

On a small-gain approach to distributed event-triggered control

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Abstract: In this paper the problem of stabilizing large-scale systems by distributed controllers, where the controllers exchange information via a shared limited communication medium is addressed. An event-triggered sampling scheme is proposed, in which each system decides when to transmit new information across the network based on the crossing of some error threshold. Stability of the interconnected large-scale system is inferred by applying a generalized small-gain theorem.

Keywords: Large scale systems, asymptotic stability, Lyapunov function, nonlinear control systems, input-to-state stability, event-triggered sampling, stability analysis.

1. INTRODUCTION

We consider large-scale systems stabilized by distributed controllers, which communicate over a limited shared medium. In this context it is of interest to reduce the communication load. An approach in this direction is event-triggered sampling, which attempts only to send "relevant" data. In order to treat the large-scale case, a combination of ISS small-gain results with ideas from event-triggering is presented.

The stability (or stabilization) of large-scale interconnected systems is an important problem which has attracted much interest. In this context the small-gain theorem was extended to the interconnection of several \mathcal{L}_p -stable subsystems. Early accounts of this approach are Vidyasagar (1981) (see also Šiljak (1978)) and references therein. For instance, in Vidyasagar (1981), Theorem 6.12, the influence of each subsystem on the others is measured via an \mathcal{L}_p -gain, $p \in [1, \infty]$ and the \mathcal{L}_p -stability of the interconnected system holds provided that the spectral radius of the matrix of the gains is strictly less than unity. In other words, the stability of interconnected \mathcal{L}_p -stable systems holds under a condition of weak coupling.

In the nonlinear case a notion of robustness with respect to exogenous inputs is input-to-state stability (ISS) (Sontag (1989)). If in a large-scale system each subsystem is ISS, then the influence between the subsystems is typically modeled via nonlinear gain functions. Small-gain theorems have been developed for ISS systems as well (Jiang et al. (1994, 1996); Teel (1996)) and more recently they have been extended to the interconnection of several

ISS subsystems (Dashkovskiy et al. (2007, 2010)). For a recent comprehensive discussion about the literature on ISS small-gain results see Liu et al. (2010).

In the literature on large-scale systems we have discussed so far, the communication aspect does not play a role. If however, a shared communication medium leads to significant further restrictions, concepts like event-triggering become of interest. We speak of event-triggering if the occurrence of predefined events, as e.g. the violation of error bounds, triggers a communication attempt. Using this approach a decentralized way of stabilizing large-scale networked control systems has been proposed in Wang and Lemmon (2009a, 2011). In these papers each subsystem broadcasts information when a locally-computed error signal exceeds a state-dependent threshold. Similar ideas are presented in Tabuada (2007); Wang and Lemmon (2009b). Numerical experiments e.g., Wang and Lemmon (2009b) show that event-triggered stabilizing controllers can lead to less information transmission than standard sampled-data controllers.

One drawback of the proposed event-triggered sampling scheme is the need for constantly checking the validity of an inequality. A related approach which tries to overcome this issue is termed self-triggered sampling (see e.g., Anta and Tabuada (2010); Mazo Jr et al. (2010)).

From a more general perspective, the way in which the subsystems access the medium must be carefully designed. In this paper we do not discuss the problem of collision avoidance. This problem is addressed for instance in the literature on medium access protocols, such as the round-robin and the try-once-discard protocol. E.g., in Nešić and Teel (2007) a large class of medium access protocols are treated as dynamical systems and the stability analysis in the presence of communication constraints is carried

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out by including the protocols in the closed-loop system. This allows to give an estimate on the maximum allowable transfer interval (MATI), that is the maximum interval of time between two consecutive transmissions which the system can tolerate without going into instability. The advantage of event-triggering lies in the possibility of reducing overall communication load. However, if events occur simultaneously at several subsystems the problem of collision avoidance remains. We will discuss this in future work.

The purpose of this paper is to explore event-triggered distributed controllers for systems, which are given as an interconnection of a large number of ISS subsystems. We assume that the gains measuring the degree of interconnection satisfy a generalized small-gain condition. To simplify presentation, it is assumed furthermore that the graph modeling the interconnection structure is strongly connected. This assumption can be removed as in Dashkovskiy et al. (2010). Since our event-triggered implementation of the control laws introduces disturbances into the system, the ISS small-gain results available in the literature are not applicable. An additional condition is required for general nonlinear systems using event triggering. This condition is explicitly given in the presented general small-gain theorem. Moreover, the functions which are needed to design the state-dependent triggering conditions are explicitly designed in such a way that the triggering events which supervise the broadcast by a subsystem only depend on local information. As an introductory example we explicitly discuss the special case of linear systems, although for this class of systems the techniques of Wang and Lemmon (2009a, 2011) are applicable. As distributed event-triggered controllers can potentially require transmission times which accumulate in finite time, we also discuss a variation of the proposed small-gain event-triggered control laws which prevents the occurrence of the Zeno phenomenon. For consensus problems, Zeno-free event-triggered controllers are studied in Dimarogonas and Johansson (2009). A related paper is also Mazo and Tabuada (2010).

Section 2 presents the class of system we focus our attention on, along with a number of preliminary notions and standing assumptions. In Section 4 the notion of ISS-Lyapunov functions is presented. Based on this notion small-gain event-triggered distributed controllers are discussed in Section 5. The results are particularized to the case of linear systems in Section 3 along with a few simulation results in Section 6. The Zeno-free distributed event-triggered controllers are proposed in Section 7. The last section contains the conclusions of the paper.

Notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. \mathbb{R}_+ denotes the set of nonnegative real numbers, and \mathbb{R}_+^n the nonnegative orthant, i.e. the set of all vectors of \mathbb{R}^n which have all the entries nonnegative. By $\|\cdot\|$ we denote the Euclidean norm of a vector or a matrix.

A function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a class- \mathcal{K} function if it is continuous, strictly increasing and zero at zero. If it is additionally unbounded, i.e. $\lim_{r \rightarrow +\infty} \alpha(r) = \infty$, then α is said to be a class- \mathcal{K}_∞ function. We use the notation $\alpha \in \mathcal{K}$ ($\alpha \in \mathcal{K}_\infty$) to say that α is a class- \mathcal{K} (class- \mathcal{K}_∞) function. The symbol $\mathcal{K} \cup \{0\}$ ($\mathcal{K}_\infty \cup \{0\}$) refers to the set of functions which include all the class- \mathcal{K} (class- \mathcal{K}_∞) functions and the

function which is identically zero. A function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is positive definite if $\alpha(r) = 0$ if and only if $r = 0$.

