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An Approach to Supply Chain Design based on Robust Capacity Allocation of Production Locations

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Abstract

Global supply chains are complex dynamical systems. Due to nonlinear dependencies between production locations, already a perturbation at one location can change the dynamic behavior of the whole supply chain. As a consequence the supply of customers in time may be at risk. In this paper we present an approach to supply chain design that maximizes the robustness of a supply chain in regard to perturbations of the production processes. To this end we consider the fluid approximation of a dynamic supply chain and assume processor sharing as the production strategy applied at the various locations. The robustness of the supply chain can be measured by the stability radius that reflects the smallest perturbation that destabilizes the network. Based on results concerning the stability radius we set up an optimization problem for the capacity allocation at each production facility. The capabilities of the approach are demonstrated using a test case.

Keywords: Supply Chain Design, Capacity Management, Global Supply Chains, Integrated Logistics

Track: Global Operations, Strategic Supply Chain Management

1 Introduction

In the last decades several factors (e.g. technological innovation, advanced information and communication technologies as well as logistics services available world-wide) have driven the development towards global supply chains. These large-scale networks are composed of external suppliers, several plants manufacturing intermediate and/or finished products, distribution and/or sales centers and transportation assets. In addition to this structural complexity the underlying procurement, manufacturing and distribution processes are dynamic. The resulting behavior of the network is often complex and shaped by nonlinear dynamics. Furthermore, the evolution of internal and external parameters that determines structural and dynamic properties of global supply chains is not certain. Especially in the case of supply chain design the anticipated customer demand as well as the dynamics of the supply chain have a significant impact on the performance and sustainability of the network. A robust supply chain design is given by the capability of the network to cope with several possible future scenarios in an efficient manner. In this context not only locations and transportation links of the supply chain are chosen but also production capacities at each location for the processed products. Stochastic programming and robust optimization are two methods in order to the set up a robust plan. Nevertheless the planning result strongly depends on the arbitrary chosen deterministic future scenarios. In this paper we present a new approach to capacity allocation of production locations during the process of supply chain design. To this end we consider a fluid approximation of a stochastic multiclass queueing network that can be used to model supply chains. The fluid model facilitates stability and robustness analysis for such networks. In particular we use the stability radius to quantify the robustness. The stability radius is given by the smallest perturbation that destabilizes the network. During the process of network design we aim to maximize the robustness of a supply chain in regard to perturbations of the production processes. Hence, we seek for a capacity allocation that enables the network to handle more workload than anticipated. This can be modeled by an increased customer demand. Based on findings in regard to the stability radius we set up a mathematical program formulation that allows to find an optimal production capacity allocation.

The outline of the paper is as follows. In Section 2 we provide a literature review that comprises applied planning systems and methods as well as an introduction to a fluid network model under proportional processor sharing. This is complemented by an illustrative example, that is used throughout the paper in order to explain the notions and to valided the results with a test case. Section 3 presents the theoretical basis for the mathematical optimization formulation that is used to derive a robust capacity allocation during the network design process. Here, stability of fluid models is defined and the essential stability characterization by the nominal workload condition is explained. Moreover, the stability radius is defined and calculated for the test scenario. The optimization problem is introduced in Section 4 and followed by the computational analysis in Section 5. The paper closes with some conclusions and an outlook.

2 Literature review

2.1 Advanced planning systems and methods

A sustainable creation of value is paramount aim of global supply chains and mainly determined by the network design. This goal is fostered by advanced planning systems (APS) that are applied to such networks. The underlying structure of APS's is illustrated by the Supply Chain Planning Matrix [15] (Figure 1).

