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On Stability of Infinite-Dimensional Discrete Inclusions*

Fabian Wirth

Abstract

For discrete inclusions in Banach spaces we study stability questions. First it is shown that the Bohl exponent of a time-varying discrete time system can be characterized via the spectral radius of an associated operator on the space of p-th order summable sequences. The main result is that for discrete inclusions on a reflexive Banach space various characteristic exponents characterizing different concepts of stability coincide. Using this result it is shown that the convexification of an exponentially stable discrete inclusions is exponentially stable. It is examined to what extent these results can be carried over to the time-varying case.

Key words: discrete inclusion, time-varying, discrete-time, infinite dimensional, Bohl exponents, Lyapunov exponents, stability

AMS Subject Classifications: 34D08, 93D05, 93D09, 93C50, 93C55

1 Introduction

In this work we are concerned with stability properties of time-varying discrete-time systems and of discrete inclusions. Several results that have been obtained for finite dimensional systems are extended to infinite dimensions.

Time-varying linear systems have been at the center of active research in recent years. Also discrete inclusions have appeared under many guises in the literature. De Blasi and Schinas [9] are the first to characterize exponential stability of finite dimensional discrete inclusions. Later Barabanov in a series of papers [2], [3], [4] uses an approach via Lyapunov exponents to obtain results on stability. In particular he shows that for discrete inclusions given by irreducible sets of matrices the dynamics of the system

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induce a norm which is a Lyapunov function for the system. Using the idea of representing time-varying discrete-time systems as one or two sided block weighted shifts that Ben-Artzi and Gohberg obtain a number of results on dichotomies of such systems. See [5], [6] and references therein for results in the finite dimensional as well as in the infinite dimensional case. Also Gurvits studies stability of discrete linear inclusions in the remarkable papers [11], [12] giving in particular several examples showing which results do not carry over from finite to infinite dimensions.

The study of infinite-dimensional time-varying discrete-time systems has been carried out to a large extent by Przyłuski in his series of articles [19], [20], [21], [22] and the work of Przyłuski and Rolewicz [23]. In these papers it has also been noted that the study of discrete time systems on Banach space is an appropriate setting for the study of delay differential equations. Robustness of stability was studied for discrete time systems in infinite dimensions by Hinrichsen and the author in [25]. As a discrete inclusion may be interpreted as a time invariant system with a specified region of uncertainty this paper extends and complements the results of the previous work on robust stability.

An important concept in our study of stability is the introduction of different characteristic exponents. The idea to characterize stability via exponents is as old as the study of stability itself. The Bohl exponent is the latest invention in this direction as it has been introduced by Bohl only in 1913 [7]. It can be said to have been rediscovered for discrete-time systems by Przyłuski and Rolewicz, who termed it generalized spectral radius.

Section 2 is devoted to the definition of various concepts of stability for time-invariant and time-varying systems. For exponential stability of a time-varying system it is not enough to require that all trajectories decay exponentially. The fact as such has already been known to Bohl [7] and has been discussed in the book on stability by Daleckii and Krein [8], and the ideas are easily transferred to the discrete-time case.

In Section 3 Lyapunov and Bohl exponents of time-varying systems are introduced. The first notion characterizes exponential stability of trajectories while the second characterizes exponential stability of evolution operators. Thus the maximal Lyapunov exponent may be strictly smaller than the Bohl exponent. We give two example to this effect. One with an unbounded and one with a bounded operator sequence defining the time-varying system.

We will show that the Bohl exponent of an operator sequence $A(\cdot) \in \ell^\infty(\mathbb{N}, \mathcal{L}(X))$ can be represented as the logarithm of the spectral radius of an associated one sided block weighted shift operator $\hat{A}$ on $\ell^p(\mathbb{N}, X)$, $1 \leq p \leq \infty$. Using this kind of representation we discuss some the properties of the Bohl exponent that were shown by Przyłuski, which turn out not to be surprising as they are a simple reformulation of results well-known for time-invariant systems. Also in terms of the block-weighted shift necessary and sufficient conditions for equality of maximal Lyapunov and Bohl exponents are given.

After this discussion of characteristic exponents we turn to the study of discrete inclusions in the following Section 4. For discrete inclusions on reflexive Banach spaces it is shown that the supremal Lyapunov exponent, the supremal Bohl exponent and a further uniform exponential growth rate coincide. In the final Section 5 time-varying discrete inclusions are considered and stability concepts are studied. These class of systems forms a generalization for time-varying systems as well as discrete inclusions and this section therefore encompasses all the results obtained up to that point.

2 Time-varying Systems

Let X be a Banach space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. $\mathcal{L}(X)$ denotes the Banach algebra of bounded linear operators from X to X. The norm on X and the induced operator norm on $\mathcal{L}(X)$ are both denoted by $\|\cdot\|$.

A Banach space X is called reflexive if the range of the natural embedding of X into X^{**} is X^{**}. X is reflexive iff the unit ball in X is compact in the weak topology which is in turn equivalent to the weak compactness of the unit ball in $\mathcal{L}(X)$ (see [10] Theorem V.4.7 and Exercise VI.9.6). Recall that a net $\{A_n\} \subset \mathcal{L}(X)$ converges weakly to A iff for all $x \in X$ and $f \in X^*$ it holds that $< A_n x, f >$ converges to $< Ax, f >$. Weak convergence is denoted by $\mathrm{w} - \lim_n A_n = A$. The weak closure of a set V is denoted by $\mathrm{w} - \mathrm{cl}\, V$.

We consider time-varying linear discrete-time systems of the form

$$x(t+1) = A(t)x(t), \quad t \in \mathbb{N}, \tag{1}$$

where $A(\cdot) = (A(t))_{t\in\mathbb{N}} \in \mathcal{L}(X)^{\mathbb{N}}$ is a sequence of bounded linear operators on X. The evolution operator associated to this system is defined by

$$\Phi_{A(\cdot)}(t,t) = I_X, \quad \Phi_{A(\cdot)}(t,s) = A(t-1)\cdot \ldots \cdot A(s), \qquad s,t \in \mathbb{N}, \quad t > s, \tag{2}$$

where we drop the subscript if this can cause no confusion. For time-varying systems of this form various notions of stability have been introduced.

Definition 2.1 (Stability) *System (1) is called*

(i) stable, *if for every $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that*

$$\|x_0\| < \delta \ \Rightarrow\ \|\Phi(t,t_0)x_0\| < \varepsilon \quad \text{for all } t \geq t_0\,,$$

(ii) asymptotically stable, *if it is stable and for every* $x_0 \in X$, $t_0 \in \mathbb{N}$

$$\lim_{t\to\infty} \Phi(t,t_0)x_0 = 0. \tag{3}$$

(iii) exponentially stable, *if there are constants* $c, \beta > 0$ *such that*

$$\|\Phi(t,s)\|_{\mathcal{L}(X)} \le c\, e^{-\beta(t-s)}, \qquad s,t \in \mathbb{N}, \quad t \ge s \tag{4}$$

holds.