2. PRELIMINARIES

Consider the interconnection of N systems described by equations of the form:

$$\begin{aligned} \dot{x}_i &= f_i(x, u_i) \\ u_i &= g_i(x + e), \end{aligned} \quad (1)$$

where $i \in \mathcal{N} \triangleq \{1, 2, \dots, N\}$, $x = (x_1^T \dots x_N^T)^T$, with $x_i \in \mathbb{R}^{n_i}$, is the state vector and $u_i \in \mathbb{R}^{m_i}$ is the i th control input. The vector e , with $e = (e_1^T \dots e_N^T)^T$ and $e_i \in \mathbb{R}^{n_i}$, is an error affecting the state. We shall assume that the maps f_i satisfy appropriate conditions which guarantee existence and uniqueness of solutions for \mathcal{L}_∞ inputs e .

The interconnection of each system i with another system j is possible in two ways. One way is that the system j influences the dynamics of the system i directly, meaning that the state variable x_j appears non trivially in the function f_i . The other way is that the controller i uses information from the agent j . In this case, the state variable x_j appears non trivially in the function g_i (and affects indirectly the dynamics of the system i).

In this paper we adopt the notion of ISS-Lyapunov functions (Sontag and Wang (1995)) to model the interconnection among the systems.

Definition 1. A smooth function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is called an *ISS-Lyapunov* function for system $\dot{x} = f(x, u)$ if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3, \chi \in \mathcal{K}$, such that for any $x \in \mathbb{R}^n$

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

and the following implication holds for all $x \in \mathbb{R}^n$ and all admissible u

$$V(x) \geq \chi(\|u\|) \Rightarrow \nabla V(x) \leq -\alpha_3(\|x\|).$$

It is well known that a system as in Definition 1 is ISS if and only if it admits an ISS-Lyapunov function. If there are more than one input present in the system, the question how to compare the influence of the different inputs arises. To answer this question we preliminarily recall the notion of monotone aggregate function from Dashkovskiy et al. (2010):¹

Definition 2. A continuous function $\mu : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is a *monotone aggregation function* if:

- (i) $\mu(v) \geq 0$ for all $v \in \mathbb{R}_+^n$ and $\mu(v) > 0$ if $s \gneq 0$;
- (ii) $\mu(v) > \mu(z)$ if $v > z$;
- (iii) If $\|v\| \rightarrow \infty$ then $\mu(v) \rightarrow \infty$.

The space of monotone aggregate functions (MAFs in short) with domain \mathbb{R}_+^n is denoted by MAF_n . Moreover, it is said that $\mu \in MAF_n^m$ if for each $i = 1, 2, \dots, m$, $\mu_i \in MAF_n$.

Monotone aggregate function are used in the following assumption to specify the way in which systems are interconnected and how controllers use information about the other systems:

¹ In the definition below, for any pair of vectors $v, z \in \mathbb{R}^n$, the notations $v \geq z$, $v > z$ are used to express the property that $v_i \geq z_i$, $v_i > z_i$ for all $i = 1, 2, \dots, n$. Moreover, the notation $v \gneq z$ indicates that $v \geq w$ and $v \neq w$.

Assumption 1. For $i = 1, 2, \dots, N$, there exists a differentiable function $V_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$, and class- \mathcal{K}_∞ functions α_{i1}, α_{i2} such that

$$\alpha_{i1}(\|x_i\|) \leq V_i(x_i) \leq \alpha_{i2}(\|x_i\|).$$

Moreover there exist functions $\mu_i \in \text{MAF}_{2N}$, $\gamma_{ij}, \eta_{ij} \in \mathcal{K}_\infty \cup \{0\}$, $j \neq i$, $\gamma_i \in \mathcal{K} \cup \{0\}$, α_i positive definite such that

$$\begin{aligned} V_i(x_i) &\geq \mu_i(\gamma_{i1}(V_1(x_1)), \dots, \gamma_{iN}(V_N(x_N))), \\ \eta_{i1}(\|e_1\|), \dots, \eta_{iN}(\|e_N\|) &\Rightarrow \\ \nabla V_i(x_i) f_i(x, g_i(x+e)) &\leq -\alpha_i(\|x_i\|). \end{aligned} \quad (2)$$

Loosely speaking, the function γ_{ij} describes the overall influence of system j on the dynamics of system i , while the function η_{ij} describes the influence of the system j on system i via the controller g_i . In particular, $\eta_{ij} \neq 0$ if and only if the controller u_i is using information from the system j . In this regard η_{ij} describes the influence of imperfect knowledge of the state of system j on system i caused by e.g., measurement noise. On the other hand, if $i \neq j$ and $\gamma_{ij} \neq 0$, then the system j influences system i (either explicitly or implicitly). We assume that $\gamma_{ii} = 0$ for any i . Observe that if system i is not influenced by any other system $j \neq i$, and there is no error e_i on the state information x_i used in the control u_i , then the assumption amounts to saying that system i is input-to-state stabilizable via state feedback.

3. THE CASE OF LINEAR SYSTEMS

To get acquainted with the assumption above, we examine in the following example the case in which the systems are linear.

Example 1. Consider the interconnection of N linear sub-systems

$$\begin{aligned} \dot{x}_i &= \sum_{j=1}^N A_{ij} x_j + B_i u_i \\ u_i &= \sum_{j=1}^N K_{ij} (x_j + e_j). \end{aligned}$$

For each index i , we assume that the pairs (A_{ii}, B_i) are stabilizable and we let the matrix K_{ii} be such that $\bar{A}_{ii} := A_{ii} + B_i K_{ii}$ is Hurwitz. Then for each $Q_i = Q_i^T > 0$ there exists a matrix $P_i = P_i^T > 0$ such that $\bar{A}_{ii}^T P_i + P_i \bar{A}_{ii} = -Q_i$. We consider now the expression $\nabla V_i(x_i) \dot{x}_i$ where

$$\begin{aligned} \dot{x}_i &= \sum_{j=1}^N (A_{ij} + B_i K_{ij}) x_j + \sum_{j=1}^N B_i K_{ij} e_j \\ &=: \sum_{j=1}^N \bar{A}_{ij} x_j + \sum_{j=1}^N \bar{B}_{ij} e_j, \end{aligned}$$

with $\bar{B}_{ij} := B_i K_{ij}$.

