Stability of fluid network models and Lyapunov functions

Michael Schönlein and Fabian Wirth

***Abstract*— We consider the class of closed generic fluid networks (GFN) models. This class contains for example fluid networks under general work-conserving and priority disciplines. Within this abstract framework a Lyapunov method for stability of GFN models was proposed by Ye and Chen. They proved that stability of a GFN model is equivalent to the property that for every path of the model a Lyapunov like function is decaying. In this paper we construct state-dependent Lyapunov functions in contrast to pathwise functionals. We first show by counterexamples that closed GFN models do not provide sufficient information that allow for a converse Lyapunov theorem with state-dependent Lyapunov functions. To resolve this problem we introduce the class of strict closed GFN models by forcing the closed GFN model to satisfy a concatenation and a perfectness condition and define a state-dependent Lyapunov function. We show that for the class of strict closed GFN models a converse Lyapunov theorem holds. Finally, it is shown that common fluid network models, like general work-conserving and priority fluid network models as well as certain linear Skorokhod problems define strict closed GFN models.**

I. INTRODUCTION

An effective tool to model complex manufacturing systems, computer systems or telecommunication networks is the family of multiclass queueing networks. An example for occurs in semiconductor fabrication, where production lines are modeled as reentrant lines, which are a special case of multiclass queueing networks. Especially in the pursuit of deriving good control strategies for multiclass queueing networks the question of stability arises. For a long period a common belief was that a sufficient condition for stability is that the traffic intensity is strictly less then one. But in 1993 Kumar and Seidman [1] presented a network with two stations processing four types of jobs which is unstable although the traffic intensity at each station is less than one. This example inspired a number of examples with different service disciplines, like first-in-first-out (FIFO) and priority, that have surprising properties. In the literature they are known as the Lu-Kumar network, the Rybko-Stolyar network or the Bramson network, see e.g. [2] or [3], [4] and [5]. In recent years further disciplines like maximum pressure and join-the-shortest-queue are investigated [6] ,[7] ,[8]. Rybko and Stolyar [5] and Dai [9] pursued the strategy of rescaling the stochastic processes that describe the dynamics of a multiclass queueing network and considered the limit of the scaling. This limit is called the fluid limit model for the queueing network and is a continuous deterministic analogue. Of course, deterministic models are much easier to investigate. The great benefit of this approach is, that the stability of the corresponding fluid limit model is sufficient for the stability of a multiclass queueing network [9]. A discussion of the relationship between queueing networks and fluid models can be found in [2].

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Due to this fact the question arises, under which conditions fluid limit models are stable. A fluid model is called stable if the fluid level process Q with unit initial level one is drained to zero in a finite time τ and remains zero beyond τ. Of course, conditions that guarantee stability depend on the service discipline of the network. In [10] Chen states necessary and sufficient conditions for stability of general work-conserving fluid networks. Stability conditions for fluid networks under FIFO and priority discipline have been derived by Chen and Zhang [11], [12]. Often the strategy to prove these conditions is to use a Lyapunov function. There a locally Lipschitz function $g : \mathbb{R}_+^K \to \mathbb{R}_+$ such that $g(x) = 0$ if and only if $x = 0$ is called a Lyapunov function, if there exists a constant $\varepsilon > 0$ such that for each fluid model solution it holds that

$$\frac{d\,[g(Q(t))]}{dt} \le -\varepsilon$$

whenever $Q(t) \neq 0$ and t is a regular point for $g(Q(\cdot))$. For more details see [13]. Within this framework linear Lyapunov functions of the form

$$g(Q(t)) = h^T Q(t)$$

where h is some positive vector in $\mathbb{R}_+^K$ are used to establish a sufficient condition for the stability of fluid network models under a priority discipline [12]. The special case for this where $h = (1, ..., 1)^T$ is used the show that a fluid model of a re-entrant line operating under last-buffer-first-served (LBFS) service discipline is stable, if the usual traffic condition $\rho_j < 1$ is satisfied for all stations j [13]. This special case is also used to prove a stability condition for fluid networks under join-the-shortest-queue discipline [8]. Ye and Chen investigated fluid networks under priority disciplines by using piecewise linear Lyapunov functions of the form

$$g(Q(t)) = \max_{1 \le j \le N} x_j^T Q(t)$$

for some nonnegative vectors $x_1, ..., x_N$ that arise from the stability condition, for details see [14]. This approach yields a sharper stability condition for fluid networks under priority discipline than in [12]. Furthermore, in the verification of a

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stability condition for fluid networks under general work-conserving disciplines a quadratic Lyapunov function

$$g(Q(t)) = Q^T(t)\, A\, Q(t)$$

with a strictly copositive matrix A is used [10]. What all the works mentioned above have in common is that the existence of Lyapunov functions are only shown to be sufficient for stability.

Before we investigate the question whether the existence of a Lyapunov function is also necessary for the stability of a fluid network, we recall briefly the basic idea of a Lyapunov function from the theory of dynamical systems. For a detailed description the reader is referred e.g. to [15], [16]. Given a time-varying dynamical system

$$\dot{x} = f(t, x), \qquad x \in \mathbb{R}^n,\, t \in [0, \infty) \tag{1}$$

with initial condition $x(t_0) = x_0$ and continuous f, where the origin is an equilibrium position, i.e. $f(t, 0) = 0$ for almost every $t \geq 0$. A real map $V(t, x) : [0, \infty) \times B_r \subset \mathbb{R}^n \to \mathbb{R}$ that is positive definite and decrescent is called a strict Lyapunov function for (1), if there exists a continuous and strictly increasing function $w : [0, r) \to [0, \infty)$ with $w(0) = 0$ such that for every solution $x(\cdot)$ and each interval $I \subset [0, \infty)$ one has

$$V(t_2, x(t_2)) - V(t_1, x(t_1)) \leq -\int_{t_1}^{t_2} w(||x(t)||)dt \tag{2}$$

for each $t_1 < t_2 \in I$ provided that $x(\cdot)$ is defined on I and $x(t) \in B_r$ for $t \in I$. It is well known that the origin is uniformly locally asymptotically stable if and only if there is a strict Lyapunov function.

