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A Multichannel IOpS Small-Gain Theorem for Large Scale Systems

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***Index Terms*— Nonlinear systems, Input-to-state stability, Lyapunov function, small gain condition**

***Abstract*— This paper extends known results on the stability analysis of interconnected systems. In particular, a small-gain theorem for the interconnection of an arbitrary number of systems via communication channels is presented. Here the communication between the subsystems is over a delayed, possibly lossy communication channel. To this end, a notion of input-to-output stability for functional differential equations is applied.**

I. INTRODUCTION

The ongoing progress in all kinds of fields of engineering leads to an increase of the complexity of dynamical systems under consideration. For the analysis of such large scale systems it is often helpful to consider them as an interconnection of many subsystems of smaller size. To this end, small-gain type theorems have proved valuable.

In the late eighties of the last century Eduardo Sontag introduced the notion of ISS [1]. Beside being of major interest for the control community on its own, it made nonlinear extensions of the linear small-gain theorem possible.

In [2] a small-gain theorem was presented which deals with the interconnection of two nonlinear systems in a feedback manner. It has been generalized to deal with the interconnection of several systems in [3].

In [4] the small-gain condition from [3] is used to construct an ISS-Lyapunov function for the overall system with the help of the ISS-Lyapunov functions from the subsystems. Similar ideas can be found in [5], but [5] is based on the so called cyclic-small-gain condition.

Work that uses also the cyclic-small-gain condition and is closer to the spirit of the presented paper is [6]. Beside using the cyclic-small-gain condition the main difference between [6] and this paper is that in [6] no "multichannel" setup for the communication is used.

Furher notable contributions dealing with small-gain type conditions are [7] respectively [8].

There exist many stability notions which are related to ISS (see e.g., [9]). One of them is the notion of input-to-output-practical-stability (IOpS) introduced in [2].

Andrew Teel derived Razumikhin-type theorems for functional differential equations (FDE) in [10] based on the ISS small-gain theorems from [2].

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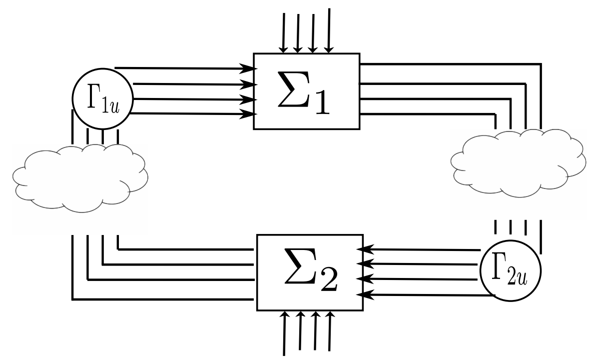

Fig. 1. Interconnection of two systems (Σ_1, Σ_2) in a multichannel way. The clouds represent the communication which can be delayed or even lossy. The arrows pointing upwards respectively downwards represent the external influences.

Polushin *et al.* presented in [11] a small-gain theorem which guarantees the IOpS property of the interconnection of two IOpS systems. In this particular work the communication of the subsystems is over multiple channels, where each of those channels can be delayed. The basic setup of the systems under consideration in [11] can be found in Figure 1.

The motivation for a setup where the communication is over multiple channels with different delays in each channel is manifold. Consider for instance a teleoperation in which a human (master) controls a robot (slave). The robot tries to maintain formation with several other robots. If one of these robots approaches an obstacle, it should send a forcefeedback to the slave, which sends the force information to the master. So that the human operator can "feel" whenever the environmental circumstances change for any of those robots.

In [12] we relaxed the condition from [11] with the help of the small-gain theorem from [3].

In this paper we extend the results from [12] to the case of an arbitrary number of subsystems which can form an arbitrary topology. The basic setup of the interconnection of the system under consideration is depicted in Figure 2.

In the original work [11] there was some kind of hierarchy between the two subsystems and it is not obvious how to extend the small-gain condition from [11] to several subsystems.

