On a small gain theorem for ISS networks in dissipative Lyapunov form

Sergey Dashkovskiy, Hiroshi Ito and Fabian Wirth

***Abstract*— In this paper we consider several interconnected ISS systems supplied with ISS Lyapunov functions defined in the dissipative form. Our aim is to construct an ISS Lyapunov function for the interconnection. We provide a condition of a small gain type under which this construction is possible and describe a method of an explicit construction of such an ISS Lyapunov function.**

I. INTRODUCTION

Interconnections of nonlinear systems appear in many applications such as logistic problems, biologic systems, power networks and others. Stability analysis of these systems is an important issue for their performance and control. Such interconnections can be studied in different frameworks such as passivity, dissipativity [21], [7], [15], [17], input-to-state stability (ISS) [18] and others. Since we consider systems with inputs we will use the notion of ISS for our purposes. There are several equivalent ways to define this property. Originally [18] it was defined in terms of a bound for the trajectories of a system, where the bound depends on the initial condition and the input function. This property can be equivalently stated in terms of an ISS-Lyapunov function. The latter formulation can again be defined in two essentially equivalent ways: in the so-called implication form and with the help of a dissipation inequality and a supply rate, see [19] for details and discussions of the different ISS formulations. In this paper we concentrate on the dissipative ISS formulation it is our aim to derive a small gain result for general interconnected systems in this framework. This complements recent results in [4], [6], where small gain results have been achieved in the trajectory formulation as well as for the implication form of the ISS Lyapunov formulation.

Considering the ISS property of the interconnections of two ISS systems, the pioneering papers used the definition in terms of trajectories [13] and Lyapunov functions with the definition in the implication form [12]. These results were recently extended to the case of interconnections of n systems, see [4], [6], [14], [5]. A small gain theorem for two systems with ISS-Lyapunov functions satisfying the dissipative inequality was obtained in [8]. It is worth noting that this definition has the advantage that it unifies the definition of ISS and integral ISS (iISS) systems. The latter set of systems is larger and contains the ISS systems as a subset. The small gain theorem for two iISS systems was proved in [9], [11]. Moreover the construction of the corresponding Lyapunov function is given in a smooth way in contrast to the constructions given in [12] and [6], [5].

In this note, we consider n ISS systems with given ISS-Lyapunov functions defined by dissipative inequalities. It is of interest 1) to obtain a corresponding small gain theorem in the dissipation formulation and 2) to construct an ISS-Lyapunov function. Here we will make an essential step in this direction. Namely, for general ISS systems, this paper achieves 1) by constructing Lipschitz continuous ISS-Lyapunov function for the interconnection of n systems. A smooth construction is shown under stronger assumptions. For a special class of dissipation inequalities, the construction is given under essentially weaker assumptions.

The paper is organized as follows. The ensuing section introduces the necessary notations and gives a precise statement of the problem. Section III explains the main idea of our approach in the simpler case of linear supply rate functions. In this case the result follows from an application of the Perron-Frobenius theorem. The idea for the proof of the main results follows a similar pattern. Their discussion are given in Section IV for the nonlinear case. We draw conclusions and outline directions of future work in Section V.

II. PROBLEM STATEMENT

We use the following notation. $(\cdot)^T$ denotes the transposition of a vector. For any vectors $a, b \in \mathbb{R}^n$ the relation $a \geq b$ is defined by $a_i \geq b_i$ for all $i = 1, \ldots, n$. The relations $>, \leq, <$ for vectors are defined in the same manner. That is, we are using the partial order on $\mathbb{R}^n$ induced by the positive orthant $\mathbb{R}^n_+$. The negation of $a \geq b$ is denoted by $a \not\geq b$ and this means that there exists an $i \in \{1, \ldots, n\}$ such that $a_i < b_i$. By $a \cdot b$ we denote the scalar product of two vectors and by $A \circ B$ we denote the composition of operators A and B. To use standard formulations of input-to-state stability, we recall, that a function α is said to be of class $\mathcal{K}$ if α is continuous, $\alpha(0) = 0$ and α is strictly increasing, if in addition it is unbounded, we say it is of class $\mathcal{K}_\infty$. A continuous function $\alpha : [0, \infty) \to [0, \infty)$ is called positive definite if $\alpha(x) = 0$ if and only if $x = 0$.