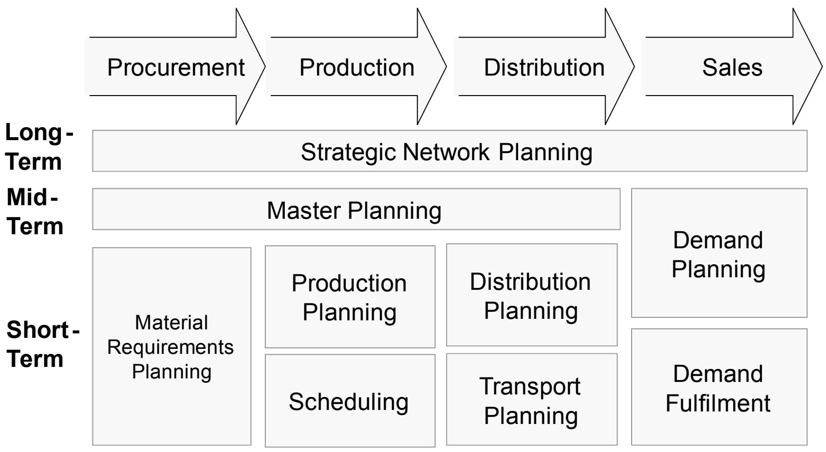

Figure 1: Supply Chain Planning Matrix.

The matrix comprises modules for the planning tasks that are characterized by time horizon and involved business functions. The degree of detail increases and the planning horizon decreases by shifting from the long-term to the short-term. These consecutive modules allow to handle the complexity of an overall planning process [6]. The modules are often based on mathematical programming formulations or heuristics that assume deterministic planning information [16]. However, it can be shown that such approaches fail to cope with a dynamic environment and the considerable uncertainty of the underlying planning information [12]. In particular strategic network planning depends on the evolution of customer demand and the duration of the product life cycle. As a consequence the supply of customers in time may be at risk when either the network configuration or the chosen production capacities are insufficient. Stochastic programming and robust optimization address uncertainty of relevant parameters [13] and are used for robust planning. They are based on a set of deterministic future scenarios of relevant parameters [8] instead of point estimates. In this context a plan is considered robust as it remains close to a desired solution for each scenario. These approaches are applied to network design problems [10], [14], [18], [7] and master planning [1], [4]. Nevertheless, the obtained planning results depend on the arbitrary chosen scenarios and do not pay attention to the dynamic properties of the network.

2.2 Description of the fluid model

In this paper we present an approach to supply chain design that maximizes the robustness of a supply chain in regard to perturbations. This approach incorporates knowledge about the dynamic behavior of supply chains as well as a consideration of uncertainty. The dynamics of such complex production networks can be modelled by multiclass queueing networks. Roughly speaking queueing networks are stable, if the queue length process remains bounded for all times. Dai presented in [5] an approach to investigate the stability of queueing networks using the so-called fluid limit model. This fluid approximation model is a continuous deterministic analogue of the discrete stochastic model. The stability of a corresponding fluid limit model implies the stability of the original queueing network [5]. In comparison to a queueing model the stability of a fluid model can be determined more easily. Since the evolution of relevant external and internal parameters is not know prior we embed a measure for robustness of a fluid model with respect to perturbations of the customer demand into our planning approach. First capabilities of this measure were presented in [17]. There the concept of the stability radius for dynamical systems [9] is adapted to the case of fluid network models. Perturbations that exceed the stability radius might lead to instability of the network. In particular this means that the customer demand cannot be fulfilled. To this end we consider the fluid approximation of a dynamic supply chain and assume processor sharing as the production strategy applied at the various locations. The following model description is a composition of [3, 19]. The considered network consists of locations $\mathcal{S}_j$ with $j \in \mathcal{J} = \{1, 2, ..., J\}$ and different types of products $\mathcal{P}_k$ with $k \in \mathcal{K} = \{1, 2, ..., K\}$. Every type of product is processed exclusively at one location. The mapping $s : \{\mathcal{P}_1, ..., \mathcal{P}_K\} \longrightarrow \{\mathcal{S}_1, ..., \mathcal{S}_J\}$ determines which type of product is processed by which location and generates the so-called constituency matrix C, with $c_{jk} = 1$ if $s(\mathcal{P}_k) = \mathcal{S}_j$ or $c_{jk} = 0$ else. For every location the set $C(\mathcal{S}_j) := \{\mathcal{P}_k \in \{\mathcal{P}_1, ..., \mathcal{P}_K\} : s(\mathcal{P}_k) = \mathcal{S}_j\}$ is nonempty. Further every type of product $\mathcal{P}_k$ has the exogenous arrival rate α_k and the process rate μ_k of products per time unit. After a product of type $\mathcal{P}_k$ has been processed at location $s(\mathcal{P}_k)$ it either leaves the network or becomes a product of another type $\mathcal{P}_l$, with $l \in \mathcal{K}$. Further p_{lk} denotes the proportion of processed products of type $\mathcal{P}_l$ that become products of type $\mathcal{P}_k$. Hence $1 - \sum_{l=1}^{K} p_{lk}$ is the part that leaves the network. The corresponding $K \times K$ matrix P is referred to as the transition matrix. It is assumed P has (the) spectral radius strictly less than one, i.e. all products leave the network. The initial amount of products is represented through the K dimensional vector $Q(0)$. The model of the network is given by (α, μ, P, C) and $Q(0)$.