The paper is organized as follows. The problem setup as well as the notion of IOpS for FDEs is presented in Section II. The main contribution of this paper is presented in Section III. We will conclude our note with some remarks

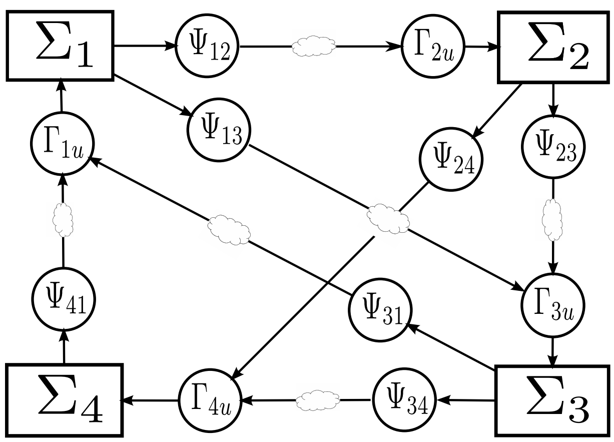

Fig. 2. Interconnection of several subsystems. Again, the clouds represent the communication in a multichannel way. The operator Ψ_{ij} describes the influence of the outputs of the i-th subsystem to the inputs of the j-th subsystem. The gain operators Γ_{iu} describes the influence from the inputs to the outputs of subsystem i.

in Section IV.

II. Preliminaries

A. Notations

In this section we introduce the class of systems under consideration and the stability notion we investigate. To this end we have to define some functional classes and their multi-dimensional extensions.

A continuous function $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is said to belong to class $\mathcal{G}$, if it is nondecreasing and satisfies $\gamma(0) = 0$. A function $\gamma \in \mathcal{G}$ is of class $\mathcal{K}$ if it is strictly increasing. If γ is of class $\mathcal{K}$ and unbounded, it is said to belong to $\mathcal{K}_\infty$.

We write $a < b,\ a, b \in \mathbb{R}^k$ if and only if $a_i < b_i\ \forall i = 1, .., k$ ($\leq, >, \geq$ are defined analogously). Note that we are comparing vectors, therefore we also have to use the negations $\ngeq, \nless, \ngtr, \nleq$. The inequality $a \ngeq b$ means that there must be at least one component i of a which is strictly less than the corresponding component of b i.e., $a \ngeq b :\Leftrightarrow\ \exists i$ such that $a_i < b_i$. The other negations are defined in a similar manner.

A mapping $T : \operatorname{dom} T \subset \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is called a *continuous and monotone operator on* $\operatorname{dom} T$, if 1. T is continuous and 2. for all $u, v \in \operatorname{dom} T$, $u \leq v$ implies $T(u) \leq T(v)$.

A matrix $\Gamma = (\gamma_{ij}),\ \gamma_{ij} \in \mathcal{K}$ or $\gamma_{ij} = 0$ for $i, j = 1, \ldots, n$ defines a continuous and monotone operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by

$$\Gamma(s) = \left(\max_j \gamma_{1j}(s_j), \ldots, \max_j \gamma_{nj}(s_j)\right)^T, \quad \text{for } s \in \mathbb{R}^n_+.$$

For such an operator we will say that $\Gamma \in \mathcal{G}^{n\times n}$. The class of these matrix-induced operators has some nice properties. Most relevant is the fact that any finite composition of matrix-induced operators gives again a matrix-induced operator and that matrix-induced operators commute with the $\max$-operation, where the $\max$-operator is understood elementwise.

The IOpS notion mentioned in the introduction usually deals with ordinary differential equations. In this note we are interested in the interconnection of many systems, where the inputs of the subsystems are delayed. An appropriate mathematical object to model such a situation are the so called *functional differential equations* or FDEs. A FDE is a differential equation where the right hand side depends on a function rather than a single point in the state space for every time instance t. For a detailed introduction to FDEs see [13]. To be more specific, the systems under consideration belong to a subclass of FDEs which are referred to as *time delay systems*.

More precisely, the systems we consider are of the form:

$$\begin{aligned} \dot{x}(t) &= f(x_d, u_d^1, \ldots, u_d^l, t) \\ y^1(t) &= h^1(x_d, u_d^1, \ldots, u_d^l, t) \\ &\ \ \vdots \\ y^r(t) &= h^r(x_d, u_d^1, \ldots, u_d^l, t), \end{aligned} \tag{1}$$

where x is the state of the system, $u^j,\ j = 1, \ldots, l$ are the inputs and $y^k, k = 1, \ldots, r$ the outputs of the system all lying in appropriate real finite-dimensional vector spaces. The subscript d denotes a retarded argument in the following way $x_d(t) := \{(s, x(s+t)), s \in [-t_d(t), 0]\},\ t_d : \mathbb{R} \mapsto \mathbb{R}_+$. We assume sufficient regularity of the maps f and h^k. Define $\|x_d(t)\| := \sup_{s\in[t-t_d(t),t]} |x(s)|$ ($\|y_d\|, \|u_d\|, \|w_d\|$ are defined analogously), where $|\cdot|$ is the maximum norm. To ease the presentation we introduce $u_d^+ := (\|u_d^1\|, \ldots, \|u_d^l\|)^T$, $y_d^+ := (\|y_d^1\|, \ldots, \|y_d^r\|)^T$, and $y^+ := (|y^1|, \ldots, |y^r|)^T$.

