A small gain condition for interconnections of ISS systems with mixed ISS characterizations

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***Abstract*— We consider interconnected nonlinear systems with external inputs. Each of the subsystems is assumed to be input-to-state stable (ISS). Sufficient conditions of small-gain type are provided guaranteeing that the interconnection is ISS. To this end we extend recently obtained small gain theorems to a more general type of interconnections. The small gain theorem proved here is applicable to situations where the ISS conditions are formulated differently for each subsystem and are either given in the maximization or the summation sense. An example shows the advantages of our results in comparison with the known ones.**

I. INTRODUCTION

In this paper we consider nonlinear interconnected systems with inputs and use the notion of input-to state-stability (ISS) as a framework for stability analysis of such networks. This notion was introduced by E. Sontag in 1989, see [10]. Our main question in this paper is whether an interconnection of ISS systems is again ISS. It is known that cascades of ISS systems are ISS, however, a feedback interconnection of two ISS systems is in general unstable. Some conditions applied on the gains of both systems can assure that their feedback is ISS. The first result of the small gain type was proved in [7] for a feedback of two ISS systems. The Lyapunov version of this result is given in [6]. Here we would like to note the difference between the small gain conditions in these papers. One of them states in [6] that the composition of both gains should be less then identity. The second condition in [7] is similar but it involves the composition of both gains and further functions of the form $(\mathrm{id}+\alpha_i)$. This difference is due to the use of different definitions of ISS in both papers. Both definitions are equivalent but the gains enter as a maximum in the first definition, and a sum of the gains is taken in the second one. The results of [7] and [6] were generalized for an interconnection of $n \geq 2$ systems in [1], [3]. It was pointed out that the similar difference in the small gain conditions remains, i.e., if the gains of different inputs enter as a maximum of gains in the ISS definition or a sum of them is taken in the definition. Moreover, it was shown that the auxiliary functions $(\mathrm{id}+\alpha_i)$ are essential in the summation case and cannot be omitted [1]. A more general definition of ISS for the case of many inputs was introduced in [9], [2] and [4]. For recent results on the small-gain conditions for a wider class of interconnections we refer to [8], [5].

In some applications it may happen that the gains of a part of systems of an interconnection are given in maximization terms while the gains of another part are given in a summation formulation. This motivates the question: do we need functions $(\mathrm{id}+\alpha_i)$ and how many of them in the small gain condition to assure stability in this case? In this paper we consider this case and answer this question. Namely we consider n interconnected ISS systems, such that in the ISS definition of the first $k \leq n$ systems the gains enter additively. For the rest of systems the definition with maximum is used. We will see that the small gain condition provided in this paper is less conservative then the one used for the general definition in [9]. Our result contains the known small gain conditions from [1] as a special case $k = 0$ or $k = n$, i.e., if only one type of ISS definition is used. An example in the end of the paper shows the advantages of our results in comparison with the known ones.

This paper is organized as follows. In section II we present necessary notation and definitions. Section III provides some auxiliary lemmas. The main result presenting the new small gain condition is given in section IV, an example in the end of this section demonstrates the novelty of this result. Section V contains conclusions.

II. PRELIMINARIES AND PROBLEM STATEMENT

In the following we denote $\mathbb{R}_+ := [0, \infty)$. $\mathbb{R}^n_+$ is the positive orthant $\{x \in \mathbb{R}^n : x \geq 0\}$. x^T stands for the transpose of a vector $x \in \mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, we use the standard partial order induced by the positive orthant. It is given by

$$\begin{aligned} x \geq y &\iff x_i \geq y_i, \quad i = 1, \dots, n, \\ x > y &\iff x_i > y_i, \quad i = 1, \dots, n. \end{aligned}$$

We write $x \not\geq y \iff \exists i \in \{1, \dots, n\} : x_i < y_i$. For a nonempty index set $I \subset \{1, \dots, n\}$ we denote by $|I|$ the number of elements of I. We write y_I for restriction $y_I := (y_i)_{i \in I}$ of vectors $y \in \mathbb{R}^n_+$. Let P_I denote the projection of $\mathbb{R}^n_+$ onto $\mathbb{R}^{|I|}_+$ and R_I be the anti-projection $\mathbb{R}^{|I|}_+ \to \mathbb{R}^n_+$, defined by

$$x \mapsto \sum_{k=1}^{|I|} x_k e_{i_k},$$

where $\{e_k\}_{k=1,\dots,n}$ denotes the standard basis in $\mathbb{R}^n$ and $I = \{i_1, \dots, i_{|I|}\}$, with $i_k < i_{k+1}$, $k = 1, \dots, |I| - 1$.