Remark 2.2 *An immediate consequence of the definition is that if (1) is exponentially stable, then* $A(\cdot) \in \ell^\infty(\mathbb{N}, \mathcal{L}(X))$.

For the time-invariant case $A(\cdot) \equiv A \in \mathcal{L}(X)$ let us discuss the relationship between stability of (1) and properties of the spectrum of A.

Let X be a Banach space over $\mathbb{C}$. For an operator $A \in \mathcal{L}(X)$ we define the spectrum

$$\sigma(A) = \{\lambda \in \mathbb{C}; \quad \lambda I - A \text{ is not invertible in } \mathcal{L}(X)\}, \tag{5}$$

the point-spectrum

$$\sigma_p(A) = \{\lambda \in \mathbb{C}; \text{ there exists } x \in X,\ x \neq 0 \text{ such that } Ax = \lambda x\}, \tag{6}$$

and the spectral radius

$$r(A) = \max\{|\lambda|; \lambda \in \sigma(A)\}. \tag{7}$$

If X is a real Banach space then we regard A as an operator in the complexification $\hat{X}$ of X and call the spectrum of $A \in \mathcal{L}(\hat{X})$ the spectrum of A. Note that the complexification $\hat{X}$ may be endowed with a norm such that the norm of the complexification of any operator in $\mathcal{L}(X)$ coincides with the norm of the operator in $\mathcal{L}(\hat{X})$, see [24] Chapter 1.

If X is finite dimensional it is well-known that in the time-invariant case asymptotic stability is equivalent to exponential stability which in turn is equivalent to $r(A) < 1$ (See [1] Theorem 5.5.1 and Remark 5.5.3). Also by the Gelfand-Levitan formula $r(A) = \lim_{t\to\infty} \|A^t\|^{\frac{1}{t}}$ ([18] Theorem 4.1.13) it follows easily that in a Banach space X a time-invariant system is exponentially stable iff $r(A) < 1$. A basic difference between the finite and the infinite dimensional case is that even for time-invariant systems asymptotic and exponential stability are not equivalent, and that asymptotic stability is not characterized by properties of the spectrum. In [19] two examples show that $r(A) < 1$ is not necessary for asymptotic stability and also the fact that $\sigma_p(A)$ is contained in the open unit circle combined with $r(A) \le 1$ is not sufficient for asymptotic stability. If time-varying systems are considered asymptotic and exponential stability are also not equivalent and exponential stability is not characterized by the spectrum of the transition matrices $A(t)$ even in the finite dimensional case, see e.g. [17] Chapter 4.4.

The following example is an adaption of an example in [8] to the discrete-time case. It shows that for exponential stability it is not sufficient that all trajectories of a system of the form (1) decay exponentially

Example 2.3 *Let* $X = \mathbb{R}$ *and*

$$A(t) = e^{-2+(t+1)\sin\sqrt{t+1}-t\sin\sqrt{t}}, \quad t \in \mathbb{N}. \tag{8}$$

Then

$$\Phi(t,s) = e^{-2(t-s)+t\sin\sqrt{t}-s\sin\sqrt{s}}, \quad t,s \in \mathbb{N},\ t > s. \tag{9}$$

So that we have for all $x_0 \in \mathbb{R},\ t_0 \in \mathbb{N}$

$$\|\Phi(t,t_0)x_0\| \le e^{t_0(1-\sin\sqrt{t_0})} \cdot e^{-(t-t_0)}\|x_0\|, \quad t \in \mathbb{N},\ t > t_0. \tag{10}$$

Hence every solution corresponding to an initial condition $(t_0, x_0) \in \mathbb{N} \times \mathbb{R}$ *goes exponentially fast to zero. However, (1) is not exponentially stable.*

By the mean value theorem we have for some $\tau \in [t, t+1]$

$$(t+1)\sin\sqrt{t+1} - t\sin\sqrt{t} = \sin\sqrt{\tau} + \frac{\sqrt{\tau}\cos\sqrt{\tau}}{2}. \tag{11}$$

It is thus easy to see that the left hand side is not uniformly bounded from above and therefore $\|A(t)\|$ *is not uniformly bounded. By Remark 2.2 it follows that the time-varying system determined by (8) is not exponentially stable.*

3 Characteristic Exponents

To characterize exponential stability the concepts of Lyapunov and Bohl exponents have been introduced. The largest exponential growth rate of system (1) is given by the discrete time version of the (upper) *Bohl* exponent [8] (named *generalized spectral radius* in [21]). In the following definition we do not assume (1) to be exponentially stable and let $(A(t))_{t\in\mathbb{N}}$ be an arbitrary sequence in $\mathcal{L}(X)$.

Definition 3.1 (Bohl exponent) *Given a sequence* $(A(t))_{t\in\mathbb{N}}$ *in* $\mathcal{L}(X)$ *the* (upper) Bohl exponent *of the system (1) is*

$$\beta(A(\cdot)) = \inf\{\beta \in \mathbb{R}; \exists c_\beta \ge 1 : t \ge s \ge 0 \Rightarrow \|\Phi(t,s)\| \le c_\beta e^{\beta(t-s)}\}, \tag{12}$$

where $\inf\emptyset = \infty$.

$\beta(A(\cdot))$ may be infinite, but if $\|A(t)\| \leq \gamma$ for all $t \in \mathbb{N}$ then it follows from (2) that

$$\|\Phi(t,s)\| \leq \|A(t-1)\| \cdot \ldots \cdot \|A(s)\| \leq \gamma^{t-s}, \tag{13}$$

hence $\beta(A(\cdot)) \leq \log \gamma$. Thus $\beta(A(\cdot)) < \infty$ if and only if $(A(t))_{t\in\mathbb{N}}$ is bounded.

In contrast Lyapunov exponents focus on the exponential growth rates of trajectories.

Definition 3.2 (Lyapunov exponent) *Given a sequence* $(A(t))_{t\in\mathbb{N}}$ *in* $\mathcal{L}(X)$ *and an initial condition* $(t_0, x_0) \in \mathbb{N} \times (X \setminus \{0\})$ *the* Lyapunov exponent *corresponding to* (t_0, x_0) *is defined by*

$$\lambda(t_0, x_0) = \inf\{\lambda \in \mathbb{R}; \exists c_\lambda \geq 1 : t \geq t_0 \Rightarrow \|\Phi(t, t_0)x_0\| \leq c_\lambda e^{\lambda(t-t_0)}\|x_0\|\}. \tag{14}$$

Furthermore we define the supremal Lyapunov exponent by

$$\kappa(A(\cdot)) := \sup\{\lambda(t_0, x_0); (t_0, x_0) \in \mathbb{N} \times (X \setminus \{0\})\}. \tag{15}$$

Remark 3.3

(i) From the definition it is immediate that negativity of the Bohl exponent characterizes exponential stability of (1), while negativity of the Lyapunov exponent characterizes exponential decay of a single trajectory. Indeed, it holds that $\beta(A(\cdot)) < 0$ *iff (1) is exponentially stable. For if* $\beta(A(\cdot)) < 0$*, then there exists a* $\varepsilon > 0$ *such that* $\beta := \beta(A(\cdot)) + \varepsilon < 0$ *and* $c_\beta, -\beta$ *satisfy (4), while if (4) is satisfied for* $(A(t))_{t\in\mathbb{N}}$ *and* $c, \beta > 0$*, then by definition* $\beta(A(\cdot)) < -\beta < 0$*. By the same argument we may treat the case of the Lyapunov exponents.*

It should be noted that Lyapunov and Bohl exponents do not characterize asymptotic stability.