The performance is described by the K dimensional product level process $\{Q(t) : t \geq 0\}$ and the K dimensional allocation process $\{T(t) : t \geq 0\}$. The amount of products $\mathcal{P}_k$ in the network at time t is denoted by $Q_k(t)$ and the total amount of time in the interval $[0, t]$ that location $s(k)$ has devoted to processing products of type $\mathcal{P}_k$ is denoted by $T_k(t)$. For brevity and to keep a clear representation we omit the capital calligraphy letters in the subscript. The next step is to fix a policy that rules the order how the arriving products are processed at each location.

We use the so-called head-of-the-line proportional processor discipline (HLPPS). Under this discipline all nonempty product types present at a location are produced simultaneously proportional to their product level. Head-of-the-line means that a location processes only one product of each type at a each time. The location allocation rate $\dot{T}_k(t)$ for type k products is proportional to the fluid level of each product type k present at time t. That is,

$$\dot{T}_k(t) = \frac{Q_k(t)}{\sum_{l \in C(j)} Q_l(t)} \qquad \text{when} \qquad \sum_{l \in C(j)} Q_l(t) > 0.$$

We note that even when the location is empty, $\dot{T}_k(t)$ may still be positive. Finally the idle time process $Y = \{Y(t) : t \geq 0\}$ is introduced, i.e. $Y_k(t)$ denotes the cumulative time that location $s(\mathcal{P}_k)$ idles in the interval $[0, t]$. With $M = \text{diag}(\mu)$ the dynamics of the fluid network under HLPPS discipline can be summarized as follows

$$Q(t) = Q(0) + \alpha\, t - (I - P^T) M T(t) \geq 0, \tag{1}$$

$$W(t) = C\, M^{-1}\, Q(t), \tag{2}$$

$$Y(t) = et - C\, T(t), \tag{3}$$

$$Y_j(t) \text{ can only increase when } W_j(t) = 0, \text{ for all } j \in \mathcal{K}, \tag{4}$$

$$\dot{T}_k(t) = \frac{Q_k(t)}{\sum_{l \in C(j)} Q_l(t)} \qquad \text{when} \qquad \sum_{l \in C(j)} Q_l(t) > 0. \tag{5}$$

Equation (4) describes the work-conserving property of the network. That is, the idle time for a product type $\mathcal{P}_k$ increases if and only if $Q_k(t) = 0$, i.e. there is no product of type $\mathcal{P}_k$ in the network waiting for being processed. Relation (1) is called the flow balance relation. Any pair $(Q(t), T(t))$ that satisfies (1)-(4) is called a fluid solution of the HLPPS fluid network. The set of all feasible fluid level processes is denoted as

$$\Phi = \{Q(t) : \exists\, T(t) \text{ such that } (Q(t), T(t)) \text{ is a fluid solution}\}.$$

The total fluid mass in the network at time t is given by the sum of the fluid level processes of every type of product. Since the fluid level processes $Q_k(t)$ of each product type $\mathcal{P}_k$ is nonnegative we use the following notation

$$||Q(t)||_1 = \sum_{k=1}^{K} Q_k(t).$$

Example 1 *Throughout this paper we consider a network where three types of products are produced at two locations. A schematic illustration is given in Figure 2. The parameters for the test scenario are*

$$\alpha = \begin{pmatrix} 0.15 \\ 0.15 \\ 0.10 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0.6 \\ 0.9 \\ 0.5 \end{pmatrix}, \quad P = \begin{pmatrix} 0.25 & 0.15 & 0.20 \\ 0.05 & 0.25 & 0.15 \\ 0.20 & 0.25 & 0.10 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

