## CONVERGENCE OF THE VALUE FUNCTIONS OF DISCOUNTED INFINITE HORIZON OPTIMAL CONTROL PROBLEMS WITH LOW DISCOUNT RATES

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For autonomous, nonlinear, smooth optimal control systems on n-dimensional manifolds we investigate the relationship between the discounted and the average yield optimal value of infinite horizon problems.

It is shown that the value functions of discounted problems converge to the value function of the average yield problem as the discount rate tends to zero, if there exist approximately optimal solutions satisfying some periodicity conditions. In general, the discounted value functions cannot be expected to converge, which is shown by a counterexample. A connection to geometric control theory is then made to establish a result of uniform convergence on compact subsets of the interior of control sets, if optimal trajectories do not leave a compact subset of the interior of these control sets.

**0.** Introduction. In this paper we are concerned with convergence properties of value functions of discounted optimal control problems where the discount rates tend to zero. While the discounted value functions converge to a value function, which represents the maximization of the present value, if the discount rate tends to infinity (Sieveking 1986), convergence of the discount rate to zero is often interpreted as passing over to the average yield problem. The connection between the average yield and low discount rates has been extensively studied for Markov decision chains and stochastic games; see (Veinott 1975, Bewley and Kohlberg 1976) and references therein. Here we will study this problem in the following setting:

Consider a connected  $C^{\infty}$ -manifold M of dimension n and an optimal control system on M satisfying the following conditions.

(0.1) 
$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{d} u_i(t) X_i(x(t)) = X(x(t), u(t))$$
 a.a.  $t \ge 0$ ;

(0.2) 
$$x(0) = x_0 \in M;$$

(0.3)  $X_0, X_1, \dots, X_d$  are  $C^{\infty}$ -vector fields on M;

(0.4) 
$$u(\cdot) \in \mathfrak{l} = \{u: \mathbf{R}_+ \to U | \text{measureable} \};$$

(0.5) 
$$U \subset \mathbf{R}^d$$
 compact with nonempty interior;

(0.6) 
$$h: M \times U \to \mathbf{R}$$
 continuous on  $M \times U$ ;

(0.7) 
$$0 \le h(x, u) \le H \quad \text{for all } (x, u) \in M \times U;$$

(0.8) For all 
$$x \in M$$
 and all  $u(\cdot) \in \mathfrak{U}$  the trajectory  $\varphi(t, x, u(\cdot))$  exists for all  $t \ge 0$ .

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The  $\lambda$ -discounted yield starting from a point x using a certain control  $u(\cdot)$  is defined as:

(0.9) 
$$J_{\lambda}(x,u(\cdot)) := \int_0^\infty e^{-\lambda t} h(\varphi(t,x,u(\cdot)),u(t)) dt$$

whereas the average yield from x using  $u(\cdot)$  is

(0.10) 
$$J_0(x, u(\cdot)) := \limsup_{T \to \infty} \frac{1}{T} \int_0^T h(\varphi(t, x, u(\cdot)), u(t)) dt$$

The associated value functions are then given by:

(0.11) 
$$V_{\lambda}(x) \coloneqq \sup_{u(\cdot) \in \mathbb{U}} J_{\lambda}(x, u(\cdot))$$

(0.12) 
$$V_0(x) \coloneqq \sup_{u(\cdot) \in \mathbb{N}} J_0(x, u(\cdot))$$

REMARKS. (a) Instead of condition (0.7) we can assume that h is uniformly bounded, as positivity of h can be easily obtained by addition of an adequate constant.

(b) By (0.7), (0.9) and (0.10), we have  $0 \le V_0(x) \le H$  and  $0 \le \lambda V_{\lambda} \le H$  for all  $x \in M$ .

(c) (0.8) holds for instance, if M is compact or the supports of the vector fields  $X_i$ , i = 0, ..., d are compact (Coddington and Levinson 1955, Chapter 2, Theorem 1.3).

In the theory of Laplace-transforms one formulation of Abel's Theorem states

$$\lim_{t \to \infty} f(t) = c \quad \text{implies} \quad \lim_{\lambda \to 0} \lambda \int_0^\infty e^{-\lambda t} f(t) \, dt = c$$

(Doetsch 1976, Satz 34.2). In our context this can be interpreted in the following way: If  $u(\cdot)$  is a control steering the system asymptotically to an equilibrium position (x, u), then

$$\lim_{\lambda \to 0} \lambda J_{\lambda}(x_0, u(\cdot)) = h(x, u) = J_0(x_0, u(\cdot)).$$

The second equality is an immediate consequence of Lemma 1.1 below. If  $u(\cdot)$  is optimal for all problems, then  $\lim_{\lambda \to 0} \lambda V_{\lambda}(x) = V_0(x)$  holds.

In §1 we will introduce some conditions which guarantee the convergence of  $\lambda V_{\lambda}(x)$  as the discount rate  $\lambda$  approaches zero in a more general situation. Although it might be expected that a decrease in the discount rate will also diminish the difference between the discounted optimal value and the optimal average yield, such that  $\lambda V_{\lambda}(x)$  always converges to  $V_0(x)$ , an example will show that  $\lambda V_{\lambda}(x)$  need not converge at all as  $\lambda$  goes to zero.

Nevertheless convergence can be shown, if there exist approximately optimal solutions satisfying two conditions:

(a) they become periodic after finite time  $T_{\lambda}$ ;

(b) the length of the periods and the  $T_{\lambda}$  do not increase too fast as  $\lambda$  goes to zero.

As "existence of approximately optimal, periodic solutions" seems a somewhat technical and impractical condition, §2 will strive to improve the results obtained up to that point. Periodicity can only be expected in so-called control sets studied in

geometric control theory. We will therefore introduce control sets and quote some of the basic properties of variant and invariant control sets, the proof of which can be found in Colonius and Kliemann (1989) and Kliemann (1987).

