is to derive conditions under which the network Σ in total is iISS with respect to input r and state x through construction of an iISS Lyapunov function for the overall network. We want to cover ISS as a special case. To this end, we define operators A, Γ :

$s \in \mathbb{R}_+^n \mapsto z \in \mathbb{R}_+^n$ by

$$z = A(s) = [\alpha_1(s_1), \alpha_2(s_2), \dots, \alpha_n(s_n)]^T,$$

$$z = \Gamma(s) = \left[\sum_{j \neq 1} \gamma_{1j}(s_j), \sum_{j \neq 2} \gamma_{2j}(s_j), \dots, \sum_{j \neq n} \gamma_{nj}(s_j) \right]^T.$$

The operator $K: \tau \in \mathbb{R}_+ \mapsto z \in \mathbb{R}_+^n$ is defined by

$$z = K(\tau) = [\kappa_1(\tau), \kappa_2(\tau), \dots, \kappa_n(\tau)]^T.$$

The following vectors are also defined:

$$V(x) = [V_1(x_1), V_2(x_2), \dots, V_n(x_n)]^T,$$

$$\dot{V}(x) = [\dot{V}_1(x_1), \dot{V}_2(x_2), \dots, \dot{V}_n(x_n)]^T,$$

where $\dot{V}_i = dV_i/dt$ is the time derivative along the trajectories $x_i(t) \in \mathbb{R}^{N_i}$. Then, the dissipation inequalities (1) can be compactly written as

$$\dot{V}(x) \leq (-A + \Gamma) \circ V(x) + K(|r|). \quad (2)$$

Recall that the relation \leq for vectors used in (2) is interpreted componentwise. The goal of this paper is to find a positive definite and radially unbounded function $V_{cl}: \mathbb{R}^N \rightarrow \mathbb{R}_+$ satisfying the dissipation inequality

$$\dot{V}_{cl}(x) \leq -\alpha_{cl}(V_{cl}(x)) + \kappa_{cl}(|r|) \quad (3)$$

along the trajectories $x(t)$ of the network Σ for some $\alpha_{cl} \in \mathcal{P}$ and $\kappa_{cl} \in \mathcal{K} \cup \{0\}$. The property (3) guarantees that the network Σ is iISS with respect to input r and state x . Furthermore, the network Σ is ISS if $\alpha_{cl} \in \mathcal{K}_\infty$.

3 Nonlinear transformation

In this preliminary section we discuss nonlinear transformations of iISS Lyapunov functions. The techniques will be essential in the constructions of the following sections. Consider \mathcal{C}^1 functions $W_i: \mathbb{R}^{N_i} \rightarrow \mathbb{R}_+$ given by

$$W_i(x_i) = \int_0^{V_i(x_i)} \lambda_i(\tau) d\tau, \quad i = 1, 2, \dots, n \quad (4)$$

for continuous functions $\lambda_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+$. This is nothing but a nonlinear transformation of $V_i(x_i)$ with a continuously differentiable function. We assume that

$$\lambda_i(s_i) > 0, \quad \forall s_i \in (0, \infty), \quad i = 1, 2, \dots, n, \quad (5)$$

$$\int_1^\infty \lambda_i(s_i) ds_i = \infty, \quad i = 1, 2, \dots, n, \quad (6)$$

$$\{\alpha_i \in \mathcal{K} \setminus \mathcal{K}_\infty \wedge \kappa_i \in \mathcal{K} \Rightarrow \limsup_{s_i \rightarrow \infty} \lambda_i(s_i) < \infty\},$$

$$i = 1, 2, \dots, n \quad (7)$$

hold. Consider the operator $F: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ defined as

$$F(s) = [\zeta_1(s_1), \zeta_2(s_2), \dots, \zeta_n(s_n)]^T,$$

where we assume that

$$\zeta_i \in \mathcal{K}_\infty, \quad \mathbf{Id} - \zeta_i \in \mathcal{K}_\infty, \quad i = 1, 2, \dots, n. \quad (8)$$

We will discuss the adequate choice of λ_i and ζ_i in the next sections.

Using these functions, we define the vectors

$$W(x) = [W_1(x_1), W_2(x_2), \dots, W_n(x_n)]^T,$$

$$\dot{W}(x) = [\dot{W}_1(x_1), \dot{W}_2(x_2), \dots, \dot{W}_n(x_n)]^T$$

along the trajectories $x_i(t)$ and the matrices

$$H(V(x)) = \begin{bmatrix} \lambda_1(V_1(x_1)) & 0 & \cdots & 0 \\ 0 & \lambda_2(V_2(x_2)) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n(V_n(x_n)) \end{bmatrix},$$

$$G(|r|) = \begin{bmatrix} \eta_1(|r|) & 0 & \cdots & 0 \\ 0 & \eta_2(|r|) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \eta_n(|r|) \end{bmatrix},$$

where the non-decreasing continuous functions $\eta_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $i = 1, 2, \dots, n$, are given by

$$\bar{\lambda}_i(\tau) = \max_{w \in [0, \tau]} \lambda_i(w), \quad (9)$$

$$\eta_i(\tau) = \begin{cases} \bar{\lambda}_i \circ \alpha_i^{-1} \circ \zeta_i^{-1} \circ \kappa_i(\tau) \\ \quad, \text{ if } \lim_{w \rightarrow \infty} \zeta_i \circ \alpha_i(w) > \kappa_i(\tau), \\ \lim_{w \rightarrow \infty} \bar{\lambda}_i(w), \text{ otherwise.} \end{cases} \quad (10)$$

Note that the assumption (7) renders the function $\eta_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by (10) well-defined. With the help of these definitions, combining the two cases $\zeta_i \circ \alpha_i(V_i(x_i)) > \kappa_i(|r|)$ and $\zeta_i \circ \alpha_i(V_i(x_i)) \leq \kappa_i(|r|)$ in (2) proves that (2) implies

$$\dot{W}(x) \leq H(V(x)) \{ -(\mathbf{Id} - F) \circ A + \Gamma \} (V(x)) \\ + G(|r|) K(|r|). \quad (11)$$

Alternatively, the inequality (11) can be expressed as

$$\dot{W}(x) \leq H(V(x)) \{ -(\mathbf{Id} + E)^{-1} \circ A + \Gamma \} (V(x)) \\ + G(|r|) K(|r|), \quad (12)$$

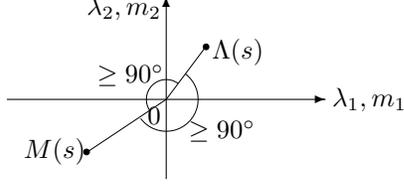


Fig. 1. Geometrical interpretation of (17): Obtuse angle.

where E is defined by

$$\begin{aligned} (\mathbf{Id} + E)(s) &:= [s_1 + \varepsilon_1(s_1), s_2 + \varepsilon_2(s_2), \dots, s_n + \varepsilon_n(s_n)]^T \\ &:= (\mathbf{Id} - F)^{-1}(s). \end{aligned} \quad (13)$$

Note that $\varepsilon_i \in \mathcal{K}_\infty$ holds since

$$\begin{aligned} (\mathbf{Id} + \varepsilon_i) \circ (s_i - \zeta_i(s_i)) - s_i \\ = -\zeta_i(s_i) + \varepsilon_i \circ (s_i - \zeta_i(s_i)) = 0 \end{aligned}$$

and $\zeta_i, \mathbf{Id} - \zeta_i \in \mathcal{K}_\infty$. The relation (13) defines a bijection between $\zeta_i \in \mathcal{K}_\infty$ and $\varepsilon_i \in \mathcal{K}_\infty$, i.e., F and E .