Standard calculations lead to the following:

$$\begin{aligned} \nabla V_i(x_i) \dot{x}_i &\leq -c_i \|x_i\|^2 + \\ 2\|x_i\| \|P_i\| &\left(\sum_{j=1, j \neq i}^N \|\bar{A}_{ij}\| \|x_j\| + \sum_{j=1}^N \|\bar{B}_{ij}\| \|e_j\| \right), \end{aligned}$$

where $c_i = \lambda_{\min}(Q_i)$. Moreover, for any $0 < \tilde{c}_i < c_i$ the inequality

$$\|x_i\| \geq \frac{2\|P_i\|}{\tilde{c}_i} \left(\sum_{j=1, j \neq i}^N \|\bar{A}_{ij}\| \|x_j\| + \sum_{j=1}^N \|\bar{B}_{ij}\| \|e_j\| \right)$$

implies that

$$\nabla V_i(x_i) \dot{x}_i \leq -(c_i - \tilde{c}_i) \|x_i\|^2.$$

The former inequality is implied by:

$$\begin{aligned} V_i(x_i) &\geq \|P_i\| \cdot \\ &\left[\frac{2\|P_i\|}{\tilde{c}_i} \left(\sum_{j=1, j \neq i}^N \frac{\|\bar{A}_{ij}\|}{[\lambda_{\min}(P_j)]^{1/2}} V_j(x_j)^{1/2} + \right. \right. \\ &\quad \left. \left. \sum_{j=1}^N \|\bar{B}_{ij}\| \|e_j\| \right) \right]^2. \end{aligned}$$

We conclude that (2) holds with

$$\left. \begin{aligned} \gamma_{ii} &= 0 \\ \gamma_{ij}(r) &= \frac{2\|P_i\|^{3/2}}{\tilde{c}_i} \frac{\|\bar{A}_{ij}\|}{[\lambda_{\min}(P_j)]^{1/2}} r^{1/2} \\ \eta_{ij}(r) &= \frac{2\|P_i\|^{3/2}}{\tilde{c}_i} \|\bar{B}_{ij}\| r \\ \mu_i(s) &= \left(\sum_{j=1}^{2n} s_j \right)^2 \\ \alpha_i(r) &= (c_i - \tilde{c}_i) r^2. \end{aligned} \right\} \quad (3)$$

It is important to remember that not all the functions γ_{ij} and η_{ij} are non-zero. Namely, γ_{ij} is non-zero if and only if A_{ij} or $B_i K_{ij}$ are non-zero matrices. In what follows, we shall refer to the set of indices j for which either one of the two matrices above are non-zero as the set \mathcal{N}_i , and to the systems corresponding to these indices $j \in \mathcal{N}_i$ as the neighbors of system i . The subset of indices j for which $B_i K_{ij}$ is non-zero is denoted as $\mathcal{N}_i^{(e)}$, and clearly $\mathcal{N}_i^{(e)} \subseteq \mathcal{N}_i$. \triangleleft

4. ISS LYAPUNOV FUNCTIONS FOR LARGE-SCALE SYSTEMS

In this section we review a general procedure for the construction of ISS Lyapunov functions. In particular, we extend recent results to a more general case that covers the case of event-triggered control.

Condition (2) can be used to naturally build a graph which describes how the systems are interconnected. Let us introduce the matrix of functions $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$ defined as

$$\Gamma = \begin{pmatrix} 0 & \gamma_{12} & \gamma_{13} & \dots & \gamma_{1N} \\ \gamma_{21} & 0 & \gamma_{23} & \dots & \gamma_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{N1} & \gamma_{N2} & \gamma_{N3} & \dots & 0 \end{pmatrix}.$$

Following Dashkovskiy et al. (2010), we associate to Γ the adjacency matrix $A_\Gamma = [a_{ij}] \in \{0, 1\}^{N \times N}$ whose entry a_{ij} is zero if and only if $\gamma_{ij} = 0$, otherwise it is equal to 1. A_Γ can be interpreted as the adjacency matrix of the graph which has a set \mathcal{N} of N nodes, each one of which is associated to a system of (1), and a set of edges $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ with the property that $(j, i) \in \mathcal{E}$ if and only if $a_{ij} = 1$. Recall that a graph is strongly connected if and only if the associated adjacency matrix is irreducible. In the present case, if the adjacency matrix A_Γ is irreducible, then we

say that Γ is irreducible. In other words, the matrix of functions Γ is said to be irreducible if and only if the graph associated to it is strongly connected.

For later use, given $\mu_i \in \text{MAF}_{2N}$, $\gamma_{ij}, \eta_{ij} \in \mathcal{K}_\infty \cup \{0\}$, it is useful to introduce the map $\bar{\Gamma}_\mu : \mathbb{R}_+^{2N} \rightarrow \mathbb{R}_+^N$ defined as

$$\bar{\Gamma}_\mu(r, s) = \begin{pmatrix} \mu_1(\gamma_{11}(r_1), \dots, \gamma_{1N}(r_N), \eta_{11}(s_1), \dots, \eta_{1N}(s_N)) \\ \vdots \\ \mu_n(\gamma_{n1}(r_1), \dots, \gamma_{nN}(r_N), \eta_{n1}(s_1), \dots, \eta_{nN}(s_N)) \end{pmatrix}.$$

Moreover, we set $\Gamma_\mu(r) := \bar{\Gamma}_\mu(r, 0)$. Since the functions which describe the interconnection of the system are in general nonlinear, the topological property of graph connectivity may not be sufficient to ensure stability properties of the interconnected system. There must also be a way to quantify the degree of coupling of the systems. In this paper, this is done using the following notion:

Definition 3. A map $\sigma \in \mathcal{K}_\infty^N$ is an Ω -path with respect to Γ_μ if:

- (i) for each i , the function σ_i^{-1} is locally Lipschitz continuous on $(0, \infty)$;
- (ii) for every compact set $K \subset (0, \infty)$ there are constants $0 < c < C$ such that for all $i = 1, 2, \dots, n$ and all the points of differentiability of σ_i^{-1} we have:

$$0 < c \leq (\sigma_i^{-1})'(r) \leq C, \quad \forall r \in K;$$

- (iii) $\Gamma_\mu(\sigma(r)) < \sigma(r)$ for all $r > 0$.

Condition (iii) in the definition above amounts to a small-gain condition for large-scale non-linear systems (in other words, condition (iii) requires the degree of coupling among the different subsystems to be weak. For a more thorough discussion on condition (iii) see Dashkovskiy et al. (2010)). To familiarize with the condition, take the case $N = 2$ and $\mu_1 = \mu_2 = \max$ (it is not difficult to see that the function $\max_{1 \leq i \leq N} r_i$ belongs to MAF_N). Then

$$\Gamma_\mu(r) = \begin{pmatrix} \gamma_{12}(r_2) \\ \gamma_{21}(r_1) \end{pmatrix}.$$

We want to show that there exists $\sigma \in \mathcal{K}_\infty^2$ such that $\Gamma_\mu(\sigma(s)) < \sigma(s)$ for all $s > 0$ if and only if $\gamma_{12} \circ \gamma_{21}(r) < r$ for all $r > 0$ (the latter can be viewed as a small-gain condition for the interconnection of ISS-subsystems). To this purpose, choose

$$\sigma(r) = \begin{pmatrix} r \\ \sigma_2(r) \end{pmatrix},$$

where $\gamma_{21} < \sigma_2 < \gamma_{12}^{-1}$. As a consequence of this choice, $\Gamma_\mu(\sigma(s))$ becomes:

$$\Gamma_\mu(\sigma(s)) = \begin{pmatrix} \gamma_{12}(\sigma_2(s)) \\ \gamma_{21}(s) \end{pmatrix}.$$

By construction, $\gamma_{12}(\sigma_2(s)) < s = \sigma_1(s)$ and $\gamma_{21}(s) < \sigma_2(s)$, i.e. $\Gamma_\mu(\sigma(s)) < \sigma(s)$ for all $s > 0$. Strong connectivity of Γ and an additional condition implies a weak coupling among all the systems, in the following sense (see Dashkovskiy et al. (2010) for a proof and a more complete statement):

Theorem 1. Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$ and $\mu \in \text{MAF}_{2N}^N$. If Γ is irreducible and $\Gamma_\mu \not\geq id$ ² then there exists an Ω -path σ with respect to Γ_μ .