For a function $v : \mathbb{R}_+ \mapsto \mathbb{R}^m$ we define its restriction to the interval $[s_1, s_2]$ by

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$$v_{[s_1,\, s_2]}(t) = \begin{cases} v(t), & \text{if } t \in [s_1, s_2], \\ 0, & \text{otherwise.} \end{cases}$$

A function $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. Note that for $\alpha \in \mathcal{K}_\infty$ α^{-1} always exists and $\alpha^{-1} \in \mathcal{K}_\infty$.

A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is non-increasing and tends to zero for $t \to \infty$.

By id we denote the identity operator in an appropriate space. Let $|\cdot|$ denote some norm in $\mathbb{R}^n$, and let in particular $|x|_{\max} = \max_i |x_i|$ be the maximum norm. The essential supremum norm of a measurable function ϕ is denoted by $\|\phi\|_\infty$. L_∞ is the set of measurable functions for which this norm is finite.

Consider the system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \tag{1}$$

and assume that the system is forward complete. This means that for all initial values $x(0) \in \mathbb{R}^n$ and all essentially bounded measurable inputs u solutions exist for all positive times. Assume also that for any initial value $x(0)$ and input u the solution is unique.

The following notions of stability are used in the remainder of the paper.

Definition 2.1: The system (1) is *input-to-state stable* (ISS), if there exist functions β of class $\mathcal{KL}$ and γ of class $\mathcal{K}$, such that the inequality

$$|x(t)| \le \beta(|x(0)|, t) + \gamma(\|u\|_\infty) \tag{2}$$

holds for all initial conditions $x(0) \in \mathbb{R}^n$ and inputs $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$, $t \ge 0$.

Definition 2.2: The system (1) is *globally stable* (GS), if there exist functions σ, $\hat{\gamma}$ of class $\mathcal{K}$, such that the inequality

$$|x(t)| \le \sigma(|x(0)|) + \hat{\gamma}(\|u\|_\infty) \tag{3}$$

holds for all $x(0) \in \mathbb{R}^n$, $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$, $t \ge 0$.

Definition 2.3: The system (1) has the *asymptotic gain* (AG) property, if there exists a function $\widetilde{\gamma}$ of class $\mathcal{K}$, such that the inequality

$$\limsup_{t\to\infty} |x(t)| \le \widetilde{\gamma}(\|u\|_\infty) \tag{4}$$

holds for all $x(0) \in \mathbb{R}^n$ and $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$.

Remark 2.4: Instead of sum of the terms on the right-hand side of (2) one can take the maximum of these terms:

$$|x(t)| \le \max\{\beta(|x(0)|, t), \gamma(\|u\|_\infty)\}. \tag{5}$$

This leads to an equivalent definition of ISS. Note that functions β, γ in (5) are in general different from those in (2). A similar equivalent definition can be written for the GS systems.

Remark 2.5: In [11] it was shown that a system is ISS if and only if it is GS and has the AG property.

Consider n interconnected control systems given by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n, u_1) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n, u_n) \end{aligned} \tag{6}$$

where $x_i \in \mathbb{R}^{N_i}$, $u_i \in \mathbb{R}^{m_i}$ and the functions $f_i : \mathbb{R}^{\sum_{j=1}^n N_j + m_i} \to \mathbb{R}^{N_i}$ are continuous and for all $r \in \mathbb{R}$ are locally Lipschitz continuous in $x = ({x_1}^T, \ldots, {x_n}^T)^T$ uniformly in u_i for $|u_i| \le r$.

The interconnection (6) can be written as (1) with $x = (x_1, \ldots, x_n)^T$, $u = (u_1, \ldots, u_n)^T$ and

$$f(x, u) = \begin{pmatrix} f_1(x_1, \ldots, x_n, P_1(u)) \\ \vdots \\ f_n(x_1, \ldots, x_n, P_n(u)) \end{pmatrix}.$$

If we consider individual subsystems, we treat the state $x_j, j \ne i$ as an independent input for the ith subsystem.

Let subsystems of (6) be ISS, i.e., there exist functions β_i of class $\mathcal{KL}$, $\gamma_{ij}, \gamma_i \in \mathcal{K}_\infty \cup \{0\}$ such that for all initial values $x_i(0)$ and inputs $u \in \mathbb{R}^m$ there exists a unique solution $x_i(t)$ satisfying for all $t \ge 0$

$$|x_i(t)| \le \beta_i(|x_i(0)|, t) + \sum_{j=1}^n \gamma_{ij}(\|x_{j[0,t]}\|_\infty) + \gamma_i(\|u\|_\infty) \tag{7}$$

for $i = 1, \ldots, k$ and

$$|x_i(t)| \le \max\{\beta_i(|x_i(0)|, t), \max_j\{\gamma_{ij}(\|x_{j[0,t]}\|_\infty)\}, \gamma_i(\|u\|_\infty)\} \tag{8}$$

for $i = k+1, \ldots, n$, where $k \in \{0, \ldots, n\}$.

We say that the gains in (7) are of sum type and the gains in (8) are of max type.