In Colonius (1989) it is shown that optimal trajectories of discounted problems converge to an optimal trajectory of the average yield problem, if the trajectories stay in some compact subset of the interior of an "invariant control set." Under similar assumptions we will show  $\lambda V_{\lambda} \rightarrow V_0$  uniformly on compact subsets of the interior of control sets.

1. Pointwise convergence of  $\lambda V_{\lambda}$ . We will first give sufficient conditions for pointwise convergence of  $\lambda V_{\lambda}(x)$  to  $V_0(x)$  as  $\lambda$  tends to 0. Afterwards we will show in an example that  $\lambda V_{\lambda}(x)$  need not converge at all.

Let us begin by noting three simple properties of  $V_0$  and  $\lambda V_{\lambda}$ . The proofs are straightforward and therefore left to the reader.

LEMMA 1.1. For all  $u(\cdot) \in \mathfrak{U}$  and all  $\tau > 0$ :

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T h(\varphi(t, x, u(\cdot)), u(t)) dt$$
$$= \limsup_{T \to \infty} \frac{1}{T} \int_{\tau}^T h(\varphi(t, x, u(\cdot)), u(t)) dt. \quad \Box$$

LEMMA 1.2. Suppose  $f: \mathbf{R} \to \mathbf{R}$  is continuous and uniformly bounded, and there are sequences  $\{\lambda_n\}, \{T_n\}$  and  $\{t_n\}$  such that:

$$\lim_{n\to\infty}\lambda_n=\lim_{n\to\infty}\lambda_nT_n=\lim_{n\to\infty}\lambda_nt_n=0.$$

Then

$$\lim_{n\to\infty}\int_{t_n}^{T_n}\lambda_n e^{-\lambda_n t}f(t)\ dt=0.\quad \Box$$

LEMMA 1.3. Suppose  $f: \mathbf{R} \to \mathbf{R}$  is continuous, and there are T, s > 0 such that f(t + s) = f(t) for all t > T, then:

$$\int_{T}^{\infty} \lambda e^{-\lambda t} f(t) dt = \frac{\lambda}{1 - e^{-\lambda s}} \int_{T}^{T + s} e^{-\lambda t} f(t) dt. \quad \Box$$

We will now calculate the average yield and the discounted return for periodic trajectories.

**PROPOSITION 1.4.** Let  $x \in M$ ,  $u(\cdot) \in \mathbb{U}$  and s, T > 0, satisfy for all t > T:

$$u(t) = u(t+s)$$
 and  $\varphi(t, x, u(\cdot)) = \varphi(t+s, x, u(\cdot)).$ 

Then:

$$J_0(x, u(\cdot)) = \frac{1}{s} \int_T^{T+s} h(\varphi(t, x, u(\cdot)), u(t)) dt = \lim_{\lambda \to 0} \lambda J_\lambda(x, u(\cdot)).$$

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PROOF.

(a) 
$$J_0(x, u(\cdot)) = \limsup_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} h(\varphi(t, x, u(\cdot)), u(t)) dt.$$

By Lemma 1.1 we know:

$$J_0(x, u(\cdot)) = \limsup_{\tau \to \infty} \frac{1}{\tau} \int_T^\tau h(\varphi(t, x, u(\cdot)), u(t)) dt.$$

Using the periodicity of  $u(\cdot)$  and  $\varphi(t, x, u(\cdot))$ , we obtain:

$$J_0(x, u(\cdot)) = \limsup_{\substack{\nu \to \infty \\ \nu \in \mathbb{N}}} \frac{1}{T + \nu s} \int_T^{T + \nu s} h(\varphi(t, x, u(\cdot)), u(t)) dt$$
$$= \limsup_{\substack{\nu \to \infty \\ \nu \in \mathbb{N}}} \frac{\nu}{T + \nu s} \int_T^{T + s} h(\varphi(t, x, u(\cdot)), u(t)) dt$$
$$= \frac{1}{s} \int_T^{T + s} h(\varphi(t, x, u(\cdot)), u(t)) dt.$$

(b) 
$$\lim_{\lambda \to 0} \lambda J_{\lambda}(x, u(\cdot)) = \lim_{\lambda \to 0} \int_{0}^{\infty} \lambda e^{-\lambda t} h(\varphi(t, x, u(\cdot)), u(t)) dt$$
$$= \lim_{\lambda \to 0} \left[ \int_{0}^{T} \lambda e^{-\lambda t} h(\varphi(t, x, u(\cdot)), u(t)) dt + \int_{T}^{\infty} \lambda e^{-\lambda t} h(\varphi(t, x, u(\cdot)), u(t)) dt \right]$$

Applying Lemma 1.2 to the first term and Lemma 1.3 to the second using the periodicity of h in t, we obtain

$$\lim_{\lambda \to 0} \lambda J_{\lambda}(x, u(\cdot)) = \lim_{\lambda \to 0} \left[ \frac{\lambda}{1 - e^{-\lambda s}} \int_{T}^{T + s} e^{-\lambda t} h(\varphi(t, x, u(\cdot)), u(t)) dt \right]$$
$$= \frac{1}{s} \int_{T}^{T + s} h(\varphi(t, x, u(\cdot)), u(t)) dt.$$

The last equality is obtained by applying the rule of de l'Hospital.

Theorem 1.5 deals with the pointwise convergence of  $\lambda V_{\lambda}$  to  $V_0$ . Using Proposition 1.4 we could expect to prove that  $\lambda V_{\lambda}(x)$  converges to  $V_0(x)$ , if there exist approximately optimal periodic solutions. It turns out, however, that we need some further restrictions regarding the length of one period and the time which elapses before periodicity is achieved.

Note that in part (a) of the proof of Proposition 1.4 we actually prove that the average yield converges, so that the result holds as well, if the average yield is defined via lim inf in (0.10). A version of Theorem 1.5 using this definition could be proved using the same assumptions and the same proof.