Remark 1. In fact, the irreducibility condition on Γ is a purely technical assumption. A way how to relax it can be found in Dashkovskiy et al. (2010).

The small gain condition stated above is not sufficient to infer stability of the overall system in the case in which error inputs are present in the system. Then an additional condition is required:

Assumption 2. There exist an Ω -path σ with respect to Γ_μ and a map $\varphi \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$ such that:

$$\bar{\Gamma}_\mu(\sigma(r), \varphi(r)) < \sigma(r), \quad \forall r > 0, \quad (4)$$

where

$$\bar{\Gamma}_\mu(\sigma(r), \varphi(r)) := \begin{pmatrix} \mu_1(\gamma_{11}(\sigma_1(r)), \dots, \gamma_{1n}(\sigma_n(r)), \varphi_{11}(r), \dots, \varphi_{1N}(r)) \\ \vdots \\ \mu_N(\gamma_{N1}(\sigma_1(r)), \dots, \gamma_{NN}(\sigma_N(r)), \varphi_{N1}(r), \dots, \varphi_{NN}(r)) \end{pmatrix}.$$

5. MAIN RESULTS

In our first result it is shown that a Lyapunov function V and a set of decentralized conditions exist which guarantee that V decreases along the trajectories of the system:

Theorem 2. Let Assumptions 1 and 2 hold. Let $V(x) = \max_{i \in \mathcal{N}} \sigma_i^{-1}(V_i(x_i))$ and, for each $i \in \mathcal{N}$, define:

$$\chi_i = \sigma_i \circ \hat{\eta}_i, \quad \text{with } \hat{\eta}_i = \max_{j \in \mathcal{N}} \varphi_{ji}^{-1} \circ \eta_{ji}.$$

Then the condition

$$V_i(x_i) \geq \chi_i(\|e_i\|), \quad \forall i \in \mathcal{N} \quad (5)$$

implies

$$\langle p, f(x, g(x+e)) \rangle \leq -\alpha(\|x\|), \quad \forall p \in \partial V(x),$$

where ∂V denotes the Clarke generalized gradient³ and

$$f(x, e) = \begin{pmatrix} f_1(x, g_1(x+e)) \\ \vdots \\ f_n(x, g_n(x+e)) \end{pmatrix}.$$

Remark 2. If $\varphi_{ji} = 0$, then we set conventionally $\varphi_{ji}^{-1} = 0$. We also remark that when computing $\max_{j \in \mathcal{N}} \varphi_{ji}^{-1} \circ \eta_{ji}$ the indices j to consider are those corresponding to the agents which use information from agent i to compute their control law: only for these agents j it is true that $\eta_{ji} \neq 0$.

Proof: For each x , let $\mathcal{N}(x) \subseteq \mathcal{N}$ be the set of indices i for which $V(x) = \sigma_i^{-1}(V_i(x_i))$. Let $i \in \mathcal{N}(x)$ and set $r = V(x)$. Then

$$\begin{aligned} V_i(x_i) &= \sigma_i(r) > \bar{\Gamma}_{\mu,i}(\sigma(r), \varphi(r)) \\ &= \mu_i(\gamma_{i1}(\sigma_1(r)), \dots, \gamma_{iN}(\sigma_N(r)), \\ &\quad \varphi_{i1}(r), \dots, \varphi_{iN}(r)). \end{aligned} \quad (6)$$

² $\Gamma_\mu \not\geq id$ means that for all $s > 0$ $\Gamma_\mu(s) \not\geq s$, i.e. for all $s \in \mathbb{R}_+^N$ such that $s > 0$ there exists $i \in \mathcal{N}$ for which $\mu_i(s_1, \dots, s_N, 0, \dots, 0) < s_i$.

³ We recall that by Rademacher's theorem the gradient ∇V of a locally Lipschitz function V exists almost everywhere. Let N be the set of measure zero where ∇V does not exist and let S be any measure zero subset of the state space where V lives. Then $\partial V(x) = \text{co}\{\lim_{i \rightarrow +\infty} \nabla V(x_i) : x \rightarrow x_i, x_i \notin N, x_i \notin S\}$.

Observe first that by definition of $V(x)$, for any $i \in \mathcal{N}(x)$ and any $j \in \mathcal{N}$,

$$\begin{aligned} \gamma_{ij}(\sigma_j(r)) &\geq \gamma_{ij}(\sigma_j(V(x))) \geq \\ &\gamma_{ij}(\sigma_j(\sigma_j^{-1}(V_j(x_j)))) = \gamma_{ij}(V_j(x_j)). \end{aligned}$$

Moreover, since for any $j \in \mathcal{N}$,

$$V_j(x_j) \geq \chi_j(\|e_j\|), \quad \chi_j = \sigma_j \circ \hat{\eta}_j$$

we have,

$$\begin{aligned} \varphi_{ij}(r) &= \varphi_{ij}(V(x)) \geq \varphi_{ij}(\sigma_j^{-1}(V_j(x_j))) \\ &\geq \varphi_{ij}(\sigma_j^{-1} \circ \sigma_j(\hat{\eta}_j(\|e_j\|))) \\ &\geq \varphi_{ij}(\sigma_j^{-1} \circ \sigma_j(\varphi_{ij}^{-1} \circ \eta_{ij}(\|e_j\|))) \\ &= \eta_{ij}(\|e_j\|). \end{aligned} \quad (7)$$

Observe that $\mu_i(v) \geq \mu_i(z)$ for all $v \geq z \in \mathbb{R}_+^{2n}$ since $\mu_i \in \text{MAF}_{2N}$ and as a consequence of Definition 2, (ii). Since $r = V(x) \geq \sigma_i^{-1}(V_i(x_i))$ for all $i \in \mathcal{N}$, by (7),

$$\begin{aligned} \mu_i(\gamma_{i1}(\sigma_1(r)), \dots, \gamma_{iN}(\sigma_N(r)), \varphi_{i1}(r), \dots, \varphi_{iN}(r)) \\ \geq \mu_i(\gamma_{i1}(V_1(x_1)), \dots, \gamma_{iN}(V_N(x_N)), \\ \eta_{i1}(\|e_1\|), \dots, \eta_{iN}(\|e_N\|)). \end{aligned}$$

The inequality above and (6) yield that for each $i \in \mathcal{N}(x)$

$$\begin{aligned} V_i(x_i) &> \mu_i(\gamma_{i1}(\sigma_1(r)), \dots, \gamma_{iN}(\sigma_N(r)), \varphi_{i1}(r), \dots, \varphi_{iN}(r)) \\ &\geq \mu_i(\gamma_{i1}(V_1(x_1)), \dots, \gamma_{iN}(V_N(x_N)), \\ &\quad \eta_{i1}(\|e_1\|), \dots, \eta_{iN}(\|e_N\|)). \end{aligned}$$

Hence, by (2),

$$\nabla V_i(x_i) f_i(x, g_i(x+e)) \leq -\alpha_i(\|x_i\|)$$

for all $i \in \mathcal{N}(x)$.