In order to get a so called converse Lyapunov theorem for fluid networks Ye and Chen followed a different, more general approach [17]. They collected the characteristic properties of fluid networks and defined a generic fluid network (GFN) model as set Φ of functions $Q : \mathbb{R}_+ \to \mathbb{R}_+^K$ that satisfy a few natural properties. A precise description of a GFN model is given in Section II. They proved that stability of a GFN model is equivalent to the property that for every function $Q \in \Phi$ a Lyapunov like function $v : \mathbb{R}_+ \to \mathbb{R}_+$ is decaying along Q. In particular, v can be chosen as

$$v(t) = \int_t^{\infty} ||Q(s)||ds. \tag{3}$$

It can be seen that this result differs from that from the theory of dynamical systems in the sense that the Lyapunov like function is not state-dependent but a functional that depends on $Q(\cdot)$. This is undesirable, because the benefit of Lyapunov's second method is that trajectories need not be known, whereas the method of Ye and Chen requires the knowledge of all solutions. In this paper we define a state-dependent Lyapunov function and prove a converse Lyapunov theorem.

This paper is organized as follows. In the Section II we recall the definition of a GFN model from [17]. Further we discuss counterexamples to emphasize that the class of (closed) GFN models is too general to provide a converse Lyapunov theorem with state-dependent Lyapunov functions. In the Section III we introduce the class of strict closed GFN models by forcing the closed GFN models to satisfy additionally a concatenation and a perfectness property. For this model class we define a state-dependent Lyapunov function and prove that within this framework the stability of a strict closed GFN model is equivalently characterized by the existence of a state-dependent Lyapunov function. In Section IV we recall some results from differential inclusions and viability theory that will be useful in Section V. There we show that general work-conserving and priority fluid networks as well as certain Skorokhod problems define strict closed GFN models.

We collect some notations that will be used throughout the paper. By $\mathbb{R}_+^K$ we denote the positive orthant $\{x \in \mathbb{R}^K : x \geq 0\}$, where $\geq$ has to be understood component-wise. Throughout the paper we mostly consider the space $(\mathbb{R}_+^K, ||\cdot||)$ with $||x|| := \sum_{i=1}^K |x_i|$. Let $C^K(\mathbb{R}_+)$ denote the space of continuous functions $f : \mathbb{R}_+ \to \mathbb{R}_+^K$. A sequence $(f_n(t))_n$ in $C^K(\mathbb{R}_+)$ is said to converge uniformly on compact sets (u.o.c.) to a continuous function $f(t) \in C^K(\mathbb{R}_+)$, if for any $T > 0$

$$\lim_{n\to\infty} \sup_{t\in[0,T]} ||f_n(t) - f(t)|| = 0.$$

We say that a function $g : \mathbb{R}_+^K \to \mathbb{R}$ is upper semi-continuous in $a \in \mathbb{R}_+^K$, if $g(a) \geq \limsup_{x\to a} g(x)$. Of course, g called upper semi-continuous if it is upper semi-continuous for every $a \in \mathbb{R}_+^K$. Further a function $g : \mathbb{R}_+^K \to \mathbb{R}_+$ is lower semi-continuous at $a \in \mathbb{R}_+^K$ if $-g$ is upper semi-continuous in $a \in \mathbb{R}_+^K$ and g is called lower semi-continuous if g is lower semi-continuous in every point. Finally, by $\mathcal{K}$ we denote the set of continuous functions $w : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfy $w(0) = 0$ and are strictly increasing.

II. GENERIC FLUID NETWORK MODELS

In this section we consider generic fluid network models introduced by Ye and Chen. In [17] they present a Lyapunov method for characterizing the stability of fluid networks. The disadvantage of their approach is that it is trajectory based. That is, the Lyapunov function depends on the path of the closed GFN model. First we recall from [17] the definition of a closed generic fluid network (closed GFN) model and the conditions for a function to be a Lyapunov function. Then we define a candidate for a Lyapunov function that does not depend on the path and show that in the setting it is not continuous in general. Further we give a counterexample that shows that the concatenation of two paths is not automatically contained in a closed GFN model.

Definition 1: A nonempty set Φ of functions $Q(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+^K$ is said to be a GFN model, if

(a) there is a $L > 0$, such that for any $Q(\cdot) \in \Phi$ and $t, s \in \mathbb{R}_+$ it holds that

$$||Q(t) - Q(s)|| \leq L\,|t - s|.$$

(b) $Q(\cdot) \in \Phi$ implies $\frac{1}{r}Q(r\cdot) \in \Phi$ for all $r > 0$.

(c) $Q(\cdot) \in \Phi$ implies $Q(s + \cdot) \in \Phi$ for all $s \geq 0$.

Furthermore, if the following condition is also satisfied, then we call Φ a closed GFN model.

(d) if a sequence $(Q_n)_n \subset \Phi$ converges to Q_* u.o.c, then $Q_* \in \Phi$.

Condition (a) states that the functions $Q(\cdot)$ are Lipschitz continuous, where condition (b) is a scaling property and condition (c) is a shift property. The content of condition (d) is that the set Φ is closed in the weak$*$ topology of L^∞. Any element $Q(\cdot)$ of Φ is called a path (of Φ) and the set of paths with initial level one is denoted by $\Phi(1) = \{Q(\cdot) \in \Phi : ||Q(0)|| = 1\}$. A further notation that will be used later is for $x \in \mathbb{R}_+^K$ the set $\Phi_x = \{Q(\cdot) \in \Phi : Q(0) = x\}$. Moreover we adapt from [17] the definition of stability of a GFN model.

Definition 2: A GFN model Φ is said to be stable, if there exists a $\tau > 0$, such that $Q(\tau + \cdot) \equiv 0$ for any path $Q(\cdot) \in \Phi(1)$.

