On the construction of ISS Lyapunov functions for networks of ISS systems

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***Abstract*— We consider a finite number of nonlinear systems interconnected in an arbitrary way. Under the assumption that each subsystem is input-to-state stable (ISS) regarding the states of the other subsystems as inputs we are looking for conditions that guarantee input-to-state stability of the overall system. To this end we aim to construct an ISS-Lyapunov function for the interconnection using the knowledge of ISS-Lyapunov functions of the subsystems in the network. Sufficient conditions of a small gain type are obtained under which an ISS Lyapunov function can be constructed. The ISS-Lyapunov function is then given explicitly, and guarantees that the network is ISS.**

***Keywords*— Nonlinear systems, Input-to-state stability, ISS Lyapunov function, small gain condition.**

I. INTRODUCTION

The property of input-to-state stability (ISS) has been introduced by Sontag [12] at the end of the last century and is now a commonly used tool to study stability properties of control systems. One of the strengths of the theory is that it naturally provides a framework in which the effect of interconnection of families of systems can be studied. This raises the question to which extent the study of large scale systems as available in the book by Šiljak [11] can be extended to the nonlinear setting. This paper endeavours to provide a contribution to this project by treating the problem of constructing ISS Lyapunov functions for a large scale system based on the knowledge of Lyapunov functions for the subsystems. The main condition to make such a construction possible is of the small gain type.

The notion of nonlinear gains of perturbed nonlinear systems has been shown to be a useful tool studying the stability of feedback systems, see e.g. [12], [7], [9]. In particular, several types of small-gain theorems for the stability of a feedback of two systems have been obtained by different authors, e.g., [8], [9], [6]. In [3] an arbitrary interconnection of more than two ISS systems is considered, and a generalized small-gain theorem is obtained. In this paper we continue the investigation of this problem and we wish to obtain statements concerning the construction of ISS Lyapunov functions from known ISS Lyapunov functions for the subsystems. This problem was also treated in [9] where an ISS Lyapunov function was constructed for a feedback loop of two ISS systems provided a small gain condition holds. We wish to extend this result for the case of more than two systems and present some preliminary results in this direction.

The paper is organized as follows. In the following Section II we recall the definitions of the basic concepts of input-to-state stability and of ISS Lyapunov functions. While it is in general only necessary to use smooth Lyapunov functions in the framework of the theory, we rely at one stage on a few results from nonsmooth analysis, so that ISS Lyapunov functions are also defined in a nonsmooth fashion. In Section III we introduce the interconnected systems under consideration in this paper and define the properties of the ISS Lyapunov functions we consider. In this construction the Lyapunov gains describing the effect of the subsystems on each other play a crucial role. The collection of these gains defines a monotone operator from the positive orthant of $\mathbb{R}^n$ to itself. Roughly speaking, the small gain condition already used in [3] states that this monotone operator should be robustly nowhere increasing. If the Lyapunov gains are linear functions a general construction of ISS Lyapunov functions is possible using this property. This case is treated in Section III-A.

The general nonlinear case is treated in Section IV. The desirable result would be that from the small gain condition it follows that an ISS Lyapunov function can be constructed. Unfortunately, we are only able to show this in the case $n = 2, 3$. The remaining problem is that from the small gain condition it follows that there exists an unbounded subset of the positive orthant on which the gain operator is strictly decreasing. For the construction of the ISS Lyapunov function we require the existence of a continuously differentiable, strictly increasing curve contained in the set on which the gain operator is strictly decreasing. We conjecture that the small gain condition is equivalent to the existence of such a curve, but this topological problem remains open in this paper.

II. BASIC DEFINITIONS

Let x be a vector in $\mathbb{R}^n$ and x^T its transpose. Let $|\cdot|$ denote the usual Euclidean norm in $\mathbb{R}^n$ and $\|\cdot\|$ the L_∞-norm. For $x, y \in \mathbb{R}^n_+$ the relation $x \geq y$ ($x > y$) means that $x_i \geq y_i$ ($x_i > y_i$) holds for $i = 1, \dots, n$. By $(\dot{})$ we denote the time derivative. Let

$$\dot{x} = f(x,u), x \in \mathbb{R}^n, u \in \mathbb{R}^m, f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \quad (1)$$

be a nonlinear dynamical system with a continuous function f such that for any $r > 0$ it is locally Lipschitz in x uniformly for all inputs u with $\|u\| < r$. The input functions u in (1) are assumed to be elements of

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$L_\infty(0,\infty)$. We say that $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a function of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. If, in addition, it is unbounded then it is of class $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed t and for each fixed s decreases in t with $\lim_{t\to\infty} \beta(s, t) = 0$.

