Comments on “A multichannel IOS Small Gain Theorem for Systems With Multiple Time-Varying Communication Delays”

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Abstract

The small-gain condition presented by Polushin *et al.* may be replaced by a strictly weaker one to obtain essentially the same result. The necessary minor modifications of the proof are given. Using essentially the same arguments, a global version of the result is also presented.

Index Terms

Networked control systems, generalized small-gain condition, Lyapunov stability, time-varying communication delays, input-to-output stability

I. Introduction

In [1] Polushin *et al.* have presented a small-gain type condition that ensures input-output stability for networked systems in the presence of time delays. In this note we show that the small-gain condition by Polushin *et al.* can be replaced by a less restrictive one. As an extension we obtain a global version of the result, with a global small-gain condition resembling that one of Dashkovskiy *et al.* [2]. By means of an example we show that the modified small-gain

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conditions are indeed less restrictive than the original one. For brevity we adopt the problem formulation and notations from [1].

II. The generalized small-gain theorem

Based on the setup and notation in [1] we formulate our generalized small-gain condition in a very compact form, thereby using [1].X to reference equation/assumption/ or result X in [1].

A. Modified notation

We need a few notations before we can state our main theorem. We write

$$\left.\begin{array}{c} \Gamma_U = \begin{pmatrix} \Gamma_{1u} & 0 \\ 0 & \Gamma_{2u} \end{pmatrix},\ \Gamma_W = \begin{pmatrix} \Gamma_{1w} & 0 \\ 0 & \Gamma_{2w} \end{pmatrix}, \\ B(x_d^+(t)) = \begin{pmatrix} \beta_1(|x_{1d}(t)|) \\ \beta_2(|x_{2d}(t)|) \end{pmatrix},\ \hat{y}^+ = \begin{pmatrix} \hat{y}_1^+ \\ \hat{y}_2^+ \end{pmatrix}, \\ \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix},\ \Psi = \begin{pmatrix} 0 & \Psi_2 \\ \Psi_1 & 0 \end{pmatrix}, \\ u^+(t) = \begin{pmatrix} u_1^+(t) \\ u_2^+(t) \end{pmatrix},\ \Delta_u = \begin{pmatrix} \Delta_{u1} \\ \Delta_{u2} \end{pmatrix}. \end{array}\right\} \tag{1}$$

Rewriting inequality [1].(3) and [1].(4) with this notation yields

$$\sup_{t\geq t_0} y^+ \leq \max\left\{B(x_d^+(t_0)), \sup_{t\geq t_0}\Gamma_U(u_d^+), \sup_{t\geq t_0}\Gamma_W(w_d^+), \delta\right\} \tag{2}$$

and, respectively,

$$\limsup_{t\to\infty} y^+ \leq \left\{\limsup_{t\to\infty}\Gamma_U(u_d^+), \limsup_{t\to\infty}\Gamma_W(w_d^+), \delta\right\}. \tag{3}$$

The interconnection of both subsystem can be described as

$$u^+(t) \equiv 0 \quad \forall t < T_0 \tag{4}$$

and

$$u^+(t) \leq \Psi(\hat{y}^+(t)) \quad \forall t \geq T_0. \tag{5}$$

To formulate subsequent statements in a precise way, it is useful to introduce the concept of monotone operators:

Definition 2.1: A mapping $T : \operatorname{dom} T \subset \mathbb{R}_+^n \to \mathbb{R}_+^n$ is called a *continuous and monotone operator on* $\operatorname{dom} T$, if 1. T is continuous and 2. for all $u, v \in \operatorname{dom} T$, $u \leq v$ implies $T(u) \leq T(v)$. •

A matrix $\Gamma = (\gamma_{ij}) \in \mathcal{G}^{n \times n}$ defines a continuous and monotone operator $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$\Gamma(s) = \big(\max_j \gamma_{1j}(s_j), \ldots, \max_j \gamma_{nj}(s_j)\big)^T, \quad \text{for } s \in \mathbb{R}_+^n.$$

The class of these matrix-induced operators has some nice properties, some of which are given in the appendix. Most relevant is the fact that any finite composition of matrix-induced operators gives again a matrix-induced operator and that matrix-induced operators commute with the max-operation (for vectors, defined element-wise).