The Lyapunov method to investigate stability of closed GFN models presented in [17] is as follows. A GFN model Φ is said to satisfy the L-condition, if there exist three class $\mathcal{K}$-functions $w_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, 3$ such that for any GFN path $Q \in \Phi$ there exists an absolutely continuous function $v(t)$ such that the following conditions hold for almost all $t \geq 0$.

$$w_1(||Q(t)||) \leq v(t) \leq w_2(||Q(t)||) \tag{4}$$

$$\dot{v}(t) \leq -w_3(||Q(t)||) \tag{5}$$

The corresponding converse Lyapunov theorem is then.

Theorem 1: A GFN model Φ is stable if and only if the L-condition is satisfied. In particular, given Q the function v can be chosen as

$$v(t) = \int_t^\infty ||Q(s)||ds. \tag{6}$$

As mentioned in the introduction the drawback of this definition is that the Lyapunov function is path-dependent and does not depend on the state of the network, which is the basic idea of a Lyapunov function for a dynamical system. We denote by $\mathcal{A}(\Phi) = \{x \in \mathbb{R}_+^K : \exists\, Q(\cdot) \in \Phi, Q(0) = x\}$ and define the following candidate $V : \mathcal{A}(\Phi) \to \mathbb{R}_+$ by

$$V(x) = \sup_{Q(\cdot)\in\Phi_x} \int_0^\infty ||Q(s)||ds. \tag{7}$$

In the sequel we assume that $\mathcal{A}(\Phi) = \mathbb{R}_+^K$. The function V can be interpreted as a measurement of the state x in the sense that $V(x)$ represents the total possible fluid mass that the network has to deal with. This raises the question of the regularity of V. Of course, we aim for the Lyapunov function continuous dependence on the state. In the frame of Definition 1 the following can be stated.

Proposition 1: If Φ is a stable closed GFN model, then the function V is upper semi-continuous.

Proof: Let $x \in \mathbb{R}_+^K$ and $(x_n)_n \subset \mathbb{R}_+^K$ be a sequence that converges to x. As Φ is stable the set $\{V(x_n) : n \in \mathbb{N}\}$ is bounded. Hence there exists a subsequence $(x_{k_l})_l$ such that

$$\limsup_{n\to\infty} V(x_n) = \lim_{l\to\infty} V(x_{n_l}) = \lim_{l\to\infty} \int_0^\infty ||Q_{n_l}(s)||ds$$

with $Q_{n_l}(0) = x_{n_l}$. Now we consider the family $\{Q_l(\cdot) : l \in \mathbb{N}\}$. Since Φ is stable the family $\{Q_l(\cdot) : l \in \mathbb{N}\}$ is bounded. By condition (a) in Definition 1 there is a single Lipschitz constant for any path $Q_l(\cdot)$ of the family $\{Q_l(\cdot) : l \in \mathbb{N}\}$ and thus the family is equicontinuous. By the theorem of Arzela-Ascoli there exists a subsequence which converges u.o.c. to some $Q_*(\cdot)$ with $Q_*(0) = x$. Since the model is closed it follows that $Q_*(\cdot) \in \Phi$. Hence by the definition of V it holds that

$$\begin{aligned}\limsup_{n\to\infty} V(x_n) &= \lim_{l\to\infty} \int_0^\infty ||Q_{n_l}(s)||ds = \int_0^\infty ||Q_*(s)||ds \\ &\leq V(x).\end{aligned}$$

■

We are interested in the continuity of V. Hence we need to look whether V is also lower semi-continuous.

Example 1: Let $K = 2$ and

$$\Phi = \left\{ \begin{pmatrix} (x_1 - t)^+ \\ (x_2 - t)^+ \end{pmatrix}, \begin{pmatrix} (c - \frac{1}{2}t)^+ \\ (c - \frac{1}{2}t)^+ \end{pmatrix}, x_1, x_2, c \in \mathbb{R}_+ \right\}.$$

It is easy to check that Φ is a closed GFN model. We consider $x_0 = (1, 1)^T$ and $x_n = (1 + \frac{1}{n}, 1 - \frac{1}{n})^T$. It holds that

$$\begin{aligned}\lim_{n\to\infty} V(x_n) &= \lim_{n\to\infty} \int_0^\infty (1 + \frac{1}{n} - t)^+ + (1 - \frac{1}{n} - t)^+ dt \\ &= \lim_{n\to\infty} \frac{1}{2}\left((1 + \frac{1}{n})^2 + (1 - \frac{1}{n})^2 \right) = 1 \\ &< 2 = \int_0^2 2(1 - \frac{1}{2}t)dt = V(x_0).\end{aligned}$$

So V defined by (7) with asymptotically stable fixed point is not lower semi-continuous.

Remark 1: The example shows that in the frame of Definition 1 our candidate V is not continuous in general.

A further major property of a Lyapunov function V for a dynamical system is, that V is decreasing along trajectories. The trajectories in the context of closed GFN models are the paths. The next example addresses this problem.

Example 2: Let $K = 2$ and

$$\Phi = \left\{ \begin{pmatrix} (x_1 - 2\,t)^+ \\ (x_2 - 4\,t)^+ \end{pmatrix}, \begin{pmatrix} (c - 2\,t)^+ \\ (c - 2\,t)^+ \end{pmatrix}, x_1, x_2, c \in \mathbb{R}_+ \right\}.$$

It is easy to check that Φ is a closed GFN model. Now consider the following two paths

$$Q_1(t) = \begin{pmatrix} (20 - 2\,t)^+ \\ (22 - 4\,t)^+ \end{pmatrix}, \quad Q_2(t) = \begin{pmatrix} (20 - 2\,t)^+ \\ (20 - 2\,t)^+ \end{pmatrix}.$$

It holds that $Q_1(1) = Q_2(1) = (18, 18)^T$ and further

$$V(Q_1(0)) = 130\frac{1}{4},\ V(Q_1(1)) = 162.$$

Thus it holds that

$$V(Q_1(1)) - V(Q_1(0)) = 162 - 130\frac{1}{4} > 0.$$

Remark 2: The second example shows that in the framework of Definition 1 there is a path of a stable closed GFN model on that our candidate is not decreasing although the closed GFN model is stable.

