

Exponential output-to-state stability of linear systems with bounded output operators[★]

Qiaoling Chen^{*} Andrii Mironchenko^{**} Fabian Wirth^{*}

^{*} Faculty of Computer Science and Mathematics, University of Passau, 94032 Passau, Germany (e-mail: qiaoling.chen@uni-passau.de, fabian.lastname@uni-passau.de).

^{**} Department of Mathematics, University of Bayreuth, 95447, Bayreuth, Germany (e-mail: andrii.mironchenko@uni-bayreuth.de)

Abstract: For linear infinite-dimensional systems, exponential detectability is known to be stronger than exponential zero-detectability, and both these concepts are hard to extend to the nonlinear setting. In this work, we show that there is a broad hierarchy of detectability notions between these two concepts, centered around the concept of exponential output-to-state stability (eOSS). We study eOSS for linear infinite-dimensional systems with bounded output operators. It is shown that exponential detectability guarantees the existence of a coercive eOSS Lyapunov function and therefore eOSS. Several counterexamples demonstrate the fundamental differences between the detectability concepts employed in this work. However, if the system has only finitely many unstable eigenvalues, then all the concepts coincide. The results are illustrated with an example.

Copyright © 2026 The Authors. This is an open access article under the CC BY-NC-ND license (<https://creativecommons.org/licenses/by-nc-nd/4.0/>)

Keywords: output-to-state stability, detectability, linear systems, Lyapunov methods, infinite-dimensional systems

AMS Subject Classification (2020): 93C05, 93B07, 37L15

1. INTRODUCTION

The concept of output-to-state stability (OSS) was introduced in (Sontag and Wang, 1996a), (Sontag and Wang, 1996b) to study detectability of nonlinear systems. It is conceptually analogous to the well-known notion of input-to-state stability (ISS). While ISS describes robust stability of a system with respect to external inputs, OSS addresses whether vanishing outputs lead to the decay of the system state. OSS provides a useful perspective for analyzing how state information can be recovered from output behavior.

For linear ordinary differential equations (ODEs) without inputs, OSS is equivalent to classical detectability as well as to zero-detectability, which states that if the output remains zero, then the state converges to the origin (Sontag, 1998). For nonlinear systems, Lyapunov methods offer a constructive approach to verifying OSS. For instance, for nonlinear ODEs, OSS is equivalent to the existence of an OSS Lyapunov function in dissipation form (Sontag and Wang, 1996a). In Krichman et al. (2001), this result was extended to nonlinear ODEs with inputs and outputs by showing that input-output-to-state stability (IOSS) is equivalent to the existence of a smooth IOSS Lyapunov function.

The Lyapunov characterization of IOSS establishes a relation between IOSS and dissipativity of control systems. Namely, viewing an IOSS Lyapunov function as a storage function, in the discrete-time setting in (Höger and Grüne, 2019), it was shown that IOSS is equivalent to strict dissipativity for non-negative supply rates. Another motivation to study OSS and IOSS notions is their fundamental role in observer design. In particular, incremental IOSS is a necessary condition for the existence of a robust observer (Allan et al., 2021). Last but not least, ISS can be characterized as a combination of input-to-output stability and IOSS, as shown for finite-dimensional systems by (Jiang et al., 1994) and extended to infinite-dimensional settings by (Bachmann et al., 2026).

During the last decade, ISS of infinite-dimensional control systems has been intensively studied, see (Mironchenko and Wirth, 2018b; Jacob et al., 2018; Schwenninger, 2020) for linear systems, and (Jayawardhana et al., 2008; Karafyllis and Krstic, 2016, 2019; Mironchenko and Wirth, 2018a; Mironchenko and Prieur, 2020; Zheng and Zhu, 2021, 2026; Chaillet et al., 2023) for the nonlinear case.

In contrast to the progress in ISS theory, OSS remains largely unexplored for infinite-dimensional systems, and faces several challenges. The state space is typically a Banach or Hilbert space, where compactness arguments, spectral techniques, and standard observability tools from finite-dimensional systems may no longer be directly applicable. Solutions may lack sufficient regularity, so classical derivatives are often unavailable, and Lyapunov methods

[★] This work was partially supported by German Research Foundation (DFG), grants: MI 1886/2-2 and MI 1886/3-1.

must be adapted to weak or mild formulations. Moreover, zero-detectability is far weaker than OSS, in contrast to the finite-dimensional setting. These difficulties remain major obstacles to the development of the general OSS theory in infinite dimensions.