We use this "multichannel" formulation to model and analyze the effects of certain inputs on certain outputs. A second advantage of this approach is to have the possibility of different delays in every "channel" as we will see in the next section.

The ensuing definition is borrowed from [11].

Definition 2.1: A system of the form (1) is input-to-output-practical-stable (IOpS) at $t = t_0$ with $t_d(t) \geq 0$, $\beta \in \mathcal{K}_\infty^{r\times 1}$, IOpS gains $\Gamma \in \mathcal{G}^{r\times l}$, restrictions $\Delta_x \in \mathbb{R}_+$, $\Delta_u \in \mathbb{R}^l_+$ and offset $\delta \in \mathbb{R}^r_+$ if the conditions $\|x_d(t_0)\| \leq \Delta_x$ and $\sup_{t\geq t_0} u_d^+ \leq \Delta_u$, imply that the solution of (1) are well-defined for $t \geq t_0$ and the following inequalities hold:

$$\sup_{t\geq t_0} y^+ \leq \max\{\beta(\|x_d(t_0)\|), \Gamma(\sup_{t\geq t_0} u_d^+), \delta\}$$

and

$$\limsup_{t\to\infty} y^+ \leq \max\{\Gamma(\limsup_{t\to\infty} u_d^+), \delta\}$$

again componentwise.

Remark 2.2: The first inequality in Definition 2.1 resembles the global stability property and the second the so called asymptotic gain property (see e.g. [9]). In the case of a trajectory based formulation the asymptotic gain property together with the global stability property is equivalent to the ISS property.

Motivated by this we use the terminology IOpS although we provide no proof nor are we aware of a proof of an equivalence to a trajectory based formulation of the concept.

B. Problem Setup

Consider n systems of FDEs $\Sigma_i,\ i = 1, 2, \ldots, n,\ n \in \mathbb{N}$ of the form

$$\begin{aligned} \dot{x}_i &= f_i(x_{id}, u_{id}^1, \ldots, u_{id}^{l_i}, w_{id}^1, \ldots, w_{id}^{v_i}, t) \\ y_i^1 &= h_i^1(x_{id}, u_{id}^1, \ldots, u_{id}^{l_i}, w_{id}^1, \ldots, w_{id}^{v_i}, t) \\ &\ \vdots \\ y_i^{r_i} &= h_i^{r_i}(x_{id}, u_{id}^1, \ldots, u_{id}^{l_i}, w_{id}^1, \ldots, w_{id}^{v_i}, t). \end{aligned} \tag{2}$$

Here we distinguish between the controlled inputs u and disturbances w. The dimensions of the state spaces and the input spaces are as follows $x_i \in \mathbb{R}^{n_i}$, $u_i^j \in \mathbb{R}^{p_{ij}},\ j = 1, .., l_i$ and $w_i^j \in \mathbb{R}^{q_{ij}},\ j = 1, .., v_i$. The output spaces are $y_i^j \in \mathbb{R}^{k_{ij}},\ j = 1, \ldots, r_i$.
Similar to the last section we have to introduce $y_i^+ := (|y_i^1|, \ldots, |y_i^{r_i}|)^T$, $u_{id}^+ := (\|u_{id}^1\|, \ldots, \|u_{id}^{l_i}\|)^T$, and $w_{id}^+ := (\|w_{id}^1\|, \ldots, \|w_{id}^{v_i}\|)^T$.

Assumption 2.3: The systems $\Sigma_i,\ i = 1, 2, \ldots, n$ are IOpS at $t = t_0$ with $t_{id}(t) > 0$ and restrictions $\Delta_{xi} \in \mathbb{R}$, $\Delta_{ui} \in \mathbb{R}^{l_i}$, $\Delta_{wi} \in \mathbb{R}^{v_i}$ and offsets $\delta_i \in \mathbb{R}^{r_i}$. More precisely, there exist $\beta_i \in \mathcal{K}^{r_i \times 1}$, $\Gamma_{iu} \in \mathcal{G}^{r_i \times l_i}$ and $\Gamma_{iw} \in \mathcal{G}^{r_i \times v_i}$, such that for each $i = 1, 2, \ldots, n$ and each $t_0 \in \mathbb{R}$ the condition $\|x_{id}(t_0)\| \leq \Delta_{xi}$, $\sup_{t \geq t_0} u_{id}^+ \leq \Delta_{ui}$ and $\sup_{t \geq t_0} w_{id}^+ \leq \Delta_{iw}$ imply that the corresponding solution of Σ_i is well-defined for all $t \geq t_0$ and the following inequalities hold