Consider a finite set of interconnected systems with state $x = \left(x_1^T, \ldots, x_n^T\right)^T$, where $x_i \in \mathbb{R}^{N_i}$, $i = 1, \ldots, n$ and $N := \sum N_i$. For $i = 1, \ldots, n$ the dynamics of the i-th

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S. Dashkovskiy is with the Department of Mathematics and Computer Science, University of Bremen, Germany dsn@math.uni-bremen.de

H. Ito is with the Department of Systems Design and Informatics, Kyushu Institute of Technology, Japan hiroshi@ces.kyutech.ac.jp

F. Wirth is with the Institute of Mathematics, University of Würzburg, Germany wirth@mathematik.uni-wuerzburg.de

subsystem is given by

$$\Sigma_i: \ \dot{x}_i = f_i(x_1, \dots, x_n, u), \tag{1}$$

where $x \in \mathbb{R}^N$, $u \in \mathbb{R}^M$, $f_i : \mathbb{R}^{N+M} \to \mathbb{R}^{N_i}$. For each i we assume unique existence of solutions and forward completeness of Σ_i in the following sense. If we interpret the variables x_j, $j \neq i$, and u as unrestricted inputs, then system (1) is assumed to have a unique solution defined on $[0, \infty)$ for any given initial condition $x_i(0) \in \mathbb{R}^{N_i}$ and any L^∞-inputs $x_j : [0, \infty) \to \mathbb{R}^{N_j}, j \neq i$, and $u : [0, \infty) \to \mathbb{R}^M$. This can be guaranteed for instance by suitable Lipschitz conditions on the f_i. It will be no restriction to assume that all systems have the same (augmented) external input u. This interconnection can be depicted as a network or a graph, see Figure 1.

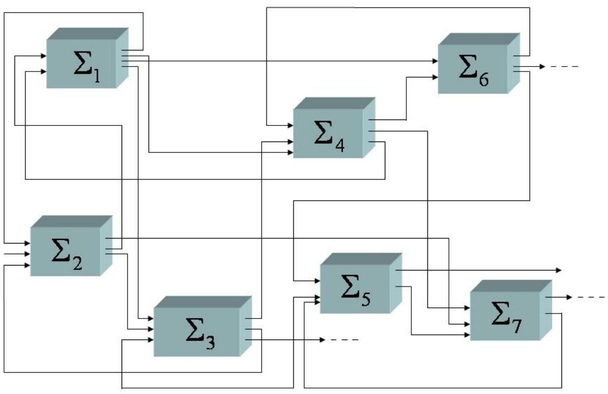

Fig. 1. An interconnection Σ

We write the interconnection of the subsystems (1) as

$$\Sigma: \ \dot{x} = f(x, u), \quad f : \mathbb{R}^{N+M} \to \mathbb{R}^N \tag{2}$$

where $x = (x_1^T, \dots, x_n^T)^T$ is the state of the overall system and $f = (f_1^T, \dots, f_n^T)^T$ is defined correspondingly.

We assume that each of the subsystems in (1) satisfies an ISS condition in the dissipative formulation, i.e., there are Lyapunov functions $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ and functions $\alpha_i, \gamma_{iu} \in \mathcal{K}_\infty$ and $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$, $i, j = 1, \dots, n$ such that

$$\dot{V}_i(x_i) \leq -\alpha_i(V_i(x_i)) + \sum_{i \neq j} \gamma_{ij}(V(x_j)) + \gamma_{iu}(\|u\|) \tag{3}$$

for all $x_i \in \mathbb{R}^{N_i}, i = 1, \dots, n$ and all $u \in \mathbb{R}^M$.

The right hand side in (3) consisting of the functions α_i, γ_{iu} and γ_{ij} is called the supply rate of the dissipation inequality. In the sequel we will always assume that $\gamma_{ii} \equiv 0$. We will also assume that the Lyapunov functions V_i as well as the functions α_i are continuously differentiable, which poses no real restriction.