The study of OSS for evolution equations on Banach spaces was initiated in (Chen et al., 2025), establishing a direct OSS Lyapunov theorem, introducing the notion of uniform global asymptotic stability modulo output, and analyzing its relationship with OSS and the vanishing output vanishing state property. These results provide a theoretical foundation and useful insight for studying OSS of infinite-dimensional linear systems.

In this paper, our contribution is threefold. In Sections 2-3, we study exponential output-to-state stability (eOSS) for linear infinite-dimensional systems with bounded output operators and establish *sufficient conditions for eOSS* based on exponential detectability and coercive eOSS Lyapunov functions formulated in both dissipation and implication form.

In Section 4, we construct several counterexamples showing that *exponential detectability is in general stronger than the existence of an eOSS Lyapunov function in dissipation form, which in turn is (rather surprisingly) stronger than the existence of an eOSS Lyapunov function in implication form*. Another counterexample shows that *exponential zero-detectability does not imply eOSS*.

In Section 5, we establish equivalence under the assumption of a spectrum separation at 0 with a finite-dimensional unstable subspace. Finally, in Section 6, we apply the proposed framework to a parabolic partial differential equation (PDE) with mixed boundary conditions and demonstrate eOSS using both Lyapunov and spectral approaches, confirming the equivalence of detectability, Lyapunov conditions, and eOSS derived in Section 5.

Our findings show that there is a whole world between the classical concepts of exponential detectability and exponential zero-detectability - centered around the notion of output-to-state stability. As this concept can be naturally formulated for nonlinear systems, we believe that OSS can serve as a canonical detectability property for infinite-dimensional systems, as it is in the setting of nonlinear ODEs (Sontag, 2008). In future work, we will use these findings together with the results in (Chen et al., 2025) to prove (or disprove) the converse Lyapunov result in implication form for (e)OSS of (non)linear distributed parameter systems.

We omit the proofs of all results due to page restrictions.

Notation: Let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$ be the sets of real numbers, complex numbers, and positive integers, respectively. We also set $\mathbb{R}_+ := [0, \infty)$. Given topological spaces X, Y , the space of continuous functions from X to Y is denoted by $C(X, Y)$ and $C^2(0, 1)$ denotes the space of real-valued functions that are twice continuously differentiable on the interval $(0, 1)$. Let X and Y be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y . We use the standard classes of comparison functions $\mathcal{P}$, $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$, see, e.g., (Mironchenko, 2023).

2. PROBLEM SETTING

Let X and Y be Banach spaces (the state space and the space of output values, respectively). Consider the system

$$\dot{x}(t) = Ax(t), \quad y(t) = Cx(t), \quad (1)$$

where $A : D(A) \subset X \rightarrow X$ generates a C_0 -semigroup $T(\cdot)$ on X , and $C \in \mathcal{L}(X, Y)$ is the output operator.

We denote by $\phi(t, x)$ the mild solution of system (1) at time $t \geq 0$ corresponding to the initial condition $x \in X$, i.e., $\phi(t, x) = T(t)x$. For each $x \in X$, the function $\phi(\cdot, x)$ belongs to the vector space $C(\mathbb{R}_+, X)$ of continuous X -valued functions on $\mathbb{R}_+$.

Choose the space of output trajectories as the vector space

$$\mathcal{Y} := C(\mathbb{R}_+, Y).$$

Slightly abusing notation, we denote by $y(\cdot, x) := CT(\cdot)x$ the output trajectory corresponding to the initial condition $x \in X$.

For $t \in \mathbb{R}_+$, we denote by $y(\cdot, x)|_{[0, t]}$ the restriction of the output trajectory $y(\cdot, x)$ to $[0, t]$, that is,

$$(y(\cdot, x)|_{[0, t]})(s) = y(s, x), \quad s \in [0, t].$$

We use the notation $\|y(\cdot, x)|_{[0, t]}\|_{\mathcal{Y}}$ for the corresponding supremum norm on $C([0, t], Y)$

$$\|y(\cdot, x)|_{[0, t]}\|_{\mathcal{Y}} := \|y(\cdot, x)|_{[0, t]}\|_{C([0, t], Y)} = \max_{s \in [0, t]} \|y(s, x)\|_Y.$$

2.1 Exponential output-to-state stability

In this subsection, we first recall the notion of output-to-state stability as introduced in (Sontag and Wang, 1996a) as well as its exponential version.