Since ISS implies GS and AG property, there exist functions $\sigma_i, \hat{\gamma}_{ij}, \hat{\gamma}_i \in \mathcal{K} \cup \{0\}$, such that for any initial value $x_i(0)$ and input $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ there exists a unique solution $x_i(t)$ and

$$|x_i(t)| \le \sigma_i(|x_i(0)|, t) + \sum_{j=1}^n \hat{\gamma}_{ij}(\|x_{j[0,t]}\|_\infty) + \hat{\gamma}_i(\|u\|_\infty) \tag{9}$$

for $i = 1 \ldots, k$ and

$$|x_i(t)| \le \max\{\sigma_i(|x_i(0)|), \max_j\{\hat{\gamma}_{ij}(\|x_{j[0,t]}\|_\infty)\}, \hat{\gamma}_i(\|u\|_\infty)\} \tag{10}$$

for $i = k+1, \ldots, n$ for all $t \ge 0$ and there exist functions $\widetilde{\gamma}_{ij}, \widetilde{\gamma}_i \in \mathcal{K} \cup \{0\}$, such that for any initial value $x_i(0)$ and input $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ there exists a unique solution $x_i(t)$ and

$$\limsup_{t\to\infty} |x_i(t)| \le \sum_{j=1}^n \widetilde{\gamma}_{ij}(\|x_{j[0,t]}\|_\infty) + \widetilde{\gamma}_i(\|u\|_\infty) \tag{11}$$

for $i = 1, \ldots, k$ and

$$\limsup_{t\to\infty} |x_i(t)| \le \max\{\max_j\{\widetilde{\gamma}_{ij}(\|x_{j[0,t]}\|_\infty)\}, \widetilde{\gamma}_i(\|u\|_\infty)\} \tag{12}$$

for $i = k+1, \ldots, n$.

Let us collect the gains γ_{ij} in a matrix $\Gamma = (\gamma_{ij})_{n\times n}$, denoting $\gamma_{ii} \equiv 0,\ i = 1,\dots,n$. The operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is defined by:

$$\Gamma(s) := \begin{pmatrix} \gamma_{12}(s_2) + \cdots + \gamma_{1n}(s_n) \\ \vdots \\ \gamma_{k1}(s_1) + \cdots + \gamma_{kn}(s_n) \\ \max\{\gamma_{k+1,1}(s_1),\dots,\gamma_{k+1,n}(s_n)\} \\ \vdots \\ \max\{\gamma_{n1}(s_1),\dots,\gamma_{n,n-1}(s_{n-1})\} \end{pmatrix} \tag{13}$$

for $s \in \mathbb{R}^n_+$. Interconnections of such systems were considered in [1] for $k = 0$ and $k = n$. In [9], [4] more general formulations of ISS are considered, which encompass the case studied in this paper. However, in these references the specific results available for the structure considered here are not provided.

Our main question is whether the interconnection (6) is ISS from u to x. It is known that even if all subsystems are ISS their interconnection need not be ISS. Recall the small gain conditions for $k = 0$ and $k = n$ assuring ISS property of such interconnections from [1]:

$$\Gamma \circ D(s) \not\geq s,\ \forall s \in \mathbb{R}^n_+\backslash\{0\} \tag{14}$$

for some $D := \mathrm{diag}_n(\mathrm{id} + \alpha)$, $\alpha \in \mathcal{K}_\infty$ for $k = n$ and

$$\Gamma(s) \not\geq s,\ \forall s \in \mathbb{R}^n_+\backslash\{0\} \tag{15}$$

for $k = 0$. In case $0 < k < n$ we can use

$$\max_{i=1,\dots,n}\{x_i\} \leq \sum_{i=1}^{n} x_i \leq n \max_{i=1,\dots,n}\{x_i\} \tag{16}$$

to pass to the situation with $k = 0$ or $k = n$. But this leads to more conservative gains. To avoid this conservativeness we are going to obtain a new small gain condition for $0 < k < n$. Expressions in (14), (15) prompt us to consider the following small gain condition: For some $\alpha \in \mathcal{K}_\infty$ let $D_k(\alpha) := \mathrm{diag}_k(\mathrm{id}+\alpha)$ and id_{n-k} be the identity on $\mathbb{R}^{n-k}$. Define

$$D := \begin{pmatrix} D_k(\alpha) & 0 \\ 0 & id_{n-k} \end{pmatrix}. \tag{17}$$

The small-gain condition on the operator Γ is then

$$\Gamma \circ D(s) \not\geq s,\ \forall s \in \mathbb{R}^n_+\backslash\{0\}, \tag{18}$$

where the map $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is defined by

$$D(s) := ((\mathrm{id} + \alpha)(s_1),\dots,(\mathrm{id} + \alpha)(s_k), s_{k+1},\dots,s_n)^T.$$

In this paper we will prove that this small gain condition guarantees the ISS-property of the interconnection (6).