Definition 1: If there exist $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that for any initial point $x(0)$ and any L_∞-input u the trajectory $x(t)$ of the system (1) satisfies

$$|x(t)| \le \beta(|x(0)|, t) + \gamma(\|u\|),\ \forall\, t \in \mathbb{R}_+, \qquad (2)$$

then the system (1) is called ISS from u to x and γ is called nonlinear gain function or briefly gain.

Definition 2: A smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an ISS-Lyapunov function of (1) if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, a positive definite function α and $\chi \in \mathcal{K}$ with

$$\psi_1(|x|) \le V(x) \le \psi_2(|x|), x \in \mathbb{R}^n, \qquad (3)$$

$$V(x) \ge \chi(|u|) \Rightarrow \nabla V(x) f(x, u) \le -\alpha(V(x)). \qquad (4)$$

The function χ is then called Lyapunov-gain.

It is known that the ISS property of (1) is equivalent to the existence of an ISS-Lyapunov function for (1), see [13]. But note that the gain in (2) and the Lyapunov-gain in (4) are in general different functions.

For our construction we will need the notions of proximal subgradient and nonsmooth ISS-Lyapunov function, cf. [2], [1].

Definition 3: A vector $\zeta \in \mathbb{R}^n$ is called a proximal subgradient of a function $\phi : \mathbb{R}^n \to (-\infty, \infty]$ at $x \in \mathbb{R}^n$ if there exists a neighborhood $U(x)$ of x and a number $\sigma \ge 0$ such that

$$\phi(y) \ge \phi(x) + \langle \zeta, y - x\rangle - \sigma|y - x|^2 \quad \forall y \in U(x).$$

The set of all proximal subgradients at x is called proximal subdifferential of ϕ at x and is denoted by $\partial_P \phi(x)$.

Definition 4: A continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be a nonsmooth ISS-Lyapunov function of the system (1) $\dot{x} = f(x, u)$, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ if

1) V is proper and positive-definite, that is, there exist functions ψ_1, ψ_2 of class $\mathcal{K}_\infty$ such that

$$\psi_1(|x|) \le V(x) \le \psi_2(|x|), \quad \forall x \in \mathbb{R}^n; \qquad (5)$$

2) there exists a positive-definite function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ and a class $\mathcal{K}$-function χ, such that

$$\sup_{u:\ V(x)\ge\chi(|u|)} \langle f(x, u), \zeta\rangle \le -\alpha(V(x)), \qquad (6)$$

$\forall \zeta \in \partial_P V(x), \forall x \ne 0.$

See also [2], p. 188 and Theorem 4.6.3.

III. A NETWORK OF INTERCONNECTED SYSTEMS

Now consider the interconnected systems

$$\dot{x}_i = f_i(x_1, \ldots, x_n, u_i), x_i \in \mathbb{R}^{N_i}, u_i \in \mathbb{R}^{m_i}, i = 1, \ldots, n \qquad (7)$$

with $f_i, i = 1, \ldots, n$ having the same continuity properties as f in (1). Each of them is assumed to be ISS from $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, u_i)$ to x_i and hence to have an ISS-Lyapunov function V_i, i.e., there are $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ and $\chi_{ij}, \gamma_i, \alpha_i \in \mathcal{K}, i, j = 1, \ldots, n,\ j \ne i$; (we set $\chi_{ii} = 0, i = 1, \ldots, n$) with

$$\psi_{i1}(|x_i|) \le V_i(x_i) \le \psi_{i2}(|x_i|), x_i \in \mathbb{R}^{N_i}, \qquad (8)$$

$$\begin{aligned} &V_i(x_i) \ge \sum_{j=1}^{n} \chi_{ij}(V_j(x_j)) + \gamma_i(|u_i|) \\ &\Rightarrow \nabla V_i(x) f_i(x, u_i) \le -\alpha_i(V_i(x_i)), \end{aligned} \qquad (9)$$

where we denote $x = (x_1^T, \ldots, x_n^T)^T \in \mathbb{R}^N$ with $N := N_1 + \cdots + N_n$.