(ii) In the theory of continuous-time time-varying systems it is customary to define Lyapunov exponents only for the initial time $t_0 = 0$*. This is justified in this setting because all evolution operators* $\Phi(t,s)$ *are invertible. As it is easy to show that*

$$\lambda(0, x_0) = \lambda(t_0, \Phi(t_0, 0)x_0), \tag{16}$$

it follows that in this case $\{\lambda(0, x_0); x_0 \in X\}$ *comprises already the whole spectrum of Lyapunov exponents. However, in the discrete-time case* $\Phi(t,s)$ *may be singular and so we have to consider initial times different from 0.*

From the definition we immediately obtain the following relation between Bohl exponent and Lyapunov spectrum for any $A(\cdot) \in \mathcal{L}(X)^{\mathbb{N}}$:

$$\beta(A(\cdot)) \geq \kappa(A(\cdot)). \tag{17}$$

In general the inequality in (17) may be strict. We return to Example 2.3 to exhibit this phenomenon.

Example 3.4 *We continue to use the notations of Example 2.3. Recall that in (10)*

$$\|\Phi(t, t_0)x_0\| \leq e^{t_0(1-\sin\sqrt{t_0})} \cdot e^{-(t-t_0)}\|x_0\|, \quad t \in \mathbb{N},\ t > t_0. \tag{18}$$

Therefore $\kappa(A(\cdot)) \leq -1$. *On the other hand,* $\|A(t)\|$ *is unbounded and hence* $\beta(A(\cdot)) = \infty$.

The preceding example is in a sense a worst case example as the Bohl exponent is infinite while the supremal Lyapunov-exponent is finite. It has the disadvantage, however, of leaving open the problem whether the inequality in (17) can also occur for bounded sequences of operators. To clarify this question consider the following example.

Example 3.5 *Let* $X = \mathbb{R}$ *and consider the sequence* $A(\cdot)$ *given by*

$$A(\cdot) = (\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, \dots).$$

It is easy to see that $\beta(A(\cdot)) = 0$ *while* $\kappa(A(\cdot)) = -\frac{1}{2}\log 2$.

Both Bohl- and Lyapunov-exponents have asymptotic expressions that are easily shown from the definition. For the Bohl exponent it is derived in [21].

Proposition 3.6 *Let* $A(\cdot) \in \mathcal{L}(X)^{\mathbb{N}}$, *then*

(i) If $\beta(A(\cdot)) < \infty$ *then*

$$\beta(A(\cdot)) = \limsup_{s, t-s\to\infty} \frac{1}{t-s} \log \|\Phi(t, s)\|.$$

(ii) For every $(t_0, x_0) \in \mathbb{N} \times X$

$$\lambda(t_0, x_0) = \limsup_{t\to\infty} \frac{1}{t-t_0} \log \|\Phi(t, t_0)x_0\|. \tag{19}$$

In [21] Bohl exponents have been named generalized spectral radius, as properties of the spectral radius for time-invariant systems coincide with properties of the Bohl exponent for time-varying systems. Let us now explain why the relation between these concepts is such a close one.

Let $1 \le p \le \infty$ and $\hat{X}_p = \ell^p(\mathbb{N}, X)$ be the space of all sequences $(x_i)_{i\in\mathbb{N}} \subset X$ satisfying $\sum_{i\in\mathbb{N}} \|x_i\|^p < \infty$ if $p < \infty$ or that are uniformly bounded if $p = \infty$. $\hat{X}_p$ is a Banach space if it is endowed with the norm $\|(x_i)_{i\in\mathbb{N}}\|_{\hat{X}_p} = (\sum_{i\in\mathbb{N}} \|x_i\|^p)^{\frac{1}{p}}$, respectively if $p = \infty$ $\|(x_i)_{i\in\mathbb{N}}\|_{\hat{X}_\infty} = \sup_{i\in\mathbb{N}} \|x_i\|$. On $\hat{X}_p$ we define the block-weighted shift operator $\hat{A}$ by defining for $x = (x_0, x_1, \dots) \in \hat{X}_p$

$$\hat{A}(x) := (0, A(0)x_0, A(1)x_1, \dots). \tag{20}$$

From the definition it is immediate that

$$\hat{A} \in \mathcal{L}(\hat{X}_p)$$

iff

$$A(\cdot) \in \ell^\infty(\mathbb{N}, \mathcal{L}(X)).$$

The following relation holds between the spectral radius of $\hat{A}$ and the Bohl exponent of $A(\cdot)$. Implicitly this has been noted for $p = \infty$ in [15] and for $p = 2$ in [21]. In [14] the spectral criterion for $p = \infty$ has been used to analyze stabilization of time-varying discrete-time systems.

Proposition 3.7 *Let $1 \le p \le \infty$, $A(\cdot) \in \ell^\infty(\mathbb{N}, \mathcal{L}(X))$ and $\hat{A}$ on $\hat{X}_p$ be defined by (20), then*

$$e^{\beta(A(\cdot))} = r(\hat{A}), \tag{21}$$

where we use the convention $e^{-\infty} = 0$.

Proof: Let $(x_i)_{i\in\mathbb{N}} \in \hat{X}_p$. By (20) we see that

$$\hat{A}^t x = (0, \dots, 0, \Phi(t,0)x_0, \Phi(t+1,1)x_1, \dots), \tag{22}$$

where the entries from $i = 0$ to $t-1$ are zero. Thus by the Gelfand-Levitan formula

$$r(\hat{A}) = \lim_{t\to\infty} \sup_{s\in\mathbb{N}} \|\Phi(t+s,s)\|^{\frac{1}{t}} \ge \limsup_{t,s\to\infty} \|\Phi(t+s,s)\|^{\frac{1}{t}} = e^{\beta(A(\cdot))}. \tag{23}$$

To prove the converse inequality note that by Proposition 3.6 (i) for every $\varepsilon > 0$ there exists $M \geq 1$ such that $\|\Phi(t,s)\| \leq Me^{(\beta(A(\cdot))+\varepsilon)(t-s)}$. Thus

$$r(\hat{A}) = \lim_{t\to\infty} \sup_{s\in\mathbb{N}} \|\Phi(t+s,s)\|^{\frac{1}{t}} \leq \lim_{t\to\infty} (Me^{(\beta(A(\cdot))+\varepsilon)t})^{\frac{1}{t}} = e^{\beta(A(\cdot))+\varepsilon} .$$

This shows the assertion. □

As the Bohl exponent can be expressed as the spectral radius of the operator $\hat{A} \in \ell^p(\mathbb{N}, X)$ we have obtained a new way to prove the following results that first appear in [21] and [23].