III. AUXILIARY RESULTS

Before we proceed to main theorems we prove some auxiliary results for the operators satisfying small gain condition (18).

Lemma 3.1: The small gain condition (18) is equivalent to $D \circ \Gamma(v) \not\geq v$ for all $v \in \mathbb{R}^n_+\backslash\{0\}$.

Proof: Note that D is always invertible and

$$D^{-1}(v) := \begin{pmatrix} (\mathrm{id} + \alpha)^{-1} \circ (v_1) \\ \vdots \\ (\mathrm{id} + \alpha)^{-1} \circ (v_k) \\ v_{k+1} \\ \vdots \\ v_n \end{pmatrix}$$

For every $v \in \mathbb{R}^n_+$ there exists a unique $w \in \mathbb{R}^n_+$ such that $v = D(w)$ and vice versa. By monotonicity of D and D^{-1} we have $D \circ \Gamma(v) \not\geq v$ if and only if $\Gamma(v) \not\geq D^{-1}(v)$. For any $w \in \mathbb{R}^n_+$ define $v = D(w)$. Then $\Gamma \circ D(w) \not\geq w$. This proves the equivalence. ■

For convenience let us introduce $\mu : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\mu(w, v) := \begin{pmatrix} w_1 + v_1 \\ \vdots \\ w_k + v_k \\ \max\{w_{k+1}, v_{k+1}\} \\ \vdots \\ \max\{w_n, v_n\} \end{pmatrix},\ w \in \mathbb{R}^n_+,\ v \in \mathbb{R}^n_+.$$

The following counterpart of Lemma 13 in [1] provides the main technical step in the proof of the main results.

Lemma 3.2: Let Γ satisfy $\Gamma \circ D(s) \not\geq s$ for any $s \in \mathbb{R}^n_+\backslash\{0\}$. Then there exists a $\phi \in \mathcal{K}_\infty$ such that for all $w, v \in \mathbb{R}^n_+$,

$$w \leq \mu(\Gamma(w), v) \tag{19}$$

implies $\|w\| \leq \phi(\|v\|)$.

Proof: Fix $v \in \mathbb{R}^n_+$. We first show, that for those $w \in \mathbb{R}^n_+$ satisfying (19) at least some components have to be bounded.

Let $\widetilde{D} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be defined by

$$\widetilde{D}(s) :=$$

$$\left(s_1 + \alpha^{-1}(s_1),\dots,s_k + \alpha^{-1}(s_k), s_{k+1},\dots,s_n\right)^T,\ s \in \mathbb{R}^n_+$$

and let

$$s^* := \widetilde{D}(v).$$

Assume there exists $w = (w_1,\dots,w_n)^T$ satisfying (19) and such that $w_i > s^*_i,\ i = 1,\dots,n$. In particular, for $i = 1,\dots,k$ we have

$$s^*_i < w_i \leq \gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n) + v_i \tag{20}$$

and hence from the definition of s^*:

$$s^*_i = v_i + \alpha^{-1}(v_i) < \gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n) + v_i.$$

Then

$$v_i < \alpha(\gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n)).$$

From (20) it follows

$$\begin{aligned} w_i &\leq \gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n) + v_i \\ &< (id + \alpha) \circ (\gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n)). \end{aligned} \tag{21}$$

Similarly, by the construction of w we have for $i = k+1, \ldots, n$

$$s_i^* < w_i \le \max\{\gamma_{i1}(w_1), \ldots, \gamma_{in}(w_n), v_i\}. \quad (22)$$

From the definition of s^* we have

$$s_i^* = v_i < w_i \le \max\{\gamma_{i1}(w_1), \ldots, \gamma_{in}(w_n), v_i\}. \quad (23)$$

Hence,

$$w_i \le \max\{\gamma_{i1}(w_1), \ldots, \gamma_{in}(w_n)\}. \quad (24)$$

From (21), (24) we get

$$w \le \begin{pmatrix} (\mathrm{id}+\alpha)\circ(\gamma_{12}(w_2)+\ldots+\gamma_{1n}(w_n)) \\ \vdots \\ (\mathrm{id}+\alpha)\circ(\gamma_{k1}(w_1)+\ldots+\gamma_{kn}(w_n)) \\ \max\{\gamma_{k+11}(w_1), \ldots, \gamma_{k+1n}(w_n)\} \\ \vdots \\ \max\{\gamma_{n1}(w_1), \ldots, \gamma_{nn-1}(w_{n-1})\} \end{pmatrix},$$

i.e., $w \le D\circ\Gamma(w)$. This contradicts the condition $\Gamma\circ D(w) \not\ge w$ of our lemma which is equivalent to $D\circ\Gamma(w) \not\ge w$ by Lemma 3.1. Hence some components of w are bounded. Iteratively we will prove that all components of w are bounded.