We remark that in [9] instead of sum in (9) there is $\max$ used, what leads to a slightly different small-gain condition than we have below.

The question is, under which conditions the interconnection (7) is ISS and how to construct an ISS-Lyapunov function for it. To study this point we introduce the following nonlinear operator: The gain operator Γ of the interconnection (7) is defined on the positive orthant $\mathbb{R}^n_+$ by

$$\begin{aligned} \Gamma(s) :=& \left(\sum_{j=1}^{n} \chi_{1j}(s_j), \ldots, \sum_{j=1}^{n} \chi_{nj}(s_j) \right)^T, \\ s =& (s_1, \ldots, s_n)^T \in \mathbb{R}^n_+, \end{aligned} \qquad (10)$$

where functions $\chi_{ij}, i, j = 1, \ldots, n$ are the Lyapunov-gains of (7) defined in (9). This operator was introduced in [5], where the authors also study its properties.

A. Linear Lyapunov-gains

To demonstrate the idea of the construction of the ISS-Lyapunov function for the interconnection (7), consider first the case, where χ_{ij} are linear functions. In this case the sum $\sum_{j=1}^{n} \chi_{ij}(V_j(x_j))$ in (9) is nothing but a matrix-vector product $\Gamma V(x)$ of Γ defined as the matrix of constants χ_{ij} and $V(x) = (V_1(x_1), \ldots, V_n(x_n))^T$.

Under the condition

$$\rho(\Gamma) < 1, \qquad (11)$$

where $\rho(\Gamma)$ denotes the spectral radius of Γ, there exists a vector $s \in \mathbb{R}^n_+$ with positive components satisfying

$$s_i > \sum_{j=1}^{n} \chi_{ij} s_j, \quad i = 1, \ldots, n. \qquad (12)$$

In case of an irreducible Γ the vector s may be taken to be a Perron-Frobenius eigenvector $s \in \mathbb{R}^n_+$ of Γ. For a reducible Γ the existence of such s with (12) follows from [10], Theorem 15.3.1 and the continuity of the spectral radius of a matrix on its elements. One can namely increase each element of Γ to become positive in such way that the spectral radius remains less than one.

Lemma 5: Let V_i be an ISS-Lyapunov function for the i-th system from (7) satisfying (8) and (9) with linear gains $\chi_{ij},\ i, j = 1, \ldots, n$. Let $\Gamma = (\chi_{ij})_{i,j=1,\ldots,n}$ of (7) satisfy (11), then the interconnection (7) is ISS. Furthermore there

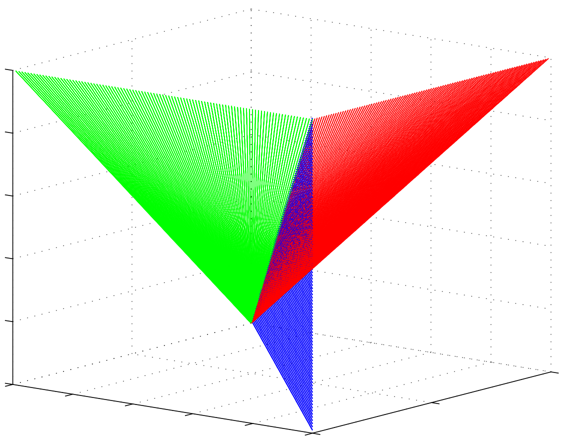

Fig. 1. Domains M_1, M_2, M_3 in $\mathbb{R}^3$

exists $s \in \mathbb{R}^n_+$ with positive components satisfying (12) and an ISS-Lyapunov function of (7) is given by

$$V(x) := \max_i \frac{V_i(x_i)}{s_i}. \qquad (13)$$

Proof In the following we show that there exists a positive definite function α and $\gamma \in \mathcal{K}$ such that:

$$\sup_{u:\, V(x) \geq \gamma(\|u\|)} \langle f(x,u), \zeta \rangle \leq -\alpha(x) \qquad (14)$$

$\forall \zeta \in \partial_P V(x), \forall x \neq 0$.