The technique applied to the iISS network in this section is essentially the same as the technique of changing ISS supply rates proposed in [24].

Remark 2 *The choice of $\zeta_i(s) \equiv 0$, $\varepsilon_i(s) \equiv 0$ and $\eta_i(s) \equiv 0$ is also valid in (11) and (12) when $\kappa_i(s) \equiv 0$.*

4 Sum-type construction

This section considers Lyapunov functions in the form of

$$V_{cl}(x) = \sum_{i=1}^n W_i(x_i) \quad (14)$$

and presents a condition under which the network Σ is guaranteed to be iISS. In order to select functions λ_i with which the sum-type Lyapunov function (14) establishes the stability of the network, we define mappings from $s \in \mathbb{R}_+^n$ to \mathbb{R}_+^n by

$$\Lambda(s) = [\lambda_1(s_1), \lambda_2(s_2), \dots, \lambda_n(s_n)]^T, \quad (15)$$

$$D(s) = [s_1 + \beta_1(s_1), s_2 + \beta_2(s_2), \dots, s_n + \beta_n(s_n)]^T \quad (16)$$

and obtain the following theorem.

Theorem 3 *Suppose that there exist continuous functions $\lambda_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $i = 1, 2, \dots, n$, such that (5), (6) (7) and*

$$\Lambda(s)^T \Gamma(s) \leq \Lambda(s)^T D^{-1} \circ A(s), \quad \forall s \in \mathbb{R}_+^n \quad (17)$$

are satisfied for some $\beta_1, \beta_2, \dots, \beta_n \in \mathcal{K}_\infty$. Then the network Σ is iISS with respect to input r and state x . If

$$\alpha_i \in \mathcal{K}_\infty, \quad i = 1, 2, \dots, n, \quad (18)$$

$$\liminf_{s_i \rightarrow \infty} \lambda_i(s_i) > 0, \quad i = 1, 2, \dots, n \quad (19)$$

are satisfied additionally, the network Σ is ISS. Furthermore, an iISS (ISS) Lyapunov function is given by (14).

Proof. Let θ_i be defined with

$$\mathbf{Id} - \theta_i = (\mathbf{Id} + \beta_i)^{-1}. \quad (20)$$

The property $\theta_i \in \mathcal{K}_\infty$ follows from $\beta_i \in \mathcal{K}_\infty$ and

$$\begin{aligned} (\mathbf{Id} - \theta_i) \circ (\mathbf{Id} + \beta_i)(s_i) - s_i \\ = \beta_i(s_i) - \theta_i \circ (\mathbf{Id} + \beta_i(s_i)) = 0. \end{aligned}$$

Pick $\zeta_i \in \mathcal{K}_\infty$ satisfying $\theta_i - \zeta_i \in \mathcal{K}_\infty$. Substituting (17) for (12), we obtain for V_{cl} defined by (14)

$$\dot{V}_{cl}(x) \leq -\Lambda(V)^T [((\mathbf{Id} + E)^{-1} - D^{-1}) \circ A(V)] + \sum_{i=1}^n \hat{\kappa}_i(|r|),$$

where $\hat{\kappa}_i := \eta_i \kappa_i \in \mathcal{K} \cup \{0\}$. Then, from

$$(\mathbf{Id} + \varepsilon_i)^{-1} - (\mathbf{Id} + \beta_i)^{-1} = \theta_i - \zeta_i \quad (21)$$

we obtain

$$\begin{aligned} \dot{V}_{cl}(x) &\leq -\sum_{i=1}^n \lambda_i(V_i(x_i)) [(\theta_i - \zeta_i) \circ \alpha_i(V_i(x_i))] \\ &\quad + \sum_{i=1}^n \hat{\kappa}_i(|r|) \\ &\leq -\sum_{i=1}^n \hat{\alpha}_i(V_i(x_i)) + \sum_{i=1}^n \hat{\kappa}_i(|r|), \end{aligned}$$

where $\hat{\alpha}_i(s) = \lambda_i(s) [(\theta_i - \zeta_i) \circ \alpha_i(s)]$. Since $\alpha_i \in \mathcal{K}$ implies $(\theta_i - \zeta_i) \circ \alpha_i \in \mathcal{K}$, the property (5) yields $\hat{\alpha}_i \in \mathcal{P}$, $i = 1, 2, \dots, n$. Let $\chi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ denote the definite integral in (4), i.e., $W_i(x_i) = \chi_i(V_i(x_i))$. The properties (5) and (6) ensure $\chi_i \in \mathcal{K}_\infty$. Hence, $W_i(x_i)$ is positive definite and radially unbounded. Defining $\mathcal{W}(s) = \{w \in \mathbb{R}_+^n : s = \sum_{i=1}^n w_i\}$, we arrive at (3) with

$$\alpha_{cl}(s) = \min_{w \in \mathcal{W}(s)} \sum_{i=1}^n \hat{\alpha}_i \circ \chi_i^{-1}(w_i), \quad \kappa_{cl}(s) = \sum_{i=1}^n \hat{\kappa}_i(s)$$

for $\alpha_{cl} \in \mathcal{P}$ and $\kappa_{cl} \in \mathcal{K} \cup \{0\}$. From $\chi_i \in \mathcal{K}_\infty$, $i = 1, 2, \dots, n$, it follows that V_{cl} in (14) is positive definite and radially unbounded. Thus, the function V_{cl} is an iISS Lyapunov function of the network Σ . If (18) and

(19) hold additionally, we choose $\hat{\alpha}_i(s) = \underline{\lambda}_i(s)[(\theta_i - \zeta_i) \circ \alpha_i(s)]$, where

$$\underline{\lambda}_i(s) = \inf_{\tau \in [s, \infty)} \lambda_i(\tau).$$

The properties (18), (19) and (5) imply $(\theta_i - \zeta_i) \circ \alpha_i \in \mathcal{K}_\infty$ and $\hat{\alpha}_i \in \mathcal{K}_\infty$. Therefore, the property $\alpha_{cl} \in \mathcal{K}_\infty$ guarantees that V_{cl} is an ISS Lyapunov function. \square

A geometrical interpretation can be given to (17) in Theorem 3 in terms of the two vector-valued functions $\Lambda(s)$ and $M(s) := -D^{-1} \circ A(s) + \Gamma(s)$. The angles enclosed by the two vectors are greater than or equal to 90° for all $s \in \mathbb{R}_+^n$ if and only if (17) holds. Such vectors are illustrated in Fig.1, where $M(s) = [m_1(s), m_2(s)]^T$ and $\Lambda(s) = [\lambda_1(s_1), \lambda_2(s_2)]^T$.