As in one of our constructions we end up with a locally Lipschitz continuous Lyapunov function for the whole system (2), we note that in case that the V_i are only locally Lipschitz continuous, then it is sufficient to let (3) hold almost everywhere to characterize input-to-state stability.

Note that if in (3) we only require that α_i is an element of the larger set of positive definite functions, then the i-th system is integral input-to-state stable (iISS) [20]. The set of iISS systems is essentially larger than the set of ISS systems. In particular in the iISS framework results of a small gain type and a corresponding Lyapunov construction were developed in [9], [11].

The aim of this paper is to find conditions on the data of the dissipation formulation that guarantee ISS of the interconnected system (2) and to provide a construction of an ISS-Lyapunov function for the interconnection under these conditions. We will also discuss how iISS-results may be obtained in this way for a special class of systems.

III. The linear case

We begin by studying the linear case, because here the conditions are much easier to analyze and it gives an idea how the general procedure should work, even though for practical applications the linearity assumption is very often much too restrictive.

We assume that the ISS-Lyapunov formulation is given in a linear form. Here linear means, that the $\mathcal{K}_\infty$ -functions $\alpha_i, \gamma_{iu} \in \mathcal{K}_\infty$ and $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ $i, j = 1, \dots, n$ are linear. Thus let $\alpha_i > 0, \gamma_{ij} \in [0, \infty)$ be positive resp. nonnegative numbers which represent the corresponding linear functions.

Define the matrices

$$A := \text{diag}\,(\alpha_1, \dots, \alpha_n)\,, \quad \Gamma := (\gamma_{ij})_{i,j=1,\dots,n} \tag{4}$$

and the vectors

$$\begin{aligned} \dot{V}_{vec}(x) &:= \begin{pmatrix} \dot{V}_1(x_1) & \dots & \dot{V}_n(x_n) \end{pmatrix}^T, \\ V_{vec}(x) &:= \begin{pmatrix} V_1(x_1) & \dots & V_n(x_n) \end{pmatrix}^T. \end{aligned} \tag{5}$$

Then the inequalities (3) can be compactly written as

$$\dot{V}_{vec}(x) \leq (-A + \Gamma) V_{vec}(x) + \gamma_u(\|u\|)$$

with the obvious definition of γ_u. In the previous equation $\leq$ is to be interpreted componentwise as defined in the preliminaries.

We note that $(-A + \Gamma)$ is a Metzler matrix, thus a matrix for which Perron-Frobenius type results are available. An overall Lyapunov function may be defined using the following lemma

Lemma III.1 *Consider the matrices A and Γ defined in* (4). *There exists a vector $\mu \in \mathbb{R}^n_+$, $\mu > 0$ such that*

$$\mu^T(-A + \Gamma) < 0 \tag{6}$$

if and only if the following spectral radius condition holds

$$r(A^{-1}\Gamma) < 1. \tag{7}$$

Proof: Note that $A = A^T$ as it is of diagonal form and A is invertible, because in (3) the functions $\alpha_i \in \mathcal{K}_\infty$,

$i = 1, \ldots, n$. Define $\eta := A\mu$, so that $\mu^T = \eta^T A^{-1}$. Then $\mu^T(-A+\Gamma) < 0$ is equivalent to

$$0 > \eta^T A^{-1}(-A+\Gamma) = \eta^T(-I + A^{-1}\Gamma)\,.$$

If $r(A^{-1}\Gamma) < 1$, then by the Perron-Frobenius theorem there exists a vector $\eta > 0$ such that

$$\eta^T(A^{-1}\Gamma) < \eta^T$$

or equivalently $\eta^T(-I + A^{-1}\Gamma) < 0$, as desired.