Definition 2.1. System (1) is called *output-to-state stable (OSS)*, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $x \in X$ and all $t \geq 0$ it holds that

$$\|\phi(t, x)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|y(\cdot, x)|_{[0, t]}\|_{\mathcal{Y}}). \quad (2)$$

System (1) is called *exponentially output-to-state stable (eOSS)*, if there exist $M, \mu > 0$ and $\gamma \in \mathcal{K}$ such that for all $x \in X$ and all $t \geq 0$ it holds that

$$\|\phi(t, x)\|_X \leq Me^{-\mu t}\|x\|_X + \gamma(\|y(\cdot, x)|_{[0, t]}\|_{\mathcal{Y}}). \quad (3)$$

If in addition γ can be chosen as a linear function, then we call (1) *OSS* (respectively, *eOSS*) *with linear gain*.

If in (3) we can choose $\gamma = 0$, then the semigroup $T(\cdot)$ is called *(uniformly) exponentially stable (UES)*.

If the inequality (3) holds, then bounded outputs lead to bounded trajectories. Furthermore, vanishing outputs imply vanishing states; see (Chen et al., 2025).

Although the four notions above are distinct in general, we show next that these notions are equivalent for (1).

Proposition 2.2. The following statements are equivalent:

- (i) System (1) is OSS.
- (ii) System (1) is OSS with linear gain.
- (iii) System (1) is eOSS.
- (iv) System (1) is eOSS with linear gain.

2.2 Lyapunov functions

To describe growth properties of functions that are not necessarily differentiable, we use Dini derivatives as a substitute for classical derivatives.

For a continuous function $q : \mathbb{R} \rightarrow \mathbb{R}$, define the (upper right-hand) *Dini derivative* of q at t by

$$D^+q(t) := \overline{\lim}_{\tau \rightarrow +0} \frac{q(t+\tau) - q(t)}{\tau}.$$

In what follows, we apply this notion to the function $t \mapsto V(\phi(t, x))$, whenever this function is continuous. In addition, define the *Lie derivative* of V at $x \in X$ by

$$\dot{V}(x) := D^+V(\phi(t, x))|_{t=0} = \overline{\lim}_{\tau \rightarrow +0} \frac{V(\phi(\tau, x)) - V(x)}{\tau},$$

provided that $t \mapsto V(\phi(t, x))$ is continuous at $t = 0$.

Lyapunov functions serve as a useful tool for studying the eOSS property, with their evolution along trajectories described by the Dini derivative. We begin by defining eOSS Lyapunov functions.

Definition 2.3. A function $V : X \rightarrow \mathbb{R}_+$ is called a *coercive eOSS Lyapunov function in dissipation form* for (1) if:

- (i) There exist $k_1, k_2, p > 0$ such that for every $x \in X$:

$$k_1\|x\|_X^p \leq V(x) \leq k_2\|x\|_X^p. \quad (4)$$
- (ii) For every $x \in X$, the map $t \mapsto V(\phi(t, x))$, $t \geq 0$, is continuous.
- (iii) There are $\alpha > 0$ and $\sigma \in \mathcal{K}_\infty$ so that the following dissipation inequality holds with p as in (i)

$$\dot{V}(x) \leq -\alpha\|x\|_X^p + \sigma(\|Cx\|_Y), \quad x \in X. \quad (5)$$

In the special case $\sigma \equiv 0$, V is called a *coercive UES Lyapunov function*.

The function V is called a *coercive eOSS Lyapunov function in implication form* for (1) if items (i) and (ii) hold, and (iii) is replaced by

- (iii') there exist $\tilde{\alpha} > 0$ and $\chi \in \mathcal{K}_\infty$ such that for every $x \in X$, we have with p as in (i)

$$\|x\|_X \geq \chi(\|Cx\|_Y) \Rightarrow \dot{V}(x) \leq -\tilde{\alpha}\|x\|_X^p. \quad (6)$$

The next proposition provides an equivalent characterization of coercive eOSS Lyapunov functions in dissipation form in terms of the implication form supplemented by an additional estimate on the complementary region.

Proposition 2.4. The following statements are equivalent:

- (i) V is a coercive eOSS Lyapunov function in dissipation form.
- (ii) V is a coercive eOSS Lyapunov function in implication form with some $p, \tilde{\alpha} > 0$ and $\chi \in \mathcal{K}_\infty$, and there exists $\eta \in \mathcal{K}_\infty$ such that for every $x \in X$

$$\|x\|_X \leq \chi(\|Cx\|_Y) \Rightarrow \dot{V}(x) \leq \eta(\|Cx\|_Y). \quad (7)$$

Remark 2.5. Proposition 4.2 below provides an eOSS system which admits a coercive eOSS Lyapunov function in implication form, but does not admit any eOSS Lyapunov function in dissipation form. This shows that the additional assumption (7) cannot be omitted.