We now provide a bound to the product $\langle p, f_i(x, g_i(x+e)) \rangle$ for each $p \in \partial\sigma_i^{-1}(V_i(x_i))$ and $i \in \mathcal{N}(x)$. Observe that σ^{-1} is only locally Lipschitz and Clarke generalized gradient must be used for $\sigma_i^{-1}(V_i(x_i))$. Fix x_i and let $\rho > 0$ be such that $\|x_i\| = \rho$. Define the compact set $K_\rho = \{V_i(x_i) \in \mathbb{R}_+ : \rho/2 \leq \|x_i\| \leq 2\rho\}$, and let

$$c_\rho = \min_{r \in K_\rho} (\sigma_i^{-1})'(r), \quad C_\rho = \max_{r \in K_\rho} (\sigma_i^{-1})'(r),$$

where $c_\rho > 0$ by definition of Ω -path σ . Bearing in mind that $\|x_i\| = \rho$, for each $p \in \partial\sigma_i^{-1}(V_i(x_i))$ there exists $\gamma_\rho \in [c_\rho, C_\rho]$ such that $p = \gamma_\rho \nabla V_i(x_i)$, and $\langle p, f_i(x, g_i(x+e)) \rangle = \gamma_\rho \nabla V_i(x_i) \cdot f_i(x, g_i(x+e)) \leq -\gamma_\rho \alpha_i(\rho) \leq -c_\rho \alpha_i(\rho)$. Set $\tilde{\alpha}_i(\rho) := c_\rho \alpha_i(\rho)$, which is a positive function for all positive ρ . Also set

$$\alpha(r) := \min\{\tilde{\alpha}_i(\|x_i\|) : r = \|x_i\|, i \in \mathcal{N}(x)\}.$$

Then, for each $p \in \partial\sigma_i^{-1}(V_i(x_i))$, $\langle p, f_i(x, g_i(x+e)) \rangle \leq -\tilde{\alpha}_i(\|x_i\|) \leq -\alpha(\|x\|)$. This in turn implies (Dashkovskiy et al. (2010)) that for each $p \in \partial V(x)$ $\langle p, f(x, g(x+e)) \rangle \leq -\alpha(\|x\|)$. \square

In the rest of the section we discuss an event-triggered control scheme (Tabuada (2007)) for the system

$$\dot{x}_i = f_i(x, u_i), \quad i \in \mathcal{N}.$$

Before we can state a stability result for the proposed event-triggered sampling scheme we need some more notations. Let $g_i, i \in \mathcal{N}$, be the map defined in Assumption 1, and let $u_i(t) = g_i(\hat{x}(t))$ be the control laws, where the vector $\hat{x}(t) = (\hat{x}_1(t)^T, \dots, \hat{x}_N(t)^T)^T$ is defined next.

For each $i \in \mathcal{N}$, let $\{t_k^i\}_{k \in \mathbb{N}_0}$ be a sequence of times with $t_0^i = 0$, $t_0^i < t_1^i < t_2^i < \dots$, and let $\hat{x}_i(t) = x_i(t_k^i)$, for all $t \in [t_k^i, t_{k+1}^i)$, for all $k \in \mathbb{N}_0$. Hence, the controllers use the samples $x_i(t_k^i)$ instead of the states $x_i(t)$. The

sampling times t_k^i are decided according to the following rule. Set $e_i(t) = \hat{x}_i(t) - x_i(t)$. Observe that $e_i(t_k^i) = 0$, since $\hat{x}_i(t_k^i) = x_i(t_k^i)$. The next time t_{k+1}^i in the sequence is computed as the minimal time greater than t_k^i such that $V_i(x_i(t)) \geq \chi_i(\|e_i(t)\|)$, where χ_i are the functions defined in Theorem 2.

Theorem 3. Let Assumptions 1 and 2 hold. Consider the interconnected system

$$\dot{x}_i(t) = f_i(x(t), g_i(\hat{x}(t))), \quad i \in \mathcal{N}, \quad (8)$$

where for each $i \in \mathcal{N}$, $\hat{x}_i(t) = x_i(t_k^i)$ for $t \in [t_k^i, t_{k+1}^i)$, and $\{t_k^i\}_{k \in \mathbb{N}_0}$ is the sequence of sampling times such that $t_0^i = 0$, and, for each $k \in \mathbb{N}_0$, $t_{k+1}^i > t_k^i$ is the smallest time t such that $\chi_i(\|\hat{x}_i(t) - x_i(t_k^i)\|) \geq V_i(x_i(t))$. Then the origin is a globally uniformly asymptotically stable equilibrium for (8).

Proof: To analyze the event-based control scheme introduced above, we define the time-varying map $\tilde{f}(t, x) = f(x, g(x+e(t)))$. The map $\tilde{f}(t, x)$ satisfies the Carathéodory conditions for the existence of solutions (see e.g. Bacciotti and Rosier (2005), Section 1.1). Because of the conditions on f (see Section 2), the solution exists and is unique. Along the solutions of $\dot{x} = \tilde{f}(t, x)$, the locally Lipschitz positive definite and radially unbounded Lyapunov function $V(x)$ introduced in Theorem 2 satisfies

$$V(x(t'')) - V(x(t')) = \int_{t'}^{t''} \frac{d}{dt} V(x(t)) dt$$

for each pair of times $t'' \geq t'$ belonging to the interval of existence of the solution. Moreover, by a property of the Clarke generalized gradient (Ceragioli (1999), Section 2.3, Proposition 4), for almost all $t \in \mathbb{R}_+$, there exists $p \in \partial V(x(t))$ such that:

$$\frac{d}{dt} V(x(t)) = \langle p, \tilde{f}(t, x(t)) \rangle.$$

By definition of $\tilde{f}(t, x)$, and recalling Theorem 2, it is true that (see Sanfelice et al. (2007), Section IV.B, for similar arguments)

$$V(x(t'')) - V(x(t')) = - \int_{t'}^{t''} \alpha(\|x(t)\|) dt.$$

We can now apply Bacciotti and Rosier (2005), Theorem 3.2, to conclude that the origin of $\dot{x} = \tilde{f}(t, x)$, and therefore of $\dot{x} = f(x, g(\hat{x}))$, is uniformly globally asymptotically stable. \square

The accumulation in finite time of the sampling times may affect event-triggered controllers. In Section 7 we discuss a variation of the previous distributed event-triggered control for which this phenomenon does not occur.