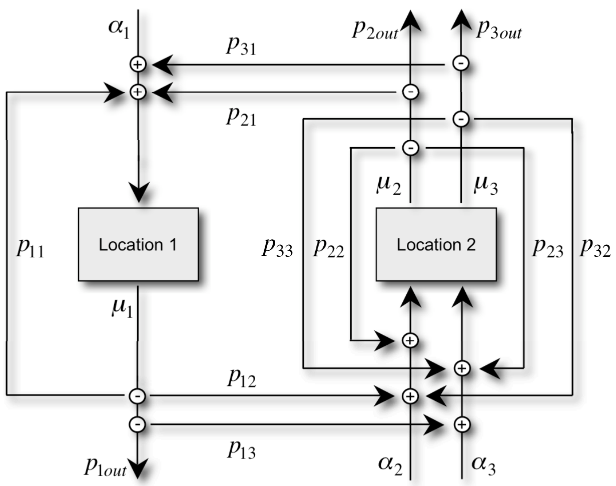

Figure 2: Fluid network with two locations under HLPPS discipline.

3 Stability and robustness analysis

Stability of a fluid network model provides a sufficient condition for the stability of the corresponding multiclass queueing network. Queueing networks are stable, if the queue length process remains bounded for all times. As fluid networks are deterministic there are three possibilities for the fluid level processes. It becomes (identically) zero past some finite time τ, remains constant or tends to infinity.

Definition 1 *A fluid network Φ is said to be stable, if there exists a finite time $\tau \geq 0$ such that*

$$Q(\tau + \cdot) \equiv 0$$

for any $Q(\cdot) \in \Phi$ with $||Q(0)||_1 = 1$.

As you can see in Figure 2 every location $\mathcal{S}_j$ has not only products to serve that are arriving from outside but also products that transit within the network. This so-called effective arrival rate for each class is denoted by λ_k and is given by

$$\lambda_k = \alpha_k + \sum_{l=1}^{K} \lambda_l \, p_{lk}. \tag{6}$$

As the spectral radius of P is less than one, this can be written as $\lambda = (I - P^T)^{-1} \alpha$. Corresponding to this location $\mathcal{S}_j$ has the so-called nominal workload

$$\rho_j = \sum_{k \in C(j)} \frac{\lambda_k}{\mu_k}, \tag{7}$$

that can be rewritten as $\rho := CM^{-1}\lambda$. Putting this together one obtains the following representation of the nominal workload that will be used throughout the paper

$$\rho = C \, M^{-1} \, (I - P^T)^{-1} \, \alpha. \tag{8}$$

Sometimes ρ is also referred to as the traffic intensity. Clearly a necessary condition for the stability of a fluid network is that for every location $\mathcal{S}_j$ the nominal workload is strictly less than one. By using the J-dimensional vector $e = (1, ..., 1)^T$ this is

$$\rho < e. \tag{9}$$

Here $<$ has to be understood componentwise. Condition (9) is necessary for fluid networks under any service discipline. However, a sufficient condition depends on the service discipline, i.e. fluid networks may be stable under some discipline but not under another, see [11]. The following theorem states that condition (9) is also sufficient for HLPPS fluid networks [2].

Theorem 1 *A HLPPS fluid network is stable if and only if* $\rho < e$.

The nominal workload for the test scenario in Example 1 is

$$\rho = \begin{pmatrix} 0.5396 \\ 0.8217 \end{pmatrix}.$$

As you can see the nominal workload of station $\mathcal{S}_1$ is 54%. This suggests that this station can deal with a higher exogenous arrival rate than $\alpha_1 = 0.15$. But at the same time as α_1 increases the total arrival rates λ_2 and λ_3 for the product types $\mathcal{P}_2$ and $\mathcal{P}_3$ are increasing as well as the nominal workload ρ_2 of station $\mathcal{S}_2$. For this reasoning there are disturbances for α_1 that are manageable for the network, but the application of this disturbances to α_2 or α_3 might already lead to instability. The stability radius quantifies the minimal magnitude of admissible perturbations of the external arrival rate that destabilize the network. To obtain this we perturb the arrival rate by adding a vector $\delta \in \mathbb{R}^K_+$ and consider the fluid network $(\alpha_\delta, \mu, P, C)$, where $\alpha_\delta := \alpha + \delta$. The quantity of interest is then $||\delta||_1$, i.e. the smallest perturbation that already destabilizes the network. Thus according to [17] we define the following.