From a different viewpoint, the scalar condition (17) imposed on vectors in \mathbb{R}_+^n states that a nonlinear combination of the sums of $\gamma_{ij}(s_j)$'s is less than or equal to a nonlinear combination of $\alpha_i(s_i)$'s. In this sense, the functions γ_{ij} in total are required to be small in comparison to the functions α_i in total, which looks like a small-gain condition. In the case of $n = 2$, we can make this observation precise and we can explicitly obtain a vector $\Lambda(s)$ solving the geometrical problem and fulfilling (5), (6), (7) and (19) as explained in the following.

Theorem 4 *Let $n = 2$. Suppose that*

$$\{\alpha_i \in \mathcal{K} \setminus \mathcal{K}_\infty \Rightarrow \gamma_{3-i,i} \in \mathcal{K} \setminus \mathcal{K}_\infty \cup \{0\}\}, \quad i = 1, 2 \quad (22)$$

holds. If there exist $\beta_1, \beta_2 \in \mathcal{K}_\infty$ satisfying

$$D \circ \Gamma(s) \not\leq A(s), \quad \forall s \in \mathbb{R}_+^2 \setminus \{0\}, \quad (23)$$

there exists a continuous function $\Lambda : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ of the form (15) such that (5), (6), (7), (17) and (19) hold.

Proof. It can be verified that the condition (23) is equivalent to the logical sum of

$$(\mathbf{Id} + \beta_1) \circ \gamma_{12} \circ \alpha_2^{-1} \circ (\mathbf{Id} + \beta_2) \circ \gamma_{21}(\tau) < \alpha_1(\tau), \quad \forall \tau \in (0, \infty) \quad (24)$$

and

$$(\mathbf{Id} + \beta_2) \circ \gamma_{21} \circ \alpha_1^{-1} \circ (\mathbf{Id} + \beta_1) \circ \gamma_{12}(\tau) < \alpha_2(\tau), \quad \forall \tau \in (0, \infty) \quad (25)$$

Note that the expression (24) (resp. (25)) implicitly requires $\lim_{\tau \rightarrow \infty} \alpha_2(\tau) \geq \lim_{\tau \rightarrow \infty} \gamma_{21}(\tau)$ (resp. $\lim_{\tau \rightarrow \infty} \alpha_1(\tau) \geq \lim_{\tau \rightarrow \infty} \gamma_{12}(\tau)$). The existence of $\beta_1, \beta_2 \in \mathcal{K}_\infty$ achieving the above logical sum is the same as the existence of $\bar{\beta}_1, \bar{\beta}_2 \in \mathcal{K}_\infty$ achieving \leq for $s \in \mathbb{R}_+$ instead of $<$ for $s \in \mathbb{R}_+ \setminus \{0\}$ in the logical sum of (24)

and (25). Indeed, the substitution $\beta_i = \bar{\beta}_i/2$ allows us to change \leq into $<$. Hence, the condition (23) is equivalent to the iISS small-gain condition presented in [14]. The function V_{cl} in (14) is identical with the one employed in [11,14], and the corresponding inequality (17) is the same as the one solved in [11,14]. Moreover, the property (22) implies that one of the properties

$$\begin{aligned} (A1) \quad & \lim_{\tau \rightarrow \infty} \alpha_1(\tau) = \infty \quad \wedge \quad \lim_{\tau \rightarrow \infty} \alpha_2(\tau) = \infty, \\ (A2) \quad & \lim_{\tau \rightarrow \infty} \alpha_1(\tau) = \infty \quad \wedge \quad \lim_{\tau \rightarrow \infty} \gamma_{12}(\tau) < \infty, \\ (A3) \quad & \lim_{\tau \rightarrow \infty} \alpha_2(\tau) = \infty \quad \wedge \quad \lim_{\tau \rightarrow \infty} \gamma_{21}(\tau) < \infty, \\ (A4) \quad & \lim_{\tau \rightarrow \infty} \gamma_{12}(\tau) < \infty \quad \wedge \quad \lim_{\tau \rightarrow \infty} \gamma_{21}(\tau) < \infty, \end{aligned}$$

is satisfied. It is also verified that

$$\begin{aligned} (24) \wedge (A2) \wedge \neg(A1) & \Rightarrow (A4) \\ (25) \wedge (A3) \wedge \neg(A1) & \Rightarrow (A4) \end{aligned}$$

hold. Hence, the non-decreasing functions $\lambda_1(s_1)$ and $\lambda_2(s_2)$ derived in [14] achieve (5), (6), (7), (17) and (19) for $n = 2$. If $\gamma_{i,j}(s_j) \equiv 0$ holds for some $i \neq j$, we can always use sufficiently small $\gamma_{i,j} \in \mathcal{K}$ when we invoke [14]. \square

In the $n = 2$ case, the components $\lambda_1(s_1)$ and $\lambda_2(s_2)$ of $\Lambda(s)$ are derived explicitly in [11,14], and the property (23) agrees with the small-gain condition in [14]. To give a geometrical insight into the condition (23), define the open set

$$\Omega^{--} := \{s = [s_1, s_2]^T \in \mathbb{R}_+^2 : -A(s) + \Gamma(s) < 0\} \quad (26)$$

whose closure contains $\{0\}$. The boundary layer is given by the two curves $l_1: \alpha_1(s_1) = \gamma_{12}(s_2)$ and $l_2: \alpha_2(s_2) = \gamma_{21}(s_1)$. By the definition of $\not\leq$, the property (23) implies $\Omega^{--} \neq \emptyset$ and it is a connected set dividing $\mathbb{R}_+^2 \setminus \{0\}$ into two disjoint sets. The existence of $\beta_1, \beta_2 \in \mathcal{K}_\infty$ requires that the thickness of Ω^{--} does not shrink to zero as we go far away from the origin $s = 0$. In other words, the class \mathcal{K}_∞ property of β_1 and β_2 implies that the distance between the two curves l_1 and l_2 increases unboundedly as the distance from the origin increases. The interpretation is illustrated by Fig.2, which shares the idea of topological separation with the classical input-output approach [27,21]. The obtuse angle problem posed by Theorem 3 is recast into the topological separation with Ω^{--} by Theorem 4 in the case of $n = 2$. The converse direction holds in the following sense.

Proposition 5 *Let $n = 2$. Suppose that there exist continuous functions $\lambda_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $i = 1, 2, \dots, n$, such that (5) and*

$$\Lambda(s)^T \Gamma(s) < \Lambda(s)^T A(s), \quad \forall s \in \mathbb{R}_+^n \setminus \{0\} \quad (27)$$

is satisfied. Then, the set Ω^{--} is non-empty, connected and unbounded.

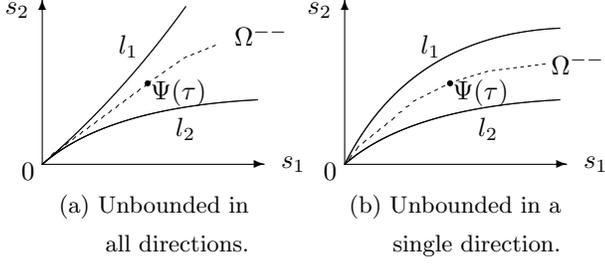


Fig. 2. Geometrical interpretation of (23) and (37): Topological separation.