Corollary 2.6. If V is a coercive eOSS Lyapunov function in dissipation form with linear gain and $p = 1$, then V is a coercive eOSS Lyapunov function in implication form with linear gain and $p = 1$.

3. LYAPUNOV THEOREMS AND DETECTABILITY

In this section, we study eOSS of linear systems using Lyapunov-based methods and the notion of exponential detectability. Sufficient conditions for eOSS are derived based on the existence of eOSS Lyapunov functions, both in implication and dissipation form. Moreover, we show that exponential detectability implies the existence of a coercive eOSS Lyapunov function, and hence eOSS itself.

3.1 Detectability and eOSS Lyapunov functions

Exponential OSS can be proved by the construction of an eOSS Lyapunov function.

Theorem 3.1. (Direct eOSS Lyapunov theorem). If the system (1) admits a coercive eOSS Lyapunov function (in dissipation form or in implication form), then it is eOSS.

While Theorem 3.1 shows that the existence of a suitable eOSS Lyapunov function implies eOSS, it is also important to identify structural system properties that guarantee the existence of such functions. One such property is exponential detectability, as defined, e.g., in (Curtain and Zwart, 2020, Definition 8.1.1).

Definition 3.2. System (1) is called *exponentially detectable* if there exists $L \in \mathcal{L}(Y, X)$ such that $A + LC$ generates an exponentially stable semigroup $T_{LC}(\cdot)$.

The construction of the following coercive eOSS Lyapunov function is motivated by (Mironchenko and Wirth, 2018b), where converse ISS Lyapunov results for linear systems with bounded input operators have been derived.

Proposition 3.3. Assume that system (1) is exponentially detectable. Let $T_{LC}(\cdot)$ be as in Definition 3.2 satisfying (3) for some $\mu > 0$, $\gamma = 0$. For any $\eta \in (0, \mu)$, define

$$V^\eta(x) := \max_{s \geq 0} \|e^{\eta s} T_{LC}(s)x\|_X. \quad (8)$$

Then V^η is a coercive eOSS Lyapunov function in dissipation form with linear σ and $p = 1$ for system (1).

Motivated by the definition of zero-detectability given in (Krichman et al., 2001), we strengthen zero-detectability by requiring exponential convergence, and define exponential zero-detectability as follows.

Definition 3.4. System (1) is called *exponentially zero-detectable*, if there exist $M, \omega > 0$ such that for all $x \in X$

$$y(\cdot, x) \equiv 0 \Rightarrow \|\phi(t, x)\|_X \leq Me^{-\omega t} \|x\|_X, \quad t \geq 0.$$

The main result of this section is

Theorem 3.5. Consider the following statements:

- (i) System (1) is exponentially detectable.
- (ii) System (1) admits a coercive eOSS Lyapunov function in dissipation form with linear gain.
- (iii) System (1) admits a coercive eOSS Lyapunov function in implication form with linear gain.
- (iv) System (1) is eOSS with linear gain.
- (v) System (1) is eOSS.
- (vi) System (1) is OSS with linear gain.
- (vii) System (1) is OSS.
- (viii) System (1) is exponentially zero-detectable.

Then

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii) \Rightarrow (viii).$$

The implication structure is summarized in Fig. 1, where “LF” means “Lyapunov function” and “LG” is “linear gain”. The crossed arrows indicate implications which fail in general; counterexamples are given in Section 4.

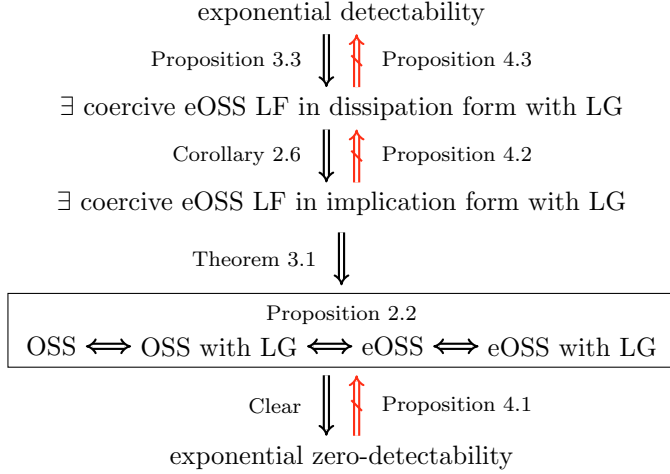


Fig. 1. Summary of the statements of Theorem 3.5.