III. MAIN RESULTS

In this section we present a way out of the dilemma by adding two conditions to the closed GFN model and defining the Lyapunov function in a slightly different way. The main result is that our definition of a Lyapunov function and the already introduced candidate V are appropriate to prove a converse Lyapunov theorem for strict closed GFN models. The road map is as follows. First we present the two additional conditions for the closed GFN model. Then we state our definition for a Lyapunov function. After that we show that under this conditions our candidate (7) is continuous. In the sequel we prove the main theorem.

Definition 3: A set $\mathcal{Q}$ of functions $Q(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+^K$ is said to be a strict closed GFN model, if

- (*a*) it is a closed GFN model
- (e) for any $Q_* \in \mathcal{Q}$ holds that if $Q_n(0) \to Q_*(0)$ there exists a $Q_n(\cdot) \in \mathcal{Q}$ that converges u.o.c. to $Q_*(\cdot)$.
- (f) for two GFN paths $Q_1(\cdot), Q_2(\cdot) \in \mathcal{Q}$ that coincide for a time $t_* \in \mathbb{R}_+$, the concatenation at time t_* is also a path of $\mathcal{Q}$.

Now we define a Lyapunov function that does not depend on the path of the strict closed GFN model.

Definition 4: Given a strict closed GFN model $\mathcal{Q}$ a continuous function $V : \mathcal{A}(\mathcal{Q}) \to \mathbb{R}_+$ is said to be a Lyapunov function, if there exist class $\mathcal{K}$ functions $w_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, 3$ such that

$$w_1(||x||) \le V(x) \le w_2(||x||) \tag{8}$$

$$V(Q(t_2)) - V(Q(t_1)) \le -\int_{t_1}^{t_2} w_3(||Q(s)||)\, ds \tag{9}$$

for all $0 \le t_1 \le t_2 \in \mathbb{R}_+$ and all paths $Q(\cdot) \in \mathcal{Q}$.

Next we show that condition (e) is the appropriate condition for continuity of our candidate V. To be precise, condition (e) closes the gap from upper semi-continuity to continuity.

Proposition 2: If $\mathcal{Q}$ is a stable strict closed GFN model, then V is continuous.

Proof: We show that V is lower semi-continuous as the continuity of V follows then together with Theorem 1. Let $x_* \in \mathbb{R}_+^K$ and $Q_*(\cdot) \in \mathcal{Q}$ such that $Q_*(0) = x_*$. Further let $(x_n)_n$ be a sequence that converges to x_*. By condition (e) in Definition 3 there exists a sequence $(Q_n(\cdot))_n$ of paths in $\mathcal{Q}$ with $Q_n(0) = x_n$ that converges u.o.c. to $Q_*(\cdot)$. As $\mathcal{Q}$ is stable and using the same arguments as in the proof of Theorem 1,

$$\begin{aligned} V(x_*) = \int_0^\infty ||Q_*(s)||ds &= \lim_{n\to\infty} \int_0^\infty ||Q_n(s)||ds \\ &\le \liminf_{n\to\infty} V(x_n). \end{aligned}$$

That is, V is lower semi-continuous. ■

Now we state the main theorem, its proof is very close to that in [17].

Theorem 2: A strict closed GFN model $\mathcal{Q}$ is stable if and only if it admits a Lyapunov function. In particular V can be chosen as

$$V(x) = \sup_{Q(\cdot)\in\mathcal{Q}_x} \int_0^\infty ||Q(s)||ds.$$

Proof: First we show that the existence of a Lyapunov function is sufficient for stability. Let V be a Lyapunov function for $\mathcal{Q}$. From (8) it follows that $V(Q(t)) \ge 0$ and inequality (9) implies that

$$V(Q(t_2)) - V(Q(t_1)) \le 0$$

for all $t_1 \le t_2 \in \mathbb{R}_+$. So $V(Q(\cdot))$ is monotone decreasing and bounded. In order to show that $V(Q(t))$ tends to zero as t goes to infinity assume that there exists a $c \in \mathbb{R}_+$ with $c > 0$ such that

$$\lim_{t\to\infty} V(Q(t)) = c.$$

Then for all $t \ge 0$ it holds that

$$0 < c \le V(Q(t)) \le w_2(||Q(t)||) \tag{10}$$

and further $0 < w_2^{-1}(c) \le ||Q(t)||$. It also holds that $0 < w_3(w_2^{-1}(c)) \le w_3(||Q(t)||)$. Now observe that from (9) it follows that

$$\begin{aligned} V(Q(t)) - V(Q(0)) &\le -\int_0^t w_3(||Q(s)||)ds \\ &\le -\int_0^t w_3(w_2^{-1}(c))ds \\ &\le -w_3(w_2^{-1}(c))\, t \end{aligned}$$

and hence $\lim_{t\to\infty} V(Q(t)) = -\infty$, which is a contradiction to (10). Consequently

$$\lim_{t\to\infty} V(Q(t)) = 0. \tag{11}$$

By (8) it follows that

$$\lim_{t\to\infty} ||Q(t)|| = 0.$$

Now fix an $r \in (0, 1)$ and consider a $Q \in \mathcal{Q}(1)$. Since $Q(\cdot)$ tends to zero, there exists an $s > 0$ such that $||Q(s)|| = r$ and $||Q(t)|| > r$ for all $t \in [0, s)$. Observe that from (8) and (9) it follows that

$$\begin{aligned} 0 &\le V(Q(s)) = V(Q(0)) + V(Q(s)) - V(Q(0)) \\ &\le w_2(1) - \int_0^s w_3(||Q(t)||)\, dt \le w_2(1) - \int_0^s w_3(r)dt \\ &\le w_2(1) - w_3(r) \cdot s \end{aligned}$$