 $\lambda \rightarrow$ 

THEOREM 1.5. Let  $x \in M$  be fixed. Assume the following:

(i) For all  $\epsilon > 0$  there exist a control  $u_0^{\epsilon}(\cdot)$  as well as  $t_0^{\epsilon}$ ,  $T_0^{\epsilon} > 0$ , such that:

(a)  $V_0(x) - J_0(x, u_0^{\epsilon}(\cdot)) < \epsilon;$ 

(b) For all  $t > T_0^{\epsilon}$  we have:

$$u_0^{\epsilon}(t) = u_0^{\epsilon}(t+t_0^{\epsilon}) \quad and \quad \varphi(t,x,u_0^{\epsilon}(\cdot)) = \varphi(t+t_0^{\epsilon},x,u_0^{\epsilon}(\cdot)).$$

(ii) For all  $\lambda > 0$  and all  $\epsilon > 0$  there exist controls  $u_{\lambda}^{\epsilon}(\cdot)$  as well as  $t_{\lambda}^{\epsilon}, T_{\lambda}^{\epsilon} > 0$ , such that:

(a)  $\lambda V_{\lambda}(x) - \lambda J_{\lambda}(x, u_{\lambda}^{\epsilon}(\cdot)) < \epsilon;$ 

(b) For all  $t > T_{\lambda}^{\epsilon}$  we have:

$$u_{\lambda}^{\epsilon}(t) = u_{\lambda}^{\epsilon}(t+t_{\lambda}^{\epsilon}) \quad and \quad \varphi(t,x,u_{\lambda}^{\epsilon}(\cdot)) = \varphi(t+t_{\lambda}^{\epsilon},x,u_{\lambda}^{\epsilon}(\cdot))$$

(c)  $\lim_{\lambda \to 0} \lambda t_{\lambda}^{\epsilon} = 0$ , for all  $\epsilon > 0$ ;

(d)  $\lim_{\lambda \to 0} \lambda T_{\lambda}^{\epsilon} = 0$ , for all  $\epsilon > 0$ . Then  $\lim_{\lambda \to 0} \lambda V_{\lambda}(x) = V_0(x)$ .

**PROOF.** (a) Fix  $\epsilon > 0$ . Choose  $u_0(\cdot)$  such that condition (i) is satisfied. Then by Proposition 1.4:

$$V_0(x) \leq J_0(x, u_0^{\epsilon}(\cdot)) + \epsilon = \lim_{\lambda \to 0} \lambda J_\lambda(x, u_0^{\epsilon}(\cdot)) + \epsilon \leq \liminf_{\lambda \to 0} \lambda V_\lambda(x) + \epsilon.$$

(b) It remains to show  $\limsup_{\lambda \to 0} \lambda V_{\lambda}(x) \leq V_0(x)$ .

Fix  $\epsilon > 0$ . Choose controls  $u_{\lambda}^{\epsilon}(\cdot)$  in accordance with condition (ii) for all  $\lambda > 0$ . We then obtain:

$$\begin{split} \limsup_{\lambda \to 0} \lambda V_{\lambda}(x) &\leq \limsup_{\lambda \to 0} \lambda J_{\lambda}(x, u_{\lambda}^{\epsilon}(\cdot)) + \epsilon \\ &= \limsup_{\lambda \to 0} \int_{0}^{\infty} \lambda e^{-\lambda t} h(\varphi(t, x, u_{\lambda}^{\epsilon}(\cdot)), u_{\lambda}^{\epsilon}(t)) dt + \epsilon \\ &\leq \limsup_{\lambda \to 0} \left[ \int_{0}^{T_{\lambda}^{\epsilon}} \lambda e^{-\lambda t} H dt + \int_{T_{\lambda}^{\epsilon}}^{\infty} \lambda e^{-\lambda t} h(\varphi(t, x, u_{\lambda}^{\epsilon}(\cdot)), u_{\lambda}^{\epsilon}(t)) dt \right] + \epsilon. \end{split}$$

We can apply Lemmas 1.2 and 1.3 to the first and second term respectively and obtain in all:

$$\limsup_{\lambda \to 0} \lambda V_{\lambda}(x) \leq \limsup_{\lambda \to 0} \left[ \frac{\lambda}{1 - e^{-\lambda t_{\lambda}^{\epsilon}}} \int_{T_{\lambda}^{\epsilon}}^{T_{\lambda}^{\epsilon} + t_{\lambda}^{\epsilon}} e^{-\lambda t} h(\varphi(t, x, u_{\lambda}^{\epsilon}(\cdot)), u_{\lambda}^{\epsilon}(t)) dt \right] + \epsilon.$$

A simple calculation shows:

$$\limsup_{\lambda \to 0} \lambda V_{\lambda}(x) \leq \limsup_{\lambda \to 0} \left[ \left( \frac{1}{t_{\lambda}^{\epsilon}} + \lambda \right) \int_{T_{\lambda}^{\epsilon}}^{T_{\lambda}^{\epsilon} + t_{\lambda}^{\epsilon}} e^{-\lambda t} h(\varphi(t, x, u_{\lambda}^{\epsilon}(\cdot)), u_{\lambda}^{\epsilon}(t)) dt \right] + \epsilon.$$

Applying Lemma 1.2 again this yields:

$$\limsup_{\lambda \to 0} \lambda V_{\lambda}(x) \leq \limsup_{\lambda \to 0} \left[ \frac{1}{t_{\lambda}^{\epsilon}} \int_{T_{\lambda}^{\epsilon}}^{T_{\lambda}^{\epsilon} + t_{\lambda}^{\epsilon}} e^{-\lambda t} h(\varphi(t, x, u_{\lambda}^{\epsilon}(\cdot)), u_{\lambda}^{\epsilon}(t)) dt \right] + \epsilon.$$

By Proposition 1.4, (ii)(c) and (d):

$$\limsup_{\lambda \to 0} \lambda V_{\lambda}(x) \leq \limsup_{\lambda \to 0} J_{0}(x, u_{\lambda}^{\epsilon}(\cdot)) + \epsilon \leq V_{0}(x) + \epsilon.$$

This completes the proof, as  $\epsilon > 0$  is arbitrarily small.  $\Box$ 

The following example is designed to show in certain cases  $\lambda V_{\lambda}(x)$  need not converge at all if  $\lambda \to 0$ . Since the proof of Theorem 1.5 uses only the periodicity of the running cost  $h(\varphi(\cdot, x, u(\cdot)), u(\cdot))$  in t, we will construct an example where this does not hold.