We write $\Gamma \not\geq \mathrm{id}$, to denote that $\Gamma(s) \not\geq s$ for all $s \in \mathbb{R}_+^n$, $s \neq 0$, i.e., that for every such s there exists an i such that $\Gamma(s)_i < s_i$.

Now formally define

$$\Gamma = \Gamma_U \circ \Psi \text{ and } \mathbb{G} = \max_{k \geq 0} \Gamma^k .$$

While Γ clearly is a matrix-induced continuous and monotone operator, $\mathbb{G}$ is not necessarily well-defined. We will see subsequently that a small-gain type condition is precisely what is needed to assure that $\mathbb{G}$ is well-defined on a subset of $\operatorname{dom} \mathbb{G} \subset \mathbb{R}_+^n$. In this case, $\mathbb{G}$ can be represented as a matrix-induced monotone and continuous operator on $\operatorname{dom} \mathbb{G}$.

B. Main results

Now we provide two generalized versions of the result in [1], with the original small-gain condition replaced by a more general condition. The first result is of a local nature, resembling the original result in [1], the second one is a corresponding global version.

To avoid confusion, we denote the vectors appearing in the small-gain condition by $\delta_{\mathrm{SGC}}, \Delta_{\mathrm{SGC}}$, whereas in [1] they have been denoted by δ, Δ. Unfortunately, δ has also the meaning of an offset in the definition of IOS. Our subscript notation aims to avoid this clash, here δ without subscript refers to the IOS offset given in (1).

Theorem 2.2: Suppose the systems [1].(2)–[1].(7) satisfy Assumptions [1].1 and [1].2 and that there exist $\delta_{\mathrm{SGC}}, \Delta_{\mathrm{SGC}} \in \mathbb{R}_+^{p+q}$, $0 \leq \delta_{\mathrm{SGC}} < \Delta_{\mathrm{SGC}}$, such that the following *local small-gain condition* holds:

$$\Gamma(\Delta_{\mathrm{SGC}}) \leq \Delta_{\mathrm{SGC}} \quad \text{and} \quad \limsup_{k \to \infty} \Gamma^k(\Delta_{\mathrm{SGC}}) \leq \delta_{\mathrm{SGC}} . \tag{6}$$

Then the following assertions hold: If

$$\Psi(\Delta^{**}) \le \Delta_u \tag{7}$$

then $\mathbb{G}$ is well-defined on the order interval $[\delta_{\mathrm{SGC}}, \Delta_{\mathrm{SGC}}]$. If in addition $\Delta_{\mathrm{SGC}} > \Delta^{**}$, where

$$\Delta^{**} = \mathbb{G}\left(\max\{B(\Delta_x), \Gamma_W(\Delta_w), \delta\}\right) , \tag{8}$$

then system [1].(2)–[1].(7) is IOS at $t = T_0$ in the sense of Definition [1].1 with

$$\begin{aligned} t_d(T_0) = t_{1d}(T_0) + t_{2d}(T_0) + \tau^*(T_0) \\ + \tau^*(T_0 - \tau^*(T_0)) \,. \end{aligned} \tag{9}$$

More precisely, the conditions $x_d^+(T_0) \le \Delta_x \,, \sup_{t \ge T_0} w_d^+ \le \Delta_w$ imply that the following inequalities hold

$$\begin{aligned} \sup_{t \ge T_0} y^+ \le \max \Big\{ \mathbb{G}\Big(\max \big\{ B(x_d^+(T_0)), \\ \Gamma_W(\sup_{t \ge T_0} w_d^+), \delta \big\} \Big), \delta_{\mathrm{SGC}} \Big\}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \limsup_{t \to \infty} y_d^+ \le \max \Big\{ \mathbb{G}\Big(\max \big\{ \Gamma_w(\limsup_{t \to \infty} w_d^+), \\ \delta \big\} \Big), \delta_{\mathrm{SGC}} \Big\}. \quad \bullet \end{aligned} \tag{11}$$

The proof is essentially the same as the corresponding version in [1], with the important difference that all applications of the original small-gain condition [1].(12) be replaced by an application of Lemma A.5.