Proposition 3.8 *Let* $(A(t))_{t\in\mathbb{N}} \in l^\infty(\mathbb{N};\mathcal{L}(X))$. *Then*

(i) The function $\beta : (l^\infty(\mathbb{N};\mathcal{L}(X)), \|\cdot\|_\infty) \to \mathbb{R}\cup\{-\infty\}$ *is upper semi-continuous.*

(ii) If $A(t) \equiv A \in \mathcal{L}(X)$ *is constant in* $t \in \mathbb{N}$ *then*

$$e^{\beta(A)} = \exp(\lim_{t\to\infty} \frac{1}{t} \log \|A^t\|) = r(A)$$

is the spectral radius of A.

(iii) The following statements are equivalent:

(a) (1) is exponentially stable.

(b) $\beta(A(\cdot)) < 0$.

(c) $r(\hat{A}) < 1$.

(d) For every $1 \leq p < \infty$ *there exists a* $\gamma > 1$ *such that* $\forall s \in \mathbb{N}\ \forall x_0 \in X : \sum_{t=s}^{\infty} \|\Phi(t,s)x_0\|^p \leq \gamma^p \|x_0\|^p$.

Proof:

(i) This is an immediate consequence of the fact that the spectral radius is upper semi-continuous, (see [16] Chapter 4 §3.2), and the observation that the map

$$l^\infty(\mathbb{N};\mathcal{L}(X)) \to \mathcal{L}(\hat{X}_p)$$

$$A(\cdot) \mapsto \hat{A}$$

is continuous.

(ii) This follows as $r(A) = r(\hat{A})$ if $\hat{A}$ is generated by $A(t) \equiv A$.

(iii) The equivalence of (a), (b) and (c) follows from Remark 3.3(i) and Proposition 3.7. Let $1 \leq p < \infty$. If (b) holds then for all $s \in \mathbb{N}$ and $x_0 \in X$ it holds that for suitable constants $c, \beta > 0$

$$\sum_{t=s}^{\infty} \|\Phi(t,s)x_0\|^p \leq \sum_{t=s}^{\infty} c^p e^{-p\beta(t-s)} \|x_0\|^p = \frac{c^p}{1-e^{-p\beta}} \|x_0\|^p \,, \tag{24}$$

so that (d) is satisfied. If (d) holds then it follows also that for any $T \geq 1$ and all $x_0 \in X$

$$\begin{aligned} \sum_{t=s+T}^{\infty} \|\Phi(t,s)x_0\|^p &= \sum_{t=s+T-1}^{\infty} \|\Phi(t,s)x_0\|^p - \|\Phi(s+T-1,s)x_0\|^p \\ &\leq (1-\gamma^{-p}) \sum_{t=s+T-1}^{\infty} \|\Phi(t,s)x_0\|^p \leq \ldots \leq \\ &\leq (1-\gamma^{-p})^T \sum_{t=s}^{\infty} \|\Phi(t,s)x_0\|^p \leq (1-\gamma^{-p})^T \gamma^p \|x_0\|^p \,. \end{aligned} \tag{25}$$

As we may see from (22) it holds for any $\hat{x} = (x_i)_{i\in\mathbb{N}} \in \hat{X}_p$ that

$$\|\hat{A}^t \hat{x}\|_{\hat{X}_p}^p = \sum_{i=0}^{\infty} \|\Phi(t+i,i)x_i\|^p \,,$$

and hence for $T \in \mathbb{N}$ by (25)

$$\|\sum_{t=T}^{\infty} \hat{A}^t \hat{x}\|_{\hat{X}_p}^p \leq \sum_{i=0}^{\infty}\sum_{t=T}^{\infty} \|\Phi(t+i,i)x_i\|^p \leq (1-\gamma^{-p})^T \gamma^p \|\hat{x}\|_{\hat{X}_p}^p \,.$$

Now it follows that $\| \sum_{t=0}^{\infty} \hat{A}^t \|$ is a bounded operator and the infinite series converges uniformly. Thus we may apply von Neumann's theorem (see [13], Satz 12.4) and it follows that $r(\hat{A}) < 1$. This completes the proof.

□

Let us also note how the information on Lyapunov exponents is contained in the operator $\hat{A}$. In the statement of the following proposition $S : \ell^p(\mathbb{N}, X) \to \ell^p(\mathbb{N}, X)$ denotes the unilateral shift given by $S(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$.

Proposition 3.9 *Let* $(A(t))_{t\in\mathbb{N}} \in l^\infty(\mathbb{N};\mathcal{L}(X))$ *and let* $\hat{A}$ *be the corresponding operator defined by (20) on* $\hat{X}_p$, $1 \le p \le \infty$. *Let* $0 \neq x_0 \in X$. *For the corresponding Lyapunov exponent* $\lambda(t_0, x_0)$ *at time* t_0 *it holds that*

(i) $S^{t_0}(x_0, 0, 0, \dots) \in \operatorname{Im}(sI - \hat{A})$ *for all* $s \in \mathbb{C}, |s| > e^{\lambda(t_0,x_0)}$.

(ii) $S^{t_0}(x_0, 0, 0, \dots) \notin \operatorname{Im}(sI - \hat{A})$ *for all* $s \in \mathbb{C}, 0 < |s| < e^{\lambda(t_0,x_0)}$.

Proof: Let us denote $\lambda := \lambda(t_0, x_0)$. First of all note that an easy calculation yields that for $S^{t_0}(x_0, 0, 0, \dots) = (sI - \hat{A})y$ to hold y must be of the form

$$y = s^{-1} S^{t_0}(x_0, s^{-1}A(t_0)x_0, s^{-2}A(t_0+1)x_0, \dots)\,.$$

(i) If $|s| > e^\lambda$ then for any $\varepsilon > 0$ there exists an $M \ge 1$ such that $\|s^{-k}A(t_0 + k - 1)x_0\| \le Me^{(\lambda+\varepsilon)k}|s|^{-k}$. Choosing ε small enough we see that $y \in \hat{X}_p$.

(ii) If $|s| < e^\lambda$ choose $\varepsilon > 0$ such that $(1+\varepsilon)|s| < e^\lambda$. Then for every constant $M \ge 1$ there exists a subsequence $(k_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ such that $\|A(t_0 + k_l - 1)x_0\| \ge M(1+\varepsilon)^{k_l}|s|^{k_l}$. In this case the sequence y defined above is not an element of $\hat{X}_p$.