$$\sup_{t \geq t_0} y_i^+ \leq \max\left\{\beta(\|x_{id}(t_0)\|), \sup_{t \geq t_0} \Gamma_{iu}(u_{id}^+), \sup_{t \geq t_0} \Gamma_{iw}(w_{id}^+), \delta_i\right\} \tag{3}$$

and

$$\limsup_{t \to \infty} y_i^+ \leq \max\left\{\limsup_{t \to \infty} \Gamma_{iu}(u_{id}^+), \limsup_{t \to \infty} \Gamma_{iw}(w_{id}^+), \delta_i\right\}. \tag{4}$$

If we introduce the following notation

$$B(x_d^+(t_0)) = (\beta_1(\|x_{1d}(t_0)\|)^T, \ldots, \beta_n(\|x_{nd}(t_0)\|)^T)^T,$$

$$\Gamma_U = \text{diag}\{\Gamma_{1u}, \ldots, \Gamma_{nu}\}, \quad \Gamma_W = \text{diag}\{\Gamma_{1w}, \ldots, \Gamma_{nw}\}$$

$\delta_{off} = \left(\delta_1^T, \ldots, \delta_n^T\right)^T$, $u_d^+ = \left((u_{1d}^+)^T, \ldots, (u_{nd}^+)^T\right)^T$ and $w_d^+ = \left((w_{1d}^+)^T, \ldots, (w_{nd}^+)^T\right)^T$ we can collect the inequalities from (3) and (4) to obtain

$$\sup_{t \geq t_0} y^+ \leq \max\left\{B(x_d^+(t_0)), \sup_{t \geq t_0} \Gamma_U(u_d^+), \sup_{t \geq t_0} \Gamma_W(w_d^+), \delta_{off}\right\}, \tag{5}$$

and, respectively,

$$\limsup_{t \to \infty} y^+ \leq \max\left\{\limsup_{t \to \infty} \Gamma_U(u_d^+), \limsup_{t \to \infty} \Gamma_W(w_d^+), \delta_{off}\right\}. \tag{6}$$

Before we can describe the interconnection of the n subsystems, we have to introduce a delayed versions of y_i^+. To this end define

$$\hat{y}_i^+(t) = (|y_i^1(t - \tau_i^1(t))|, \ldots, |y_i^{r_i}(t - \tau_i^{r_i})|)^T, \quad i = 1, \ldots, n$$

where $\tau_i^j : \mathbb{R} \to \mathbb{R}_+,\ i = 1 \ldots n,\ j = 1, \ldots r_i$ are Lebesgue measurable functions. The functions τ_i^j describe the delay of the j-th component of the output of the i-th subsystem.
To shorten the notation, define

$$\hat{y}^+(t) = \left(\hat{y}_1^+(t)^T, \ldots, \hat{y}_n^+(t)^T\right)^T.$$

Assumption 2.4: For the interconnection of the n subsystems we assume that the following holds

$$u_d^+(t) \equiv 0,\ \forall t < T_0 \tag{7}$$

$$u_d^+(t) \leq \Psi(\hat{y}^+(t)),\ \forall t \geq T_0 \tag{8}$$

where the operator $\Psi :\ \mathbb{R}_+^{\sum_{i=1}^n r_i} \to \mathbb{R}_+^{\sum_{i=1}^n l_i},\ s \mapsto \Psi(s) = \left(\max_j \Psi_{1j}(s_j), \ldots, \max_j \Psi_{nj}(s_j)\right)^T,\ \Psi_{ij} \in \mathcal{G}^{l_i \times r_j},\ s_j \in \mathbb{R}_+^{r_j}$ for all $i, j = 1, \ldots, n$ is of the form

$$\Psi = \begin{pmatrix} 0 & \Psi_{12} & \cdots & \Psi_{1n} \\ \Psi_{21} & 0 & \cdots & \Psi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{n1} & \Psi_{n2} & \cdots & 0 \end{pmatrix}.$$

Remark 2.5: Assumption 2.4 states that there exists a $T_0 \in \mathbb{R}$ which is the first time instance a connections has been established. Before that time the input is constant 0. After T_0 the operator Ψ_{ij} describes how the output of the jth subsystem influences the input of the ith subsystem. See Figure 2 for a sketch of the interconnection structure.
To ensure that communication between the subsystems happens at least sometime, we have to make the following assumption on the delays.