and further

$$s \le \frac{w_2(1)}{w_3(r)} =: t_r.$$

This t_r satisfies that for all $Q \in \mathcal{Q}(1)$ it hold that

$$\min\{t \ge 0 : ||Q(t)|| = r\} \le t_r. \tag{12}$$

Next we prove that for all $Q \in \mathcal{Q}(1)$ there exists a $\tau \in [0, \infty)$ such that $\min\{t \ge 0 : ||Q(t)|| = 0\} \le \tau$. So we construct for any given $Q \in \mathcal{Q}(1)$ a sequence $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$ such that

$$||Q(\tau_k)|| = r^k \qquad \text{for } k = 1, 2, \ldots\, .$$

In the beginning we use the scale and shift property and (12) to conclude that there is a $t_1 \leq t_r$ such that

$$\left|\left|\frac{1}{r}Q(r \cdot 0 + t_1)\right|\right| = 1.$$

So we choose $\tau_1 = t_1$. By repeating this, i.e. by choosing $t_2 \leq t_r$ such that

$$\left|\left|\frac{1}{r}\left(\frac{1}{r}Q(rt_2 + t_1)\right)\right|\right| = 1$$

and definig $\tau_2 := rt_2 + t_1$ we get $||Q(\tau_2)|| = r^2$. Through successive continuation in this manner we obtain

$$\tau_k = r^{k-1}t_k + r^{k-2}t_{k-1} + ... + t_1$$

such that $||Q(\tau_k)|| = r^k$. The sequence $(\tau_k)_k$ is Cauchy, as

$$\tau_k - \tau_{k-1} = r^{k-1}\,t_k \leq r^{k-1}\,t_r \qquad \text{for } k = 1, 2, \dots .$$

and consequently has the limit $\tau := \lim_{k\to\infty} \tau_k$. For the limit τ the following is true

$$||Q(\tau)|| = ||Q(\lim_{k\to\infty} \tau_k)|| = \lim_{k\to\infty} ||Q(\tau_k)|| = \lim_{k\to\infty} r^k = 0$$

for all $Q(\cdot) \in \mathcal{Q}(1)$ as (12) holds for all $Q(\cdot) \in \mathcal{Q}(1)$. Thus it remains to show that any $Q(t) \in \mathcal{Q}(1)$ stays zero for all $t \geq \tau$. So consider a $t \geq \tau$. As $w_i \in \mathcal{K}$ for $i = 1, 2$ and as V is a Lyapunov function for $\mathcal{Q}$ the following holds

$$\begin{aligned}0 \leq w_1(||Q(t)||) \leq V(Q(t)) \leq V(Q(\tau))\\ \leq w_2(||Q(\tau)||) = w_2(0) = 0.\end{aligned}$$

This shows the stability of the GFN model.

Conversely let $\mathcal{Q}$ be stable. Then there is a $\tau > 0$ such that $Q(\tau + \cdot) \equiv 0$ for all paths $Q(\cdot) \in \mathcal{Q}(1)$. We define the following comparison functions

$$w_1(r) := \frac{r^2}{2L}, \quad w_2(r) := r^2\,(1 + L\tau)\,\tau, \quad w_3(r) := r$$

and show that our candidate

$$V(x) = \sup_{Q(\cdot)\in\mathcal{Q}_x} \int_0^\infty ||Q(s)||ds$$

is a Lyapunov function. As $\mathcal{Q}$ satisfies the Lipschitz condition (a) it follows that

$$||\widetilde{Q}(s)|| \geq ||\widetilde{Q}(t)|| - L(s - t) \tag{13}$$

for all $s \geq t$. In particular for $t = 0$ that is

$$||\widetilde{Q}(s)|| \geq ||\widetilde{Q}(0)|| - Ls. \tag{14}$$

Using the last inequality we get the following estimate from below

$$\begin{aligned}V(x) &= \sup_{\widetilde{Q}(\cdot)\in\mathcal{Q}_x} \int_0^\infty ||\widetilde{Q}(s)||\,ds \geq \sup_{\widetilde{Q}(\cdot)\in\mathcal{Q}_x} \int_0^{\frac{||x||}{L}} ||\widetilde{Q}(s)||\,ds\\ &\geq \sup_{\widetilde{Q}(\cdot)\in\mathcal{Q}_x} \int_0^{\frac{||x||}{L}} (||x|| - Ls)\;ds\\ &= \sup_{\widetilde{Q}(\cdot)\in\mathcal{Q}_x} \left\{||x||\frac{||x||}{L} - \frac{||x||^2}{2L}\right\} = \frac{||x||^2}{2L} = w_1(||x||).\end{aligned}$$

To obtain an estimate from above consider $\widetilde{Q}(\cdot) \in \mathcal{Q}_x$. Note that by the scale property it follows that $\frac{1}{||x||}\widetilde{Q}(||x||s) \in \mathcal{Q}(1)$ and further the stability of $\mathcal{Q}$ implies that

$$\widetilde{Q}(s) = 0 \qquad \forall\, s \geq ||x||\tau. \tag{15}$$

The triangle inequality together with the Lipschitz condition imply that for all $s \in [0, ||x||\tau]$ it holds that

$$||\widetilde{Q}(s)|| \leq ||\widetilde{Q}(0)|| + L||x||\tau = ||x||\,(1 + L\tau). \tag{16}$$

With (15) and (16) an estimate from above is derived as follows

$$\begin{aligned}V(x) &= \sup_{\widetilde{Q}(\cdot)\in\mathcal{Q}_x} \int_0^{||x||\tau} ||\widetilde{Q}(s)||\,ds\\ &\leq \sup_{\widetilde{Q}(\cdot)\in\mathcal{Q}_x} \int_0^{||x||\tau} ||x||\,(1 + L\,\tau)\,ds\\ &= ||x||^2\,(1 + L\tau)\,\tau = w_2(||x||).\end{aligned}$$

Now consider the decrease condition

$$\begin{aligned}&V(Q(t_2)) - V(Q(t_1)) =\\ &\sup_{\widetilde{Q}(\cdot)\in\mathcal{Q}_{Q(t_2)}} \int_0^\infty ||\widetilde{Q}(s)||\,ds - \sup_{\widetilde{Q}(\cdot)\in\mathcal{Q}_{Q(t_1)}} \int_0^\infty ||\widetilde{Q}(s)||\,ds.\end{aligned}$$