Let us denote $s^1 := s^*$ from the first step. We have already proved that $w \not> s^1$ for all w satisfying (19). Fix such a w, then there exists an index set $I_1 \subset \{1, \ldots, n\}$, possibly depending on w, such that $w_i > s_i^1$, $i \in I_1$ and $w_i \le s_i^1$, for $i \in I_1^c = \{1, \ldots, n\} \setminus I_1$. Note that by the first step I_1^c is nonempty. We now renumber the coordinates so that

$$w_i > s_i^1 \text{ and } w_i \le \sum_{j=1}^{n} \gamma_{ij}(w_j) + v_i,\ i = 1, \ldots, k_1, \quad (25)$$

$$w_i > s_i^1 \text{ and } w_i \le \max\{\max_j \gamma_{ij}(w_j), v_i\},\ i = k_1+1, \ldots, n_1, \quad (26)$$

$$w_i \le s_i^1 \text{ and } w_i \le \sum_{j=1}^{n} \gamma_{ij}(w_j) + v_i, \quad (27)$$

$i = n_1+1, \ldots, n_1+1+k_2$

$$w_i \le s_i^1 \text{ and } w_i \le \max\{\max_j \gamma_{ij}(w_j), v_i\}, \quad (28)$$

$i = n_1+k_2+2, \ldots, n$, where $n_1 = |I_1|$, $k_1+k_2 = k$. Using (27), (28) in (25), (26) we get:

$$w_i \le \sum_{j=1}^{n_1} \gamma_{ij}(w_j) + \sum_{j=n_1+1}^{n} \gamma_{ij}(s_j^1) + v_i, \quad (29)$$

$i = 1, \ldots, k_1$, and

$$w_i \le \max\{\max_{j=1,\ldots,n_1} \gamma_{ij}(w_j), \max_{j=n_1+1,\ldots,n} \gamma_{ij}(s_j^1), v_i\}, \quad (30)$$

$i = k_1+1, \ldots, n_1$. Define v^1 by

$$v_i^1 := \sum_{j=n_1+1}^{n} \gamma_{ij}(s_j^1) + v_i,\ i = 1, \ldots, k_1 \text{ and}$$

$$v_i^1 := \max\{\max_{j=n_1+1,\ldots,n} \gamma_{ij}(s_j^1), v_i\},\ i = k_1+1, \ldots, n_1.$$

Now (29), (30) take the form:

$$w_i \le \sum_{j=1}^{n_1} \gamma_{ij}(w_j) + v_i^1,\ i = 1, \ldots, k_1, \quad (31)$$

$$w_i \le \max\{\max_{j=1,\ldots,n_1} \gamma_{ij}(w_j), v_i^1\},\ i = k_1+1, \ldots, n_1. \quad (32)$$

Let us represent Γ as $\Gamma = \begin{pmatrix} \Gamma_{I_1I_1} & \Gamma_{I_1I_1^c} \\ \Gamma_{I_1^cI_1} & \Gamma_{I_1^cI_1^c} \end{pmatrix}$ and let us define the maps $\Gamma_{I_1I_1} : \mathbb{R}_+^{n_1} \to \mathbb{R}_+^{n_1}$, $\Gamma_{I_1I_1^c} : \mathbb{R}_+^{n-n_1} \to \mathbb{R}_+^{n_1}$, $\Gamma_{I_1^cI_1} : \mathbb{R}_+^{n_1} \to \mathbb{R}_+^{n-n_1}$ and $\Gamma_{I_1^cI_1^c} : \mathbb{R}_+^{n-n_1} \to \mathbb{R}_+^{n-n_1}$ analogous to the Γ. Let

$$D_{I_1}(s) := ((id+\alpha)(s_1), \ldots, (id+\alpha)(s_k), s_{k+1}, \ldots, s_{n_1})^T.$$

From $\Gamma\circ D(s) \not\ge s$ for all $s \ne 0$, $s \in \mathbb{R}_+^n$ it follows that $\Gamma_{I_1I_1}\circ D_{I_1}(z) \not\ge z$ for all $z \ne 0$, $z \in \mathbb{R}_+^{n_1}$. Using the same approach as for $w \in \mathbb{R}_+^n$ it can be proved that some components of $w^1 = (w_1, \ldots, w_{n_1})^T$ are bounded.