Assumption 2.6: There exists $\tau_* > 0$ and a piecewise continuous function $\tau^* :\ \mathbb{R} \mapsto \mathbb{R}_+$ with $\tau^*(t_2) - \tau^*(t_1) \leq t_2 - t_1$ for all $t_2 \geq t_1$ such that

$$\tau_* \leq \min_{\substack{i=1,\ldots,n \\ j=1,\ldots,r_i}} \{\tau_i^j(t)\} \leq \max_{\substack{i=1,\ldots,n \\ j=1,\ldots,r_i}} \{\tau_i^j(t)\} \leq \tau^*(t), \tag{9}$$

and

$$t - \max_{\substack{i=1,\ldots,n \\ j=1,\ldots,r_i}} \{\tau_i^j(t)\} \to \infty \text{ as } t \to \infty \tag{10}$$

for all $t \geq 0$.

Remark 2.7: The inequalities (9) say that the delays should be bounded from above by $\tau^*(t)$ and from below by τ_*. Because of the propagation delay of any physical system the existence of a lower bound τ_* is guaranteed. Basically, (10) states that the delay should not grow faster than the time itself. In the literature an assumption in the kind of $\dot{\tau}^*(t) < 1$ can be found to ensure that property. To account for possible information losses we have to adopt the more general Assumption 2.6. In [11] a methodology to satisfy Assumption 2.6 either by timestamping or by sequence numbering can be found. Time stamping refers to an approach where each piece of information (a packet if for instance TCP is used as a communication protocol) is tagged with the time information when the information was sent.
In sequence numbering each outgoing message is given an unique number in such a way that the receiver can reconstruct the correct order (sequence) in which they had been sent.

III. Main Result

We find it convenient to define

$$\Gamma = \Gamma_U \circ \Psi \text{ and } \mathbb{G} = \max_{k\geq 0} \Gamma^k \,. \tag{11}$$

Let $m = \sum_{i=1}^n r_i$. Overall, the operator $\Gamma : \mathbb{R}_+^m \to \mathbb{R}_+^m$ is of the form

$$\Gamma(s) = \Big(\big(\max_{j\neq 1}\{\Gamma_{1u} \circ \Psi_{1j}(s_j)\}\big)^T,$$
$$\big(\max_{j\neq 2}\{\Gamma_{2u} \circ \Psi_{2j}(s_j)\}\big)^T, \ldots, \big(\max_{j\neq n}\{\Gamma_{nu} \circ \Psi_{nj}(s_j)\}\big)^T \Big)^T$$

with $s = (s_1, \ldots, s_n)^T$, $s_i \in \mathbb{R}_+^{r_i}$.
While $\Gamma \in \mathcal{G}^{m\times m}$ clearly is a matrix-induced continuous and monotone operator, $\mathbb{G}$ is not necessarily well-defined. We will see in the next lemma that a small-gain type condition is precisely what is needed to assure that $\mathbb{G}$ is well-defined.

Lemma 3.1: Let Γ be defined as in (11) and assume there exist $\delta, \Delta \in \mathbb{R}_+^m$, $\delta < \Delta$, such that

$$\limsup_{k\to\infty} \Gamma^k(\Delta) \leq \delta \,, \tag{12}$$

then $\mathbb{G}(s)$ is well-defined for all $s \leq \mathbb{G}(\Delta)$

Proof: Define $\Delta^* = \mathbb{G}(\Delta) = \max_{l\geq 0}\{\Gamma^l(\Delta)\}$. It follows from (12) that there exists an $l^* \in \mathbb{N}$ such that $\Delta^* = \max_{0\leq l\leq l^*}\{\Gamma^l(\Delta), \delta\}$. Because the maximum is over a finite set $\Delta^* < \infty$ exists. Clearly, $\Gamma(\Delta^*) \leq \Delta^*$. From the monotonicity of Γ we can deduce

$$0 \leq \Gamma^k(\Delta^*) \leq \Gamma^{k-1}(\Delta^*) \leq \cdots \leq \Gamma(\Delta^*) \leq \Delta^*.$$