For the special case that $\Delta_{\mathrm{SGC}} = \infty$ and $\delta_{\mathrm{SGC}} = 0$ and by utilizing Lemma A.3, we have a corresponding global version. This version is applicable in case that the IOS restrictions $\Delta_{ui}, \Delta_{wi}, \Delta_{xi}$, $i = 1, 2$, of the subsystems are infinite. Notable is the similarity of the small-gain condition to the one given in [2]:

Remark 2.3 (A note on the local small-gain condition (6)*):* Condition (6) implies

$$\Gamma(s) \not\geq s \text{ for all } s \in [\delta_{\mathrm{SGC}}, \Delta_{\mathrm{SGC}}],\ s \ne \delta_{\mathrm{SGC}} \,.$$

The argument is similar to the proof of Lemma A.3. Notably, the converse does not hold *locally*, but it does *globally*, as is emphasized by the following global extension of the main result. $\bullet$

Corollary 2.4: Suppose the systems [1].(2)–[1].(7) satisfy Assumptions [1].1 and [1].2 with infinite restrictions $\Delta_{ui}, \Delta_{wi}, \Delta_{xi}$, $i = 1, 2$, and that one of the following equivalent conditions holds:

1) Γ satisfies the small-gain condition $\Gamma \ngeq \mathrm{id}$;
2) $\Gamma^k(s) \to 0$ as $k \to \infty$ for all $s \geq 0$;
3) all minimal cycles (and hence all cycles) in Γ are contractions (see Lemma A.3 for the meaning of this condition).

Then system [1].(2)–[1].(7) is IOS at $t = T_0$ in the sense of Definition [1].1 with infinite restrictions and

$$t_d(T_0) = t_{1d}(T_0) + t_{2d}(T_0) + \tau^*(T_0) + \tau^*(T_0 - \tau^*(T_0))\,.$$

More precisely, boundedness of $x_d^+(T_0)$, $\sup_{t\geq T_0} w_d^+$ implies that the following inequalities hold

$$\sup_{t\geq T_0} y^+ \leq \mathbb{G}\Big(\max\big\{B(x_d^+(T_0)), \Gamma_W(\sup_{t\geq T_0} w_d^+), \delta\big\}\Big),$$

$$\limsup_{t\to\infty} y_d^+ \leq \mathbb{G}\Big(\max\big\{\Gamma_w(\limsup_{t\to\infty} w_d^+), \delta\big\}\Big)\,. \quad \bullet$$

Remark 2.5: Note that [1].(17) can be interpreted as $\mathbb{G} \circ B$ and $\mathbb{G} \circ \Gamma_W$. Indeed, condition [1].(12) guarantees that the maximum in (10) respectively (11) is already attained when restricting to $k \leq 2$ (recall that $\mathbb{G} = \max_{k\geq 0} \Gamma^k$). Unsuprisingly, in the more general case the obtained estimates are more restrictive. $\bullet$

The next lemma relates our new small-gain condition, which can essentially be formulated as

$$\Gamma(s) \ngeq s \quad \forall s \neq 0, \tag{12}$$

to the old one, [1].(12): Since $\Gamma < \mathrm{id}$ in particular implies $\Gamma \ngeq \mathrm{id}$, but not *vice versa*, we see that condition (12) is indeed weaker than [1].(12).