Let M_i be open domains in $\mathbb{R}^n_+$ defined by

$$M_i := \Big\{ (v_1, \dots, v_n)^T \in \mathbb{R}^n_+ : \ \frac{v_i}{s_i} > \max_{j \neq i} \Big\{ \frac{v_j}{s_j} \Big\} \Big\}. \qquad (15)$$

From this definition follows that

$$M_i \bigcap M_j = \emptyset, \ i \neq j, \text{ and } \bigcup_{i=1}^n \overline{M}_i = \mathbb{R}^n_+,$$

where $\overline{M}_i$ is the closure of M_i. Note that V defined by (13) is continuous in $\mathbb{R}^N_+$ and can fail to be differentiable only at those points where $\frac{V_i(x_i)}{s_i} = \frac{V_j(x_j)}{s_j}$ for some $i \neq j$.

Now take any $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n) \in \mathbb{R}^N$ with $(V_1(\hat{x}_1), \dots, V_n(\hat{x}_n)) \in M_i$ then it follows that in some neighborhood U of $\hat{x}$ we have $V(x) = \frac{V_i(x_i)}{s_i}$ for all $x \in U$, so that V is differentiable in $x \in U$. Our aim is to show that there exists a positive definite function $\tilde{\alpha}_i$ and $\phi \in \mathcal{K}$ such that $V(x) > \phi(\|u\|)$ implies $\nabla V(x) f(x,u) < -\tilde{\alpha}_i(V(x))$.

Consider the inequality

$$V(x) > \frac{\gamma_i(\|u\|)}{s_i - \sum \chi_{ij} s_j}, \qquad (16)$$

i.e.,

$$\frac{V_i(x_i)}{s_i} \Big(s_i - \sum_{j=1}^n \chi_{ij} s_j \Big) > \gamma_i(\|u\|)$$

or

$$V_i(x_i) > \frac{V_i(x_i)}{s_i} \sum_{j=1}^n \chi_{ij} s_j + \gamma_i(\|u\|).$$

By the definition of M_i this implies

$$V_i(x_i) > \sum_{j=1}^n \frac{V_j(x_j)}{s_j} \chi_{ij} s_j + \gamma_i(\|u\|)$$

$$= \sum_{j=1}^n \chi_{ij} V_j(x_j) + \gamma_i(\|u\|).$$

Then from (9) it follows that

$$\begin{aligned} \nabla V(x) f(x,u) &= \frac{1}{s_i} \nabla V_i(x_i) f_i(x,u) \\ &\leq -\frac{1}{s_i} \alpha_i(V_i(x_i)) < -\tilde{\alpha}_i(V(x)), \end{aligned} \qquad (17)$$

where $\tilde{\alpha}_i$ is positive-definite function, since s_i is a positive constant. Then (14) follows with $\gamma(r) = \max_i \frac{\gamma_i(r)}{s_i - \sum \chi_{ij} s_j}$ and $\alpha(r) := \min_i \{\tilde{\alpha}_i(r)\}$.

It remains to consider $x \in \mathbb{R}^n$ such that $(V_1(x_1), \dots, V_n(x_n)) \in \overline{M_i} \cap \overline{M_j}$, where $V(x)$ may be not differentiable.

For this purpose we use some results from [2]. For smooth functions g_i, $i = 1, \dots, n$ it follows that $g(x,u) = \max_i \{g_i(x,u)\}$ is Lipschitz continuous and Clarke's generalized gradient of g is given by , cf. [2],

$$\begin{aligned} \partial_{Cl} g(x) &= co \Big\{ \bigcup_{i \in M(x)} \nabla_x g_i(x,u) \Big\}, \\ M(x) &= \{ i : g_i(x,u) = g(x) \}, \end{aligned} \qquad (18)$$

where co denotes the convex hull. In our case

$$\partial_{Cl} V(x) = co \Big\{ \frac{1}{s_i} \nabla V_i(x) : \ \frac{1}{s_i} V_i(x) = V(x) \Big\}. \qquad (19)$$