Proof. Define

$$\overline{\Omega}^{++} := \{s = [s_1, s_2]^T \in \mathbb{R}_+^2 : -A(s) + \Gamma(s) \geq 0\}. \quad (28)$$

Unless the set Ω^{--} is non-empty, connected and unbounded, the set $\overline{\Omega}^{++} \setminus \{0\}$ is not empty. In such a situation, we have $\Lambda(s)^T[-A(s) + \Gamma(s)] \geq 0$ for $s \in \overline{\Omega}^{++} \setminus \{0\}$ due to (5). This contradicts (27). \square

Proposition 5 can be also observed from the fact that the angles enclosed by $\Lambda(s)$ and $-A(s) + \Gamma(s)$ cannot be greater than 90° for $s \in \overline{\Omega}^{++} \setminus \{0\}$. Note that the property (27) is implied by the existence of $\beta_1, \beta_2 \in \mathcal{K}_\infty$ satisfying (17). Theorem 4 precisely clarifies the converse of Proposition 5 with such $\beta_1, \beta_2 \in \mathcal{K}_\infty$ in the case of $n = 2$.

Remark 6 When we consider only 0-GAS (i.e., global asymptotic stability of $x = 0$ for $r(t) \equiv 0$ or $K(s) \equiv 0$), the requirement $\beta_i \in \mathcal{K}_\infty$ in Theorem 3 can be relaxed into $\beta_i \in \mathcal{P}$ and $\mathbf{Id} + \beta_i \in \mathcal{K}_\infty$ for $i = 1, 2, \dots, n$. Note that using $<$ in (17) with $D = \mathbf{Id}$ cannot always ensure 0-GAS since it cannot exclude the no-gap case [1]. It is known that in the no-gap case information on α_i and γ_{ij} is not sufficient to conclude 0-GAS. Property (17) with positive definite β_i 's ensures that the no-gap case does not occur. Likewise, inequality (23) with $D = \mathbf{Id}$ cannot guarantee the 0-GAS. In order to avoid the no-gap case, we need to add an assumption as in [14].

Remark 7 Since we do not assume $\alpha_i \in \mathcal{K}_\infty$ in this paper, we do not resort to A^{-1} . This contrasts vividly with the developments in [5] applicable exclusively to ISS subsystems. The difference appears in (17) and (23) in a natural manner. Theorem 4 not only proves a conjecture made in [5] for $n = 2$, but also covers iISS subsystems.

5 Max-type construction

In this section, we define a locally Lipschitz function $V_{cl} : \mathbb{R}^N \rightarrow \mathbb{R}_+$ by

$$V_{cl}(x) = \max_{i=1,2,\dots,n} W_i(x_i). \quad (29)$$

Alternatively, we can write the above V_{cl} as

$$V_{cl}(x) = \max_{i=1,2,\dots,n} \psi_i^{-1}(V_i(x_i)), \quad (30)$$

where $\psi_i \in \mathcal{K}_\infty$ is given by

$$\psi_i^{-1}(s_i) = \int_0^{s_i} \lambda_i(\tau) d\tau. \quad (31)$$

Note that the right hand side of the above equation is guaranteed to be of class \mathcal{K}_∞ by (5) and (6). An apparent feature of the max-type Lyapunov function (29) is its Lipschitz continuity, while the sum-type Lyapunov function (14) is continuously differentiable.

For interconnected ISS systems, some studies derive Lyapunov functions of the form (30), e.g., [15,7,9,18]. The following theorem demonstrates that the max-type Lyapunov function is not useful if at least one subsystem is only iISS.

Theorem 8 Let V_{cl} be defined by (30), and let $V_{cl}^\circ(x; \dot{x})$ denote the Clarke generalized derivative at x in the direction of \dot{x} . If there exist continuously differentiable $\psi_i \in \mathcal{K}_\infty$, $i = 1, 2, \dots, n$, such that all differentiable trajectories³ $x(t) \in \mathbb{R}^N$ fulfilling (1) with $\alpha_i \in \mathcal{K}$, $\gamma_{ij} \in \mathcal{K} \cup \{0\}$ for $r(t) \equiv 0$ satisfy

$$V_{cl}^\circ(x; \dot{x}) \leq 0, \quad \forall x \in \mathbb{R}^N, \quad (32)$$

then

$$\sum_{j \neq i} \lim_{\tau \rightarrow \infty} \gamma_{ij}(\tau) \leq \lim_{\tau \rightarrow \infty} \alpha_i(\tau), \quad i = 1, 2, \dots, n. \quad (33)$$

Proof. To prove the claim by contradiction, suppose that

$$\sum_{j \neq i} \lim_{\tau \rightarrow \infty} \gamma_{ij}(\tau) > \lim_{\tau \rightarrow \infty} \alpha_i(\tau) \quad (34)$$

holds for some $i = p \in \{1, 2, \dots, n\}$. Let

$$M_p := \{x \in \mathbb{R}^N : \psi_p^{-1}(V_p(x_p)) > \psi_j^{-1}(V_j(x_j)), \forall j \neq p\},$$

$$L_p := \left\{ x \in \mathbb{R}^N : \sum_{j \neq p} \gamma_{pj}(V_j(x_j)) > \lim_{\tau \rightarrow \infty} \alpha_p(\tau) \right\}.$$

Since the ψ_i^{-1} 's are of class \mathcal{K}_∞ , the set M_p is unbounded in all directions, i.e., M_p contains a sequence $\{x^{p,k} \in \mathbb{R}^N\}$, $k = 1, 2, \dots$, such that $V_i(x_i^{p,k}) \rightarrow \infty$ for all $i =$

³ Here, the trajectories are not necessarily associated with differential equations of the form $\dot{x} = f(x, r)$. Using the technique developed in [14], we can also address the existence of a corresponding differential equation in Theorem 8.

1, 2, ..., n when $k \rightarrow \infty$. This fact and (34) ensure $M_p \cap L_p \neq \emptyset$. Recall that if V_{cl} is differentiable at x , then its Clarke generalized derivative coincides with the usual directional derivative at this point x . Also note that V_{cl} is differentiable in M_p by the definition (30). Property (1) with $r(t) \equiv 0$ yields $\dot{V}_{cl}(x) \leq \xi(x)$ for $x \in M_p$, where

$$\xi(x) := \lambda_p(V_p(x_p)) \left\{ -\alpha_p(V_p(x_p)) + \sum_{j \neq p} \gamma_{pj}(V_j(x_j)) \right\}.$$

By assumption, in the set M_p , the function $\xi(x)$ is the smallest upper bound of $\dot{V}_{cl}(x)$ covering all trajectories $x(t) \in \mathbb{R}^N$ defined with (1). The definition of L_p implies

$$\xi(x) > 0, \quad \forall x \in M_p \cap L_p. \quad (35)$$

On $M_p \cap L_p$ the function V_{cl} in the form of (30) is differentiable. Since the Clarke generalized derivative agrees with the directional derivative of V_{cl} at differentiable points, the property (35) contradicts (32). \square

The property (33) means that each subsystem Σ_i is ISS with respect to input $x_j, j \neq i$ and state x_i [25]. Theorem 8 can be interpreted as follows: In the construction of a Lyapunov function of the form (30), the function ψ_i^{-1} needs to ensure that if the maximum of (30) is attained for the i -th subsystem, then the decay of the particular subsystem appears as the decrease of the function V_{cl} . Thus, the max-type construction requires that each subsystem be decaying when its state is large. However, this property is not guaranteed when a subsystem is iISS.