□

As we have seen in the previous Examples 3.4 and 3.5 it is possible that Bohl and maximal Lyapunov exponent are different. Let us now characterize when they are equal.

Theorem 3.10 *Let* $\mathbb{K} = \mathbb{R}, \mathbb{C}$. *Let* X *be a Banach space over* $\mathbb{K}$. *For a sequence* $(A(t))_{t\in\mathbb{N}} \in l^\infty(\mathbb{N}, \mathcal{L}(X))$ *the following statements are equivalent*

(i) $\beta(A(\cdot)) = \kappa(A(\cdot))$.

(ii) For all $\varepsilon > 0$:

$$M_\varepsilon := \sup_{t_0\in\mathbb{N}, x_0\in X} \inf\{M;\ \|\Phi(t,t_0)x\| \le Me^{(\kappa(A(\cdot))+\varepsilon)(t-t_0)}\|x_0\|\} < \infty\,.$$

(iii) For every $\varepsilon > 0$ *sufficiently small there exists a* $t_0 \in \mathbb{N}$ *and an* $x_0 \in X$ *such that*

$$S^{t_0}(x_0, 0, 0, 0, \dots) \notin \operatorname{Im}((r(\hat{A}) - \varepsilon)I - \hat{A})\,.$$

Proof: $(i) \Leftarrow (ii)$: If for $\varepsilon > 0$ it holds that $M_\varepsilon < \infty$ then we obtain that for all $t \geq s \geq 0$ it holds that $\|\Phi(t,s)\| \leq M_\varepsilon e^{(\kappa(A(\cdot))+\varepsilon)(t-t_0)}$ and hence $\beta(A(\cdot)) \leq \kappa(A(\cdot)) + \varepsilon$. Now (17) implies the assertion.
$(i) \Rightarrow (ii)$: Assume there exists an $\varepsilon > 0$ such that $M_\varepsilon = \infty$. Hence there exist sequences $(s_k)_{k\in\mathbb{N}}, (t_k)_{k\in\mathbb{N}}$ and $(M_k)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$ $t_k > s_k$ and $\lim_{k\to\infty} M_k = \infty$ and furthermore

$$\|\Phi(t_k, s_k)\| \geq M_k e^{(\kappa(A(\cdot))+\varepsilon)(t_k - s_k)} .$$

By definition this implies $\beta(A(\cdot)) \geq \kappa(A(\cdot)) + \varepsilon$.
The equivalence of (i) and (iii) follows from Proposition 3.7 and Proposition 3.9. □

4 Discrete Inclusions

Let X be a Banach space and $\mathcal{M} \subset \mathcal{L}(X)$ be a bounded set. We consider the discrete inclusion

$$x(t+1) \in \{Ax(t) \; ; \; A \in \mathcal{M}\} \quad t \in \mathbb{N}. \tag{26}$$

A sequence $\{x(t)\}_{t\in\mathbb{N}}$ is called solution of (26) with initial condition $x_0 \in X$ if $x(0) = x_0$ and for all $t \in \mathbb{N}$ there exists an $A(t) \in \mathcal{M}$ such that $x(t+1) = A(t)x(t)$.

We may introduce two immediate concepts as regards characteristic exponents of (26).

$$\bar{\kappa}(\mathcal{M}) := \sup\{\kappa(A(\cdot)) \; ; \; A(\cdot) \in \mathcal{M}^{\mathbb{N}}\} , \tag{27}$$

$$\bar{\beta}(\mathcal{M}) := \sup\{\beta(A(\cdot)) \; ; \; A(\cdot) \in \mathcal{M}^{\mathbb{N}}\} . \tag{28}$$

A further quantity that will be of interest is given by the uniform exponential growth rate

$$\bar{\delta}(\mathcal{M}) := \limsup_{t\to\infty} \frac{1}{t} \log \sup_{A(\cdot)\in\mathcal{M}^{\mathbb{N}}} \|\Phi_{A(\cdot)}(t,0)\| . \tag{29}$$

Corresponding to these definitions we introduce the following concepts of stability:

Definition 4.1 *Let X be a Banach space and $\mathcal{M} \subset \mathcal{L}(X)$ be a bounded set. The discrete time inclusion of the form (26) given by $\mathcal{M}$*

(i) has exponentially decaying trajectories *if $\bar{\kappa}(\mathcal{M}) < 0$.*

(ii) is called exponentially stable *if* $\bar{\beta}(\mathcal{M}) < 0$.

(iii) is called uniformly exponentially stable *if* $\bar{\delta}(\mathcal{M}) < 0$.

Let us note that for every $\mathcal{M}$ bounded every $A(\cdot) \in \mathcal{M}^{\mathbb{N}}$ satisfies:

$$\beta(A(\cdot)) = \limsup_{s,t-s\to\infty} \frac{1}{t-s} \log \|\Phi_{A(\cdot)}(t,s)\|$$

$$= \limsup_{s,t-s\to\infty} \frac{1}{t-s} \log \|\Phi_{A(s+\cdot)}(t-s,0)\|$$

$$\leq \limsup_{s,t-s\to\infty} \frac{1}{t-s} \log \sup_{A(\cdot)\in\mathcal{M}^{\mathbb{N}}} \|\Phi_{A(\cdot)}(t-s,0)\| = \bar{\delta}(\mathcal{M}) .$$

This together with (17) implies that

$$\bar{\kappa}(\mathcal{M}) \leq \bar{\beta}(\mathcal{M}) \leq \bar{\delta}(\mathcal{M}) . \tag{30}$$

It is our goal to show that the three quantities in (30) are in fact equal for discrete inclusions on reflexive Banach spaces. In order to do this we need the following proposition.

Proposition 4.2 *Let X be a reflexive Banach space and let $\mathcal{M} \subset \mathcal{L}(X)$ be weakly compact. Then for every sequence $\{x_k(t)\}$ of solutions of (26) with bounded initial condition, i.e. $\|x_k(0)\| \leq c$ for all $k \in \mathbb{N}$, there exists a subsequence $\{x_{k_l}(t)\}$ of solutions and a solution $\{x(t)\}$ of (26) satisfying*

$$\mathrm{w}-\lim_{l\to\infty} x_{k_l}(t) = x(t) , \quad \textit{for all } t \in \mathbb{N}. \tag{31}$$

Proof: As every closed ball $B(0,r)$ is weakly compact in X, we may choose a subsequence $\{x_{k_l^0}(t)\}$ of the original sequence such that $\mathrm{w}-\lim_{l\to\infty} x_{k_l^0}(0) = x_0$ for an appropriate $x_0 \in X$. Now choose a subsequence $\{x_{k_l^1}(t)\}$ of the sequence $\{x_{k_l^0}(t)\}$ such that $\mathrm{w}-\lim_{l\to\infty} A_{k_l^1}(0) = A(0)$ for some appropriate $A(0) \in \mathcal{M}$. Continue this process inductively and consider the diagonal sequence $\{x_{k_l}(t)\} := \{x_{k_l^l}(t)\}$ and the solution $\{x(t)\}$ of the discrete inclusion (26) determined by the initial condition x_0 and the operator sequence $(A(0), A(1), \dots) \in \mathcal{M}^{\mathbb{N}}$.