EXAMPLE 1.6. Consider the following optimal control problem on R:

(1.1)  $\dot{x} = 2 - u, \quad u \in [0, 1],$ 

$$(1.2) x(0) = 0 \in \mathbf{R},$$

(1.3)  $0 \le h(x) \le 2, \quad \forall x \ge 0$ 

Maximize

$$\int_0^\infty e^{-\lambda t} h(\varphi(t,0,u(\cdot))) dt.$$

By (1.1) and (1.2) our ability of steering is very limited. As we can easily see for all  $t \ge 0$  and all  $u(\cdot) \in \mathbb{1}$  the following holds:

(1.4) 
$$t \leqslant \varphi(t, 0, u(\cdot)) \leqslant 2t.$$

Furthermore for all  $\lambda > 0$ :

(1.5) 
$$\lambda T \leq \log(8/7) \Rightarrow \int_0^T \lambda e^{-\lambda t} h(\varphi(t, x, u(\cdot))) dt \leq \frac{1}{4},$$

(1.6) 
$$\lambda T \ge \log(8) \Rightarrow \int_T^\infty \lambda e^{-\lambda t} h(\varphi(t, x, u(\cdot))) dt \le \frac{1}{4}.$$

For short we will write  $a = \log(8/7)$  and  $b = \log(8)$ .

Consider the sequence  $\{\lambda_n\}$  and a sequence of intervals  $I_n$ , defined as follows:

$$\lambda_{0} = 1 \qquad I_{0} = [0, b],$$

$$\lambda_{1} = \frac{a}{b} \qquad I_{1} = \left(b, 2\frac{b^{2}}{a}\right),$$

$$\vdots$$

$$\lambda_{2n} = \frac{a^{2n}}{2^{n}b^{2n}} \qquad I_{2n} = \left(2^{n}\frac{b^{2n}}{a^{2n-1}}, 2^{n}\frac{b^{2n+1}}{a^{2n}}\right),$$

$$\lambda_{2n+1} = \frac{a^{2n+1}}{2^{n}b^{2n+1}} \qquad I_{2n+1} = \left(2^{n}\frac{b^{2n+1}}{a^{2n}}, 2^{n+1}\frac{b^{2n+2}}{a^{2n+1}}\right).$$

Obviously  $\lim_{n\to\infty} \lambda_n = 0$  holds.

Denoting the control  $u(t) \equiv 1$  by 1 for short, we have by (1.4) for all  $n \ge 0$ :

$$\varphi(t,0,1) \in I_{2n}, \text{ if } t \in [a/\lambda_{2n}, b/\lambda_{2n}]$$

$$\varphi(t,0,u(\cdot)) \in I_{2n+1} \quad \forall u(\cdot) \in \mathfrak{U}, \text{ if } t \in [a/\lambda_{2n+1}, b/\lambda_{2n+1}].$$

Define  $\tilde{h}$  by

$$\tilde{h}(x) = \begin{cases} 2, x \in I_{2n}, & n \ge 0, \\ 0, x \in I_{2n+1}, & n \ge 0, \end{cases}$$

and we obtain by (1.5) and (1.6) for all  $n \ge 0$ :

$$\lambda_{2n}V_{\lambda_{2n}}(0) \geq \int_{a/\lambda_{2n}}^{b/\lambda_{2n}} \lambda_{2n} e^{-\lambda_{2n}t} \tilde{h}(\varphi(t,0,1)) dt = \frac{3}{2}.$$

On the other hand, we have by (1.5) and (1.6):

$$\lambda_{2n+1}V_{\lambda_{2n+1}}(0) \leq \sup_{u(\cdot)\in\mathbb{U}} \int_{a/\lambda_{2n+1}}^{b/\lambda_{2n+1}} \lambda_{2n+1} e^{-\lambda_{2n+1}t} \tilde{h}(\varphi(t,0,u(\cdot))) dt + \frac{1}{2} = \frac{1}{2}.$$

In all we have obtained:

$$\limsup_{\lambda \to 0} \lambda V_{\lambda}(0) \geq \frac{3}{2}, \qquad \liminf_{\lambda \to 0} \lambda V_{\lambda}(0) \leq \frac{1}{2}.$$

Now we have defined the running cost  $\tilde{h}$  to be discontinuous, so that it does not satisfy condition (0.6). It can be easily seen how to construct a continuous function h, which allows for the same conclusion.  $\Box$ 

2. Convergence in control sets. We will now turn to the problem of describing situations in which the assumptions of Theorem 1.5 do indeed occur. It turns out we can construct approximately optimal solutions satisfying the assumptions of Theorem 1.5, provided there exist optimal solutions that do not leave some compact subset of the interior of a control set. The notion of control sets has been discussed in Kliemann (1987). Existence of periodic, approximately optimal solutions in control sets has already been proved in Colonius and Kliemann (1989). Here we will have to prove a more specific result, as we need some information about the length of one period and the time span elapsing before periodicity is achieved.