It is stressed that the property (33) is not a necessary condition for the stability of the network Σ . It is rather a fundamental limitation of the max-type construction of Lyapunov functions. In contrast to the max type construction, the sum-type construction presented in Section 4 can lead us to the stability of the network Σ even if the property (33) is violated. In fact, in the case of $n = 2$, the inequality (24) can be satisfied even if $\gamma_{12}(\infty) > \alpha_1(\infty)$ as long as $\gamma_{21}(\infty) < \alpha_2(\infty)$. In the same way, the inequality (25) can be satisfied even if $\gamma_{21}(\infty) > \alpha_2(\infty)$ as long as $\gamma_{12}(\infty) < \alpha_1(\infty)$.

If we restrict our attention to networks of ISS subsystems, we can derive the dissipation inequality (3) for the stability of Σ based on the max-type Lyapunov function. Using the mapping from $\mathbb{R}_+ \rightarrow \mathbb{R}_+^n$ defined by

$$\Psi(\tau) = [\psi_1(\tau), \psi_2(\tau), \dots, \psi_n(\tau)]^T, \quad (36)$$

the following demonstrates this fact.

Theorem 9 *Suppose that there exist continuous functions $\lambda_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+, i = 1, 2, \dots, n$, such that (5), (6) (7) and*

$$D \circ \Gamma(\Psi(\tau)) \leq A(\Psi(\tau)), \quad \forall \tau \in \mathbb{R}_+ \quad (37)$$

are satisfied for some $\beta_1, \beta_2, \dots, \beta_n \in \mathcal{K}_\infty$. Then the network Σ is iISS with respect to input r and state x . If (18) and (19) are satisfied additionally, the network Σ is ISS. Furthermore, an iISS (ISS) Lyapunov function is given by (29).

Proof. Suppose that $\psi_i \in \mathcal{K}_\infty, i = 1, 2, \dots, n$ fulfill all the requirements in Theorem 9. Assume for the moment that, for $x \neq 0$, the maximum in (29) is attained uniquely by the $i = p \in \{1, 2, \dots, n\}$, i.e.,

$$\psi_p^{-1}(V_p(x_p)) > \psi_j^{-1}(V_j(x_j)), \quad \forall j \neq p. \quad (38)$$

Let $[\Gamma(s)]_p$ denote the p -th component of the vector $\Gamma(s)$. Then, for $V_{cl}(x)$ defined in (29), the inequality (12) yields

$$\begin{aligned} \dot{V}_{cl}(x) &\leq \lambda_p(V_p(x_p)) \left\{ -(\mathbf{Id} + \varepsilon_p)^{-1} \alpha_p(V_p) \right. \\ &\quad \left. + [\Gamma(V)]_p \right\} + \eta_p(|r|) \kappa_p(|r|). \end{aligned} \quad (39)$$

Since the definition of Γ and (38) ensure

$$\begin{aligned} [\Gamma(V)]_p &= [\Gamma([\psi_1 \circ \psi_1^{-1}(V_1(x_1)), \dots, \psi_n \circ \psi_n^{-1}(V_n(x_n))])]_p \\ &\leq [\Gamma(\Psi \circ \psi_p^{-1}(V_p(x_p)))]_p, \end{aligned}$$

we obtain

$$\begin{aligned} \dot{V}_{cl}(x) &\leq \lambda_p(V_p(x_p)) \left\{ -(\mathbf{Id} + \varepsilon_p)^{-1} \alpha_p(\psi_p \circ \psi_p^{-1}(V_p(x_p))) \right. \\ &\quad \left. + [\Gamma(\Psi \circ \psi_p^{-1}(V_p(x_p)))]_p \right\} + \eta_p(|r|) \kappa_p(|r|) \end{aligned}$$

from the definition of A . Now, let $\theta_p \in \mathcal{K}_\infty$ be computed with (20). Pick $\zeta_p \in \mathcal{K}_\infty$ satisfying $\theta_p - \zeta_p \in \mathcal{K}_\infty$. From the p -th row of (37), $\psi_p \in \mathcal{K}_\infty$ and (21) it follows that

$$\dot{V}_{cl}(x) \leq -\lambda_p(V_p(x_p)) [(\theta_p - \zeta_p) \circ \alpha_p(V_p(x_p))] + \hat{\kappa}_p(|r|)$$

holds for $\hat{\kappa}_p := \eta_p \kappa_p \in \mathcal{K} \cup \{0\}$. Therefore, there exists $\hat{\alpha}_i \in \mathcal{P}$ such that

$$\dot{V}_{cl}(x) \leq -\hat{\alpha}_p(V_p(x_p)) + \hat{\kappa}_p(|r|) \quad (40)$$

is satisfied. The functions $(\theta_p - \zeta_p) \circ \alpha_p$ and $\hat{\alpha}_p$ are of class \mathcal{K}_∞ if (18) and (19) hold. Repeating (40) for $p \in \{1, 2, \dots, n\}$ and using $V_{cl}(x) = \psi_p^{-1}(V_p(x_p))$ implied by (38), we have

$$\dot{V}_{cl}(x) \leq -\min_i \hat{\alpha}_i \circ \psi_i(V_{cl}(x)) + \max_i \hat{\kappa}_i(|r|) \quad (41)$$

for all $x \in \mathbb{R}^N$ where the maximization in (29) is uniquely defined. The set of such points is an open and dense in \mathbb{R}^N . For the rest of the proof, we can employ the arguments in [4,3,9]. Since the locally Lipschitz continuous function V_{cl} is the maximization of \mathcal{C}^1 functions V_i , the Clarke subgradient of V_{cl} in $x \in \mathbb{R}^n$ can be

computed by the set

$$\partial_{Cl}V_{cl}(x) = \text{conv} \left\{ \nabla (\sigma_i^{-1} \circ \alpha_i \circ V_i)(x_i) : \sigma_i^{-1} \circ \alpha_i(V_i(x_i)) = V_{cl}(x) \right\},$$

where $\text{conv} \{ \cdot \}$ denotes the convex hull. As we have (41) for each of the extremal points of $\partial_{Cl}V_{cl}(x)$, the dissipation inequality (41) holds in terms of the Clarke generalized derivative for each ζ in the Clarke subgradient. Thus, the function V_{cl} given in (29) is a Lipschitz continuous iISS (ISS) Lyapunov function for the network Σ . \square

Although Theorem 9 does not explicitly state that each subsystem Σ_i is assumed to be ISS with respect to input x_j , $j \neq i$ and state x_i , it is imposed implicitly. Since ψ_i 's are class \mathcal{K}_∞ functions, the condition (37) implies (33), which amounts to the ISS of Σ_i , $i = 1, 2, \dots, n$, [25]. This fact is consistent with Theorem 8. In contrast to Theorem 9 of the max type, the sum-type construction presented in Section 4 can deal with iISS subsystems which are not ISS. The limiting value of (17) does not result in a restriction like (33) since the parameter $\Lambda(s)$ is ‘‘multiplied’’ on both sides of (17). It is worth mentioning that, to obtain the stability of the network Σ , some of the subsystems Σ_i is necessarily ISS but not all, which is proved for $n = 2$ in [14].