Hence $\lim_{k\to\infty} \Gamma^k(\Delta^*)$ exists. From (12) and the definition of $\limsup$ we can derive

$$\begin{aligned}\lim_{k\to\infty} \Gamma^k(\Delta^*) &= \lim_{k\to\infty} \Gamma^k(\max_{l\geq 0}\{\Gamma^l(\Delta)\}) \\ &= \lim_{k\to\infty} \max_{l\geq k}\{\Gamma^l(\Delta)\} = \limsup_{k\to\infty} \Gamma^k(\Delta) \leq \delta\end{aligned}$$

where we have used that $\Gamma(\max\{a,b\}) = \max\{\Gamma(a), \Gamma(b)\}$. Simple monotonicity arguments show that $\limsup_{k\to\infty} \Gamma^k(s) \leq \delta$ for all $s \leq \Delta^*$. It follows from similar arguments as above that there exists a $k^* \in \mathbb{N}$ such that $\mathbb{G}(s) = \max_{0\leq k\leq k^*}\{\Gamma^k(s), \delta\} < \infty$ for all $s \leq \Delta^*$. Hence $\mathbb{G}(s)$ is well-defined for all $s \leq \Delta^*$ and the proof is complete. ■

The next lemma is the main technical tool for the proof of our main theorem.

Lemma 3.2: Let the premise of Lemma 3.1 hold. Then for all $a, b \leq \mathbb{G}(\Delta)$,

$$a \leq \max\{b, \Gamma(a)\} \tag{13}$$

implies

$$a \leq \max_{k\geq 0}\{\Gamma^k(b), \delta\} \,. \tag{14}$$

Proof: If we apply Γ on both sides of inequality (13), we obtain

$$\Gamma(a) \leq \max\{\Gamma(b), \Gamma^2(a)\}.$$

Substituting this in (13) yields

$$a \leq \max\{b, \Gamma(b), \Gamma^2(a)\}.$$

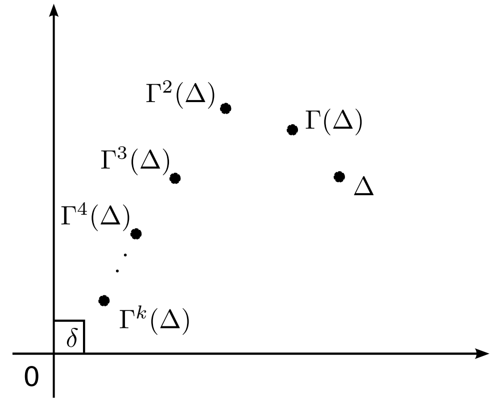

Fig. 3. Sketch of the evolution of the gain operator Γ starting from Δ. Assuming that condition (12) holds, the iteration of $\Gamma^k(\Delta)$ will end up in the small box of size δ.

Repeat this procedure k times to obtain

$$a \leq \max\{b, \Gamma(b), \Gamma^2(b), \ldots, \Gamma^k(b), \Gamma^{k+1}(a)\}.$$

If we choose k large enough we can use (12) to get

$$a \leq \max\{\mathbb{G}(b), \delta\}.$$

From Lemma 3.1 we know that $\mathbb{G}(b)$ is well-defined and we can write

$$a \leq \max_{k\geq 0}\{\Gamma^k(b), \delta\}$$

and the proof is complete. ■

Usually, small-gain type conditions compare some operator with the identity. As we will see in the next lemma, condition (12) can also be interpreted in this manner.

Lemma 3.3: Let the premise of Lemma 3.1 hold. then

$$\Gamma(s) \not\geq s \quad \forall s \in [\delta, \mathbb{G}(\Delta)],\ s \neq \delta.$$

Proof: We will prove this by contradiction. So assume there exists $s \in [\delta, \mathbb{G}(\Delta)]$ such that $\Gamma(s) \geq s$. From the monotonicity of Γ it follows readily that $\Gamma^k(\mathbb{G}(\Delta)) \geq \Gamma^k(s) \geq s \geq \delta$. Realizing that this contradicts (12) finishes the proof. ■

Remark 3.4: From the following example it can be seen that the converse of Lemma 3.3 does not hold. Consider the following operator.

$$T = \begin{pmatrix} 2\,\mathrm{id} & 0 \\ 0 & \frac{1}{2}\,\mathrm{id} \end{pmatrix}$$

It is easy to verify, that $T \in \mathcal{G}^{2\times 2}$ and $T(s) \not\geq s\ \forall s \in [\delta, \mathbb{G}(\Delta)]$, for arbitrary $\Delta,\ \delta \in \mathbb{R}^2,\ \delta > 0,\ \Delta > \delta$. On the other hand $T^k(s),\ k \to \infty$ is unbounded, contradicting (12). From the last example we see that we have to exclude the possibility of unbounded growth. The next lemma shows that a descent in one single point is needed to ensure that property.