We will first give the basic definitions and state some properties of control sets. Afterwards we can show periodic solutions with the desired properties exist in compact subsets of the interior of control sets. This can be used to prove a result of locally uniform convergence of  $\lambda V_{\lambda}$  to  $V_0$ .

DEFINITION 2.1. The positive orbit of x up to time T is given by:

$$O^+_{\leq T}(x) := \{ y \in M | \text{there are } 0 \leq t \leq T \text{ and a control } u(\cdot) \}$$

such that 
$$\varphi(t, x, u(\cdot)) = y$$
.

We now define the positive orbit by:

$$O^+(x) \coloneqq \bigcup_{T \ge 0} O^+_{\le T}(x). \quad \Box$$

DEFINITION 2.2. A subset  $D \subset M$  is called a control set, if: (i)  $D \subset \overline{O^+(x)}$  for all  $x \in D$ . (ii) D is maximal with property (i). (iii) If  $D = \{x\}$ , there is a  $u \in U$ , so that  $\varphi(t, x, u) = x$ ,  $\forall t \ge 0$ .

DEFINITION 2.3. A control set C is called invariant, if

$$\overline{C} = \overline{O^+(x)}, \quad \forall x \in C. \square$$

Note that periodic trajectories can only occur in control sets, variant or invariant.

To avoid degenerate situations the following setup is standard in geometric control theory: Let L denote the Lie-algebra generated by the vector fields  $X_i(\cdot)$ , i = 0, ..., d. Let  $\Delta_L$  denote the distribution defined through L in TM, the tangent space of M. Assume that

(2.1) 
$$\dim \Delta_{I}(x) = n \quad \text{for all } x \in M.$$

This assumption guarantees that the positive and negative orbits (defined analogously to Definition 2.1) of x up to time T have nonempty interior, a fact that will be vital for all following results. For a proof of this property see (Isidori 1989, Chapter 2, Theorem 2.7).

If L is of locally finite type or dim  $\Delta_L(x)$  is constant, then M can be partitioned into maximal integral submanifolds invariant under the vector fields  $X_i$ , which can in turn be assumed to be the new state space (Isidori 1989, Sussmann 1973). Under these circumstances condition (2.1) holds on the maximal integral submanifolds.

LEMMA 2.4. Consider a control system on M satisfying (0.1)—(0.8) and (2.1). Then: (i) If C is an invariant control set, then  $C = \overline{C}$ .

(ii) Every invariant control set has nonempty interior.

(iii) If D is a control set, then int  $D \subset O^+(x)$  for all  $x \in D$ .

(iv) If M is a compact manifold, then there exist at least one at most finitely many invariant control sets.

PROOF. Kliemann (1987, Lemmas 2.1 and 2.2.)

The definition of control sets only demands approximate reachability, i.e., existence of controls steering into any neighbourhood of a given point. There is even a finite time controllability property in the interior of control sets, which we will need for the construction of periodic solutions.

DEFINITION 2.5. We define a "first time hitting map" by:

$$k: M \times M \to \mathbf{R} \cup \{\infty\}$$

 $(x, y) \mapsto \inf \{t | \text{there is a } u(\cdot) \in \mathbb{U} \text{ such that } \varphi(t, x, u(\cdot)) = y \}. \square$ 

PROPOSITION 2.6. Consider a control system on M satisfying (0.1)—(0.8) and (2.1). Assume there exist a control set  $D \subset M$  with int  $D \neq \emptyset$  and two compact sets  $K_1 \subset D$ and  $K_2 \subset$  int D. Then there is a constant r dependent on  $K_1$  and  $K_2$ , such that:

(i)  $k(x, y) \leq r$  for all  $x \in K_1$ ,  $y \in K_2$ .

(ii)  $K_1 \subset \text{int } O_{\leq r}^-(y)$  for all  $y \in K_2$ .

(iii)  $K_2 \subset \text{int } O_{\leq r}^+(x)$  for all  $x \in K_1$ .

PROOF. Colonius and Kliemann (1989, Proposition 2.3.)

Now we can finally turn to the construction of periodic solutions. As the average yield case is far simpler, we will consider this case first. A more general statement of this Proposition can be found in Colonius and Kliemann (1989, Theorem 4.2).

PROPOSITION 2.7. Consider an optimal control system on M satisfying (0.1)—(0.8)and (2.1). Assume there are a control set  $D \subset M$ , a point  $x \in M$ , a compact subset  $K \subset \text{int } D$ , a control  $u(\cdot) \in \mathbb{U}$  and a T > 0, such that  $\varphi(t, x, u(\cdot)) \in K$  for all  $t \ge T$ . Then for every  $\epsilon > 0$  there exists a control  $u_{\epsilon}(\cdot)$ , such that

(i)  $u_{\epsilon}(T + \cdot)$  and  $\varphi(T + \cdot, x, u_{\epsilon}(\cdot))$  are periodic with the same period.

(ii)  $J_0(x, u(\cdot)) - J_0(x, u_{\epsilon}(\cdot)) < \epsilon$ .

PROOF. Fix  $\epsilon > 0$ . By Lemma 1.1 we know  $J_0(x, u(\cdot)) = J_0(\varphi(T, x, u(\cdot)), u(T + \cdot))$ , and we can therefore assume, without loss of generality, T = 0. By Proposition 2.6 there is an  $r \ge 0$ , such that  $k(x, y) \le r$  for all  $x, y \in K$ .