Remark 3.6. In (Chen et al., 2025), it was shown that OSS implies the uniform global asymptotic stability modulo output, which in turn implies the vanishing output vanishing state property. One could introduce the exponential versions of these concepts and put them into the general picture of Figure 1. But we do not follow this path here.

4. COUNTEREXAMPLES

Theorem 3.5 gives a chain of implications from exponential detectability to exponential zero-detectability. In this section, it is shown that the converse implications cannot be expected. We present three counterexamples:

- (i) exponential zero-detectability does not imply eOSS;
- (ii) eOSS does not imply the existence of an eOSS Lyapunov function in dissipation form;
- (iii) the existence of a coercive eOSS Lyapunov function with linear gain does not imply exponential detectability.

4.1 Exponential zero-detectability does not imply eOSS

The first example separates exponential zero-detectability from eOSS.

Let $X = L^2(0,1)$ be the space of square integrable Lebesgue measurable complex-valued functions on $(0,1)$ and $Y = \mathbb{C}$. Consider the system

$$\dot{x}(t) = Ax(t), \quad y(t) = Cx(t), \quad (9)$$

where $A \in \mathcal{L}(X)$ and $C \in \mathcal{L}(X, Y)$ are defined by

$$(Af)(s) = isf(s), \quad s \in (0,1); \quad Cf = \int_0^1 f(s) ds.$$

The operator A generates a C_0 -group $T(\cdot)$ given by

$$(T(t)f)(s) = e^{ist}f(s), \quad s \in (0,1), \quad t \in \mathbb{R}.$$

It is easy to see that T is isometric, i.e., $\|T(t)f\|_X = \|f\|_X$ for all $f \in X$ and all $t \geq 0$.

Proposition 4.1. The following statements hold:

- (i) System (9) is exponentially zero-detectable.
- (ii) System (9) is not eOSS.

4.2 EOSS does not imply the existence of an eOSS Lyapunov function in dissipation form

The next example shows that, in this generality, the converse eOSS Lyapunov theorem fails in dissipation form, although a coercive eOSS Lyapunov function in implication form still exists.

Let $X = C_{2\pi}(\mathbb{R}, \mathbb{C})$ be the space of all 2π -periodic, continuous maps on $\mathbb{R}$, equipped with the supremum norm $\|x\|_X := \max_{s \in \mathbb{R}} |x(s)|$. Let $Y = \mathbb{C}$. Consider the system

$$\dot{x}(t) = Ax(t), \quad y(t) = Cx(t), \quad (10)$$

where A generates the left translation group $T(\cdot)$ on X given for any $x \in X$ by

$$(T(t)x)(s) = x(s+t), \quad s \in \mathbb{R}, \quad t \in \mathbb{R}.$$

The operator $C : X \rightarrow \mathbb{C}$ is defined by

$$Cx = x(0), \quad x \in X.$$

Then X is a Banach space and $T(\cdot)$ is a group of isometries on X ; see (Engel and Nagel, 2000, p. 34-35).

Proposition 4.2. The following hold:

- (i) (10) admits a coercive eOSS Lyapunov function in implication form with linear gain and $p = 1$.
- (ii) (10) is eOSS with linear gain.
- (iii) For any given $\rho \in \mathcal{P}$, $\sigma \in \mathcal{K}_\infty$, (10) does not admit any function $V : X \rightarrow \mathbb{R}_+$, which is continuous along the trajectories (i.e., Definition 2.3(ii) holds) and satisfies the dissipation inequality

$$\dot{V}(x) \leq -\rho(\|x\|_X) + \sigma(\|Cx\|_Y), \quad x \in X. \quad (11)$$

- (iv) There is no coercive eOSS Lyapunov function in dissipation form for (10).

4.3 The existence of an eOSS Lyapunov function does not imply exponential detectability

The final example separates the existence of eOSS Lyapunov functions from exponential detectability.

Let $X = \ell^2$ be the space of all complex-valued sequences $x = (x_n)_{n \in \mathbb{N}}$ such that $\sum_{n=1}^\infty |x_n|^2 < \infty$, equipped with the norm $\|x\|_{\ell^2} := (\sum_{n=1}^\infty |x_n|^2)^{1/2}$ and let $Y = \ell^\infty$ be the space of all bounded complex-valued sequences with the norm $\|y\|_{\ell^\infty} := \sup_{n \in \mathbb{N}} |y_n|$.