Lemma 2.6: Given $\Gamma_{12} \in \mathcal{G}^{p\times q}$ and $\Gamma_{21} \in \mathcal{G}^{q\times p}$, denote $\Gamma = \begin{pmatrix} 0 & \Gamma_{12} \\ \Gamma_{21} & 0 \end{pmatrix}$. Then $\Gamma \ngeq \mathrm{id}$ if and only if $\Gamma_{12} \circ \Gamma_{21} \ngeq \mathrm{id}$ if and only if $\Gamma_{21} \circ \Gamma_{12} \ngeq \mathrm{id}$. $\bullet$

Proof: First assume that $\Gamma \ngeq \mathrm{id}$ holds. By [3, Lemma 2.1] for all $k \geq 1$, $\Gamma^k \ngeq \mathrm{id}$. In particular for $k = 2$ we have $\begin{pmatrix} \Gamma_{12} \circ \Gamma_{21}(s_1) \\ \Gamma_{21} \circ \Gamma_{12}(s_2) \end{pmatrix} \ngeq \mathrm{id}$. By considering the special cases $s = (s_1^T, 0)^T$ and $s = (0, s_2^T)^T$ separately, we conclude $\Gamma_{12} \circ \Gamma_{21} \ngeq \mathrm{id}$ and $\Gamma_{21} \circ \Gamma_{12} \ngeq \mathrm{id}$.

Now suppose $\Gamma_{12} \circ \Gamma_{21} \ngeq \mathrm{id}$ and assume there exists $s = (s_1^T, s_2^T)^T$ with $s_1 \neq 0$, such that $\Gamma(s) \geq s$. By monotonicity $\Gamma^2(s) \geq \Gamma(s) \geq s$. At the same time we have $\Gamma^2(s) = \begin{pmatrix} \Gamma_{12} \circ \Gamma_{21}(s_1) \\ \Gamma_{21} \circ \Gamma_{12}(s_2) \end{pmatrix}$, implying $\Gamma^2(s) \ngeq s$ because of $\Gamma_{12} \circ \Gamma_{21} \ngeq \mathrm{id}$, a contradiction. The other cases follow by essentially the same argument. ■

Note that (12) essentially says that Γ has to be a contraction. This is also implied by requiring $\Gamma(s) < s$, $\forall s \neq 0$, as in [1], but this requisite is much stronger.

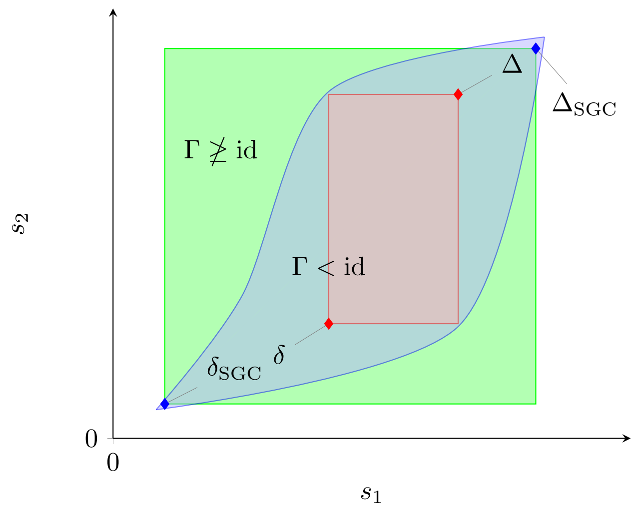

Fig. 1. Differences between the two small-gain conditions: The bubble-shaped (blue) region is where $\Gamma(s) < s$ holds. The condition in [1] requires a rectangular set contained in this region, and the corresponding order interval (δ, Δ) is denoted as the the small red box. In contrast, condition (6) may essentially include the entire bubble-shaped region, giving a less conservative estimate as the order interval $[\delta_{\mathrm{SGC}}, \Delta_{\mathrm{SGC}}]$, indicated as the large green box.