By definition of the average yield there is a  $t_{e}$ , such that:

(2.2) 
$$J_0(x,u(\cdot)) < \frac{1}{t_{\epsilon}} \int_0^{t_{\epsilon}} h(\varphi(t,x,u(\cdot)),u(t)) dt + \frac{\epsilon}{2}$$

If  $t_{\epsilon}$  is large enough, then for all  $v(\cdot) \in \mathfrak{U}$ :

(2.3) 
$$\frac{1}{t_{\epsilon}+r} \int_{0}^{t_{\epsilon}+r} h(\varphi(t,x,v(\cdot)),v(t)) dt + \frac{\epsilon}{2}$$
$$\geq \frac{1}{t_{\epsilon}} \int_{0}^{t_{\epsilon}} h(\varphi(t,x,v(\cdot)),v(t)) dt.$$

Since  $\varphi(t_{\epsilon}, x, u(\cdot)) \in K$  there are a control  $w(\cdot)$  and a  $t_1 \leq r$ , satisfying  $\varphi(t_1, \varphi(t_{\epsilon}, x, u(\cdot)), w(\cdot)) = x$ . Define  $u_{\epsilon}(\cdot)$  by:

$$u_{\epsilon}(t) := \begin{cases} u(t), & 0 \leq t \leq t_{\epsilon}, \\ w(t-t_{\epsilon}), & t_{\epsilon} < t \leq t_{\epsilon} + t_{1}. \end{cases}$$

Because of  $\varphi(t_{\epsilon} + t_1, x, u_{\epsilon}(\cdot)) = x$  we can continue  $u_{\epsilon}(\cdot) (t_{\epsilon} + t_1)$ -periodically. By Proposition 1.4 and the inequalities (2.2) and (2.3) this leads to:

$$\begin{aligned} J_0(x, u_{\epsilon}(\cdot)) &= \frac{1}{t_{\epsilon} + t_1} \int_0^{t_{\epsilon} + t_1} h(\varphi(t, x, u_{\epsilon}(\cdot)), u_{\epsilon}(t)) dt \\ &\geq \frac{1}{t_{\epsilon}} \int_0^{t_{\epsilon}} h(\varphi(t, x, u(\cdot)), u(t)) dt - \frac{\epsilon}{2} > J_0(x, u(\cdot)) - \epsilon. \quad \Box \end{aligned}$$

PROPOSITION 2.8. Consider a control system on M satisfying (0.1)—(0.8) and (2.1). Let  $x \in M$  be fixed. Assume there are a control set D, a compact subset  $K \subset \text{int } D$ , a control  $u(\cdot) \in \mathbb{1}$  and a  $T \ge 0$ , such that  $\varphi(t, x, u(\cdot)) \in K$  for all  $t \ge T$ . Then there exists a constant r = r(K) and for all  $\epsilon > 0$  and all  $\lambda > 0$  there exist positive constants  $s_1, s_2$  and a control  $w(\cdot)$  all dependent on  $\epsilon$  and  $\lambda$ , such that:

(i)  $\lambda J_{\lambda}(x, u(\cdot)) - \lambda J_{\lambda}(x, w(\cdot)) < \epsilon$ .

(ii) For all  $t \ge T + s_1$  we have:

$$w(t) = w(t + s_2)$$
 and  $\varphi(t, x, w(\cdot)) = \varphi(t + s_2, x, w(\cdot)).$ 

(iii) 
$$s_1, s_2 \leq \log\left(3\frac{H(1-e^{-\lambda r})}{\epsilon} + 1\right)/\lambda + r.$$

(iv)  $\lim_{\lambda \to 0} \lambda s_1(\epsilon, \lambda) = 0 = \lim_{\lambda \to 0} \lambda s_2(\epsilon, \lambda)$  holds for every  $\epsilon > 0$ .

**PROOF.** Let us first note (iv) is an immediate consequence of (iii). As r is independent of  $\epsilon$  and  $\lambda$  the assertion follows from:

$$\lim_{\lambda \to 0} \lambda \frac{\log \left(3 \frac{H(1 - e^{-\lambda r})}{\epsilon} + 1\right)}{\lambda} = \lim_{\lambda \to 0} \log \left(3 \frac{H(1 - e^{-\lambda r})}{\epsilon} + 1\right) = 0.$$

Fix  $\epsilon > 0$  and  $\lambda > 0$ . Define  $r := \sup\{k(x, y) | x, y \in K\} < \infty$ . For short we will write

(2.4) 
$$a \coloneqq \frac{\log\left(3\frac{H(1-e^{-\lambda r})}{\epsilon}+1\right)}{\lambda}.$$

For each  $y \in K$  define  $\mathfrak{U}(y) := \{u(\cdot) \in \mathfrak{U} | \varphi(a, y, u(\cdot)) \in K\}$ .

We know by assumption that  $\mathfrak{U}(y)$  is not empty for all  $y \in K$ . If  $a \ge r$ , that is if  $e^{-\lambda r} \ge \epsilon/3H$ , which is always true provided  $\epsilon$  and  $\lambda$  are small enough, then  $\mathfrak{U}(y) \ne \emptyset$  for all  $y \in K$ .

Choose  $\bar{x} \in K$  and  $\bar{u}(\cdot) \in \mathfrak{U}(\bar{x})$ , such that

$$\int_{0}^{a} \lambda e^{-\lambda t} h(\varphi(t, \bar{x}, \bar{u}(\cdot)), \bar{u}(t)) dt + \frac{\epsilon}{3} (1 - e^{-\lambda a})$$

(2.5)

$$\geq \sup_{x \in K} \sup_{u(\cdot) \in \mathfrak{U}(x)} \int_0^a \lambda e^{-\lambda t} h(\varphi(t, x, u(\cdot)), u(t)) dt.$$

By assumption there are a control  $w_1(\cdot)$  and  $r_1 \leq r$ , such that

$$\varphi(r_1,\varphi(T+a,x,u(\cdot)),w_1(\cdot))=\bar{x}$$

and a control  $w_2(\cdot)$  and  $r_2 < r$ , such that

$$\varphi(r_2,\varphi(a,\bar{x},\bar{u}(\cdot)),w_2(\cdot))=\bar{x}.$$

We will now define the control  $w(\cdot)$  we are looking for by first steering to K as does the original control  $u(\cdot)$ . We will continue to follow  $u(\cdot)$  for time a. Then we steer to  $\bar{x}$ , which will be the starting point of a periodic trajectory: First we steer with control  $\bar{u}(\cdot)$  for time a, to return to  $\bar{x}$  by steering with  $w_2(\cdot)$ .