Conversely, if $r(A^{-1}\Gamma) \geq 1$ then there exists a vector $z \geq 0, z \neq 0$ such that

$$(A^{-1}\Gamma - I)z \geq z\,.$$

We now fix such a vector z. So for any $\eta > 0$ we have

$$\eta^T(A^{-1}\Gamma - I)z \geq \eta^T z \geq 0$$

so that it cannot hold that $\eta^T(-I + A^{-1}\Gamma) < 0$. ■

We now assume that $r(A^{-1}\Gamma) < 1$ and choose a vector $\mu \in \mathbb{R}^n_+$, $\mu > 0$ such that (6) holds. Consider the following candidate for an ISS-Lyapunov function

$$V(x) := \mu^T V_{vec}(x) = \sum_{i=1}^n \mu_i V_i(x_i). \tag{8}$$

Then we have

$$\dot{V}(x) = \mu^T \dot{V}_{vec}(x) \leq \mu^T(-A+\Gamma)V_{vec}(x) + \mu^T\gamma_u(\|u\|)$$

and defining $0 > L := \mu^T(-A+\Gamma)$ we obtain

$$\dot{V}(x) \leq LV_{vec}(x) + \mu^T\gamma_u(\|u\|) \leq -lV(x) + \mu^T\gamma_u(\|u\|)$$

for a positive number defined by $l := -\max_i \frac{L_i}{\mu_i}$. Note that if Γ is irreducible, then $\mu > 0$ may be chosen as an eigenvector of $(-A+\Gamma)$ corresponding to the largest eigenvalue and in this case l is this largest eigenvalue.

Of course, the last equation is the desired dissipative ISS condition and in (8) we have obtained a smooth ISS-Lyapunov function for the interconnection. We have thus proved the following result.

Proposition III.2 *Consider a network of the form* (1), (2) *where each of the subsystems satisfies an ISS condition of the form* (3) *where all the functions occurring in the right hand side are linear. If for the matrices* A, Γ *defined in* (4) *we have*

$$r(A^{-1}\Gamma) < 1\,,$$

then the interconnected system (2) *is ISS with a dissipative ISS Lyapunov function given by* (8).

In the next section we will see how this idea can be used in the general nonlinear case.

IV. Main results

Unfortunately, there is no immediate extension of the previous construction to the nonlinear case. For example the matrices A and Γ contain nonlinear functions instead of numbers and the notions of eigenvalue and spectral radius are no longer available. The construction problem of an ISS-Lyapunov function becomes essentially more difficult. Here we will provide a nonsmooth construction for the nonlinear case. In some applications smoothness of a Lyapunov function can be important in implementation. We will also show a smooth construction for a special case.

We consider the interconnected system (2) and assume that the subsystems (1) are ISS with the ISS-Lyapunov functions V_i satisfying (3) where the supply rate functions can be nonlinear.

First let us note that the condition (7) can be equivalently formulated as $r(\Gamma A^{-1}) < 1$ or written as

$$\Gamma A^{-1} s \not\geq s, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\}.$$

The last condition makes sense also for nonlinear operators defined below.

The data we are working with is defined in (3). We assume from now on that the matrix

$$\Gamma := (\gamma_{ij})_{i,j=1,..,n} \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$$

is irreducible and similarly to the linear case we define the following map on $\mathbb{R}^n_+$

$$\Gamma(s) = \Big(\sum_{j=1}^n \gamma_{1j}(s_j), \ldots, \sum_{j=1}^n \gamma_{nj}(s_j)\Big)^T, \quad s \in \mathbb{R}^n_+ \tag{9}$$

and a diagonal operator acting on $s \in \mathbb{R}^n_+$ by

$$A(s) := \big(\alpha_1(s_1) \quad \ldots \quad \alpha_n(s_n)\big)^T\,. \tag{10}$$

With this notation inequalities (3) can be written in a vector form

$$\dot{V}_{vec} \leq (-A+\Gamma)(V_{vec}(x)) + \gamma_u(||u||) \tag{11}$$

with γ_u defined in the obvious way.