Lemma 3.5: Let Γ be defined as in (11). If there exist an $a \in \mathbb{R}_+^m$ and some $k \in \mathbb{N}$ such that $\Gamma^k(a) < a$, then there exists $b < a$ such that

$$\limsup_{k\to\infty} \Gamma^k(a) \leq b.$$

Proof: Choose an $l > k$. Then the following holds

$$0 \leq \Gamma^l(a) \leq \cdots \leq \Gamma^{k+1}(a) \leq \Gamma^k(a) < a.$$

This is a bounded and monotone sequence. Hence

$$\lim_{r\to\infty} \Gamma^r(a) = b$$

exists. The proof is finalized by noting that $b < a$, which follows from $\Gamma^k(a) < a$. ■

For a more detailed presentation of the topological properties of such monotone operators see [14].

Before we state the main contribution of this paper we introduce $\Delta_x = (\Delta_{x1}, \ldots, \Delta_{xn})^T$, $\Delta_u = (\Delta_{u1}^T, \ldots, \Delta_{un}^T)^T$ and $\Delta_w = (\Delta_{w1}^T, \ldots, \Delta_{wn}^T)^T$.

Theorem 3.6: Suppose the system (2), satisfies Assumptions 2.3, 2.4 and 2.6 and that there exist $\delta, \Delta \in \mathbb{R}_+^m$, $0 \leq \delta < \Delta$, such that the following *local small-gain condition* holds:

$$\limsup_{k\to\infty} \Gamma^k(\Delta) \leq \delta\,. \tag{15}$$

Then the following assertions hold: The operator $\mathbb{G}$ is well-defined on the order interval $[\delta, \Delta]$. If in addition $\mathbb{G}(\Delta) > \Delta^{**}$, where

$$\Delta^{**} = \mathbb{G}\left(\max\{B(\Delta_x), \Gamma_W(\Delta_w), \delta_{off}\}\right), \tag{16}$$

and

$$\Psi(\Delta^{**}) \leq \Delta_u \tag{17}$$

holds, then system (2) is IOpS at $t = T_0$ in the sense of Definition 2.1 with

$$t_d(T_0) = \max_{i=1,\ldots,n}\{t_{id}(T_0)\} + \tau^*(T_0) + \tau^*(T_0 - \tau^*(T_0))\,. \tag{18}$$

More precisely, the conditions $x_d^+(T_0) \leq \Delta_x$, $\sup_{t\geq T_0} w_d^+ \leq \Delta_w$ imply that the following inequalities hold

$$\sup_{t\geq T_0} y^+ \leq \max\Big\{\mathbb{G}\Big(\max\{B(x_d^+(T_0)), \Gamma_W(\sup_{t\geq T_0} w_d^+), \delta_{off}\}\Big), \delta\Big\}, \tag{19}$$

and

$$\limsup_{t\to\infty} y^+ \leq \max\Big\{\mathbb{G}\Big(\max\{\Gamma_w(\limsup_{t\to\infty} w_d^+), \delta_{off}\}\Big), \delta\Big\}. \tag{20}$$

Remark 3.7: Equation (18) takes into account the delays of the individual systems (t_{id}) coming form Assumption 2.3 as well as the maximal communication delay comming from Assumption 2.6.

Proof: [of Theorem 3.6] The fact that $\mathbb{G}$ is well-defined on $[\delta, \Delta]$ follows from Lemma 3.1. Now consider system (2) and assume

$$x_d^+(T_0) \leq \Delta_x \quad \text{and} \quad \sup_{t\in[T_0,\infty)} w_d^+ \leq \Delta_w. \tag{21}$$

It remains left to show that the restrictions on the input Δ_u hold for all positive times. To this end we will first establish that the output is bounded up to some time T_{max}. Then we will show that $T_{max} = \infty$. After showing that the trajectory of the interconnected system exists for all positive times the claim follows by an application of our small-gain argument (Lemma 3.2).