Note, that directly from the definitions of $\partial_P V(x)$ and $\partial_{Cl} V(x)$, see [2], e.g., it follows that $\partial_{Cl} V(x) \supset \partial_P V(x)$. Now for every extremal point of $\partial_{Cl} V(x)$ the decrease condition (17) is satisfied. By convexity, the same is true for every element of $\partial_{Cl} V(x)$. Now Theorems 4.3.8 and 4.5.5 of [2] show the strong invariance and attractivity of the set $\{x : V(x) \leq \gamma(\|u\|)\}$. It follows that V is an ISS-Lyapunov function for the interconnection (7). $\square$

Note that for linear Γ the condition (11) is equivalent to

$$\chi(s) \not\geq s, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\}, \qquad (20)$$

or in other words, for any $s \in \mathbb{R}^n_+$ there is at least one $i \in \{1, \dots, n\}$ such that the i-th component of $\Gamma(s)$ is strictly less then s_i. The property (20) is meaningful also for nonlinear Γ.

IV. Main result

In this section we generalize the ideas of construction of an ISS-Lyapunov function to the nonlinear case. The condition (11) makes no sense if Γ is nonlinear, however (20) still can be applied, which can also be written as $\Gamma \not\geq \mathrm{id} \quad \text{on} \quad \mathbb{R}^n_+ \setminus \{0\}$. Recall that an ISS-criterion of the interconnection (7) was obtained in [3], [5], where it was shown, that the last condition is not sufficient for the ISS of (7). The small gain condition used there is a bit stronger, namely, if there exists an auxiliary diagonal operator $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by $D = \mathrm{diag}(\mathrm{id} + \alpha)$, i.e.,

$$D(s) = (s_1 + \alpha(s_1), \ldots, s_n + \alpha(s_n))^T,$$

where $s = (s_1, \ldots, s_n)$ and $\alpha \in \mathcal{K}_\infty$, such that the gain operator Γ satisfies

$$D \circ \Gamma \not\geq \mathrm{id} \quad \text{on} \quad \mathbb{R}^n_+ \setminus \{0\}, \tag{21}$$

then (7) is ISS. Here and in the following $\circ$ denotes a composition of two operators.

See in [5] also the explanations about the changes in the small gain condition (21) in case of use of $\max$ instead of the sum in (9).

Theorem 6: Let V_i be an ISS-Lyapunov function for the i-th system in (7), $\;i = 1, \ldots, n$, i.e., (8) and (9) hold. Assume there exist continuously differentiable $\sigma_i \in \mathcal{K}_\infty$ with $\sigma_i'(s) > 0$ for all $s > 0$ such that

$$\sigma_i(t) > (\mathrm{id} + \alpha)\Big(\sum_{\mathrm{j}=1}^{\mathrm{n}} \chi_{\mathrm{ij}}(\sigma_{\mathrm{j}}(\mathrm{t}))\Big), \qquad \forall t > 0, \quad i = 1, \ldots, n, \tag{22}$$

for some $\alpha \in \mathcal{K}_\infty$. Then the interconnection (7) is ISS with ISS-Lyapunov function

$$V(x_1, \ldots, x_n) := \max_i \{\sigma_i^{-1}(V_i(x_i))\}. \tag{23}$$

The condition (22) states, that there is a curve in $\mathbb{R}^n_+$ parameterized by $\sigma_i \in \mathcal{K}_\infty, \; i = 1, \ldots, n$ such that for any point $s \neq 0$ on the curve the condition $s_i > (\mathrm{id} + \alpha)(\sum_{\mathrm{j}=1}^{\mathrm{n}} \chi_{\mathrm{ij}}(\mathrm{s_j}))$ holds for all $i = 1, \ldots, n$, which is a nonlinear version of (12). Before we prove the theorem let us consider this curve closer.

Lemma 7: The existence of $\sigma_i, i = 1, \ldots, n$, as in (22) implies that Γ satisfies (21).