We will prove (31) by induction. For $t = 0$ the assertion is clear by construction, so consider the case $t+1$, then for any $f \in X^*$ it holds that

$$< x(t+1) - x_{k_l}(t+1), f >=$$

$$< A(t)x(t) - A(t)x_{k_l}(t), f > + < A(t)x_{k_l}(t) - A_{k_l}(t)x_{k_l}(t), f > .$$

The first term of the right hand side converges to zero by the induction hypothesis, while the second term converges to zero by the weak convergence of the $A_{k_l}(t)$ to $A(t)$. □

Using the preceding fact we can show the following proposition on the boundedness of the solutions of an asymptotically stable discrete inclusion partly using ideas that were also used in [9] for the finite dimensional case.

Theorem 4.3 *Let X be a reflexive Banach space and let $\mathcal{M} \subset \mathcal{L}(X)$ be weakly compact. If for all $A(\cdot) \in \mathcal{M}^{\mathbb{N}}$ it holds that*

$$\lim_{t\to\infty} \Phi_{A(\cdot)}(t,0)x_0 = 0\,,$$

then there exists a constant $c_{\mathcal{M}} \in \mathbb{R}$ such that

$$\sup\{\|\Phi_{A(\cdot)}(t,0)\|\ ;\ t \in \mathbb{N}, A(\cdot) \in \mathcal{M}^{\mathbb{N}}\} < c_{\mathcal{M}}\,. \tag{32}$$

Proof: In order to show (32) we have to show that

$$\sup\{\|\Phi_{A(\cdot)}(t,0)x_0\|\ ;\ t \in \mathbb{N}, A(\cdot) \in \mathcal{M}^{\mathbb{N}}, \|x_0\| \le 1\} < \infty\,.$$

By the principle of uniform boundedness (see, [16], Theorem III.1.27) it is sufficient that for each $f \in X^*$ there exists a constant c_f satisfying

$$\sup\{| < \Phi_{A(\cdot)}(t,0)x_0, f > |\ ;\ t \in \mathbb{N}, A(\cdot) \in \mathcal{M}^{\mathbb{N}}, \|x_0\| \le 1\} < c_f\,. \tag{33}$$

Assume there exists an $\bar{f} \in X^*$ for which (33) does not hold, i.e. there exist sequences $\{x_n\}_{n\in\mathbb{N}}, \{t_n\}_{n\in\mathbb{N}}$ and $\{A_n(\cdot)\}_{n\in\mathbb{N}}$ such that

$$| < \Phi_{A_n(\cdot)}(t_n,0)x_n, \bar{f} > |\ \ge n+1\,. \tag{34}$$

Let us assume that the sequences have been chosen in such a way as to guarantee that for all solutions $\{x(t)\}$ of (26) with initial condition $x_0, \|x_0\| \le 1$ it holds that

$$| < x(t), \bar{f} > |\ < n+1 \quad \text{for } t = 0,1,\dots,t_n - 1\,. \tag{35}$$

As $\mathcal{M}$ is bounded it follows immediately that

$$t_1 \le t_2 \le t_3 \dots\,, \quad t_n \to \infty\,.$$

We claim that for every n it holds that

$$| < x_n(t), \bar{f} > | > 1\,, \quad t = 1,\dots,t_n\,, \tag{36}$$

for otherwise consider a $1 \le t' \le t_n$ with

$$| < x_n(t'), \bar{f} > | \le 1\,.$$

The sequence $z(t) = x_n(t + t')$ is a solution of (26) with initial condition $x_0 = x_n(t')$, $\|x_0\| \leq 1$, by (35) it follows that

$$n + 1 > | < z(t_n - t'), \bar{f} > | = | < x_n(t_n), \bar{f} > | ,$$

in contradiction to (34). Now consider a subsequence $\{x_{n_j}(t)\}$ of $\{x_n(t)\}$ such that

$$\mathrm{w} - \lim_{j \to \infty} x_{n_j}(t) = x(t) ,$$

for some solution $\{x(t)\}$ of (26). By assumption there exists a $T \in \mathbb{N}$ such that $\|x(t)\| < 1/2\|\bar{f}\|$ for all $t \geq T$ and hence

$$| < x(t), \bar{f} > | < \frac{1}{2} , \quad \text{for all } t \geq T .$$

Now for all j big enough it holds that

$$| < x_{n_j}(T), \bar{f} > | > 1 ,$$

by (36) and thus $x(T)$ is not the weak limit of the $x_{n_j}(T)$. This contradicts our construction, which completes the proof. □

Remark 4.4 *In the finite dimensional theory Theorem 4.3 can be used to show that an asymptotically stable discrete inclusion is in fact exponentially stable [9]. Note that this is false in infinite dimensions as even for time-invariant systems asymptotic stability does not imply exponential stability, see [19].*

Using Theorem 4.3 we obtain the following for growth bounds of discrete inclusions:

Theorem 4.5 *Let X be a reflexive Banach space and assume that $\mathcal{M} \subset \mathcal{L}(X)$ is weakly compact, then*

$$\bar{\kappa}(\mathcal{M}) = \bar{\beta}(\mathcal{M}) = \bar{\delta}(\mathcal{M}) . \tag{37}$$

Thus a discrete inclusion given by a weakly compact $\mathcal{M}$ has exponentially decaying trajectories iff it is exponentially stable iff it is uniformly exponentially stable.

Proof: By (30) it remains to show that $\bar{\delta}(\mathcal{M}) \leq \bar{\kappa}(\mathcal{M})$. Assume without loss of generality that $\bar{\kappa}(\mathcal{M}) < 0 < \bar{\delta}(\mathcal{M})$. By the definition of the Lyapunov exponents this implies that all solution of (26) converge to zero and hence by Theorem 4.3 there exists a constant $c_{\mathcal{M}}$ such that

$$\sup\{\|\Phi_{A(\cdot)}(t, 0)\| \; ; \; t \in \mathbb{N}, A(\cdot) \in \mathcal{M}^{\mathbb{N}}\} < c_{\mathcal{M}} .$$

This implies that $\bar{\delta}(\mathcal{M}) \leq \limsup_{t \to \infty} 1/t \log c_{\mathcal{M}} = 0$, a contradiction. □

Corollary 4.6 *Let X be a reflexive Banach space and let $\mathcal{M} \subset \mathcal{L}(X)$ be weakly compact. For every $\varepsilon > 0$ there exists a constant $c_{\mathcal{M},\varepsilon}$ such that*

$$\|\Phi_{A(\cdot)}(t,0)\| \leq c_{\mathcal{M},\varepsilon} e^{(\bar{\beta}(\mathcal{M})+\varepsilon)t}\,, \tag{38}$$

for all $A(\cdot) \in \mathcal{M}^{\mathbb{N}}$ and all $t \in \mathbb{N}$.