Theorem 9 does not require the α_i 's to be of class \mathcal{K}_∞ which are assumed in [5]. Although both Theorem 9 and the result in [5] deal with ISS subsystems, Theorem 9 allows us to get rid of transformation into $\alpha_i \in \mathcal{K}_\infty$ [24] which may give rise to unnecessary conservativeness in practice. The geometrical interpretation used in [5] for class \mathcal{K}_∞ α_i 's (originating from [9]) can be still applied to the condition (37) derived for class \mathcal{K} α_i 's. The vector-valued function $\Psi(\tau)$ is an infinite length path that starts at the origin for $\tau = 0$ and grows unboundedly as $\tau \rightarrow \infty$ inside the topological separator Ω^{--} defined by (26).

Now, we address the existence of such a path Ψ solving (37). The following theorem presents a condition guaranteeing the existence, which is a consequence of the results developed in [9].

Theorem 10 *Assume that α_i , $i = 1, 2, \dots, n$, are \mathcal{C}^1 class \mathcal{K}_∞ functions satisfying*

$$\frac{d}{d\tau} \alpha_i(\tau) > 0, \quad \forall \tau \in (0, \infty), \quad i = 1, 2, \dots, n \quad (42)$$

and that Γ is irreducible, [9]. Suppose that there exist $\beta_1, \beta_2, \dots, \beta_n \in \mathcal{K}_\infty$ satisfying

$$D \circ \Gamma(s) \not\geq A(s), \quad \forall s \in \mathbb{R}_+^n \setminus \{0\}. \quad (43)$$

Then, there exist continuously differentiable functions

$\psi_i \in \mathcal{K}_\infty$, $i = 1, 2, \dots, n$ such that (37) and

$$\frac{d}{d\tau} \psi_i(\tau) > 0, \quad \forall \tau \in (0, \infty), \quad i = 1, 2, \dots, n \quad (44)$$

are satisfied. Moreover, if

$$\lim_{\tau \rightarrow \infty} \frac{d}{d\tau} \alpha_i(\tau) > 0, \quad i = 1, 2, \dots, n \quad (45)$$

holds, the property

$$\limsup_{\tau \rightarrow \infty} \frac{d}{d\tau} \psi_i(\tau) < \infty, \quad i = 1, 2, \dots, n. \quad (46)$$

is achieved additionally.

Proof. By virtue of $\alpha_i \in \mathcal{K}_\infty$, the property (43) is equivalent to

$$D \circ \Gamma \circ A^{-1}(s) \not\geq s, \quad \forall s \in \mathbb{R}_+^n \setminus \{0\}. \quad (47)$$

The results in [9] with smoothing [10] guarantees the existence of \mathcal{C}^1 functions $\hat{\psi}_i \in \mathcal{K}_\infty$, $i = 1, 2, \dots, n$ satisfying

$$D \circ \Gamma \circ A^{-1}(\hat{\Psi}(\tau)) < \hat{\Psi}(\tau), \quad \forall \tau \in (0, \infty), \quad (48)$$

$$\frac{d}{d\tau} \hat{\psi}_i(\tau) > 0, \quad \forall \tau \in (0, \infty), \quad i = 1, 2, \dots, n, \quad (49)$$

where $\hat{\Psi} = [\hat{\psi}_1, \hat{\psi}_2, \dots, \hat{\psi}_n]^T$. Setting $\Psi(\tau) = A^{-1} \circ \hat{\Psi}(\tau)$, we arrive at (37). The properties (49) and (42) ensure the differentiability of ψ_i and (44). We obtain (46) from (45) if

$$\limsup_{\tau \rightarrow \infty} \hat{\psi}_i'(\tau) < \infty, \quad i = 1, 2, \dots, n \quad (50)$$

holds. If (50) is not satisfied by a particular $\hat{\Psi}$ achieving (48) and (49), we can always find a continuously differentiable $\rho \in \mathcal{K}_\infty$ such that replacing $\hat{\Psi}$ by $\hat{\Psi}(\rho)$ achieves (48), (49), and (50). To see this, define

$$F(\tau) := \max_i \hat{\psi}_i'(\tau).$$

Let G denote the antiderivative of F satisfying $G(0) = 0$. This function G is of class \mathcal{K}_∞ since we have

$$\hat{\psi}_1'(\tau) \leq F(\tau) \leq \sum_{i=1}^n \hat{\psi}_i'(\tau), \quad \forall \tau \in \mathbb{R}_+$$

for $\hat{\psi}_i \in \mathcal{K}_\infty$, $i = 1, 2, \dots, n$. Define $\rho(\tau) = G^{-1}(\tau)$ which is of class \mathcal{K}_∞ . Then, we have

$$\rho'(\tau) \cdot \psi_i'(\rho(\tau)) \leq \rho'(\tau) \cdot F(\rho(\tau)) = \frac{F(\rho(\tau))}{G'(\rho(\tau))} = 1 \quad i = 1, 2, \dots, n.$$

Therefore, the component functions of the vector $\hat{\Phi}(\rho)$ are of class \mathcal{K}_∞ and achieve (48), (49) and (50). \square

It is stressed that (7) is automatically satisfied by $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, \dots, n$. Note that the properties $\psi_i \in \mathcal{K}_\infty$, $i = 1, 2, \dots, n$, and (44) imply (6) and (5). The property (46) ensures (19). Hence, the above theorem guarantees the existence of solutions $\{\lambda_i\}$ to the problem posed by Theorem 9 in the case of $\alpha_1, \dots, \alpha_n \in \mathcal{K}_\infty$. Theorem 10 can be considered as the extension of the idea in Jiang et al. [15] for two subsystems to $n \geq 2$ subsystems. Theorem 10 also removes the technical assumption mentioned in Section 5 of [5]. The twofold assumption was undesirable since it is imposed on intermediate variables appearing in technical steps. The trick introduced in Section 3 to put aside the exogenous signals plays the key role in removing one part of the assumption. The other part is replaced by the explicit assumption (42) (and (45) in the presence of r).

Remark 11 When we only consider 0-GAS in Theorem 9 with $\alpha_1, \dots, \alpha_n \in \mathcal{K}_\infty$, the condition (37) can be replaced by

$$\Gamma(\Psi(\tau)) < A(\Psi(\tau)), \quad \forall \tau \in \mathbb{R}_+ \setminus \{0\} \quad (51)$$

in view of the proof of Theorem 9 with $\kappa_i = \eta_i = \zeta_i = \varepsilon_i = 0$ and $\theta_i \in \mathcal{P}$. In this case, the condition (43) is replaced by

$$\Gamma(s) \not\geq A(s), \quad \forall s \in \mathbb{R}_+^n \setminus \{0\}. \quad (52)$$

Remark 12 An operator Γ is irreducible if and only if the network is strongly connected in the sense of a directed graph. There are cascades in Σ if Γ is reducible. For such a network, we can apply Theorem 10 to each irreducible block, and then use the fact that cascades of ISS systems are ISS [24]. Alternatively, we can introduce sufficiently small γ_{ij} so that Γ becomes irreducible and the solvability of (43) remains unchanged.