We proceed inductively, defining

$$I_{j+1} \subsetneqq I_j, \quad I_{j+1} := \{i \in I_j : w_i > s_i^{j+1}\}, \quad (33)$$

with $I_{j+1}^c := I \setminus I_{j+1}$ and

$$s^{j+1} := \widetilde{D}_{I_j} \circ (\mu^j(\Gamma_{I_jI_j^c}(s_{I_j^c}^j), v_{I_j})), \quad (34)$$

where $\widetilde{D}_{I_j}$ is defined analogous to $\widetilde{D}$, the map $\Gamma_{I_jI_j^c} : \mathbb{R}_+^{n-n_j} \to \mathbb{R}_+^{n_j}$ acts analogous to Γ for vectors of the corresponding dimension, $s_{I_j^c}^j = (s_i^j)_{i\in I_j^c}$ is the restriction defined in the preliminaries and μ^j is appropriately defined similar to the definition of μ.
The nesting (33), (34) will end after at most $n-1$ steps: there exists a maximal $l \le n$, such that

$$\{1, \ldots, n\} \supsetneqq I_1 \supsetneqq \ldots \supsetneqq I_l \ne \emptyset$$

and all components of w_{I_l} are bounded by the corresponding components of s^{l+1}. Let

$$s_\varsigma := \max\{s^*, R_{I_1}(s^2), \ldots, R_{I_l}(s^{l+1})\}$$

$$:= \begin{pmatrix} \max\{(s^*)_1, (R_{I_1}(s^2))_1, \ldots, (R_{I_l}(s^{l+1}))_1\} \\ \vdots \\ \max\{(s^*)_n, (R_{I_1}(s^2))_n, \ldots, (R_{I_l}(s^{l+1}))_n\} \end{pmatrix}$$

where R_{I_j} denotes the anti-projection $\mathbb{R}_+^{|I_j|} \to \mathbb{R}_+^n$ defined above.
Let the n-fold composition $M\circ\ldots\circ M$ be denoted by $[M]^n$. By the definition of μ for all $v \in \mathbb{R}_+^n$ it holds

$$0 \le v \le \mu(\Gamma, \mathrm{id})(v) := \mu(\Gamma(v), v).$$

Applying $\widetilde{D}$ we have

$$\begin{aligned} &0 \le v \le \widetilde{D}(v) \le \widetilde{D}\circ(\mu(\Gamma, \mathrm{id}))(v) \le \ldots \\ &\le [\widetilde{D}\circ\mu(\Gamma, \mathrm{id})]^n(v). \end{aligned} \quad (35)$$

From (34) and (35) for w satisfying (19) we have

$$w \le s_\varsigma \le [\widetilde{D}\circ\mu(\Gamma, \mathrm{id})]^n(v).$$

The term on the right-hand side does not depend on any particular choice of nesting of the index sets. Hence every w satisfying (19) also satisfies

$$w \leq [\widetilde{D} \circ \mu(\Gamma, \mathrm{id})]^n (|v|_{\max}, \ldots, |v|_{\max})^T$$

and taking the maximum-norm on both sides yields

$$|w|_{\max} \leq \phi(|v|_{\max})$$

for some function ϕ of class $\mathcal{K}_\infty$. For example, ϕ can be chosen as

$$\begin{aligned} \phi(t) := \quad & \max\{([\widetilde{D} \circ \mu(\Gamma, \mathrm{id})]^n (t, \ldots, t))_1, \ldots, \\ & ([\widetilde{D} \circ \mu(\Gamma, \mathrm{id})]^n (t, \ldots, t))_n\}. \end{aligned}$$

This completes the proof of the lemma. ∎

IV. MAIN RESULTS

Now we turn back to the question of stability. In order to prove ISS of (6) we use the same approach as in [1]. The main idea is to prove that the system is GS and AG and then to use the result of [11] by which AG and GS systems are ISS.

So, let us prove at first small gain theorems for GS and AG of the system.

Theorem 4.1: Assume that each subsystem of (6) is GS. If there exists D as in (17) such that $\Gamma \circ D(x) \not\geq x$ for all $x \neq 0$ is satisfied, then the system (1) is GS.

Proof: Let us take the supremum over $\tau \in [0, t]$ on both sides of (9), (10). For $i = 1, \ldots, k$ we have

$$\|x_{i[0,t]}\|_\infty \leq \sigma_i(|x_i(0)|) + \sum_{j=1}^{n} \gamma_{ij}(\|x_{j[0,t]}\|_\infty) + \gamma_i(\|u\|_\infty) \tag{36}$$

and for $i = k+1, \ldots, n$ it follows

$$\begin{aligned} \|x_{i[0,t]}\|_\infty \leq \max\{&\sigma_i(|x_i(0)|), \max_j\{\gamma_{ij}(\|x_{j[0,t]}\|_\infty)\}, \\ &\gamma_i(\|u\|_\infty)\} \end{aligned} \tag{37}$$

Let us denote $w = \left(\|x_{1[0,t]}\|_\infty, \ldots, \|x_{n[0,t]}\|_\infty\right)^T$, $(\Gamma)_{ij} = \gamma_{ij}$,

$$\begin{aligned} v \quad &= \quad \begin{pmatrix} \sigma_1(|x_1(0)|) + \gamma_1(\|u\|_\infty) \\ \vdots \\ \sigma_k(|x_k(0)|) + \gamma_k(\|u\|_\infty) \\ \max\{\sigma_{k+1}(|x_{k+1}(0)|), \gamma_{k+1}(\|u\|_\infty)\} \\ \vdots \\ \max\{\sigma_n(|x_n(0)|), \gamma_n(\|u\|_\infty)\} \end{pmatrix} \\ &= \quad \mu(\sigma(|x(0)|), \gamma(\|u\|_\infty)) \end{aligned}$$