From condition (f) it follows that

$$\begin{aligned}V(Q(t_1)) &= \sup_{\widetilde{Q}(\cdot)\in\mathcal{Q}_{Q(t_1)}} \int_0^\infty ||\widetilde{Q}(s)||\,ds\\ &\geq \int_{t_1}^{t_2} ||Q(s)||\,ds + \sup_{\widetilde{Q}(\cdot)\in\mathcal{Q}_{Q(t_2)}} \int_0^\infty ||\widetilde{Q}(s)||\,ds\\ &= \int_{t_1}^{t_2} ||Q(s)||\,ds + V(Q(t_2)).\end{aligned}$$

and hence

$$\begin{aligned}&V(Q(t_2)) - V(Q(t_1))\\ &\quad\leq -\int_{t_1}^{t_2} ||Q(s)||\,ds = -\int_{t_1}^{t_2} w_3(||Q(s)||)\,ds.\end{aligned}$$

Thus together with Proposition 2 we see that V is a Lyapunov function. ■

IV. ELEMENTS FROM CONTROL THEORY AND DIFFERENTIAL INCLUSIONS

We want to apply the main theorem to fluid network models that work under a specific discipline. SO we have to show that the additional conditions (e) and (f) are satisfied in each case. In order to go for condition (e) we make use of concepts that are common in control theory and involve differential inclusions. Clearly a detailed description of the dynamics of a fluid network depends on the specific discipline that is used. But one part of the dynamics of fluid network models that have all service disciplines in common is the so called flow balance relation

$$Q(t) = Q(0) + \alpha t - (I - P^T)\,MT(t). \tag{17}$$

Here α represents the inflow rate, μ denotes the outflow rate, $M = \mathrm{diag}(\mu)$ and P is the transition matrix. The initial value or level of the fluid network is given by $Q(0)$. A basic property of the fluid level process Q as well as the allocation process $T(t)$ is that both processes are Lipschitz continuous [10] and hence differentiable almost everywhere. So for almost all $t \in \mathbb{R}_+$ the flow balance relation (17) can also be written as

$$\dot{Q}(t) = \alpha - (I - P^T)\, M\dot{T}(t), \qquad Q(0) = x. \tag{18}$$

Now we consider the derivative of the allocation process as the control variable, i.e. $u(t) = \dot{T}(t)$. Note that u is measurable. The allocation process is determined through the service discipline. So each service discipline has a set of admissible controls $U(Q)$, where $u \in U(Q)$ if and only if $u \in \mathbb{R}^K_+$ satisfies some allocation conditions that are specific to the discipline. As mentioned in [10] the allocation process need not be unique and so for every $t \in \mathbb{R}_+$ there are different possibilities to choose $u(t)$. But the admissible control values u depend on the fluid level process $Q(t)$ through the allocation conditions. Consequently we consider the set of admissible control values as a set $U(Q(t))$. Thus the flow balance relation (18) can also be expressed by a differential inclusion of the form

$$\dot{Q}(t) = \alpha - (I - P^T)\, Mu(t) =: f(Q(t), u(t)), \quad Q(0) = x$$

with $u(t) \in U(Q(t))$. Often U is referred to as the feedback map. By setting

$$F(Q) = f(Q, U(Q)) = \bigcup_{u \in U(Q)} f(Q, u) \tag{19}$$

we rewrite this as a closed loop differential inclusion

$$\dot{Q} \in F(Q), \quad Q(0) = x. \tag{20}$$

In the following we state some results from the theory of differential inclusions that will be useful to show that specific fluid networks satisfy the conditions (e) and (f). Let X any Y be two normed spaces and consider a differential inclusion

$$\dot{x}(t) \in F(x(t)) \tag{21}$$

with initial condition $x(0) = x_0 \in X$. Let $\mathcal{S}_F(x_0)$ denote the set of all solutions of (21) starting at $x_0 \in X$. Before we can state the existence theorem we have to define when a set-valued map is Lipschitz. A set-valued map $F : X \rightsquigarrow Y$ is called Lipschitz around $x \in X$ if there exist a positive constant λ and a neighborhood $\mathcal{U} \subset \mathrm{Dom}(F)$ of x such that

$$F(x_1) \subset F(x_2) + \lambda\, ||x_1 - x_2||\, B_Y$$

for all $x_1, x_2 \in \mathcal{U}$, where $B_Y = \{y \in Y : ||y|| \le 1\}$. The existence theorem is as follows [18, Corollary 5.3.2].

Theorem 3: Assume that $F : X \rightsquigarrow X$ is Lipschitz on the interior of its domain. Then for any $x_0 \in \mathrm{Int}(\mathrm{Dom}(F))$ and $v_0 \in F(x_0)$ there exists a $T > 0$ and a solution $x(\cdot)$ to (21) defined on $[0, T]$ and satisfying $x(0) = x_0$ and $\dot{x}(0) = v_0$.

The next theorem of Filippov [18, Theorem 5.3.1 and Corollary 5.3.3] states conditions such that the solution of the differential inclusion depends Lipschitz continuously on the initial condition.

Theorem 4: Assume that $F : X \rightsquigarrow X$ is Lipschitz with constant λ and has closed values on the interior of its domain. Let $y(\cdot) \in \mathcal{S}_F(y_0)$ be a given absolutely continuous function. Then

$$d(y(t), \mathcal{S}_F(x_0)(t)) \le ||x_0 - y_0||\, e^{\lambda t}$$

so that the solution map $\mathcal{S}_F$ is lower semi-continuous.