Proof: This follows from the definition of $\bar{\delta}(\mathcal{M})$. □

Corollary 4.7 *Let X be a reflexive Banach space and let $\mathcal{M} \subset \mathcal{L}(X)$ be weakly compact. If the discrete inclusion (26) given by $\mathcal{M}$ has a positive uniform growth rate, then there exists a trajectory with positive exponential growth rate. In particular there exists an unbounded trajectory.*

Proof: This is obvious from Theorem 4.5. □

Let us note that Theorem 4.5 also has implications on balanced convex sets. For $\mathcal{M} \subset \mathcal{L}(X)$ we denoted the balanced convexification by

$$\operatorname{bco} \mathcal{M} := \operatorname{co} \bigcup_{a \in \mathbb{K}, |a|=1} a\mathcal{M}\,.$$

Corollary 4.8 *Let X be a reflexive Banach space and assume that $\mathcal{M} \subset \mathcal{L}(X)$ is weakly compact, then*

$$\bar{\beta}(\mathcal{M}) = \bar{\beta}(\operatorname{co} \mathcal{M})\,, \tag{39}$$

and

$$\bar{\beta}(\mathcal{M}) = \bar{\beta}(\operatorname{bco} \mathcal{M})\,. \tag{40}$$

Proof: In view of the preceding Theorem 4.5 for (39) it is sufficient to show that $\bar{\delta}(\mathcal{M}) = \bar{\delta}(\operatorname{co} \mathcal{M})$ as it holds that

$$\bar{\beta}(\mathcal{M}) \leq \bar{\beta}(\operatorname{co} \mathcal{M}) \leq \bar{\delta}(\operatorname{co} \mathcal{M})\,.$$

To this end it is sufficient to note that for each $t \in \mathbb{N}$ it holds that

$$\sup_{A(\cdot) \in \mathcal{M}^{\mathbb{N}}} \|\Phi_{A(\cdot)}(t,0)\| \geq \sup_{A(\cdot) \in \operatorname{co} \mathcal{M}^{\mathbb{N}}} \|\Phi_{A(\cdot)}(t,0)\|\,,$$

due to the convexity of the norm. Now (40) follows from (39) as the matrix products considered in (40) are up to a constant of modulus one the same as in (39). □

Corollary 4.9 *Let X be a reflexive Banach space and assume that $\mathcal{M} \subset \mathcal{L}(X)$ is weakly compact, then*

$$\bar{\beta}(\mathcal{M}) = \bar{\beta}(\mathrm{w} - \mathrm{cl}\,\mathrm{bco}\,\mathcal{M})\,. \tag{41}$$

Proof: As in the proof of the preceding corollary we have to show that $\bar{\delta}(\mathcal{M}) = \bar{\delta}(\mathrm{w} - \mathrm{cl}\,\mathrm{co}\,\mathcal{M})$ and for this it is sufficient to see that $\bar{\delta}(\mathrm{co}\,\mathcal{M}) = \bar{\delta}(\mathrm{w} - \mathrm{cl}\,\mathrm{co}\,\mathcal{M})$. As for convex subsets of $\mathcal{L}(X)$ weak closure and strong closure coincide (see [10], Corollary VI.1.5) the assertion follows after noting that for a strong limit A of an operator sequence $\{A_n\}$ it holds that $\|A\| \leq \limsup_{n\to\infty} \|A_n\|$. □

5 Time-varying Discrete Inclusions

Let X be a Banach space and let for every $t \in \mathbb{N}$ the set $\mathcal{M}(t) \subset \mathcal{L}(X)$ be bounded. We consider the time-varying discrete inclusion

$$x(t+1) \in \{Ax(t)\ ;\ A \in \mathcal{M}(t)\} \quad t \in \mathbb{N}\,. \tag{42}$$

A sequence $\{x(t)\}_{t\in\mathbb{N}}$ is called solution of (42) with initial condition $x_0 \in X$ if $x(0) = x_0$ and for all $t \in \mathbb{N}$ there exists an $A(t) \in \mathcal{M}(t)$ such that $x(t+1) = A(t)x(t)$. Also we say that a sequence $A(\cdot) \in \mathcal{M}(\cdot)$ if $A(t) \in \mathcal{M}(t)$ for all $t \in \mathbb{N}$. We say that $\mathcal{M}(\cdot)$ is uniformly bounded if there exists a constant c such that $\sup\{\|A\|\ ;\ A \in \mathcal{M}(t), t \in \mathbb{N}\} < c$.

As before we introduce the characteristic exponents

$$\bar{\kappa}(\mathcal{M}(\cdot)) := \sup\{\kappa(A(\cdot))\ ;\ A(\cdot) \in \mathcal{M}(\cdot)\}\,, \tag{43}$$

$$\bar{\beta}(\mathcal{M}(\cdot)) := \sup\{\beta(A(\cdot))\ ;\ A(\cdot) \in \mathcal{M}(\cdot)\}\,. \tag{44}$$

$$\bar{\delta}(\mathcal{M}(\cdot)) := \limsup_{t\to\infty} \frac{1}{t} \log \sup_{A(\cdot)\in\mathcal{M}(\cdot), s\in\mathbb{N}} \|\Phi_{A(\cdot)}(t+s,s)\|\,, \tag{45}$$

and define the concepts of exponentially decaying trajectories, exponentially stable and uniformly exponentially stable as in Definition 4.1.

Let us note that for $\mathcal{M}(\cdot)$ uniformly bounded we have

$$\bar{\kappa}(\mathcal{M}(\cdot)) \leq \bar{\beta}(\mathcal{M}(\cdot)) \leq \bar{\delta}(\mathcal{M}(\cdot))\,, \tag{46}$$

where the first inequality may be strict. This follows from Examples 3.4 and 3.5 as time-varying systems are a special case of time-varying discrete inclusions where each $\mathcal{M}(t)$ is a singleton set. Also for time-varying systems it is clear that both $\bar{\beta}$ and $\bar{\delta}$ reduce to the Bohl exponent (by (23)) and are therefore equal. This extends to the general case.

Theorem 5.1 *Let X be a reflexive Banach space and let $\mathcal{M}(\cdot)$ be uniformly bounded. Assume furthermore that for each $t \in \mathbb{N}$ the set $\mathcal{M}(t)$ is weakly compact, then*

$$\bar{\beta}(\mathcal{M}(\cdot)) = \bar{\delta}(\mathcal{M}(\cdot))\,. \tag{47}$$

Thus a time-varying discrete inclusion is exponentially stable iff it is uniformly exponentially stable.