In all we have:

$$w(t) = \begin{cases} u(t), & 0 \leq t \leq T + a; \\ w_1(t - (T + a)), & T + a < t \leq T + a + r_1; \\ \overline{u}(t - (T + a + r_1)), & T + a + \tau_1 < t \leq T + r_1 + 2a; \\ w_2(t - (T + r_1 + 2a)), & T + \tau_1 + 2a < t \leq T + r_1 + 2a + r_2; \\ w(t - (a + r_2)), & T + \tau_1 + 2a + r_2 < t. \end{cases}$$

We have now constructed a control  $w(\cdot)$  for which assertions (ii) and (iii) are obviously true. It remains to show, that:

$$\lambda J_{\lambda}(x, u(\cdot)) - \lambda J_{\lambda}(x, w(\cdot)) < \epsilon.$$

Keeping in mind that  $s_1 = a + r_1$  and  $s_2 = a + r_2$ , we proceed as follows:

$$\lambda J_{\lambda}(x, u(\cdot)) - \lambda J_{\lambda}(x, w(\cdot)) = \int_{T+a}^{\infty} \lambda e^{-\lambda t} \left[ h(\varphi(t, x, u(\cdot)), u(t)) - h(\varphi(t, x, w(\cdot)), w(t)) \right] dt.$$

Using the periodicity of  $w(\cdot)$  starting at time  $T + s_1$ , we obtain:

$$\begin{split} \lambda J_{\lambda}(x,u(\cdot)) &- \lambda J_{\lambda}(x,w(\cdot)) \\ &\leqslant \int_{T+a}^{T+s_1} \lambda e^{-\lambda t} H \, dt + \sum_{\nu=0}^{\infty} \int_{T+s_1+\nu s_2}^{T+s_1+\nu s_2+a} \lambda e^{-\lambda t} \\ &\times \left[ h(\varphi(t,x,u(\cdot)),u(t)) - h(\varphi(t,x,w(\cdot)),w(t)) \right] \, dt \\ &+ \sum_{\nu=0}^{\infty} \int_{T+s_1+\nu s_2+a}^{T+s_1+\nu s_2+a} \lambda e^{-\lambda t} H \, dt. \end{split}$$

Remembering inequality (2.5) and using the properties  $\varphi(T + s_1 + \nu s_2, x, w(\cdot)) = \bar{x}$  as well as  $\varphi(T + s_1 + \nu s_2, x, u(\cdot)) \in K$ , we can continue:

$$\begin{split} \lambda J_{\lambda}(x, u(\cdot)) &- \lambda J_{\lambda}(x, w(\cdot)) \\ &\leqslant He^{-\lambda(T+a)} (1 - e^{-\lambda r_1}) + e^{-\lambda(T+s_1)} \sum_{\nu=0}^{\infty} e^{-\lambda \nu s_2} \frac{\epsilon}{3} (1 - e^{-\lambda a}) \\ &+ He^{-\lambda(T+s_1+a)} \sum_{\nu=0}^{\infty} e^{-\lambda \nu s_2} (1 - e^{-\lambda r_2}) \\ &\leqslant He^{-\lambda a} (1 - e^{-\lambda r_1}) + \frac{\epsilon (1 - e^{-\lambda a})}{3(1 - e^{-\lambda s_2})} + He^{-\lambda a} \frac{(1 - e^{-\lambda r_2})}{(1 - e^{-\lambda s_2})} \leqslant \epsilon \end{split}$$

The last inequality is true, because by (2.4) we know:

$$e^{-\lambda a} < \frac{\epsilon}{3H(1-e^{-\lambda r})}$$
.

We will now state the convergence result of Colonius (1989).

THEOREM 2.9. Consider a control system on M satisfying (0.1)—(0.8) and (2.1). Furthermore we will assume the running cost h can be expressed as

(2.6) 
$$h(x,u) = h_0(x) + \sum_{i=1}^{1} u_i h_i(x).$$

Assume there exist an invariant control set C, a compact subset  $K \subset \text{int } C$ , and optimal controls  $u_{\lambda}(\cdot)$ , such that  $\varphi(t, x, u_{\lambda}(\cdot)) \in K$  for all  $t \ge 0$  and all  $\lambda > 0$ . Then there exist a subsequence  $\lambda_n \to 0$  and a control  $u(\cdot)$ , such that for all T > 0:

(i)  $u_{\lambda_n}(\cdot) [[0, T]]$  converges weakly to  $u(\cdot) [[0, T]]$  in  $L^2([0, T], \mathbf{R}^d)$ . (ii)  $\varphi(\cdot, x, u_{\lambda_n}(\cdot))$  converges uniformly to  $\varphi(\cdot, x, u(\cdot))$  on [0, T]. (iii)  $V_0(x) = J_0(x, u(\cdot))$ .

PROOF. Colonius (1989, Corollary 2.7).

We can now deduce two results concerning the convergence of  $\lambda V_{\lambda}$  in control sets from Theorem 1.5. By Lemma 1.1 it is clear,  $V_0$  is constant on the interior of control sets.  $\lambda V_{\lambda}$  converges uniformly on compact subsets to this constant value, under the following assumptions:

THEOREM 2.10. Consider an optimal control system on M satisfying conditions (0.1)—(0.8) and (2.1). Assume there exist a control set  $D \subset M$ , an  $x \in \text{int } D$ , a compact subset  $K \subset \text{int } D$ , and optimal controls  $u_{\lambda}(\cdot)$  and  $u_0(\cdot)$  (optimal with respect to the  $\lambda$ -discounted and the average yield problem respectively), such that

$$\begin{split} \varphi\big(t, x, u_{\lambda}(\cdot)\big) &\in K, \qquad \forall t \ge 0, \forall \lambda > 0, \\ \varphi\big(t, x, u_{0}(\cdot)\big) &\in K, \qquad \forall t \ge 0. \end{split}$$

Then:

$$\lambda V_{\lambda} \rightarrow V_{0}$$
 uniformly on compact subsets of int D.