Proof Assume there is an $0 \neq x \in \mathbb{R}^n_+$ such that $D \circ \Gamma x \geq x$. Then the sequence $x(k), \; k \in \mathbb{N}$ defined by

$$x(k+1) := D \circ \Gamma(x(k)), \; x(0) := x, \; k \in \mathbb{N}$$

is unbounded in $\mathbb{R}^n_+$. Since $\sigma_i \in \mathcal{K}_\infty$ there is a positive number $t =: t(0)$ big enough, such that there is a point on the curve with $s(0) := \sigma(t(0)) > x(0)$ and hence $\Gamma(s(0)) \geq \Gamma(x(0))$. By (22) we have $s(0) > s(1) := D \circ \Gamma(s(0)) > D \circ \Gamma(x(0)) =: x(1)$. Note that $s(1)$ may not belong to σ, however from the continuity of Γ follows that there exists $t(1) < t(0)$ such that $s^*(1) := \sigma(t(1)) > s(1) > x(1)$. Then $s(2) := D \circ \Gamma(s^*(1)) \geq D \circ \Gamma(x(1)) =: x(2)$ and again from the continuity of Γ there is $s^*(2)$ on the curve σ such that $s^*(2) > s(2) \geq x(2)$. By iteration we obtain a bounded sequence $s^*(k), \; k \in \mathbb{N}$ which dominates the sequence $x(k), \; k \in \mathbb{N}$. This is a contradiction. The lemma is proved. □

We believe that converse is also true:

Conjecture 8: If Γ satisfies (21) then there exist $\sigma_i \in \mathcal{K}_\infty, i = 1, \ldots, n$ with (22).

Let us present some arguments that count in favor of the claim: In the linear case this curve is, for example, the ray defined by the vector s from (12). In the nonlinear case for $n = 2$ the existence of such a curve follows from [9]. Below we construct such a curve for $n = 3$. Moreover it is shown in [4], Proposition 5.6 that there is an unbounded domain $\Omega \in \mathbb{R}^n_+$, such that for any point $x \in \Omega$ the inequality $D \circ \Gamma(x) < x$ holds, and for any $r > 0$ the simplex $S_r := \{s \in \mathbb{R}^n_+ : \; s_1 + \cdots + s_n = r\}$ has a nonempty intersection $\Omega \cap S_r \neq \emptyset$ with this domain. The desired curve has to be in Ω and since Ω intersects every S_r there seems to be sufficient room to construct it. This problem however remains open.

Proof of Theorem 6 Having $\sigma_1(t), \ldots, \sigma_n(t)$ satisfying (22) the idea of the proof is essentially the same as for Lemma 5. We define

$$M_i := \Big\{(v_1, \ldots, v_n)^T \in \mathbb{R}^n_+ : \sigma_i^{-1}(v_i) > \max_{j \neq i}\{\sigma_j^{-1}(v_j)\}\Big\}. \tag{24}$$

From (22) it follows that

$$\sigma_i(t) - \sum_{j=1}^{n} \chi_{ij}(\sigma_j(t)) > \alpha\Big(\sum_{j=1}^{n} \chi_{ij}(\sigma_j(t))\Big) =: \rho(t). \tag{25}$$

Note that $\rho \in \mathcal{K}_\infty$. Now for any $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) \in \mathbb{R}^N$ with $(V_1(\hat{x}_1), \ldots, V_n(\hat{x}_n)) \in M_i$ it follows that there is a neighborhood U of $\hat{x}$ such that $V(x) = \sigma_i^{-1}(V_i(x_i))$ holds for all $x \in U$, so that V is differentiable in $x \in U$. Again we are looking for a positive definite function $\tilde{\alpha}_i$ and $\phi \in \mathcal{K}$ such that $V(x) > \phi(\|u\|)$ implies $\nabla V(x) f(x, u) < -\tilde{\alpha}_i(V(x))$.