Choose a countable dense subset $(z_k)_{k \in \mathbb{N}}$ of the unit sphere of ℓ^2 . Since

$$\|x\|_{\ell^2} = \sup_{\|z\|_{\ell^2}=1} |\langle x, z \rangle_{\ell^2}|,$$

and the map $z \mapsto \langle x, z \rangle_{\ell^2}$ is continuous, we have

$$\|x\|_{\ell^2} = \sup_{k \in \mathbb{N}} |\langle x, z_k \rangle_{\ell^2}|, \quad x \in \ell^2.$$

Define

$$C : \ell^2 \rightarrow \ell^\infty, \quad Cx := (\langle x, z_k \rangle_{\ell^2})_{k \in \mathbb{N}}.$$

Then C is linear, and

$$\|Cx\|_{\ell^\infty} = \sup_{k \in \mathbb{N}} |\langle x, z_k \rangle_{\ell^2}| = \|x\|_{\ell^2}, \quad x \in \ell^2.$$

Thus, C is a linear isometry from ℓ^2 into ℓ^∞ . Equivalently, $C : \ell^2 \rightarrow \text{Im } C$ is an isometric isomorphism.

Consider the system

$$\dot{x}(t) = 0, \quad y(t) = Cx(t). \quad (12)$$

Then $A = 0$ and $T(t) = I$.

Proposition 4.3. The following hold:

- (i) System (12) admits a coercive eOSS Lyapunov function in dissipation form with linear gain with $p = 1$.
- (ii) System (12) is not exponentially detectable.

Remark 4.4. The implication from the existence of a coercive eOSS Lyapunov function in dissipation form with linear gain to exponential detectability fails even for the Hilbert state space $X = \ell^2$ and the bounded generator $A = 0$. However, the example (12) still relies on the fact that the output space Y is not Hilbert.

For the trivial dynamics $A = 0$, this type of construction cannot arise as soon as the output space Y is Hilbert, with X merely assumed to be a Banach space. To see this, suppose that the system admits a coercive eOSS Lyapunov function in dissipation form with arbitrary $\alpha, p > 0$ and arbitrary $\sigma \in \mathcal{K}_\infty$, that is,

$$0 = \dot{V}(x) \leq -\alpha \|x\|_X^p + \sigma(\|Cx\|_Y).$$

Taking $\|x\|_X = 1$, we obtain

$$\alpha \leq \sigma(\|Cx\|_Y).$$

Since $\sigma \in \mathcal{K}_\infty$, it follows that

$$\|Cx\|_Y \geq \sigma^{-1}(\alpha) > 0, \quad \|x\|_X = 1.$$

For $x \neq 0$, it yields

$$\|Cx\|_Y = \|x\|_X \left\| C \frac{x}{\|x\|_X} \right\|_Y \geq \sigma^{-1}(\alpha) \|x\|_X.$$

This implies $\|Cx\|_Y \geq \sigma^{-1}(\alpha) \|x\|_X$ for all $x \in X$.

Thus $C(X)$ is closed in Y and $C^{-1} : C(X) \rightarrow X$ is bounded. Since Y is a Hilbert space, there is an orthogonal projection $P_{C(X)} : Y \rightarrow C(X)$. Hence, the operator

$$L = -\lambda C^{-1} P_{C(X)}$$

is for any $\lambda > 0$ bounded from Y to X and satisfies $LC = -\lambda I$. Therefore $A + LC = -\lambda I$ generates an exponentially stable semigroup.

5. SPECTRUM DECOMPOSITION AT 0 AND EXPONENTIAL DETECTABILITY

Section 4 showed that exponential zero-detectability does not imply eOSS or exponential detectability in general. Here we show that if the unstable subspace of the system is finite-dimensional, then all these properties are equivalent.

Recall that a subspace $V \subset X$ is called $T(\cdot)$ -invariant if

$$T(t)V \subset V \quad \forall t \geq 0.$$

Here and below, $X = X_u \oplus X_s$ means that X is an internal direct sum of X_u and X_s , which are closed subspaces of X , namely: $X_u \cap X_s = \{0\}$, and every $x \in X$ admits a unique decomposition $x = x_u + x_s$ with $x_u \in X_u$ and $x_s \in X_s$, see also (Curtain and Zwart, 2020, Section 8.1).

Assumption 5.1. There exist closed, $T(\cdot)$ -invariant subspaces $X_u, X_s \subset X$ such that

$$X = X_u \oplus X_s, \quad \dim X_u < \infty,$$

where X_u is the spectral subspace of A associated with the spectrum in the closed right half-plane. Moreover, there exist constants $M_s, \omega_s > 0$ such that

$$\|T(t)x_s\|_X \leq M_s e^{-\omega_s t} \|x_s\|_X \quad \forall x_s \in X_s, \quad \forall t \geq 0. \quad (13)$$

Such decompositions, which separate finitely many unstable modes from an exponentially stable infinite-dimensional part, are classical in PDE control; see (Russell, 1978). For sufficient conditions ensuring this assumption, we refer to (Curtain and Zwart, 2020, Section 2.4).