We now reformulate the small gain conditions that were introduced in [4], [6], [16] to make them suitable for our case. The nonrobust version of the small gain condition is given by

$$\Gamma \circ A^{-1}(s) \not\geq s\,, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\}\,. \tag{12}$$

It has been shown, that this condition is not quite sufficient to obtain the desired stability result. Thus the condition we now want to impose is the *robust small gain condition* which requires that for some $D = \operatorname{diag}(\operatorname{id} + \beta_1, \ldots, \operatorname{id} + \beta_n)$, $\beta_i \in \mathcal{K}_\infty$ we have

$$D \circ \Gamma \circ A^{-1}(s) \not\geq s\,, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\}\,. \tag{13}$$

To compare this with the linear case, note that in the linear case both (12) and (13) are equivalent to $r(\Gamma A^{-1}) < 1$ which is in turn equivalent to the condition $r(A^{-1}\Gamma) < 1$. In this sense this is a natural generalization of the linear small gain condition.

One of the central results of [6], [16], [5] is that in the case that Γ is irreducible and (13) holds there exists a continuously differentiable path $\sigma : \mathbb{R}_+ \to \mathbb{R}^n_+$ such that $\sigma(0) = 0$, σ is strictly increasing and unbounded in every component and so that

$$D \circ \Gamma \circ A^{-1}(\sigma(\tau)) < \sigma(\tau)\,, \quad \forall\, \tau > 0. \tag{14}$$

The existence of such a path is central in one of the constructions for a Lyapunov function we will present.

A further condition that will lead to another class of Lyapunov functions is the assumption that there are bounded positive definite functions $\eta_i, i = 1, \ldots, n$, such that $\int_0^\infty \eta_i(\tau)d\tau = \infty$ and so that for $\eta = (\eta_1, \ldots, \eta_n)^T$ we have

$$\eta(s)^T \Gamma \circ A^{-1}(s) < \eta(s)^T s, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\}. \tag{15}$$

Again a robust version of this condition is that there exists a diagonal D as before such that

$$\eta(s)^T D \circ \Gamma \circ A^{-1}(s) < \eta(s)^T s, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\}\,. \tag{16}$$

As we will see, both these conditions allow for the construction of interesting Lyapunov functions.

In the linear case the statement $r(A^{-1}\Gamma) < 1$ is equivalent to (14) as well as to (16). The latter equivalence has been shown in Lemma III.1. Interestingly, one equivalence is obtained by studying right eigenvectors, while the other uses left eigenvectors. We conjecture that (13) does not only imply (14) but also (16). The proof of the known result that (13) implies the existence of the path described in (14) uses the Knaster-Kuratowski-Mazurkiewicz theorem. We suspect that using complementary arguments (16) can be shown.

Theorem IV.1 *Consider the interconnected systems* (1) *and assume that each subsystem has a dissipative ISS-Lyapunov function as in* (3)*. Then*
(i) If the weak small gain condition (15) *is satisfied and if for each $i \in \{1, \ldots, n\}$ and $\lambda_i(\tau) := \eta_i(\alpha_i(\tau)), \tau \in \mathbb{R}_+$ we have*

$$\int_0^\infty \lambda_i(\tau)\, d\tau = \infty\,, \tag{17}$$

then the interconnection (2) is iISS with an iISS-Lyapunov function defined by

$$V(x) := \sum_{i=1}^n \int_0^{V_i(x_i)} \lambda_i(\tau) d\tau\,. \tag{18}$$

(ii) If the robust small gain condition (16) *is satisfied and* (17) *holds then the interconnection (2) is ISS with a Lyapunov function $V(x)$ again defined by* (18)*.*

Proof: First note that (17) guarantees that the function V defined in (18) is a proper function.
(i) Consider the derivative of V along the trajectories of the system (2). Defining $\lambda(V_{vec}) := (\lambda_1(V_1), \ldots, \lambda_n(V_n))$ and using (11) we obtain

$$\begin{aligned}\frac{dV}{dt} &= \lambda(V_{vec})^T \dot{V}_{vec} \\ &< \lambda(V_{vec})^T(-A(V_{vec}) + \Gamma(V_{vec}) + \gamma_u(||u||))\end{aligned} \tag{19}$$