To derive the defining inequality of ISS Lyapunov functions consider the inequality

$$V(x) > \rho^{-1}(\gamma_i(|u|)). \tag{26}$$

From this inequality it follows that $\rho(V(x)) > \gamma_i(|u|)$ or using the definition of ρ

$$\sigma_i(V(x)) - \sum_{j=1}^{n} \chi_{ij}(\sigma_j(V(x))) > \gamma_i(|u|),$$

or equivalently

$$\begin{aligned} V_i(x_i) = \sigma_i(V(x)) &> \sum_{j=1}^{n} \chi_{ij}(\sigma_j(V(x))) + \gamma_i(|u|) \\ &= \sum_{j=1}^{n} \chi_{ij}(\sigma_j(\sigma_i^{-1}(V_i(x_i)))) + \gamma_i(|u|) \\ &> \sum_{j=1}^{n} \chi_{ij}(V_j(x_j)) + \gamma_i(|u|), \end{aligned} \tag{27}$$

where we have used $(V_1(\hat{x}_1),\dots,V_n(\hat{x}_n)) \in M_i$ in the last inequality. Summarizing this shows that (26) implies

$$V_i(x_i) > \sum_{j=1}^{n} \chi_{ij}(V_j(x_j)) + \gamma_i(|u|),$$

and hence from (9) we obtain

$$\begin{aligned}\nabla V(x) f(x,u) &= (\sigma_i^{-1})'(V_i(x_i))\nabla V_i(x_i) f_i(x,u)\\ &\le -(\sigma_i^{-1})'(V_i(x_i))\alpha_i(V_i(x_i)) =: -\tilde{\alpha_i}(V(x)),\end{aligned} \tag{28}$$

where $\tilde{\alpha_i}$ is a positive definite function by definition. It remains to treat the points where V may fail to be differentiable. The argument for this case is the same as in the proof of Lemma 5. □

A. Construction of σ

Here we show how to construct $\sigma_i \in \mathcal{K}_\infty$ satisfying (22) given (21) for $n = 3$. For $n = 2$ such a curve can be constructed as in [9]

Lemma 9: Let $n = 3$ and nonlinear gains $\chi_{ij} \in \mathcal{K}_\infty$ or $\chi_{ij} = 0,\ i,j = 1,2,3,$ satisfy (21). Then there exist functions $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{K}_\infty$ satisfying (22) and $\sigma_i' > 0,\ i = 1,2,3.$

Proof First assume χ_{12}, χ_{13} are nonzero functions, i.e., $\chi_{12}, \chi_{13} \in \mathcal{K}_\infty$. Let $s_1(t), s_2(t)$ be continuous functions to be defined later. For brevity we denote $\rho = \mathrm{id} - \alpha$. Consider the set

$$\begin{aligned}M(t) = \{s_3 \in \mathbb{R}_+\ |\chi_{13}(s_3) &< \rho^{-1}(s_1(t)) - \chi_{12}(s_2(t)),\\ \chi_{23}(s_3) &< \rho^{-1}(s_2(t)) - \chi_{21}(s_1(t)),\\ s_3 &> \rho(\chi_{31}(s_1(t)) + \chi_{32}(s_2(t)))\}\end{aligned} \tag{29}$$

or equivalently $M(t)$ is the set of numbers s_3 satisfying

$$\begin{aligned}&\rho(\chi_{31}(s_1(t)) + \chi_{32}(s_2(t))) < s_3 <\\ &\min\{\chi_{13}^{-1}(\rho^{-1}(s_1(t)) - \chi_{12}(s_2(t))),\\ &\quad \chi_{23}^{-1}(\rho^{-1}(s_2(t)) - \chi_{21}(s_1(t)))\}.\end{aligned} \tag{30}$$

Note that for any $s_1 > 0$ there is exactly one $s_2 > 0$ such that

$$\chi_{13}^{-1}(\rho^{-1}(s_1) - \chi_{12}(s_2)) = \chi_{23}^{-1}(\rho^{-1}(s_2) - \chi_{21}(s_1)) \tag{31}$$

holds. This follows using monotonicity arguments. For a fixed s_1 the left hand side of (31) is strictly decreasing function of s_2 while the right hand side of (31) is strictly increasing one. Further from (21) the condition

$$(\mathrm{id} + \alpha) \circ \begin{bmatrix} 0 & \chi_{12} \\ \chi_{21} & 0 \end{bmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \not\geq \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \tag{32}$$

holds, i.e., $\rho \circ \chi_{21} \circ \rho(\chi_{12}(s)) < s$ or $\chi_{12}^{-1} \circ \rho^{-1} \circ \chi_{21}^{-1} \circ \rho^{-1}(s) < s$.