In what follows, we write

$$T_u(t) := T(t)|_{X_u}, \quad T_s(t) := T(t)|_{X_s},$$

and denote their generators by A_u and A_s , respectively. The corresponding output operators are denoted by

$$C_u := C|_{X_u}, \quad C_s := C|_{X_s}.$$

Theorem 5.2. Let Assumption 5.1 hold. The following statements are equivalent:

- (i) System (1) is exponentially detectable.
- (ii) System (1) is eOSS.
- (iii) System (1) is exponentially zero-detectable.
- (iv) The pair (A_u, C_u) is exponentially zero-detectable.
- (v) The pair (A_u, C_u) is exponentially detectable.
- (vi) The pair (A_u, C_u) is observable.

In particular, all the statements (i)-(viii) in Theorem 3.5 are equivalent.

6. EXPONENTIAL OUTPUT-TO-STATE STABILITY OF A PARABOLIC SYSTEM

In this section, the abstract results developed earlier are applied to a concrete parabolic PDE system. First, an explicit coercive UES Lyapunov function is constructed to verify UES under suitable conditions. Next, the finite-dimensional detectability result from Theorem 5.2 is used to show that the system is eOSS with linear gain for all parameter values. Since Assumption 5.1 is satisfied, the parabolic system admits an eOSS Lyapunov function and is exponentially detectable.

Let $a > 0$, and $c, q \in \mathbb{R}$. Consider the linear parabolic equation with output

$$x_t(z, t) = ax_{zz}(z, t) + cx(z, t), \quad (14a)$$

$$y(t) = \int_0^1 x(z) dz, \quad (14b)$$

where $x_t(z, t), x_{zz}(z, t)$ denote the first and second partial derivatives of $x(z, t)$ w.r.t. t respectively z . The equation is defined for $z \in (0, 1), t > 0$, endowed with boundary conditions (namely, a Robin condition on the left end and a Dirichlet condition at the right end)

$$x_z(0, t) = -qx(0, t), \quad (15a)$$

$$x(1, t) = 0, \quad (15b)$$

and an initial condition

$$x(z, 0) = x_0(z).$$

Define

$$A = a \frac{d^2}{dz^2} + c \text{ on } X = L^2(0, 1)$$

with domain

$$D(A) = \{x \in H^2(0, 1) \mid x_z(0) = -qx(0), x(1) = 0\},$$

where $H^2(0, 1)$ is the standard Sobolev space of square integrable functions with (weak) first and second derivatives that are again square integrable.

According to (Engel and Nagel, 2000, Chapter VI, Exercise 4.7), A generates an analytic semigroup.

System (14) with boundary conditions (15) is unstable if c or q is positive and sufficiently large, see (Smyshlyaev and Krstic, 2010, p. 13). However, we have the following:

Proposition 6.1. Let $X = L^2(0, 1)$ and $Y = \mathbb{R}$. If $c < \frac{a\pi^2}{4}$ and $q < \min\{1, \frac{a\pi^2 - 4c}{a(\pi^2 + 4)}\}$, then

$$V(x) := \frac{1}{2} \int_0^1 x^2(z) dz = \frac{1}{2} \|x\|_X^2 \quad (16)$$

is a coercive UES Lyapunov function for (14) with (15), and thus (14) with (15) is UES. Consequently, (14) with (15) is eOSS with linear gain.

Remark 6.2. The coercive UES Lyapunov function in Proposition 6.1 satisfies a dissipation estimate that is independent of the output. Therefore, under the same conditions as in Proposition 6.1, the system is UES and eOSS with linear gain for any nonlinear output of the form $y_p(t) = \int_0^1 |x(z, t)|^p dz$, where $0 < p \leq 2$.

Next we are going to show that the system (14) with (15) is eOSS for all parameter values using a decomposition.

Proposition 6.3. Let $X = L^2(0, 1)$ and $Y = \mathbb{R}$. System (14) with boundary conditions (15) is eOSS with linear gain for any $a > 0$ and any $c, q \in \mathbb{R}$.

Remark 6.4. In particular, (14) with (15) admits a coercive eOSS Lyapunov function, and is both exponentially detectable and exponentially zero-detectable.