From assumption (15) we have for all $x \neq 0$

$$\eta(A(V_{vec}))^T \Gamma \circ A^{-1} \circ A(V_{vec}) < \eta(A(V_{vec}))^T A(V_{vec})$$

and thus

$$-\lambda(V_{vec})^T A(V_{vec}) + \lambda(V_{vec})^T \Gamma(V_{vec}) < 0.$$

This term can be bounded from above by $-\alpha(V)$ for some positive definite function α. Further recall that the functions η_i are assumed to be bounded. Hence λ_i, $i = 1, \ldots, n$ is also bounded and there exists some positive constant M such that $\gamma_u(||u||) \cdot \lambda(V_{vec}) < M\gamma(||u||)$ for some $\gamma \in \mathcal{K}_\infty$. From (19) it follows that

$$\frac{dV(x)}{dt} \leq -\alpha(V(x)) + \gamma(||u||) \tag{20}$$

and the iISS property of the interconnection follows.
(ii) In case the stronger assumption (16) holds, then in the argument above α can be taken of class $\mathcal{K}_\infty$. Thus in case of (16) the overall system is ISS. ■

Remark IV.2 This theorem reduces the problem of a construction of a Lyapunov function to a geometrical problem of the construction of a continuous curve in $\mathbb{R}^n_+$ parameterized by η_i and satisfying (15) or respectively (16) condition. However the existence and construction of such auxiliary functions η_i may be a nontrivial problem. We hope that the small gain condition

$$D \circ \Gamma \circ A^{-1}(s) \not\geq s\,, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\}$$

implies the existence. An explicit construction of η is a matter of our future research. A similar method was used in [3] and [6], where a construction of a corresponding parameterized curve was performed on the base of the corresponding small gain condition.

The function V in (18) is smooth which is in general a desirable property. In the following we provide a nonsmooth construction of an ISS-Lyapunov function for the interconnection (2) where a corresponding auxiliary function can be explicitly constructed.

Theorem IV.3 *Let the systems given in (1) be ISS in the sense of* (3) *and assume that their supply rate functions are such that the operators A and Γ defined above satisfy the small gain condition (13). Assume further that for $\sigma_1, \ldots, \sigma_n$ given in* (14) *there are constants $0 < c < C$ such that*

$$0 < c < \frac{d}{d\tau}\sigma_i^{-1} \circ \alpha_i(\tau) < C\,, \quad \textit{for all } \tau > 0\,.$$

Then the interconnection (2) is ISS. An ISS-Lyapunov function is given by

$$V(x) := \max_{i=1,\ldots,n} \sigma_i^{-1} \circ \alpha_i(V_i(x_i))\,. \tag{21}$$

Proof: Let us assume for the moment that for a given $x \neq 0$ we have that the maximum in (21) is uniquely attained in the first component $i = 1$, i.e., $V(x) = \sigma_1^{-1} \circ \alpha_1(V_1(x_1))$. Denote by Γ_1 the first row of Γ. We obtain

$$\dot{V}(x) = \frac{d}{dt}\, \sigma_1^{-1} \circ \alpha_1(V_1(x_1)) = \left(\sigma_1^{-1} \circ \alpha_1\right)' (V_1(x_1)) \dot{V}_1(x_1)$$

and

$$\dot{V}_1(x_1) \leq [-\alpha_1(V_1(x_1)) \ + \Gamma_1(V_{vec}(x)) + \gamma_{1u}(\|u\|)]$$

We now denote $z_i = \alpha_i(V_i(x_i)), z := (z_1, \ldots, z_n)^T$ and obtain the following estimate

$$-\alpha_1(V_1(x_1)) + \Gamma_1(V_{vec}(x)) = -z_1 + \Gamma_1 \circ A^{-1}(z)$$

$$= -\sigma_1 \circ \sigma_1^{-1}(z_1) + \Gamma_1 \circ A^{-1}(\sigma_1 \circ \sigma_1^{-1}(z_1), \ldots, \sigma_n \circ \sigma_n^{-1}(z_n))$$

and as by the assumption of this first part of the proof we have $\sigma_1^{-1}(z_1) > \sigma_j^{-1}(z_j)$ for $j = 2, \ldots, n$, we obtain

$$\leq -\sigma_1 \circ \sigma_1^{-1}(z_1) + \Gamma_1 \circ A^{-1} \circ \sigma(\sigma_1^{-1}(z_1)) \qquad (22)$$