Definition 2 *The stability radius of the network* (α, μ, P, C) *is*

$$r(\alpha, \mu, P, C) = \inf\{\ ||\delta||_1 : (\alpha_\delta, \mu, P, C) \text{ is not stable } \}. \tag{10}$$

To make presentation shorter we use the notion

$$\rho(\delta) = C\, M^{-1}\, (I - P^T)^{-1}\, (\alpha + \delta). \tag{11}$$

Since the condition (9) is necessary and sufficient for the stability for HLPPS fluid networks the stability radius can be described by

$$r(\alpha, \mu, P, C) = \min \{\ ||\delta||_1\ :\ \rho(\delta) \not< 1\ \}\,. \tag{12}$$

Here $\not<$ means that there exists at least one component of ρ that is greater or equal than one. This reflects the fact that for some location $\mathcal{S}_j$ the nominal workload is at least one and thus the network is unstable.

To give a geometric interpretation of the previous definition and the equivalent representation (12) of the stability radius we focus on location $\mathcal{S}_2$ from Example 1. That is, we consider one location that processes two types of products. In Figure 3 the light grey domain represents the set of all arrival rates that satisfy condition (9). So (α_2, α_3) of the test scenario is an interior point of the light gray domain. The dark grey picmented domain represents the largest neighborhood $B_r(\alpha)$ around α that is completely in the light grey domain, where the radius of $B_r(\alpha)$ is the stability radius. To be precise, for any arrival rate from the interior of $B_r(\alpha)$ the network is stable, while for arrival rates on the boundary of $B_r(\alpha)$ the stability of the network cannot be guaranteed.

To calculate the stability radius we look for the smallest disturbance $\delta \in \mathbb{R}^K_+$ such that $\rho_j(\delta) \geq 1$ for at least one location $\mathcal{S}_j$, i.e.

$$\max_{j \in \mathcal{J}} \rho_j(\delta) \geq 1. \tag{13}$$

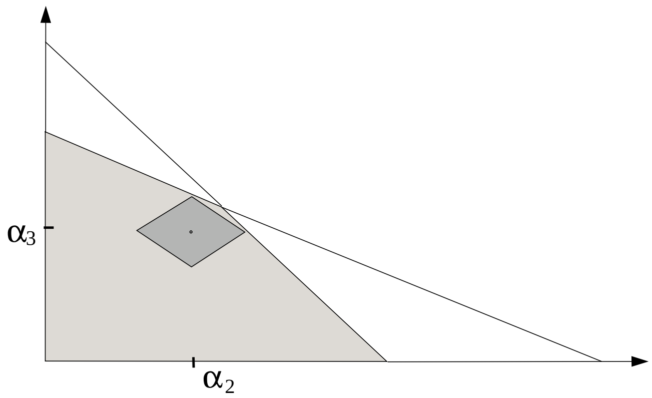

Figure 3: Illustration for the stability radius for one station serving two product types.

Consequently the stability radius can be calculated by the following optimization problem

$$\min \sum_{k=1}^{K} \delta_k \tag{14}$$

$$\text{such that } \max_{j \in \mathcal{J}} \rho_j(\delta) \geq 1. \tag{15}$$

Applying this scheme to the test scenario of the present paper leads to the stability radius $r(\alpha, \mu, P, C) = 0.0547781$ and a corresponding additional arrival rate $\delta = (0, 0, 0.0547781)^T$.

Remark 1 *This scheme enables us to quantify for a given network, i.e. for given α, μ, P and C the robustness of the network with respect to perturbation of the exogenous arrival rates. Further we note that the stability radius is a property of a given system. Consequently different values of α, μ, P or C lead to different stability radii.*