The following example shows a case where the condition of Polushin *et al.* is not applicable, but a weaker small-gain condition proposed in this note instead is:

Example 2.7: Consider two systems of FDE's as in [1] and let Assumption 1 of [1] hold. In particular, let the gains be

$$\Gamma_{1u} = \begin{pmatrix} \gamma_{13} & 0 \\ 0 & \gamma_{24} \end{pmatrix}, \quad \Gamma_{2u} = \begin{pmatrix} 0 & \gamma_{32} \\ \gamma_{41} & 0 \end{pmatrix},$$

$$\gamma_{13} = \gamma_{24} = \gamma_{32} = \mathrm{id}\,, \quad \gamma_{41} = \frac{1}{2}\,\mathrm{id}\ .$$

A small calculation shows

$$\Gamma_{1u} \circ \Gamma_{2u} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \gamma_{32} \circ \gamma_{24}(s_2) \\ \gamma_{41} \circ \gamma_{13}(s_1) \end{pmatrix} = \begin{pmatrix} s_2 \\ \frac{1}{2} s_1 \end{pmatrix} \not< \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

for, e.g., $s_1 = s_2$. Therefore the condition from [1, Theorem 1],

$$\Gamma_{1u} \circ \Psi_2 \circ \Gamma_{2u} \circ \Psi_1(s) < s \ \ \forall s \in (\delta^{\#}, \Delta^{\#}) \subset \mathbb{R}^2_+$$

with $\Psi_1 = \Psi_2 = \mathrm{id}$ is not satisfied. On the other hand, if we use notation (1) the gain matrix of this particular example is

$$\Gamma = \Gamma_U \circ \Psi = \begin{pmatrix} \Gamma_{1u} & 0 \\ 0 & \Gamma_{2u} \end{pmatrix} \circ \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \gamma_{13} & 0 \\ 0 & 0 & 0 & \gamma_{24} \\ 0 & \gamma_{32} & 0 & 0 \\ \gamma_{41} & 0 & 0 & 0 \end{pmatrix}.$$

This matrix has only one simple cycle:

$$\gamma_{13} \circ \gamma_{32} \circ \gamma_{24} \circ \gamma_{41} = \frac{1}{2}\,\mathrm{id} < \mathrm{id} \qquad \Longrightarrow \qquad \Gamma(s) \not\geq s \ \forall s \neq 0\,,$$

where the last implication follows from Lemma A.3. •

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APPENDIX

A. Technical lemmata

It might be useful to point out for applications, that the set of gain matrices as employed in this paper has some nice algebraic properties. We list a few, the proofs are not very involved and are omitted for brevity.

Lemma A.1 (Closedness under composition): Given $\Gamma_1 \in \mathcal{G}^{l\times m}$ and $\Gamma_2 \in \mathcal{G}^{m\times n}$, then there exits $\Gamma \in \mathcal{G}^{l\times n}$ satisfying $\Gamma = \Gamma_1 \circ \Gamma_2$ as operator $\mathbb{R}^n_+ \to \mathbb{R}^l_+$. •

Hence, by using induction, it is at hand that any finite composition yields again a matrix-induced operator.

Lemma A.2 (Distributive-law w.r.t. maximization): Given $\Gamma \in \mathcal{G}^{n\times m}$, and $a, b \in \mathbb{R}^m_+$, then $\Gamma\big(\max\{a,b\}\big) = \max\big\{\Gamma(a), \Gamma(b)\big\}$. •

The following lemmata are at the heart of the proof of the small-gain theorem.

Lemma A.3: Let Γ be of class $\mathcal{G}^{n\times n}$. The following are equivalent.

1) $\Gamma(s) \ngeq s \quad \forall s \in \mathbb{R}^n_+, s \neq 0$;
2) $\lim_{k\to\infty} \Gamma^k(s) = 0 \quad \forall s \in \mathbb{R}^n_+$;
3) All cycles in Γ are contractions, i.e.,

$$\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \cdots \circ \gamma_{i_k i_1} < \mathrm{id}$$

for all $(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \quad \forall k \geq 1. \quad \bullet$

See [3, Theorem 6.4] for a proof. A *minimal cycle* is a cycle that does not contain any shorter cycles. It is not difficult to see that 3) can be replaced by

3′) all minimal cycles in Γ are contractions.