6. AN EXAMPLE

Consider the interconnection of linear systems as in Section 3

$$\dot{x}_i = \sum_{j=1}^N \bar{A}_{ij} x_j + \sum_{j=1}^N \bar{B}_{ij} e_j \quad i = 1, \dots, N, \quad (9)$$

with \bar{A}_{ii} Hurwitz for $i \in \{1, \dots, N\}$. In order to apply our event-triggered sampling scheme, we first have to check the conditions of Theorem 2. As verified in Section 3,

Assumption 1 holds for system (9) with each Lyapunov function given by $V_i(x_i) = x_i^T P_i x_i$.

To check Assumption 2 we recall Lemma 7.2 from Dashkovskiy et al. (2010):

Lemma 1. Let $\alpha \in \mathcal{K}_\infty$ satisfy $\alpha(ab) = \alpha(a)\alpha(b)$ for all $a, b \geq 0$. Let $D = \text{diag}(\alpha)$, $G \in \mathbb{R}^{n \times n}$, and Γ_μ be given by

$$\Gamma_\mu(s) = D^{-1}(GD(s)).$$

Then $\Gamma_\mu \not\leq \text{id}$ if and only if the spectral radius of G is less than one.

It is easy to see that for the linear case Γ_μ from Section 4 with entries from (3) is of the form of Lemma 1 with $\alpha(r) := \sqrt{r}$ and

$$G_{ij} = \frac{2\|P_i\|^{3/2}}{\tilde{c}_i} \frac{\|\bar{A}_{ij}\|}{[\lambda_{\min}(P_j)]^{1/2}}, \quad i \neq j, \quad i, j = 1, \dots, N$$

and zeros as diagonal entries. In other words, $\gamma_{ij}(r) = G_{ij}\alpha(r)$. Let us assume that the spectral radius of G is less than one. For the case of linear systems an Ω -path is given by a half line in the direction of an eigenvector s^* of a matrix G^* which is a perturbed version of G (for details, see the proof of (Dashkovskiy et al., 2010, Lemma 7.12)). Denote this half line by $\sigma(r) := s^*r$.

To show a way to construct a φ for which

$$\bar{\Gamma}_\mu(\sigma(r), \varphi(r)) < \sigma(r), \quad \forall r > 0 \quad (10)$$

holds, consider the i th row of (10) and exploit the fact that the Ω -path is linear:

$$\left(\sum_{j=1, j \neq i}^N \frac{2\|P_i\|^{3/2}}{\tilde{c}_i} \frac{\|\bar{A}_{ij}\|}{[\lambda_{\min}(P_j)]^{1/2}} \sqrt{rs_j^*} + \sum_{j=1}^N \varphi_{ij}(r) \right)^2 < rs_i^*.$$

Bearing in mind that $\|\bar{A}_{ij}\| \neq 0$ if and only if $j \in \mathcal{N}_i \setminus i$ (see last paragraph of Example 3), the inequality rewrites as:

$$\left(\sum_{j \in \mathcal{N}_i \setminus i} \frac{2\|P_i\|^{3/2}}{\tilde{c}_i} \frac{\|\bar{A}_{ij}\|}{[\lambda_{\min}(P_j)]^{1/2}} \sqrt{rs_j^*} + \sum_{j=1}^N \varphi_{ij}(r) \right)^2 < rs_i^*. \quad (11)$$

If we make the choice $\varphi_{ij}(r) = a_{ij}\sqrt{r}$ for all $j \in \mathcal{N}_i \setminus i$ and $\varphi_{ij}(r) = 0$ otherwise, we obtain:

$$\sum_{j \in \mathcal{N}_i \setminus i} a_{ij} < \sqrt{s_i^*} - \sum_{j \in \mathcal{N}_i \setminus i} \frac{2\|P_i\|^{3/2}}{\tilde{c}_i} \frac{\|\bar{A}_{ij}\|}{[\lambda_{\min}(P_j)]^{1/2}} \sqrt{s_j^*} =: \rho_i.$$

It is worth noting that $\rho_i > 0$ by the spectral condition on G .

Assume without loss of generality that $\mathcal{N}_i \setminus i \neq \emptyset$ (if not, (11) trivially holds). Without further knowledge of the system, we choose $\varphi_{ij}(r) := \frac{\rho_i}{|\mathcal{N}_i \setminus i|} \sqrt{r}$, where $|\mathcal{N}_i \setminus i|$ denotes the cardinality of the set $\mathcal{N}_i \setminus i$, to ensure that (10) holds. Simulations suggest that it might be better to not choose the a_{ij} uniformly, but to relate them to the system matrices (in particular, to the spectral radii of the coupling matrices $\bar{B}_{ij} = B_i K_{ij}$).

Now we can calculate the trigger functions χ_i as in Theorem 2 by using the Ω -path and the φ_{ij} from above. The map $\hat{\eta}_i$ is calculated using the η_{ij} from (3).

Stability of the interconnected system is then inferred by Theorem 3.

To illustrate the feasibility of our approach we simulated the interconnection of three linear systems of dimension three. The entries of the system matrices are drawn randomly from a uniform distribution on the open interval

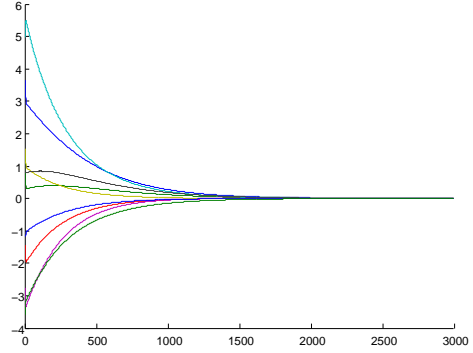


Fig. 1. Trajectories of the interconnected system with periodic sampling

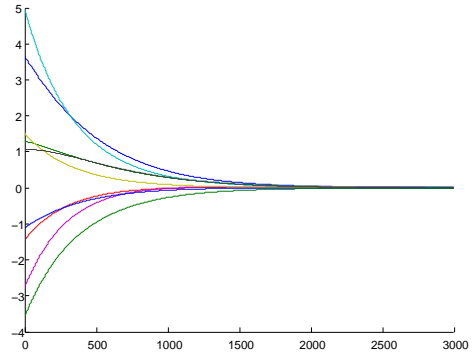


Fig. 2. Trajectories of the interconnected system with event-triggered sampling

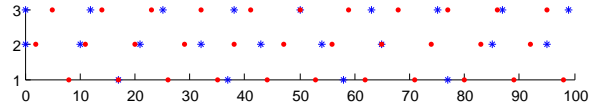


Fig. 3. 33 periodic (red dots) and 22 (blue stars) events at the beginning of the simulation