The goal of this paper is to use this scheme to derive for a network with given topology, represented by P, C, and given arrival rate α an capacity allocation setup such that the magnitude of the admissible perturbations of the arrival rate is maximized. That is, we aim to derive service rates μ_* such that the corresponding stability radius $r(\mu_*)$ has the largest value. So we have to solve the following optimization problem

$$\max\ r(\mu) \tag{16}$$

$$\text{such that } C\,M \leq z, \tag{17}$$

for a given vector $z \in \mathbb{R}_+^J$ that describes the capacity bounds of the locations in the network. As the objective function $r(\mu)$ itsself is a solution of an optimization problem we need to derive a way to calculate the stability radius that depends on μ. So we look again at the optimization problem (14), (15). For brevity we denote $B(\mu) = C\,M^{-1}(I - P^T)^{-1}$. Then the workload condition $\rho(\delta) \not< e$ can be written as

$$B(\mu)\delta \not< e - B(\mu)\alpha. \tag{18}$$

Since the $J \times K$ matrix $B(\mu)$ contains only nonnegative entries the left hand side of the above conditions is a weighted sum in δ, with nonnegative weights. So a perturbation with minimal 1-norm is of the form $\delta_* = (0, ..., 0, \delta_k, 0, ..., 0)^T$ for some $k \in \mathcal{K}$. The fact that δ_* satisfies (18) can be written as

$$\max_j \{B(\mu)_{jk}\delta_k\} \geq 1 - (B(\mu)\alpha)_j\,. \tag{19}$$

Thus another representation of the stability radius, where the dependency on μ is expressed, is

$$r(\mu) = \min_{k\in\mathcal{K}} \left\{ \frac{1 - (B(\mu)\alpha)_j}{\max_j B(\mu)_{jk}} \right\}. \tag{20}$$

4 Optimization model for the capacity allocation

In this section we setup an optimization model to maximize the stability radius. That is, for a network with given α, P and C the goal is to a find process rates μ_* such that the stability radius of the corresponding network (α, μ_*, P, C) has the largest stability radius. Based on the derived representation (20) of the stability radius that depends on μ, we make the following assumptions.

4.1 Assumptions

The stability radius is given by the smallest positive perturbation that when added solely to the arrival rate of a product type leads to instability. This holds no matter which arrival rate is disturbed. The total production capacity of each location is limited and is shared between the different product types. We aim to find a capacity allocation such that the stability radius corresponding to this network is maximized under all feasible processing rates.

4.2 Nomenclature

Sets

K product types

J locations / production facilities

set$C_{j,k}$ set, that determines which product type k is served at which location j

Parameters

α_k External arrival rate of products of type k

b_j Maximal Service rate of production facility j that is available for servicing products assigned to the considered production facility

$P_{l,k}$ Routing matrix that determines for product type l which kind of product type k it becomes after it is processed

$C_{j,k}$ Constituency matrix, that determines which product type is served be a certain production facility

$I_{l,k}$ Identity matrix of product types

R Inverse matrix of $(I - P^T)$

z Capacity allocation bound

L Large scalar (big M)

Variables

μ_k Service rate of product k at the assigned production facility

Δ Maximal perturbation of the arrival rate of each product type in the case that the other product types are not disturbed (measure for the stability radius)

$\rho_{j,k}$ Workload of each production location j in the case that the arrival rate α_k of product type k is disturbed

$CM_{j,k}$ Auxiliary matrix CM^{-1}

$A_{j,k}$ Auxiliary matrix $CM^{-1}(I - P^T)^{-1}$

Binary variables

$X_{j,k}$ Binary variable denoting that production location j has nominal workload $\rho_j = 1$ if the arrival rate α_k of product type k is disturbed

4.3 Mathematical model

First the two auxiliary matrices are calculated, where $A_{j,k}$ describes the relation between the arrival rate α and the nominal workload ρ, see (8).

$$CM_{j,k} = C_{j,k}\mu_k^{-1} \qquad (j \in \mathcal{J}, k \in \mathcal{K}) \tag{21}$$

$$A_{j,k} = \sum_j CM_{j,k} R_{l,k} \qquad (j \in \mathcal{J}, k \in \mathcal{K}) \tag{22}$$

The workload of each production facility is given as follows if the arrival rate of product type k is disturbed.

$$\rho_{j,l} = \sum_k A_{j,k}(\alpha_k + \Delta \cdot \delta_{lk}) \qquad (j \in \mathcal{J}, l \in \mathcal{K}) \tag{23}$$

Here we used the notation of the so-called Kronecker delta

$$\delta_{lk} = \begin{cases} 1 & \text{for } l = k \\ 0 & \text{else.} \end{cases}$$

Equation (24) enforces that the nominal workload of location j is greater or equal to one, if and only if exactly the exogenous arrival rate of product type k is disturbed.