V. APPLICATIONS TO SOME FLUID NETWORKS

In this section we show that our main result can be applied to some special fluid networks. In particular we show that fluid networks under general work-conserving and priority disciplines satisfy the additional conditions (e) and (f) given in Definition 3. The following description of a fluid network is taken from [17]. A fluid network consists of K different fluid classes and J stations, where the fluids are processed. Let $\mathcal{K} := \{1, ..., K\}$ and $\mathcal{J} := \{1, ..., J\}$. There is a (possibly not injective) map s that prescribes which fluid class is processed at which station. Fluid class k is exclusively processed at station $s(k)$. For every station the set $C(j) := \{k \in \mathcal{K} : s(k) = j\}$ can without loss of generality assumed to be nonempty. The corresponding $J \times K$ matrix C is called the constituency matrix, where $c_{jk} = 1$ if $s(k) = j$ and zero else. Further we introduce two nonnegative vectors $\alpha, \mu \in \mathbb{R}^K_+$ and a $K \times K$ substochastic matrix P. Where α_k denotes the exogenous inflow rate of fluid class k and μ_k denotes the potential outflow rate of fluid class k. The matrix P will be referred to as the flow transfer matrix. The element p_{kl} of P denotes the proportion of the outflow of class k which turns into fluid class l. So $1 - \sum_{l=1}^K p_{kl}$ is the part of the outflow of class k that leaves the network. The flow transfer matrix is assumed to have spectral radius strictly less than one, i.e. all fluids eventually leave the network. The initial fluid level is represented through the K-dimensional vector Q_0. The fluid network is described by (α, μ, P, C) with initial fluid level Q_0. The performance is described by the K-dimensional fluid level process $\{Q(t) : t \ge 0\}$ and the K-dimensional allocation process $\{T(t) : t \ge 0\}$, where $Q_k(t)$ denotes the amount of class k fluids in the network at time t and $T_k(t)$ denotes the total amount of time during the interval $[0, t]$ that station $s(k)$ has devoted to processing fluid class k. A precise description of the dynamics of a fluid network depends on the service discipline.

A. Fluid networks under general work-conserving disciplines

The dynamics of a fluid network under general work-conserving service discipline can be summarized as follows

$$Q(t) = Q_0 + \alpha\, t - (I - P^T) MT(t) \ge 0, \tag{22}$$

$$T(0) = 0 \text{ and } T(\cdot) \text{ is nondecreasing}, \tag{23}$$

$$I(t) = et - C\, T(t) \text{ and } I(\cdot) \text{ is nondecreasing}, \tag{24}$$

$$0 = \int_0^\infty (C\, Q(t))^T\; dI(t), \tag{25}$$

where $M = \text{diag}(\mu)$. Equation (25) describes the work-conserving property of the network and relation (22) is called the flow balance relation. In general the allocation process is not unique. Any pair $(Q(t), T(t))$ that satisfies (22)-(25) is called fluid solution of the work-conserving fluid network. The set of all feasible fluid level processes is denoted as

$$\mathcal{Q}_C = \{Q(t) : \exists T(t) : (Q(t), T(t)) \text{ is a fluid solution}\}.$$

Before we prove a theorem that guarantees the existence of such a work-conserving allocation process we bring the conditions (23)-(25) into the context of differential inclusions. That is, $\dot{T}(t) = u$ and

$$u \geq 0, \quad e - Cu \geq 0, \quad (CQ(t))^T \cdot (e - Cu) = 0. \quad (26)$$

So the set of admissible controls is

$$U_C(Q) := \left\{u \in \mathbb{R}_+^K \,:\, (26) \text{ are satisfied } \right\}.$$

We consider the following set-valued map

$$G_C(Q) = \bigcup_{u \in U_C(Q)} \left\{\alpha - (I - P^T)\, Mu\right\} \quad (27)$$

and the corresponding differential inclusion

$$\dot{Q} \in G_C(Q), \quad Q(0) = x. \quad (28)$$

Using this approach we are able to give an alternate proof for the Theorem 2.1 in [10]. That is, a fluid network under general work-conserving service discipline and given initial level has at least one solution.

Theorem 5: For any fluid network (α, μ, P, C) with initial level x the set $\mathcal{Q}_C$ is nonempty.

Proof: The set $U_C(Q)$ is compact and by condition (26) the mapping

$$Q \mapsto U_C(Q)$$

is Lipschitz. Hence, G in (28) is Lipschitz and applying Theorem 3 completes the proof. ■

In [17] it is shown that $\mathcal{Q}_C$ defines a closed GFN model. So we only have to prove that the conditions (e) and (f) are satisfied. The key tool to proving condition (e) is Theorem 4, which is often called Filippov's theorem in the literature.

Proposition 3: $\mathcal{Q}_C$ is perfect.

Proof: To show that $\mathcal{Q}_C$ satisfies condition (e) in the Definition 3 we make use of Filippov's Theorem. So what is left to show is that G_C has closed valued. This follows from the fact that $U_C(Q)$ is compact. ■

Proposition 4: $\mathcal{Q}_C$ satisfies the concatenation property.

Proof: Since for every $x \in \mathbb{R}_+^K$ the set of solutions $\mathcal{S}_{G_C}(x)$ is nonempty, it follows immediately that the concatenation property is satisfied. ■

Summarizing we obtain.

Theorem 6: The work-conserving fluid network $\mathcal{Q}_C$ is a strict closed GFN model.

B. Fluid networks under priority disciplines

The priority service discipline distributes different priorities to the fluid classes that are processed at one location. This is done via a permutation mapping $\pi : \mathcal{K} \to \mathcal{K}$. To be precise, let $s(l) = s(k)$ for $l, k \in \mathcal{K}$ then fluids of class l have higher priority than fluids of class k, if $\pi(l) < \pi(k)$. That is, fluids of class k are not processed as long as the fluid level of class l is greater than zero. For each $k \in \mathcal{K}$ the set of fluid classes that are processed at the same location $s(k)$ that have higher priority is denoted by $H_k = \{l : l \in C(s(k)), \pi(l) \leq \pi(k)\}$. To derive a description of fluid network under the priority discipline π we introduce the unused capacity process $Y(t)$. That is $Y_k(t)$ is denotes the cumulative remaining capacity of location $s(k)$ for processing fluids of classes that have strictly lower priority than fluids of class k. The dynamics can be described as follows

$$Q(t) = Q_0 + \alpha\, t - (I - P^T)MT(t) \geq 0, \quad (29)$$

$$T(0) = 0 \text{ and } T(\cdot) \text{ is nondecreasing}, \quad (30)$$

$$Y_k(t) = t - \sum_{l \in H_k} T_l(t) \text{ and } Y(\cdot) \text{ is nondecreasing}, \; k \in \mathcal{K} \quad (31)$$

$$0 = \int_0^\infty Q_k(t)\; dY_k(t), \qquad k \in \mathcal{K}. \quad (32)$$

Any pair $(Q(t), T(t))$ that satisfies (29)-(32) is called fluid solution of the fluid network under the priority discipline π. The set of all feasible fluid level processes is denoted as

$$\mathcal{Q}_P = \{Q(t) : \exists T(t) : (Q(t), T(t)) \text{ is a fluid solution}\}.$$