Assumption 2.3 together with (5), (21) as well as causality arguments imply that

$$y_d^+(T_0) \leq \max\{B(\Delta_x), \Gamma_W(\Delta_w), \delta_{off}\}.$$

With the help of (7), (8) and Assumption 2.6 we can deduce

$$\sup_{t\in[T_0-t_d(T_0),T_0+\tau_*]} u^+ \leq \Psi(\max\{B(\Delta_x), \Gamma_W(\Delta_w), \delta\}) \leq \Psi(\Delta^{**}),$$

where the last inequality follows from (16). From the last inequality together with (17) we see that the restrictions on the inputs are satisfied for $t \in [T_0 - t_d(T_0), T_0 + \tau_*]$. Hence there exists $T_{\max} > T_0 + \tau_*$ such that the solutions of (2) are well-defined for all $t \in [T_0, T_{\max})$. Now we want to show that

$$\sup_{t\in[T_0,T_{\max})} y_d^+ \leq \Delta^{**}. \tag{22}$$

We will prove (22) by contradiction. So assume there exists $T_1 \in [T_0, T_{\max} - \tau_*)$ such that

$$\sup_{t\in[T_0,T_1]} y_d^+ \leq \Delta^{**} \text{ and } \sup_{t\in[T_0,T_1+\tau_*]} y_d^+ \not\leq \Delta^{**}. \tag{23}$$

Combining (5), (18), (21) with (8) and Assumption 2.6, we obtain

$$\sup_{t\in[T_0,T_1+\tau_*]} y_d^+ \leq \max\Big\{B(\Delta_x), \Gamma_W(\Delta_w), \Gamma(\sup_{t\in[T_0,T_1]} y_d^+), \delta_{off}\Big\}.$$

From the definition of (16) it is easy to see that $\Gamma(\Delta^{**}) \leq \Delta^{**}$. Hence we can deduce with the help of the first inequality in (23)

$$\sup_{t\in[T_0,T_1+\tau_*]} y_d^+ \leq \max\{B(\Delta_x), \Gamma_W(\Delta_w), \Delta^{**}\} = \Delta^{**},$$

which contradicts the second inequality in (23). This contradiction proves (22).

Next we want to show that $T_{\max} = \infty$. Again we will prove this by contradiction. Due to the IOpS assumption on the subsystems and (21) $T_{\max} < \infty$ implies

$$\sup_{t\in[T_0,T_{\max})} u^+ \not\leq \Delta_u. \tag{24}$$

From (8) and (17) we can see that (24) implies

$$\Psi\left(\sup_{t\in[T_0,T_{\max})} \hat{y}^+\right) \not\leq \Psi(\Delta^{**}).$$

Because of the monotonicity of Ψ and the fact that $\sup \hat{y}^+ \leq \sup y_d^+$ we get

$$\sup_{t\in[T_0,T_{\max})} y_d^+ \not\leq \Delta^{**},$$

which contradicts (22), hence $T_{\max} = \infty$.
Summarizing, the restrictions on the inputs hold for all $t \in [T_0, \infty)$ and thus together with (21) we conclude that the solutions are well-defined for all positive times. Hence we can use (5) to get

$$\sup_{t \geq T_0} y_d^+ \leq \max\Big\{B(x_d^+(T_0)), \Gamma_W(\sup_{t\geq T_0} w_d^+), \Gamma(\sup_{t\geq T_0} y_d^+), \delta_{off}\Big\}.$$

Using Lemma 3.2 we conclude

$$\sup_{t \geq T_0} y_d^+ \leq \max\Big\{\max_{k\geq 0} \Gamma^k\Big(\max\{B(x_d^+(T_0)), \Gamma_W(\sup_{t\geq T_0} w_d^+)), \delta_{off}\}\Big), \delta\Big\},$$

which can be easily rewritten to get (19). Similarly we can use (6) together with Lemma 3.2 to get

$$\limsup_{t\to\infty} y_d^+ \leq \max\Big\{\max_{k\geq 0} \Gamma^k\big(\max\{\Gamma_W(\limsup_{t\to\infty} w_d^+(t)), \delta_{off}\}\big), \delta\Big\}.$$

Realizing that this can be brought into the form (20) finishes the proof. ■

IV. Conclusion and Future Work

In this paper we have continued our work on small-gain type conditions from [12]. Namely, we have presented a small-gain theorem which uses the notion of IOpS for FDEs to ensure that the interconnection of an arbitrary number of subsystems is again IOpS. In particular we have considered the case where the communication is over delayed, possible lossy communication channels.

The use of the maximum formulation of the ISS property has proved quite valuable. One of the main reasons is, that the maximum commutes with monotone operators. Sometimes it is more natural or convenient to use different types of the ISS formulation (e.g., the sum formulation) or even a mixed version of different formulations, see e.g., [15]. In [16] we will continue our work on small-gain conditions for ISS systems to handle a more general class of ISS formulations.

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