$$\rho_{j,k} \geq 1 - (1 - X_{j,k})L \qquad (j \in \mathcal{J}, k \in \mathcal{K}) \tag{24}$$

Equation (25) ensures that the nominal workload does not exceed 1.

$$\rho_{j,k} \leq 1 \qquad (j \in \mathcal{J}, k \in \mathcal{K}) \tag{25}$$

Condition (26) guarantees that the nominal workload of exactly one location equals one for a perturbation with exactly one strictly positive component.

$$\sum_j \sum_k X_{j,k} = 1 \tag{26}$$

The allocation capacity of each location j is bounded by z_j.

$$\sum_{k \in \text{set}C_{j,k}} \mu_k \leq z_j \qquad (j \in \mathcal{J}) \tag{27}$$

The objective function maximize the perturbation that can be added to each arrival rate individually before the whole network becomes unstable. This number reflects the stability radius.

$$\max \Delta \tag{28}$$

5 Computational analysis

We applied the optimization model from Section 4 to the introduced test case. The parameters of the network are chosen as follows.

$$\alpha = \begin{pmatrix} 0.15 \\ 0.15 \\ 0.10 \end{pmatrix}, \quad P = \begin{pmatrix} 0.25 & 0.15 & 0.20 \\ 0.05 & 0.25 & 0.15 \\ 0.20 & 0.25 & 0.10 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and $L = 1000$. In the case that the production capacity allocation is pre-given ($\mu = (0.6,\, 0.9,\, 0, 5)^T$) the stability radius of the test case is $\Delta = 0.055$. Table 1 shows the workload $\rho_{j,k}$ of each production location j in regard to the disturbed product type k. Column one shows for instance the workload of the two production locations if only product type 1 is disturbed by Δ. Since the stability radius reflects the smallest perturbation that leads to instability Table 1 shows that the network becomes instable at location 2 if a perturbation of magnitude $\Delta = 0.055$ is added to the arrival rate of product type 3.

$\rho_{j,k}$	$k = 1$	$k = 2$	$k = 3$
$j = 1$	0.606	0.488	0.507
$j = 2$	0.899	0.949	1.000

Table 1.

The allocation of available production capacity to the different product types at a certain production location is performed by the mathematical program of Section 4. In the following we set the maximal production capacity of location 2 to 1.4, which equals the sum of the required production capacities of product type 2 and 3 in the fixed case. Furthermore we set the available production capacity at location 1 to 1. The obtained capacity allocation by the program is $\mu_1 = 0.397, \mu_2 = 0.722$ and $\mu_3 = 0,678$. Moreover the stability radius takes a value of 0.077. The workload of each of the production locations for the individually disturbed arrival rates of the production types are given by Table 2.

$\rho_{j,k}$	$k = 1$	$k = 2$	$k = 3$
$j = 1$	1.000	0.747	0.787
$j = 2$	0.891	0.989	1.000

Table 2.

Table 2 shows that the network becomes instable when a perturbation of magnitude 0.077 is added either to the arrival rate of product type 1 or product type 3. In this context it is remarkable that the stability radius can be increase by 40% without adding additional production capacity. Moreover, the in total required capacity can be reduced by 10.15% with an advanced capacity allocation. These promising results demonstrate the capabilities of our approach for a robust capacity allocation.

6 Conclusion and outlook

In this paper we have introduced a new approach to robust capacity allocation at production locations for a sustainable supply chain design. In particular we focused on the question how the production capacity at a certain location within a supply chain should be allocated to the processed product types in order maximize the robustness of the whole network. To this end we introduced a fluid network model under proportional processor sharing that approximates a multiclass queueing network. In this context stability of fluid models was defined and the essential stability characterization by the nominal workload condition was explained. Moreover, the stability radius was introduced as a measure for the robustness of the supply chain. This measure is utilized by a mathematical program formulation that maximizes the robustness by choosing an appropriate capacity allocation at each production location to the processed product types. The obtained computational results are very promising and demonstrate the potential for an improved robustness. This kind of robustness ensures that as long as deviations from the assumed demand level stay below the stability radius the network keeps stable and hence is able to satisfy customer demand. In the future other sources of perturbations have to be considered as well as different service disciplines. Furthermore the approach might be integrated in the original network design problem.

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