For a fixed s_1 let s_2^* be the zero point of $\rho^{-1}(s_1) - \chi_{12}(s_2)$ and s_2^{**} be the zero point of $\rho^{-1}(s_2) - \chi_{21}(s_1)$ then

$$s_2^* = \chi_{12}^{-1} \circ \rho^{-1}(s_1) = \chi_{12}^{-1} \circ \rho^{-1} \circ \chi_{21}^{-1} \circ \rho^{-1}(s_2^{**})) < s_2^{**}.$$

Hence the zero point of the left hand side of (31) is greater than the one of the right side of (31). This proves that for any s_1 there is always exactly one s_2 satisfying (31).

By the continuity and monotonicity of $\chi_{12}, \chi_{21}, \chi_{13}, \chi_{22}$ it follows that s_2 depends continuously on s_1 and is strictly increasing with s_1. We can define $\sigma_1(t) = s_1(t) = t, \sigma_2(t) = s_2(t)$, where $t > 0$ and $s_2(t)$ is the solution of (31) with $s_1 = t$.

Denote $h(t) = \rho(\chi_{31}(s_1(t)) + \chi_{32}(s_2(t)))$ and $g(t) = \chi_{13}^{-1}(\rho^{-1}(s_1(t)) - \chi_{12}(s_2(t))) = \chi_{23}^{-1}(\rho^{-1}(s_2(t)) - \chi_{21}(s_1(t)))$, then we have

$$M(t) = \{s_3 | h(t) < s_3 < g(t)\}.$$

Let us show that $M(t) \neq \emptyset$ for any $t > 0$. If this is not true then there exists a t^*, such that

$$s_3^* := h(t^*) \geq g(t^*)$$

holds. Consider the point $s^* := (s_1^*, s_2^*, s_3^*) := (t^*, s_2(t^*), s_3^*)$. Then

$$D \circ \Gamma(s^*) = \rho \circ \begin{pmatrix} \chi_{12}(s_2^*) + \chi_{13}(s_3^*) \\ \chi_{21}(s_1^*) + \chi_{23}(s_3^*) \\ \chi_{31}(s_1^*) + \chi_{32}(s_2^*) \end{pmatrix} \geq \begin{pmatrix} s_1(t^*) \\ s_2(t^*) \\ s_3^* \end{pmatrix},$$

but this contradicts (21). Hence $M(t)$ is not empty for all $t > 0$.

Consider the functions $h(t)$ and $g(t)$. The question is how to choose $\sigma_3(t) \in M(t)$ to be a $\mathcal{K}_\infty$ function. Note that $h(t) \in \mathcal{K}_\infty$. Let $g^*(t) := \min_{T \geq t} g(T) \leq g(t)$. Let for $t > 0$ be $A_t := \{s \in \mathbb{R}_+ : g^*(s) = g^*(t)\}$. Since $g(t)$ is unbounded this set is compact. Denote $s^* := \max A_t$, then we have $g^*(t) = g^*(s^*) = g(s^*) > h(s^*) \geq h(t)$. Hence $h(t) < g^*(t) \leq g(t)$ for any $t > 0$ where g^* is a (not strictly) increasing function. Let $\sigma_3 := (h + g^*)/2 \in \mathcal{K}_\infty$. Now we have strictly increasing $\sigma_1, \sigma_2, \sigma_3$ satisfying (22). By standard analysis tools this curve can be regularized to satisfy $\sigma_i' > 0$.

The case where one of χ_{12}, χ_{13} is not a $\mathcal{K}_\infty$ function but zero can be treated similarly. □

V. Conclusions

We have considered a network of ISS systems with given ISS Lyapunov functions. We have shown how an ISS Lyapunov function can be constructed for the network. In special cases we have shown that the existence of an ISS Lyapunov function is guaranteed by the condition (21). We conjecture that this condition assures the existence of an ISS Lyapunov function for the general case, i.e., for n systems with nonlinear gains. This assumption is currently under investigation.

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