From (36), (37) we obtain $w \leq \mu(\Gamma(w), v)$. Then by Lemma 3.2 there exists $\phi \in \mathcal{K}_\infty$ such that

$$\begin{aligned} \|x_{[0,t]}\|_\infty \quad &\leq \quad \phi(\|\mu(\sigma(|x(0)|), \gamma(\|u\|_\infty))\|) \\ &\leq \quad \phi(\|\sigma(|x(0)|) + \gamma(\|u\|_\infty)\|) \\ &\leq \quad \phi(2\|\sigma(|x(0)|)\|) + \phi(2\|\gamma(\|u\|_\infty)\|) \end{aligned} \tag{38}$$

for some class $\mathcal{K}$ function ϕ and all $t > 0$. Hence for every initial condition and essentially bounded input u the solution of the system (1) exists for all $t \geq 0$ and is uniformly bounded, since the right-hand side of (38) does not depend on t. The estimate for GS is then given by (38). ∎

Theorem 4.2: Assume that each subsystem of (6) has the AG property and that solutions of the system (1) exist for all positive times and are uniformly bounded. If there exists a D as in (17) such that $\Gamma \circ D(x) \not\geq x$ for all $x \neq 0$, then system (1) satisfies the AG property.

Remark 4.3: The existence of solutions for all times is essential, otherwise the assertion is not true. See Example 14 in [1].

Proof: Let τ be an arbitrary initial time. From the definition of the AG property we have for $i = 1, \ldots, k$

$$\limsup_{t \to \infty} |x_i(t)| \leq \sum_{j=1}^{n} \gamma_{ij}(\|x_{j[\tau,\infty]}\|_\infty) + \gamma_i(\|u\|_\infty) \tag{39}$$

and for $i = k+1, \ldots, n$

$$\limsup_{t \to \infty} |x_i(t)| \leq \max\{\max_j\{\gamma_{ij}(\|x_{j[\tau,\infty]}\|_\infty)\}.\gamma_i(\|u\|_\infty)\} \tag{40}$$

Since all solutions of (6) are bounded Lemma 7 from [1] can be applied and we get that

$$\limsup_{t \to \infty} |x_i(t)| = \limsup_{\tau \to \infty} \|x_{i[\tau,\infty]}\|_\infty) =: l_i(x_i), i = 1, \ldots, n.$$

By this property from (39), (40) and Lemma II.1 in [11] it follows that

$$l_i(x_i) \leq \sum_{j=1}^{n} \gamma_{ij}(l_j(x_j)) + \gamma_i(\|u\|_\infty)$$

for $i = 1, \ldots, k$ and

$$l_i(x_i) \leq \max\{\max_j\{\gamma_{ij}(l_j(x_j))\}, \gamma_n(\|u\|_\infty)\}$$

for $i = k+1, \ldots, n$. Using Lemma 3.2 we conclude

$$\limsup_{t \to \infty} \|x(t)\| \leq \phi(\|u\|_\infty) \tag{41}$$

for some ϕ of class $\mathcal{K}$, which is the desired AG property. ∎

Theorem 4.4: Assume that each subsystem of (6) is ISS. If there exists a D as in (17) such that $\Gamma \circ D(x) \not\geq x$ for all $x \neq 0$, then system (1) is ISS.

Proof:

From Theorem 1 in [11] each subsystem is GS. By Theorem 4.1 the whole system (1) is GS. This implies that solution of (1) exists for all times and is uniformely bounded.

From Theorem 1 in [11] each subsystem has the AG property. Applying Theorem 4.2 the whole system (1) has the AG property.

This implies that (1) is ISS by Theorem 1 in [11]. ∎

Example 4.5: To demonstrate the advantages of Theorem 4.4 we consider the interconnected system (6) with $n = 3$, $k = 1$ where each subsystem is ISS:

$$\begin{aligned} &|x_1(t)| \leq \beta_1(|x(0)|) + \gamma_{13}(||x_{3[0,t]}||_\infty) + \gamma_1(\|u\|_\infty) \\ &|x_2(t)| \leq \max\{\beta_2(|x(0)|), \gamma_{21}(||x_{1[0,t]}||_\infty), \\ &\qquad\qquad \gamma_{23}(||x_{3[0,t]}||_\infty), \gamma_2(\|u\|_\infty)\} \\ &|x_3(t)| \leq \max\{\beta_3(|x(0)|), \gamma_{32}(||x_{2[0,t]}||_\infty), \gamma_3(\|u\|_\infty)\} \end{aligned} \tag{42}$$

with gains given by $\gamma_{13}(t) = (id+\rho)^{-1}(t), \rho \in \mathcal{K}_\infty$, $\gamma_{21}(t) = t$, $\gamma_{23}(t) = t$ and $\gamma_{32}(t) = t(1-e^{-t})$, $t \geq 0$. In this case we have the following

$$\Gamma = \begin{pmatrix} 0 & 0 & \gamma_{13} \\ \gamma_{21} & 0 & \gamma_{23} \\ 0 & \gamma_{32} & 0 \end{pmatrix}$$