Proof: Choosing $p = 2$ recall the definition of the space $\hat{X}_2$ from (20) and introduce the set

$$\hat{\mathcal{M}} := \{\hat{A}\ ;\ A(\cdot) \in \mathcal{M}(\cdot)\}\,.$$

For the discrete inclusion on $\hat{X}_2$ given by $\hat{\mathcal{M}}$ we can apply Theorem 4.5: it is quite straightforward to see that $\hat{X}_2$ is reflexive and that $\hat{\mathcal{M}}$ is weakly compact. Thus we have that

$$\bar{\kappa}(\hat{\mathcal{M}}) = \bar{\beta}(\hat{\mathcal{M}}) = \bar{\delta}(\hat{\mathcal{M}})\,.$$

Now $\bar{\beta}(\hat{\mathcal{M}}) \geq \sup_{\hat{A}\in\hat{\mathcal{M}}} \log r(\hat{A}) = \bar{\beta}(\mathcal{M}(\cdot))$ and on the other hand

$$\bar{\delta}(\mathcal{M}(\cdot)) = \limsup_{t\to\infty} \frac{1}{t} \log \sup_{A(\cdot)\in\mathcal{M}(\cdot), s\in\mathbb{N}} \|\Phi_{A(\cdot)}(t+s,s)\| =$$

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{\hat{A}(\cdot)\in\hat{\mathcal{M}}} \|\Phi_{\hat{A}(\cdot)}(t,0)\| = \bar{\delta}(\hat{\mathcal{M}})\,.$$

It therefore remains to show that $\sup_{\hat{A}\in\hat{\mathcal{M}}} \log r(\hat{A}) \geq \bar{\kappa}(\hat{\mathcal{M}})$.

Consider any $\hat{A}(\cdot) \in \hat{\mathcal{M}}^{\mathbb{N}}$ and $x_0 = (x_{0,i})_{i\in\mathbb{N}} \in \hat{X}_2$. Let $\lambda := \lambda(0, x_0)$ (where without loss of generality we set $t_0 = 0$). If $\lambda = -\infty$ there is nothing to show, as then clearly $\bar{\beta}(M(\cdot)) \geq \lambda$. Assume $\lambda \in \mathbb{R}$ and fix $\varepsilon > 0$, then for every $M \in \mathbb{N}$ there exists a $t_M \in \mathbb{N}$ such that

$$\|\Phi_{\hat{A}(\cdot)}(t_M, 0)x_0\| \geq M e^{(\lambda-\varepsilon)t_M} \|x_0\|\,.$$

Note that by uniform boundedness of $\mathcal{M}(\cdot)$ we have $t_M \to \infty$ for $M \to \infty$. If we denote $\hat{A}(t) =: (A_i(t))_{i\in\mathbb{N}}$, where $A_i(t) \in \mathcal{M}(i)$ this implies that for some index $i \in \mathbb{N}$

$$\|A_{i+t_M-1}(t_M - 1)A_{i+t_M-2}(t_M - 2) \cdot \ldots \cdot A_i(0)\| \geq M e^{(\lambda-2\varepsilon)t_M}\,.$$

Let i_M denote the smallest index for which the preceding inequality is satisfied. If the set $J = \{i_M\ ;\ M \in \mathbb{N}\}$ is bounded there exists an index

$j \in J$ such that $j = i_M$ for infinitely many $M \in \mathbb{N}$. This implies that the operator sequence

$$A(\cdot) := (*, \dots, *, A_j(0), A_{j+1}(1), A_{j+2}(2), \dots, A_{j+k}(k), \dots)$$

satisfies $\beta(A(\cdot)) \geq (\lambda - 2\varepsilon)$, where $*$ indicates that an arbitrary $A(s) \in \mathcal{M}(s)$ may be chosen for $s = 0, \dots, j-1$.

If J is not bounded we can construct an operator sequence $B(\cdot)$ with the desired property as follows: For $s = 0, \dots, i_1 - 1$ choose $B(s) \in \mathcal{M}(s)$ arbitrarily. For $s = i_1, \dots, i_1 + t_1 - 1$ let $B(s) = A_s(s - i_1)$. Now choose $M(2)$ such that $i_{M(2)} > i_1 + t_1 - 1$. For $s = i_1 + t_1, \dots, i_{M(2)} - 1$ choose $B(s) \in \mathcal{M}(s)$ arbitrarily. For $s = i_{M(2)}, \dots, i_{M(2)} + t_{M(2))} - 1$ let $B(s) = A_s(s - i_{M(2)})$. Continue this procedure inductively. For the operator sequence constructed in this way it holds that $\beta(B(\cdot)) \geq \lambda - 2\varepsilon$.

In all this shows $\bar{\beta}(M(\cdot)) \geq \bar{\kappa}(\hat{\mathcal{M}})$. □

Corollary 5.2 *Let X be a reflexive Banach space and let $\mathcal{M}(\cdot)$ be uniformly bounded. Assume furthermore that for each $t \in \mathbb{N}$ the set $\mathcal{M}(t)$ is weakly compact, then for every $\varepsilon > 0$ there exists a constant $c_{\mathcal{M}(\cdot),\varepsilon}$ such that*

$$\|\Phi_{A(\cdot)}(t+s, s)\| \leq c_{\mathcal{M}(\cdot),\varepsilon} e^{(\bar{\beta}(\mathcal{M}(\cdot))+\varepsilon)t},$$

for all $A(\cdot) \in \mathcal{M}(\cdot)$ and $t \geq s \in \mathbb{N}$.

Proof: This follows from the definition of $\bar{\delta}(\mathcal{M}(\cdot))$. □

Note that an equivalent statement to Corollary 4.7 is false, as $\mathcal{M}(t) = \{0\}$ may occur for infinitely many t while a positive uniform exponential growth rate exists.

Corollary 5.3 *Let X be a reflexive Banach space and let $\mathcal{M}(\cdot)$ be uniformly bounded. Assume furthermore that for each $t \in \mathbb{N}$ the set $\mathcal{M}(t)$ is weakly compact. Then for the time-varying discrete inclusion $\mathcal{N}(\cdot)$ given by*

$$\mathcal{N}(t) = \mathrm{w} - \mathrm{cl}\,\mathrm{bco}\,\mathcal{M}(t),$$

it holds that

$$\bar{\beta}(\mathcal{N}(\cdot)) = \bar{\beta}(\mathcal{M}(\cdot)).$$

Proof: This may be shown in a similar fashion to Corollaries 4.8 and 4.9 after noting that $\hat{\mathcal{N}} = \mathrm{w} - \mathrm{cl}\,\mathrm{bco}\,\hat{\mathcal{M}}$. □

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Institut für Dynamische Systeme
Universität Bremen, D-28344 Bremen
Germany

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