REFERENCES

- Allan, D.A., Rawlings, J., and Teel, A.R. (2021). Nonlinear detectability and incremental input/output-to-state stability. *SIAM Journal on Control and Optimization*, 59(4), 3017–3039.
- Bachmann, P., Dashkovskiy, S., and Mironchenko, A. (2026). Characterization of input-to-output stability for infinite-dimensional systems. *IEEE Transactions on Automatic Control*, 71(3), 1481–1496.
- Chaillet, A., Karafyllis, I., Pepe, P., and Wang, Y. (2023). The ISS framework for time-delay systems: a survey. *Math. Contr., Sign., and Syst.*, 35(2), 237–306.
- Chen, Q., Mironchenko, A., and Wirth, F. (2025). Lyapunov criterion for output-to-state stability of distributed parameter systems. *IFAC-PapersOnLine*, 59(8), 60–65.
- Curtain, R. and Zwart, H. (2020). *Introduction to Infinite-Dimensional Systems Theory: A State-Space Approach*. Springer, New York, NY, USA.
- Engel, K.J. and Nagel, R. (2000). *One-Parameter Semigroups for Linear Evolution Equations*. Springer-Verlag, New York.
- Höger, M. and Grüne, L. (2019). On the relation between detectability and strict dissipativity for nonlinear discrete time systems. *IEEE Control Systems Letters*, 3(2), 458–462.
- Jacob, B., Nabiullin, R., Partington, J.R., and Schwenninger, F.L. (2018). Infinite-dimensional input-to-state stability and Orlicz spaces. *SIAM Journal on Control and Optimization*, 56(2), 868–889.
- Jayawardhana, B., Logemann, H., and Ryan, E.P. (2008). Infinite-dimensional feedback systems: the circle criterion and input-to-state stability. *Communications in Information and Systems*, 8(4), 413–444.
- Jiang, Z., Teel, A., and Praly, L. (1994). Small-gain theorem for ISS systems and applications. *Mathematics of Control, Signals, and Systems*, 7(2), 95–120.
- Karafyllis, I. and Krstic, M. (2016). ISS with respect to boundary disturbances for 1-D parabolic PDEs. *IEEE Transactions on Automatic Control*, 61(12), 3712–3724.
- Karafyllis, I. and Krstic, M. (2019). *Input-to-State Stability for PDEs*. Springer-Verlag, London.
- Krichman, M., Sontag, E.D., and Wang, Y. (2001). Input-output-to-state stability. *SIAM Journal on Control and Optimization*, 39(6), 1874–1928.
- Mironchenko, A. and Prieur, C. (2020). Input-to-state stability of infinite-dimensional systems: recent results and open questions. *SIAM Review*, 62(3), 529–614.
- Mironchenko, A. and Wirth, F. (2018a). Characterizations of input-to-state stability for infinite-dimensional systems. *IEEE Transactions on Automatic Control*, 63(6), 1602–1617.
- Mironchenko, A. and Wirth, F. (2018b). Lyapunov characterization of input-to-state stability for semilinear control systems over Banach spaces. *Systems & Control Letters*, 119, 64–70.
- Mironchenko, A. (2023). *Input-to-State Stability: Theory and Applications*. Springer Nature.
- Russell, D.L. (1978). Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. *SIAM Review*, 20(4), 639–739.
- Schwenninger, F.L. (2020). Input-to-state stability for parabolic boundary control: Linear and semilinear systems. In J. Kerner, H. Laasri, and D. Mugnolo (eds.), *Control Theory of Infinite-Dimensional Systems*, 83–116. Springer.
- Smyshlyaev, A. and Krstic, M. (2010). *Adaptive Control of Parabolic PDEs*. Princeton University Press, Princeton, NJ, USA.
- Sontag, E.D. (1998). *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer-Verlag, New York.
- Sontag, E.D. and Wang, Y. (1996a). Detectability of nonlinear systems. In *Proceedings of the Conference on Information Science and Systems (CISS 96)*, 1031–1036. Princeton, NJ.
- Sontag, E. and Wang, Y. (1996b). New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control*, 41(9), 1283–1294.
- Sontag, E.D. (2008). Input to State Stability: Basic Concepts and Results. In *Nonlinear and Optimal Control Theory*, chapter 3, 163–220. Springer, Heidelberg.
- Zheng, J. and Zhu, G. (2021). Approximations of Lyapunov functionals for ISS analysis of a class of higher dimensional nonlinear parabolic PDEs. *Automatica*, 125, 109414.
- Zheng, J. and Zhu, G. (2026). Unified Lyapunov Method for ISS of PDEs: A Tutorial on Constructing Generalized Lyapunov Functionals for Parabolic and Hyperbolic Equations. *arXiv preprint arXiv:2605.01344*.