PROOF. By Proposition 2.8 and Theorem 1.5,  $\lambda V_{\lambda}(x) \rightarrow V_0(x)$ . Fix a compact subset  $Q \subset \text{int } D$ . By Proposition 2.6 there is a constant  $\infty > r = \sup\{k(x, y)|x, y \in Q \cup \{x\}\} \ge 0$ . For every  $y \in Q$  there exist a control  $u(\cdot)$  and  $T \le r$  such that  $\varphi(T, x, u(\cdot)) = y$ . By Bellman's principle (Elliot 1987) we know for all  $\lambda > 0$ :

$$\lambda V_{\lambda}(x) \geq \int_{0}^{T} \lambda e^{-\lambda t} h(\varphi(t, x, u(\cdot)), u(t)) dt + e^{-\lambda T} \lambda V_{\lambda}(y).$$

Therefore:

$$\lambda V_{\lambda}(y) - \lambda V_{\lambda}(x) \leq (1 - e^{-\lambda T}) \lambda V_{\lambda}(y) - \int_{0}^{T} \lambda e^{-\lambda t} h(\varphi(t, x, u(\cdot)), u(t)) dt$$
$$\leq H(1 - e^{-\lambda T}).$$

By symmetry, we therefore know:

$$\lambda V_{\lambda}(x) - \lambda V_{\lambda}(y) \leq H(1 - e^{-\lambda r}).$$

As  $\lim_{\lambda \to 0} H(1 - e^{-\lambda r}) = 0$  we know  $\lim_{\lambda \to 0} \lambda V_{\lambda}(x) = \lim_{\lambda \to 0} \lambda V_{\lambda}(y)$  for all  $y \in Q$ . Uniform convergence follows, since we have shown for all  $y, z \in Q$ :

$$\left|\lambda V_{\lambda}(z) - \lambda V_{\lambda}(y)\right| \leq H(1 - e^{-\lambda r}).$$

This result can be reformulated, if we assume the starting point to lie in an invariant control set C. Under these circumstances no assumption concerning the average yield problem is necessary, if we assume condition (2.6) and make use of the result by Colonius (1989).

COROLLARY 2.11. Consider an optimal control system on M satisfying conditions (0.1)—(0.8), (2.6) and (2.1). Let  $C \subset M$  be an invariant control set. Assume there exist an  $x \in \text{int } C$ , a compact subset  $K \subset \text{int } C$ , and optimal controls  $u_{\lambda}(\cdot)$ , such that

$$\varphi(t, x, u_{\lambda}(\cdot)) \in K, \quad \forall t \ge 0, \forall \lambda > 0.$$

Then:

 $\lambda V_{\lambda} \rightarrow V_0$  uniformly on compact subsets of int C.

**PROOF.** By Theorem 2.9 there exists a subsequence of  $\{u_{\lambda}(\cdot)\}$  converging weakly to a control  $u_0(\cdot)$  which is optimal as regards the average yield problem. We know  $\varphi(t, x, u_0(\cdot)) \in K$  for all  $t \ge 0$  by part (ii) of Theorem 2.9. Therefore we can now conclude as in the proof of Theorem 2.10.  $\Box$ 

REMARK 2.12. If a control system satisfying (0.1)—(0.8) and (2.1) is considered, the assumptions of Theorem 2.10 hold, for instance, if M is a compact manifold on which the system is completely controllable, because then M is the invariant control set, which is open and closed.  $\Box$ 

The following example intends to show the assertion of Theorem 2.10 is as general as can be expected in the following sense: The convergence of the discounted value functions need not be uniform on the whole of a control set, if the assumptions of Theorem 2.10 are satisfied.

EXAMPLE 2.13. Consider the optimal control problems on  $\mathbf{R}$  given by

$$\dot{x} = x - u, \qquad u \in [-1, 1],$$
$$h(x) = \begin{cases} 1, & x \leq 0, \\ 1 - x, & 0 \leq x \leq 1, \\ 0, & 1 \leq x. \end{cases}$$

Condition (2.1) concerning the Lie algebra is obviously satisfied. D = (-1, 1) is easily shown to be a variant control set of the control system. Furthermore there exist optimal trajectories for all  $\lambda > 0$  and all  $x \in D$  staying inside a compact subset of int D. The same is true for the average yield problem. For instance it is always optimal to steer to 0 by a constant control 1 or -1 and to stay there afterwards. By Theorem 2.10,  $\lambda V_{\lambda}$  converges uniformly to  $V_0$  on compact subsets of D as  $\lambda$  tends to 0. The convergence is not uniform on the whole of D: It can be easily shown, that

$$V_0(x) = \begin{cases} 1, & x < 1, \\ 0, & x \ge 1. \end{cases}$$

On the other hand we know for all  $\lambda > 0$  that  $V_{\lambda}(1) = 0$  holds, since it is impossible to steer from 1 to D. Uniform convergence would imply for every  $\epsilon > 0$  there is a  $\lambda(\epsilon)$ , such that for all  $0 < \lambda < \lambda(\epsilon)$  and all  $x \in D$ , the inequality  $|1 - \lambda V_{\lambda}(x)| < \epsilon$ holds. On the contrary, it is a well-known fact that value functions of discounted optimal control problems are continuous (Elliot 1987). Note  $\lambda V_{\lambda}(1)$  could also be constructed to be divergent in a similar example, using the techniques of Example 1.6.

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