Remark 13 For the 0-GAS, the inequality (51) of the existence of an omni-directionally unbounded path $\psi_i, \dots, \psi_n \in \mathcal{K}_\infty$ and the corresponding condition (43) of the topological separation with Ω^{--} can be obtained without resorting to the construction of a Lyapunov function V_{cl} of the network Σ . Indeed, in [20], these two conditions are derived in the framework of the monotone systems theory (i.e., Lemma 3.13 of [20]). Generalizing that kind of approach to the case of stability with respect to the external signals $r(t) \not\equiv 0$ is by no means easy as it has been commonly observed [1]. From (51) with $\psi_i \in \mathcal{K}_\infty$, $i = 1, 2, \dots, n$, the property $\lim_{s \rightarrow \infty} \alpha_i(s) > 0$, $i = 1, 2, \dots, n$, is derived in [20]. As discussed in Section 6, the condition (51) with $\psi_i \in \mathcal{K}_\infty$, $i = 1, 2, \dots, n$, is an excessive requirement in view of the 0-GAS of the network.

Remark 14 The property (52) obtained for 0-GAS and $\alpha_1, \dots, \alpha_n \in \mathcal{K}_\infty$ can be shown to be equivalent to the small-gain condition derived for ISS subsystems given in the implication form [9]. Since $\alpha_1, \dots, \alpha_n$ are of class \mathcal{K}_∞ , we can use $(\mathbf{Id} + \ell_i) \circ \alpha_i^{-1}(\sum_j s_j)$ as monotone aggregation functions (MAFs) for any $\ell_i \in \mathcal{P}$ such that $\mathbf{Id} + \ell_i \in \mathcal{K}_\infty$, $i = 1, 2, \dots, n$. Then, the components γ_{ij} of Γ become fictitious gains of the subsystems⁴. For this pair of a vector of MAFs and a gain matrix, the approach in [9] gives the sufficient condition $(\mathbf{Id} + L) \circ A^{-1} \circ \Gamma(s) \not\geq s$ for the 0-GAS of Σ , where $L(s) = [\ell_1(s_1), \dots, \ell_n(s_n)]^T$. From this condition and $\alpha_i \in \mathcal{K}_\infty$ the inequality (52) follows. If the components of Γ are unbounded, the fulfillment of (52) implies the existence of a function L as above achieving $(\mathbf{Id} + L) \circ A^{-1} \circ \Gamma(s) \not\geq s$. In the presence of an external input r , the preceding work [9] modifies (51) as (5.3) with an additional parameter representing the external input. The reduction to a matrix gain-like condition similar to (43) can be achieved under additional structural assumptions on the MAFs such as (5.10) and (M₄) of [9] or by decomposing the operator $A^{-1} \circ \Gamma$ into components with some conservativeness. It is worth noting that the approach in [9] needs $A^{-1} \circ \Gamma$ instead of (43) with no inverse maps for constructing ψ_i . This is due to the fact that the function V_{cl} constructed in this paper satisfies the dissipation inequality (3) for the network Σ while the one in [9] does not necessarily satisfy (3). It is interesting that (43) can be shown to be equivalent to $D \circ A^{-1} \circ \Gamma(s) \not\geq s$ with respect to the existence of not necessarily identical $\beta_i \in \mathcal{K}_\infty$ under appropriate assumptions on A and Γ .

6 Discussions

6.1 Max vs. sum

The condition (43) derived for the max-type construction is identical to the topological separation condition (23) for the sum-type construction in the case of $n = 2$. Hence, according to Theorem 8 that demonstrates the fundamental limitation of the max-type construction, there appears to be a fatal gap between (43) and the unbounded path condition (37) requiring $\psi_i \in \mathcal{K}_\infty$ for all $i = 1, 2, \dots, n$ when the subsystems are only iISS. We do not have such a gap between (43) and the obtuse angle condition (17). To elaborate this point, recall the set Ω^{--} whose boundaries are given by $l_1: \alpha_1(s_1) = \gamma_{12}(s_2)$ and $l_2: \alpha_2(s_2) = \gamma_{21}(s_1)$ on the s_1 - s_2 plane (see Fig.2). The unboundedness of Ω^{--} in the s_{3-k} direction is equivalent to the ISS property of Σ_k since the unboundedness is identical with $\alpha_k(\infty) \geq \gamma_{k,k-3}(\infty)$ and we can invoke [25]. Figure 2 (ii) illustrates the case where only Σ_2 is ISS and Ω^{--} is unbounded only in the s_1 direction. The topological separation property (43) does not

⁴ Each row of Γ can be considered as a combined gain with respect to the output $\alpha_i(V_i)$.

necessarily imply that the set Ω^{--} is unbounded in both s_1 and s_2 directions. However, Theorem 8 has demonstrated that, if we use the max-type Lyapunov function (29), the property (33) must hold. This means that the separator Ω^{--} is necessarily unbounded in both s_1 and s_2 directions as in Fig.2 (i). Indeed, it is straightforward to verify that the sufficient condition (37) using $\psi_1, \psi_2 \in \mathcal{K}_\infty$ for the max-type Lyapunov function (29) requires the set Ω^{--} to be unbounded in both s_1 and s_2 directions. Thus, the condition (37) with $\psi_1, \psi_2 \in \mathcal{K}_\infty$, i.e., the omni-directionally unbounded path condition, is much more demanding than the topological separation (43). In contrast, the obtuse angle condition (17) for the sum-type construction allows Ω^{--} to be bounded in some s_i direction. Indeed, Theorem 4 proves that one subsystem Σ_k of Σ_1 and Σ_2 is allowed to be non-ISS, which is identical to the boundedness of Ω^{--} in the s_{3-k} direction. In such a case, the condition (37) cannot be met if we require $\psi_{3-k} \in \mathcal{K}_\infty$.

In this way, whether the separator Ω^{--} is allowed to be bounded in a direction and whether $\psi_i \notin \mathcal{K}_\infty$ is allowed capture the difference between the capability of the sum-type and max-type construction of Lyapunov functions. As far as ISS subsystems are concerned, such a difference does not appear. As a matter of fact, if we restrict our attention to networks of two ISS subsystems, we can summarize as follows:

Proposition 15 *Let $n = 2$. Assume $\alpha_1, \alpha_2 \in \mathcal{K}_\infty \cap \mathcal{C}^1$, (42) and (45). Suppose that there exist $\beta_1, \beta_2 \in \mathcal{K}_\infty$ satisfying (23). Then, the following hold:*

- (i) *There exist continuous functions $\lambda_1, \lambda_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that (5), (6) (17) and (19) are satisfied.*
- (ii) *There exist continuous functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that (5), (6), (37) and (19) are satisfied.*

This above is a direct consequence of Theorems 4 and 10. The property (7) is satisfied by $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. As Proposition 15 demonstrates, under the unified condition (23), we can obtain ISS Lyapunov functions for the network Σ of ISS subsystems based on the two approaches. The extension of this unification to the general n subsystems case has not yet been accomplished. Proposition 15 has been first proved for the linear case with the general n in [5]. Indeed, in the case of linear A, Γ, D , both the problems posed in Theorem 3 and Theorem 9 can be solved by theorems of the Perron-Frobenius type. A necessary and sufficient condition for the solvability is $\rho(\Gamma A^{-1}) < 1$, where $\rho(\cdot)$ denotes the spectral radius [5]. The functions Λ and Ψ are obtained as a suitable left-eigenvector and a right eigenvector, respectively.