Lemma A.4: Let $\Gamma \in \mathcal{G}^{n\times n}$ satisfy the small-gain condition $\Gamma \ngeq \mathrm{id}$, then for $a, b \in \mathbb{R}^n_+$,

$$a \leq \max\big\{b, \Gamma(a)\big\} \tag{13}$$

implies

$$a \leq \max_{k\geq 0} \Gamma^k(b) \in \mathbb{R}^n_+ . \quad \bullet \tag{14}$$

Proof: We identify $\Gamma^0(b) = b$. By Lemma A.3 we have $\Gamma^k(b) \to 0$ for $k \to \infty$, so the set $\{\Gamma^k(b)\}$ must be bounded and have a least upper bound in $\mathbb{R}^n_+$. Inequality (14) now follows by recursively substituting (13) into itself and by noting that $\Gamma\big(\max\{a, b\}\big) = \max\big\{\Gamma(a), \Gamma(b)\big\}$.

■

A local version of the previous result is the following, stated for monotone and continuous operators:

Lemma A.5: Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be continuous and monotone and assume there exist $\delta_{\mathrm{SGC}}, \Delta_{\mathrm{SGC}} \in \mathbb{R}^n_+$, $\delta_{\mathrm{SGC}} < \Delta_{\mathrm{SGC}}$, such that

$$T(\Delta_{\mathrm{SGC}}) \leq \Delta_{\mathrm{SGC}} \quad \text{and} \quad \limsup_{k\to\infty} T^k(\Delta_{\mathrm{SGC}}) \leq \delta_{\mathrm{SGC}} .$$

Then for all $a, b \leq \Delta_{\mathrm{SGC}}$,

$$a \leq \max\{b, T(a)\} \implies a \leq \max_{k\geq 0}\{T^k(b), \delta_{\mathrm{SGC}}\} . \quad \bullet$$

The proof is similar to the previous one.

B. Proofs of the main results

Proof of Theorem 2.2: The proof that $\mathbb{G}$ is well-defined on $[\delta_{\rm SGC}, \Delta_{\rm SGC}]$ follows by a similar argument as in the proof of Lemma A.4. Now consider system [1].(2)-[1].(7) and suppose

$$x_d^+(T_0) \leq \Delta_x \quad \text{and} \quad \sup_{t\in[T_0,\infty)} w_d^+ \leq \Delta_w. \tag{15}$$

Assumption [1].1 together with (4), (15) as well as causality arguments imply that

$$y_d^+(T_0) \leq \max\{B(\Delta_x), \Gamma_W(\Delta_w), \delta\}.$$

With the help of (4), (5) and Assumption [1].2.i we can deduce

$$\begin{aligned} \sup_{t\in[T_0-t_d(T_0),T_0+\tau_*]} u^+ &\leq \Psi(\max\{B(\Delta_x), \Gamma_W(\Delta_w), \delta\}) \\ &\leq \Psi(\Delta^{**}), \end{aligned}$$

where the last inequality follows from (8). From the last inequality together with (7) we see that the restrictions on the inputs are satisfied for $t \in [T_0 - t_d(T_0), T_0 + \tau_*]$. Hence there exists $T_{\max} > T_0 + \tau_*$ such that the solutions of [1].(2)-[1].(7) are well-defined for all $t \in [T_0, T_{\max})$. Now we want to show that

$$\sup_{t\in[T_0,T_{\max})} y_d^+ \leq \Delta^{**}. \tag{16}$$

We will prove (16) by contradiction. So assume there exists $T_1 \in [T_0, T_{\max} - \tau_*)$ such that

$$\sup_{t\in[T_0,T_1]} y_d^+ \leq \Delta^{**} \text{ and } \sup_{t\in[T_0,T_1+\tau_*]} y_d^+ \not\leq \Delta^{**}. \tag{17}$$

Combining (2), (9), (15) with (5) and Assumption [1].2.i, we obtain

$$\begin{aligned} \sup_{t\in[T_0,T_1+\tau_*]} y_d^+ \leq \max\big\{&B(\Delta_x), \Gamma_W(\Delta_w), \\ &\Gamma(\sup_{t\in[T_0,T_1]} y_d^+), \delta\big\}. \end{aligned}$$