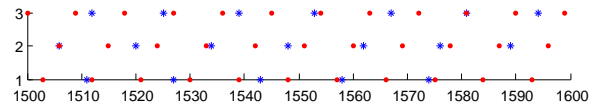


Fig. 4. 34 (red dots) periodic and 19 (blue stars) events in the middle of the simulation

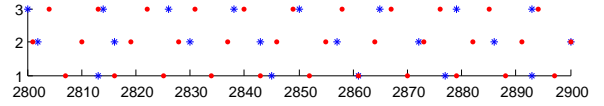


Fig. 5. 34 periodic (red dots) and 21 (blue stars) events at the end of the simulation

$(-5, 5)$. We repeat this procedure until the spectral radius of the corresponding matrix G is less than one.

In Figure 1 new information is sampled every three units of time. Which system has to transmit information is decided by a round robin protocol (i.e., first system one, then system two, system three and again system one and so on).

In Figure 2 our event-triggered sampling scheme is used. Over the range of 3000 units of time the system with periodic sampling transmitted 1000 new information, whereas in our scheme the events were triggered only 595 times.

By looking at Figure 1 and Figure 2 it seems like the periodic sampled system converges a bit faster. Indeed, the systems state norm of the event-triggered system at time 3000 is already reached by the periodic sampled system after 2486 (i.e., 828 periodic samplings) units of time. But still the number of triggered events (595) is smaller than the number of periodic events (828).

An idea when the different systems sample their state is depicted in Figures 3-5. The first picture shows the sampling behavior at the beginning of the simulation. The other two are from the middle and the end of the simulation, respectively.

7. ON ZENO-FREE DISTRIBUTED EVENT-TRIGGERED CONTROL

The aim of this section is to show that it is possible to design distributed event-triggered control schemes for which the accumulation of the sampling times in finite time does not occur. To this purpose we focus on a simpler system than (1), namely a system with $N = 2$, and where $u_i = g_i(x_i + e_i)$ for $i = 1, 2$, and $w = 0$. We rewrite the closed-loop system as:

$$\begin{aligned}\dot{x}_1 &= \hat{f}_1(x_1, x_2, e_1) \\ \dot{x}_2 &= \hat{f}_2(x_1, x_2, e_2).\end{aligned}\quad (12)$$

For this system we adopt a slight variation of the input-to-state stability property which is at the basis of our Zeno-free event-triggered control (Jiang et al. (1996), Definition 2.2 and Lemma 4.1):

Assumption 3. For $i = 1, 2$, there exist a differentiable function $V_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$, and class- \mathcal{K}_∞ functions α_{i1}, α_{i2} such that

$$\alpha_{i1}(\|x_i\|) \leq V_i(x_i) \leq \alpha_{i2}(\|x_i\|).$$

Moreover there exist functions $\mu_1, \mu_2 \in \text{MAF}_2$, $\gamma_{12}, \gamma_{21}, \eta_{11}, \eta_{22} \in \mathcal{K}_\infty$, α_1, α_2 positive definite and positive constants c_1, c_2 such that

$$\begin{aligned}V_1(x_1) &\geq \mu_1(\gamma_{12}(V_2(x_2)), \eta_{11}(\|e_1\|) + c_1) \Rightarrow \\ &\quad \nabla V_1(x_1) \hat{f}_1(x_1, x_2, e_1) \leq -\alpha_1(\|x_1\|) \\ V_2(x_2) &\geq \mu_2(\gamma_{21}(V_1(x_1)), \eta_{22}(\|e_2\|) + c_2) \Rightarrow \\ &\quad \nabla V_2(x_2) \hat{f}_2(x_1, x_2, e_2) \leq -\alpha_2(\|x_2\|).\end{aligned}\quad (13)$$

In what follows we set $\mu_1 = \mu_2 = \max$. In the words of Jiang et al. (1996) both systems are input-to-state *practically* stable. We assume that the two systems satisfy a small-gain condition of the following form:

Assumption 4. For each $r > 0$, $\gamma_{12} \circ \gamma_{21}(r) < r$.

By Lemma A.1 in Jiang et al. (1996), the small-gain property implies the existence of a function $\sigma_2 \in \mathcal{K}_\infty$, continuously differentiable on $(0, \infty)$ such that for all $r > 0$

$$\gamma_{21}(r) < \sigma_2(r) < \gamma_{12}^{-1}(r).$$

Set $\sigma_1(r) = r$. Moreover, let $\varphi \in \mathcal{K}_\infty^2$ be such that $\varphi_1(r) < r$ and $\varphi_2(r) < \sigma_2(r)$. The small-gain condition above plays the role of condition (4). In fact, we have for all $r > 0$:

$$\hat{\Gamma}_\mu(\sigma(r), \varphi(r)) := \begin{pmatrix} \mu_1(\gamma_{12}(\sigma_2(r)), \varphi_1(r)) \\ \mu_2(\gamma_{21}(\sigma_1(r)), \varphi_2(r)) \end{pmatrix} < \sigma(r). \quad (14)$$

Compared with $\bar{\Gamma}_\mu(\sigma(r), \varphi(r))$ in Assumption 2, each entry of $\hat{\Gamma}_\mu(\sigma(r), \varphi(r))$ depends on a single function of φ . This reflects the fact that each sub-system i is affected by the

error signal e_i only. We now state a new version of Theorem 2 for the system (12):

Theorem 4. Let Assumptions 3 and 4 hold. Let $V(x) = \max_{i=1,2} \sigma_i^{-1}(V_i(x_i))$ and $\mathcal{N}(x) \subseteq \{1, 2\}$ be the indices i such that $V(x) = \sigma_i^{-1}(V_i(x_i))$. Assume that, for each $i = 1, 2$,

$$\max\{\eta_{ii}^{-1}(\varphi_i \circ \sigma_i^{-1}(V_i(x_i)) - c_i), \eta_{ii}^{-1}(c_i)\} \geq \|e_i\|. \quad (15)$$

If c_1, c_2 are such that

$$\frac{1}{2} \varphi_2 \circ \sigma_2^{-1} \circ \varphi_1(2c_1) = c_2, \quad (16)$$

then

$$\langle p, \hat{f}(x, g(x + e)) \rangle \leq -\alpha(\|x\|), \quad \forall p \in \partial V(x),$$

for all $x = (x_1^T x_2^T)^T \in \{x : V(x) \geq \max_{i \in \mathcal{N}(x)} \varphi_i^{-1}(2c_i)\}$, where

$$\hat{f}(x, e) = \begin{pmatrix} \hat{f}_1(x_1, x_2, e_1) \\ \hat{f}_2(x_1, x_2, e_2) \end{pmatrix}.$$