6.2 Translating solutions for ISS subsystems

This section shows a technique to compute Λ for the sum-type construction by making use of a solution Ψ

to the max-type construction in the case of $n = 2$. It is stressed that, as demonstrated by Theorem 8, such translation is possible only when all the subsystems are ISS. For simplicity, we consider

$$\beta_i(\tau) = c_i \tau, \quad i = 1, 2. \quad (53)$$

The following presents a formula for the conversion.

Theorem 16 *Let $n = 2$. Assume $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Suppose that there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that (37) is satisfied with (53) for some $c_1, c_2 > 2$. Then, the choice*

$$\Lambda(s) = \begin{bmatrix} \lambda_1(s_1) \\ \lambda_2(s_2) \end{bmatrix} = \begin{bmatrix} -P_2 \circ \psi \circ \psi_1^{-1}(s_1) \\ -P_1 \circ \psi \circ \psi_2^{-1}(s_2) \end{bmatrix}, \quad (54)$$

$$P(s) = \begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix} = -D_H^{-1} \circ A(s) + \Gamma(s), \quad D_H(s) = \begin{bmatrix} \frac{c_1}{2} s_1 \\ \frac{c_2}{2} s_2 \end{bmatrix}$$

satisfies (5), (7), (17) and (19) with (53) for another pair of $c_1, c_2 > 1$.

Proof. The property (37) and the definition of D_H imply

$$P(\psi(\tau)) \leq -\Gamma(\psi(\tau)) \leq 0, \quad \forall \tau \in \mathbb{R}_+. \quad (55)$$

Let $\tau_i = \psi_i^{-1}(s_i)$ for each $i = 1, 2$. Then,

$$\Lambda(s)^T P(s) = -P_2(\psi(\tau_1)) P_1 \left(\begin{bmatrix} \psi_1(\tau_1) \\ \psi_2(\tau_2) \end{bmatrix} \right) - P_1(\psi(\tau_2)) P_2 \left(\begin{bmatrix} \psi_1(\tau_1) \\ \psi_2(\tau_2) \end{bmatrix} \right).$$

If $P_i([\psi_1(\tau_1), \psi_2(\tau_2)]^T) \leq 0$ holds for $i = 1, 2$, the property (55) yields

$$\Lambda(s)^T P(s) \leq 0 \quad \text{for } s = \begin{bmatrix} \psi_1(\tau_1) \\ \psi_2(\tau_2) \end{bmatrix}. \quad (56)$$

We next assume that $P_2([\psi_1(\tau_1), \psi_2(\tau_2)]^T) > 0$ and

$$P_2(\psi(\tau_1)) + P_2 \left(\begin{bmatrix} \psi_1(\tau_1) \\ \psi_2(\tau_2) \end{bmatrix} \right) \leq 0 \quad (57)$$

hold. Using $P_1(\psi(\tau_2)) \leq 0$ guaranteed by (55) again, we have

$$\Lambda(s)^T P(s) \leq -P_2(\psi(\tau_1)) \left[P_1 \left(\begin{bmatrix} \psi_1(\tau_1) \\ \psi_2(\tau_2) \end{bmatrix} \right) - P_1(\psi(\tau_2)) \right].$$

Since combination of $P_2([\psi_1(\tau_1), \psi_2(\tau_2)]^T) > 0$ and (55) implies $\tau_1 > \tau_2$, we have

$$P_1\left(\begin{bmatrix} \psi_1(\tau_1) \\ \psi_2(\tau_2) \end{bmatrix}\right) - P_1(\psi(\tau_2)) < 0.$$

Thus, we arrive at (56). In the same manner, we obtain (56) in the case where $P_1([\psi_1(\tau_1), \psi_2(\tau_2)]^T) > 0$ and

$$P_1(\psi(\tau_2)) + P_1\left(\begin{bmatrix} \psi_1(\tau_1) \\ \psi_2(\tau_2) \end{bmatrix}\right) \leq 0 \quad (58)$$

hold. Note that $P_i([\psi_1(\tau_1), \psi_2(\tau_2)]^T) > 0$ cannot hold for $i = 1, 2$ simultaneously due to (55) and the definition of P_i . The properties (58) and (57) are implied by

$$P_1(\psi(\tau_2)) + P_1\left(\begin{bmatrix} 0 \\ \psi_2(\tau_2) \end{bmatrix}\right) \leq 0,$$

$$P_2(\psi(\tau_1)) + P_2\left(\begin{bmatrix} \psi_1(\tau_1) \\ 0 \end{bmatrix}\right) \leq 0,$$

respectively. The above pair is again guaranteed by (55). Therefore, we have reached

$$\Lambda(s)P(s) \leq 0, \quad \forall s \in \mathbb{R}_+^2$$

with $c_1/2 > 1$ and $c_2/2 > 1$, which corresponds to (17). Finally, the first inequality in (55) and $\psi_i \in \mathcal{K}_\infty$ imply (5) and (19). The definition of P yields (7). \square

Although different pairs of solutions are available in [11,14], making a choice from many Lyapunov functions is sometimes advantageous in systems analysis and design. The solutions $\{\lambda_1, \lambda_2\}$ presented in [11,14] are better than the solutions obtained through Theorem 16 in the sense not only that $c_1, c_2 > 1$ for (37) is satisfactory in [11,14], but also that the solutions in [11,14] can establish the stability of the network even when some subsystems are only iISS.

7 Concluding Remarks

This paper has demonstrated that the sum-type construction not only provides us with continuously differentiable Lyapunov functions directly for networks, but also covers the class of iISS subsystems which are not ISS, while the max-type construction based on Lipschitz continuous Lyapunov functions requires the subsystems to be ISS. It should be stressed that this is neither a limitation of the small-gain approach nor a necessary condition for the stability of the network. The ISS requirement is the fundamental limitation of the max-type way to construct Lyapunov functions. Solutions fulfilling the

stability condition derived in the max-type construction are available in [15] for two subsystems, and computable by utilizing [9,18] for n subsystems. In contrast, solving the stability condition in the sum-type construction has been harder, and the formulas in [11,14] apply only to $n = 2$. However, the sum-type Lyapunov function can actually establish stability of the network even when some subsystems are not ISS. This paper has demonstrated the relationship between the nonlinear transformations in the sum-type and the max-type construction. In the $n = 2$, it has been proved that restricting to ISS subsystems allows the solvability conditions in the two types of construction to agree with each other. Finally, it is worth mentioning that Theorem 4 and Proposition 15 (i) can be also proved for $n > 2$ by making use of the result reported recently in [12] if Σ is restricted to a cycle network. They are omitted since their proof need large space. Generalization of Theorem 4 and Proposition 15 (i) to networks in general structure is not known and it is an interesting topic of future research.

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