Now for $\tau := \sigma_1^{-1}(z_1)$ we have by (14)

$$D \circ \Gamma \circ A^{-1} \circ \sigma(\tau) \leq \sigma(\tau)$$

hence

$$\Gamma \circ A^{-1} \circ \sigma(\tau) \leq D^{-1} \circ \sigma(\tau)$$

and so (recall that β_i is defined before (13)) we have from (22) for the first component that

$$\begin{aligned} &-\sigma_1(\tau) + \Gamma_1 \circ A^{-1} \circ \sigma(\tau) \\ &< ((\text{id } + \beta_1)^{-1} - \text{id }) \circ \sigma_1(\tau) \qquad (23) \\ &= -\beta_1 \circ (\text{id } + \beta_1)^{-1} \circ \alpha_1(V_1(x_1)) < 0\,. \end{aligned}$$

Hence under the assumption that $V(x) = \sigma_1^{-1} \circ \alpha_1(V_1(x_1))$ is uniquely given we obtain

$$\dot{V}(x) \leq -c\beta_1 \circ (\text{id } + \beta_1)^{-1} \circ \sigma_1(V(x)) + C\gamma_{1u}(\|u\|)\,.$$

The argument can be repeated for the indices $i = 2, \ldots, n$ in the same manner and so setting

$$\tilde{\alpha}(s) := \min_{i=1,\ldots,n} c\beta_i \circ (\text{id } + \beta_i)^{-1} \circ \sigma_i(s)$$

and

$$\gamma(s) := \max_{i=1,\ldots,n} C\gamma_{iu}(\|u\|)$$

we obtain that

$$\dot{V}(x) \leq -\tilde{\alpha}(V(x)) + \gamma(\|u\|)\,.$$

for all points $x \in \mathbb{R}^N$ where the maximizing argument in (21) is uniquely defined. As the set of such points is an open and dense subset of $\mathbb{R}^N$ and as the function V is locally Lipschitz continuous, this proves that it is a Lipschitz ISS Lyapunov function for the interconnestion.

(In this case this can also be seen directly in an easy manner, [2],[1], [6]. As V is obtained by the maximization of $\mathcal{C}^1$ functions V_i, $i = 1, 2, ..., n$, the Clarke subgradient of V in $x \in \mathbb{R}^n$ can be computed by the set

$$\partial_{Cl} V(x) = \text{conv} \{ \quad \bigtriangledown \left(\sigma_i^{-1} \circ \alpha_i \circ V_i\right)(x_i) \mid \sigma_i^{-1} \circ \alpha_i(V_i(x_i)) = V(x) \}\ ,$$

where $\text{conv}\, M$ denotes the convex hull of the set M. As we have the dissipation inequality presented above as $\dot{V} \leq -\tilde{\alpha}(V(x)) + \gamma(\|u\|)$ for each of the extremal points of $\partial_{Cl} V(x)$, the dissipation inequality holds in terms of the Clarke generalized derivative for each ζ in the Clarke subgradient.) ∎

Thus we have obtained two different ways of constructing dissipative ISS Lyapunov functions. To compare the two constructions, we briefly return to the linear case as detailed in Section III. Recall that for the matrices in (4) the necessary condition is $r(A^{-1}\Gamma) < 1$. The construction explained in Section III uses a left vector such that $\mu^T(-A + \Gamma) < 0$ and sets $V(x) := \mu^T V_{vec}(x)$. In the construction of Theorem IV.3 we choose a right vector $s \in \mathbb{R}^n_+$ such that $\Gamma A^{-1} s < s$. For $\mu := A^{-1}s$ this is equivalent to $(-A+\Gamma)\mu < 0$. We then let $V(x) := \max_{i=1,\ldots,n} \mu_i^{-1} V_i(x_i)$ and by Theorem IV.3 this is an ISS Lyapunov function. In the context of convex analysis maximization and summation are dual operations. In this sense the two construction are dual to one another.