Proof: Let $i \in \mathcal{N}(x)$. Then, as a consequence of Assumption 4, (14) holds, and we have:

$$V_i(x_i) = \sigma_i(V(x)) > \mu_i(\gamma_{i,i+1}(\sigma_{i+1}(V(x))), \varphi_i(V(x))), \quad (17)$$

where the index $i + 1$ is intended modulo 2. Suppose that $V(x) \geq \varphi_i^{-1}(2c_i)$ or, equivalently, that $V_i(x_i) \geq \sigma_i \circ \varphi_i^{-1}(2c_i)$. Then, by (15),

$$\varphi_i \circ \sigma_i^{-1}(V_i(x_i)) \geq \eta_{ii}(\|e_i\|) + c_i$$

and in turn

$$\varphi_i(V(x)) = \varphi_i \circ \sigma_i^{-1}(V_i(x_i)) \geq \eta_{ii}(\|e_i\|) + c_i. \quad (18)$$

Further observe that, since $V(x) \geq \sigma_j^{-1}(V_j(x_j))$ for $j = 1, 2$, then

$$\gamma_{i,i+1}(\sigma_{i+1}(V(x))) \geq \gamma_{i,i+1}(V_{i+1}(x_{i+1})).$$

By (17), (18) and the latter, we conclude that:

$$V_i(x_i) > \mu_i(\gamma_{i,i+1}(V_{i+1}(x_{i+1})), \eta_{ii}(\|e_i\|) + c_i),$$

and we can use (13) in Assumption 3. In other words, under the conditions of the theorem, for each $i \in \mathcal{N}(x)$, if $V(x) \geq \varphi_i^{-1}(2c_i)$, then $\nabla V_i(x_i) \hat{f}_i(x_1, x_2, e_i) \leq -\alpha_i(\|x_i\|)$. Now we can repeat the same arguments of the last part of the proof of Theorem 2, and conclude that for all x such that $V(x) \geq \max_{i \in \mathcal{N}(x)} \varphi_i^{-1}(2c_i)$ and for all $p \in \partial V(x)$, $\langle p, \hat{f}(x, g(x + e)) \rangle \leq -\alpha(\|x\|)$. \square

Remark 3. It is not difficult to choose the functions φ_1, φ_2 in such a way that (16) is fulfilled, provided that $c_1 > c_2$ (if $c_2 > c_1$, one can interchange the two indices and obtain the same result). Indeed, recall that $\varphi_1 < \text{Id}$ and $\varphi_2 < \sigma_2$, and set $\varphi_1 = \varepsilon_1 \text{Id}$, $\varphi_2 = \varepsilon_2 \sigma_2$, with $0 < \varepsilon_1, \varepsilon_2 < 1$. Then the left-hand side of (16) becomes

$$\frac{1}{2} \varphi_2 \circ \sigma_2^{-1} \circ \varphi_1(2c_1) = \varepsilon_1 \varepsilon_2 c_1,$$

and it is enough to choose $\varepsilon_1 \varepsilon_2 = \frac{c_2}{c_1}$.

Consider the interconnected system

$$\dot{x}_i(t) = f_i(x_1(t), x_2(t), g_i(\hat{x}_i(t))), \quad i = 1, 2, \quad (19)$$

We can now define the event-triggered control laws following the same arguments of Section 5, the difference being that the sequence of sampling times t_k^i is generated according to the rule (15) rather than (5). In view of the result above, one can argue that, as far as the

state $x(t)$ belongs to that region of the state space where $\max\{\varphi_1^{-1}(2c_1), \varphi_2^{-1}(2c_2)\} \leq V(x(t)) \leq V(x(0))$, the Lyapunov function computed along the trajectory decreases with a velocity which is bounded away from zero. Moreover, the rates at which the errors e_1, e_2 increase are always bounded from above. Since each error e_i must grow at least of a quantity $\varphi_i^{-1}(2c_i)$ before a new sampling can take place, the fact that the errors evolve with a limited velocity show that the length of the inter-sampling intervals is always above a finite positive quantity, i.e. the set of sampling times is locally finite. We summarize with the following statement:

Theorem 5. Let Assumptions 3 and 4 hold, and let c_1, c_2 satisfy (16). Consider the interconnected system

$$\dot{x}_i(t) = f_i(x_1(t), x_2(t), g_i(\hat{x}_i(t))), \quad i = 1, 2, \quad (20)$$

where for each $i \in \mathcal{N}$, $\hat{x}_i(t) = x_i(t_k^i)$ for $t \in [t_k^i, t_{k+1}^i)$, and $\{t_k^i\}_{k \in \mathbb{N}_0}$ is the sequence of sampling times such that $t_0^i = 0$, and, for each $k \in \mathbb{N}_0$, $t_{k+1}^i > t_k^i$ is the smallest time t such that for $i = 1, 2$

$$\max\{\eta_{ii}^{-1}(\varphi_i \circ \sigma_i^{-1}(V_i(x_i(t))) - c_i), \eta_{ii}^{-1}(c_i)\} \geq \|e_i(t)\|.$$

Then, for every initial condition, the solution converges in finite time to the level set $\{x : V(x) \leq \max_{i \in \{1,2\}} \varphi_i^{-1}(2c_i)\}$, where $V(x) = \max_{i=1,2} \sigma_i^{-1}(V_i(x_i))$. Moreover, for $i = 1, 2$ there exist positive values T_i such that $t_{k+1}^i - t_k^i \geq T_i$ for all $k \geq \mathbb{N}_0$.

Remark 4. The size of region where the state converges in finite time depends on the constants c_1, c_2 . In the case the two subsystems are input-to-state stable, that is (13) holds with $c_1 = c_2 = 0$, then in Theorem 5 it is possible to replace c_i with any positive value \hat{c}_i and the claim remains valid. The advantage is that the sampling times of the resulting event-triggered controllers do not accumulate in finite time as before, but additionally the set to which the state converges can be made arbitrarily small. In other words, under the stronger assumption of input-to-state stability for the two subsystems, the result above proves practical stability of the closed-loop system.

8. CONCLUSION

We presented an event-triggered sampling scheme for controlling interconnected systems. Each system in the interconnection decides when to send new information across the network independently of the other systems. This decision is based only on its own state and a given Lyapunov function. Stability of the interconnected system is inferred by the application of a nonlinear small-gain condition. The feasibility of our approach is presented with the help of a numerical simulation. To prevent the accumulation of the sampling times in finite time, we proposed a variation of the event-triggered sampling-scheme which guarantee practical stability of the interconnected system.

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