Again we bring this into the context of differential inclusions. The constraints for $k \in \mathcal{K}$ are here

$$u \geq 0, \quad 1 - \sum_{l \in H_k} u_l \geq 0, \quad Q_k \cdot (1 - \sum_{l \in H_k} u_l) = 0 \quad (33)$$

and the set of admissible controls is

$$U_P(Q) := \left\{u \in \mathbb{R}_+^K \,:\, (33) \text{ are satisfied } \right\}.$$

Note that $U_P(Q)$ is compact and $Q \mapsto U_P(Q)$ is Lipschitz. Thus from Filippov's Theorem the following can be concluded.

Theorem 7: The set $\mathcal{Q}_P$ is nonempty and perfect.

In order to prove that $\mathcal{Q}_P$ is a strict closed GFN model it remains to show that the concatenation property holds, as the validity of the conditions (a)-(d) is shown in [17, Lemma 3.5]. Some straightforward arguments lead to the following result.

Proposition 5: $\mathcal{Q}_P$ satisfies the concatenation property.

Thus the following holds true.

Theorem 8: The fluid network under priority discipline $\mathcal{Q}_P$ is a strict closed GFN model.

VI. THE LINEAR SKOROKHOD PROBLEM

Another possible way to approximate a multiclass queueing network is to consider the so called diffusion limit. This limit can be regarded as a semi-martingale reflected

Brownian motion (SRBM). Similar to the fluid limit, a sufficient condition for the stability of the SRBM is the stability of the linear Skorokhod problem (LSP) [19]. The following description is taken from [20] and [17]. Let R be a $J \times J$ matrix, $\theta \in \mathbb{R}^J$ and $X_0 \in \mathbb{R}^J_+$. The pair $(X, Y) \in C^J(\mathbb{R}_+)$ is said to solve the LSP (θ, R) with initial state X_0, if they jointly satisfy

$$X(t) = X_0 + \theta t + RY(t) \geq 0, \tag{34}$$

$$Y(0) = 0 \text{ and } Y(\cdot) \text{ is nondecreasing}, \tag{35}$$

$$0 = \int_0^\infty X_j(t)\, dY_j(t), \qquad j = 1, ..., J. \tag{36}$$

The first question that arises is, which conditions guarantee the existence of a solution of the LSP(θ, R). In oder to state such a condition a $J \times J$ matrix R is said to be an S-matrix, if there exists an $x \geq 0$ such that $Rx > 0$, and is said to be completely-S if all of its principal submatrices are S-matrices. The following theorem from [21, Theorem 1] contains the desired statement.

Theorem 9: The LSP(θ, R) has a solution (X, Y) if and only if the matrix R is completely-S.

Analogous to the previous subsections we define

$$\mathcal{Q}_{LSP} = \{X(t) : \exists\, Y(t) : (X(t), Y(t)) \text{ satisfy } (34) - (36)\}.$$

Note that Theorem 9 states only the existence of a solution. In general the solution is not unique, for a counterexample see e.g. [21].

Definition 5: A LSP(θ, R) is said to be stable if, for any number $\varepsilon > 0$ and any $X \in \mathcal{Q}_{LSP}$ with $||X_0|| = 1$, there exists a $\tau \geq 0$ such that $||X(\tau + \cdot)|| < \varepsilon$.

To ensure that the set $\mathcal{Q}_{LSP}$ is nonempty, Theorem 9 states that R has to be completely-S. In [17, Theorem 5.2] it is shown that in this case Definition 5 is equivalent to Definition 2. To derive a necessary and sufficient condition for stability of the linear Skorokhod problem we have to show that $\mathcal{Q}_{LSP}$ is a strict closed GFN model. The next lemma from [21, Lemma 1] or [17, Lemma 5.1] contains that $\mathcal{Q}_{LSP}$ satisfies the Lipschitz condition.

Lemma 1: If the matrix R is completely-S, then there exists a constant M such that any solution (X, Y) of LSP(θ, R) is Lipschitz continuous with constant M.

The fact that $\mathcal{Q}_{LSP}$ is closed follows from Proposition 1 in [21]. Furthermore that the scale, shift property hold is stated in [22, Section 2]. A straightforward argument shows the validity of the concatenation property.

Proposition 6: $\mathcal{Q}_{LSP}$ satisfies the concatenation condition (f).

It remains to show that $\mathcal{Q}_{LSP}$ is perfect. Again we bring the linear Skorokhod problem into the context of differential inclusions. That is, let $\dot{Y}(t) = u$ and

$$G(X) = \bigcup_{u \in U_{LSP}(X)} \{\theta + Ru\}, \tag{37}$$

where the set of admissible controls U_{LSP} is determined through the conditions

$$u \geq 0, \quad X_j\, u_j = 0, \quad \forall\, t \geq 0,\ j = 1, ..., J. \tag{38}$$

While it is clear that the set described by (38) is unbounded on the boundary of the positive orthant, Lemma 1 may be used to see that the effective set of controls is bounded. The corresponding differential inclusion is of the form

$$\dot{X}(t) \in G(X(t)), \qquad X(0) = X_0. \tag{39}$$

Using this formulation it is again possible but more involved to obtain the following result, which we do not prove here for reasons of space.

Theorem 10: $\mathcal{Q}_{LSP}$ is a strict closed GFN model.

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