A. Linearly Scaled Gains

In this subsection we specialize the results we have obtained so far to the case, where the gains are obtained by linearly scaling gain functions associated with each of the subsystems.

To be precise, we assume that there exist functions $g_i \in \mathcal{K}_\infty$ and $a_i, c_{ij} \in \mathbb{R}_+$, $a_i > 0$, $i, j = 1, \ldots, n$ such that the gain functions in (3) are given by

$$\gamma_{ij}(s) = c_{ij} g_j(s),\ \forall j, \quad \alpha_i(s) = a_i g_i(s),\ \forall i\,. \qquad (24)$$

We now let $\tilde{A} = \text{diag}\,(a_1, \ldots, a_n)$ and $C = \left(c_{ij}\right)_{i,j=1,\ldots,n}$ and we denote for $s \in \mathbb{R}^n_+$

$$g(s) := \left(g_1(s_1), \ldots, g_n(s_n)\right)^T.$$

Note that with respect to our previous notation we have

$$A(s) = \tilde{A} g(s)\,, \quad \Gamma(s) = C g(s)\,.$$

Note also that from (3) we obtain ISS of the subsystems if we have $g_i \in \mathcal{K}_\infty$, $i = 1, \ldots, n$. On the other hand if the g_i are only positive definite, then we merely have integral ISS for the subsystems

Theorem IV.4 *Consider the interconnected systems* (1) *and assume that each subsystem has a dissipative ISS-Lyapunov function as in* (3) *where the gain functions satisfy* (24)*. Assume $r(\tilde{A}^{-1}C) < 1$ and let $\mu > 0$ be a vector such that $\mu^T(-\tilde{A} + C) < 0$.*
(i) If the functions $g_i, i = 1, \ldots, n$ are positive definite, then the interconnected system is integral ISS with an integral ISS Lyapunov function given by

$$V(x) := \mu^T V_{vec}(x)\,. \qquad (25)$$

(ii) If the functions $g_i \in \mathcal{K}_\infty$ $, i = 1, \ldots, n$, then the interconnected system is ISS with an ISS Lyapunov function given by (25)*.*

Proof: First note, that the choice of μ in the formulation of the theorem is possible by Lemma III.1. We have for $V(x) := \mu^T V_{vec}(x)$ that

$$\dot{V}(x) = \mu^T \dot{V}_{vec}(x) \leq \mu^T(-\tilde{A}+C)g(V_{vec}(x)) + \mu^T\gamma_u(\|u\|)$$

and defining $0 > L := \mu^T(-\tilde{A}+C)$ we obtain

$$\dot{V}(x) \leq Lg(V_{vec}(x)) + \mu^T\gamma_u(\|u\|) \leq -l(V(x)) + \mu^T\gamma_u(\|u\|)\,,$$

where we define

$$l(s) := \min\{-Lg(V_{vec}(x)) \mid \mu^T V_{vec}(x) = s\}\,.$$

It is clear that l is positive definite if the g_i are and that $l \in \mathcal{K}_\infty$ if the g_i are. This proves the assertion. ■

It is worth mentioning that the spectral radius condition implicitly requires some subsystems in the overall system to be ISS in the case (i) of the above theorem. For instance, in the two subsystems case, $a_1 < c_{12}$ implies $a_2 > c_{21}$ which indicates that at least one subsystem needs to be ISS although the subsystem is defined by a dissipation inequality only with positive definite functions of the integral ISS type. This fact is consistent with the result in [10].

V. CONCLUSIONS

In this paper we have introduced an approach of a construction of Lyapunov functions for interconnected ISS systems. This method provides an explicit construction for a general interconnection of any number of ISS systems. Our construction is based on the existence of some auxiliary function that can be found explicitly for a nonsmooth construction. We have also shown how they can be found for a smooth construction in a special case of supply rate functions. Their construction for general supply rates is a matter of our future investigations. We also hope to relax the technical assumption $0 < c < (\sigma_i \circ \alpha_i)'(\tau) < C$ in Theorem IV.3.

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