Then the small gain condition (18) becomes

$$\begin{pmatrix} \gamma_{13}(s_3) \\ \max\{\gamma_{21} \circ (\mathrm{id}+\alpha)(s_1), \gamma_{23}(s_3)\} \\ \gamma_{32}(s_2) \end{pmatrix} \not\geq \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

for all $s \in \mathbb{R}^3_+ \backslash \{0\}$, $t \geq 0$. This condition is equivalent to

$$(\mathrm{id}+\alpha) \circ \gamma_{13} \circ \gamma_{32} \circ \gamma_{21}(t) < t \tag{43}$$

and simultaneously

$$\gamma_{23} \circ \gamma_{32}(t) < t \tag{44}$$

for all $t > 0$.
For $\alpha = \rho$ the inequality (43) is satisfied:

$$(\mathrm{id}+\alpha) \circ (id+\rho)^{-1} \circ (t(1-e^{-t}))$$
$$= (\mathrm{id}+\rho) \circ (id+\rho)^{-1} \circ (t(1-e^{-t})) = t(1-e^{-t}) < t.$$

The inequality (44) is also satisfied:

$$t(1-e^{-t}) < t.$$

Then by Theorem 4.4 system (1) is ISS.

In order to apply results from [1] we need to use (16) in (42). Then we obtain estimations of trajectories by

$$\begin{aligned} |x_1(t)| &\leq \beta_1(|x(0)|) + \gamma_{13}(||x_{3[0,t]}||_\infty) + \gamma_1(\|u\|_\infty) \\ |x_2(t)| &\leq \beta_2(|x(0)|) + \gamma_{21}(||x_{1[0,t]}||_\infty) \\ &\quad + \gamma_{23}(||x_{3[0,t]}||_\infty) + \gamma_2(\|u\|_\infty) \\ |x_3(t)| &\leq \beta_3(|x(0)|) + \gamma_{32}(||x_{2[0,t]}||_\infty) + \gamma_3(\|u\|_\infty) \end{aligned} \tag{45}$$

The small gain condition from [1] is: there exist $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, 3$ such that

$$\begin{pmatrix} \gamma_{13} \circ (\mathrm{id}+\alpha_1)(s_3) \\ \gamma_{21} \circ (\mathrm{id}+\alpha_2)(s_1) + \gamma_{23} \circ (\mathrm{id}+\alpha_2)(s_3) \\ \gamma_{32} \circ (\mathrm{id}+\alpha_3)(s_2) \end{pmatrix} \not\geq \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \tag{46}$$

for all $s \in \mathbb{R}^3_+ \backslash \{0\}$.

This condition implies for $s = \begin{pmatrix} 0 \\ t \\ \gamma_{32} \circ (\mathrm{id}+\alpha_3)(t) \end{pmatrix}$ and $t \geq 0$, that

$$\gamma_{23} \circ (\mathrm{id}+\alpha_2) \circ \gamma_{32} \circ (\mathrm{id}+\alpha_3)(t) < t$$

or the even weaker condition

$$\gamma_{23} \circ (\mathrm{id}+\alpha_2) \circ \gamma_{32}(t) < t. \tag{47}$$

Suppose such an α_2 exists. Then

$$\gamma_{23} \circ (\mathrm{id}+\alpha_2) \circ \gamma_{32}(t) = (\mathrm{id}+\alpha_2)(t(1-e^{-t})) < t.$$

Hence,

$$(t(1-e^{-t})) + \alpha_2(t(1-e^{-t})) < t \text{ or}$$

$$\alpha_2(t(1-e^{-t})) < te^{-t}.$$

This leads to a contradiction, since $\lim_{t\to\infty} \alpha_2(t(1-e^{-t})) = +\infty$ and $\lim_{t\to\infty} te^{-t} = 0$. It follows that there are no $\alpha_2 \in \mathcal{K}_\infty$ such that (47) is satisfied. So by this small gain condition we cannot conclude whether the interconnection is ISS.

A similar argument applies, if the ISS formulation is transformed to the maximum formulation throughout. Again, an inconclusive formulation results.

V. CONCLUSIONS

We have considered several ISS systems. The gains of these systems are defined in two different ways. This kind of interconnections is more general then in [1]. A new small gain condition assuring the ISS property of such interconnection is proved in this paper. An example shows the effectiveness and advantage of this condition in comparison to known results.

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