From the definition of (8) it is easy to see that $\Gamma(\Delta^{**}) \leq \Delta^{**}$. Hence we can deduce with the help of the first inequality in (17)

$$\sup_{t\in[T_0,T_1+\tau_*]} y_d^+ \leq \max\{B(\Delta_x), \Gamma_W(\Delta_w), \Delta^{**}\} = \Delta^{**},$$

which contradicts the second inequality in (17). This contradiction proves (16). Next we want to show that $T_{\max} = \infty$. Again we will prove this by contradiction. Due to the IOS assumption on the subsystems $T_{\max} < \infty$ implies

$$\sup_{t \in [T_0, T_{\max})} u^+ \not\leq \Delta_u. \tag{18}$$

From (5) and (7) we can see that (18) implies

$$\Psi(\sup_{t \in [T_0, T_{\max})} \hat{y}^+) \not\leq \Psi(\Delta^{**}).$$

Because of the monotonicity of Ψ and the fact that $\sup \hat{y}^+ \leq \sup y_d^+$ we get

$$\sup_{t \in [T_0, T_{\max})} y_d^+ \not\leq \Delta^{**},$$

which contradicts (16), hence $T_{\max} = \infty$.

Summarizing, the restrictions on the inputs hold for all $t \in [T_0, \infty)$. Hence we can use (2) to get

$$\sup_{t \geq T_0} y_d^+ \leq \max\{B(x_d^+(T_0)), \Gamma_W(\sup_{t \geq T_0} w_d^+), \Gamma(\sup_{t \geq T_0} y_d^+), \delta\}.$$

Using Lemma A.5 we conclude

$$\sup_{t \geq T_0} y_d^+ \leq \max\Big\{ \max_{k \geq 0} \Gamma^k(\max\{B(x_d^+(T_0), \Gamma_W(\sup_{t \geq T_0} w_d^+)), \delta\}), \delta_{\mathrm{SGC}} \Big\},$$

which can be easily rewritten to get (10). Similarly we can use (3) together with Lemma A.5 to get

$$\limsup_{t \to \infty} y_d^+ \leq \max\Big\{ \max_{k \geq 0} \Gamma^k\big(\max\{\Gamma_W(\limsup_{t \to \infty} w_d^+(t)), \delta\}\big), \delta_{\mathrm{SGC}} \Big\}.$$

Realizing that this can be brought into the form (11) finishes the proof. ∎

Proof of Corollary 2.4: The proof is again essentially the same. Instead of Lemma A.5 now Lemma A.4 serves as the main technical tool. Instead of Δ_w this time we have to use $\sup_{t \in [T_0, \infty)} w_d^+(t)$, which we assumed to be finite. Instead of Δ_x we use $x_d^+(T_0)$, which is also finite. Previously, we had a constant Δ^{**} independent of the particular choice of input, here we define Δ_w^{**} by

$$\Delta_w^{**} = \mathbb{G}\Big(\max\Big\{B\big(x_d^+(T_0)\big), \Gamma_W\big(\sup_{t \in [T_0, \infty)} w_d^+(t)\big), \delta\Big\}\Big),$$

which depends on the choice of input w. Using essentially the same steps as in the previous proof, thereby replacing Δ^{**} by Δ^{**}_w, we obtain from the equivalent of (16)–(17) using Lemma A.4,

$$\sup_{t\in[T_0,T_1+\tau_*]} y_d^+ \leq \max\big\{B(x_d^+(T_0)), \Gamma_W(\sup_{t\in[T_0,\infty)} w_d^+(t)), \\ \Gamma(\sup_{t\in[T_0,T_1]} y_d^+), \delta\big\} \leq \Delta^{**}_w.$$

This inequality implies $T_{\max} = \infty$. From here we obtain the desired estimates as in the proof of the theorem. ■