Stabilization of nonlinear systems with delayed data-rate-limited feedback

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***Abstract*— We consider nonlinear control systems with encoded feedback over delayed data-rate-limited communication channels. It is assumed that a smooth static state feedback exists, which renders the closed loop system ISS with respect to measurement noise, if there is no delay nor any data rate restriction. For this case we present an encoder/decoder scheme for which it can be shown that semi-global asymptotic stability is achieved despite the presence of delay and packet loss.**

I. INTRODUCTION

In recent years control applications using data-rate limited communication channels have attracted considerable attention. One approach in this area is to consider the feedback design in a first step. The knowledge of system and continuous controller is then used to treat the problems arising from the restrictions due to the communication channel. In this paper we follow this approach and consider in particular the effects of delay and packet loss in the channel.

Consider a setup as in Figure 1. If the sensor is not close to

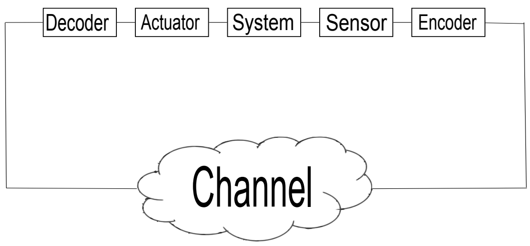

Fig. 1. The setup

the actuator, the problem of controlling over a communication channel arises. Typically with a communication channel effects of quantization, delay, packet loss, noise and out of order delivery of packets have to be taken into account. Many contributions dealing with the linear case can be found (e.g., [6] and the references therein).

This paper is a contribution to the general topic of nonlinear control with limited information. A good survey in this area is [4].

In this paper we address the problems of quantization, delay and packet loss with the help of what is known as a dynamic quantizer. The idea of a non-static quantizer was first introduced within the communication community (c.f., [2]) and was brought to the control community by [1].

We consider systems of the form

$$\dot{x} = f(x,u), \quad x \in \mathbb{R}^n, \quad u \in U \subset \mathbb{R}^m \tag{1}$$

satisfying standard assumptions and for which it holds additionally that a smooth stabilizing controller exists.

In [5] an encoding-decoding-scheme was introduced, which drives the quantization error to 0, making it possible to achieve asymptotic stability under certain input-to-state stability (ISS) assumptions (see Definition 1) and provided that the data rate is sufficiently large.

To get an idea how the dynamic quantizer works, consider

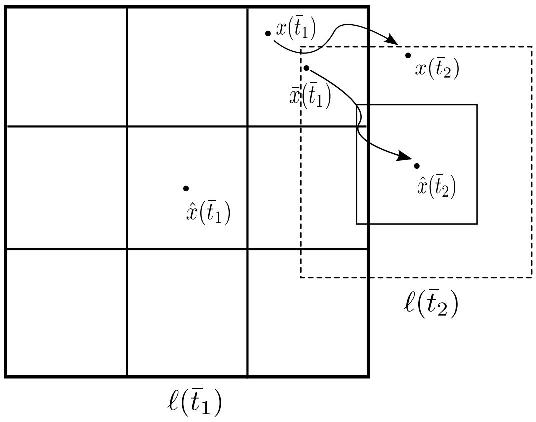

Fig. 2. Schematic representation of the dynamic quantizer

the following course of actions (Figure 2). The encoder and the decoder agree on a hypercube in which the initial state is known to lie. Initially, this hypercube is centered at the origin. The length of one of its edges is denoted by ℓ. We call this hypercube the quantization region. This quantization region is divided into N^n hypercubes, where n is the dimension of the state space. We refer to those hypercubes as subregions and to N as the number of partitions per dimension. The sensor determines the actual subregion in which the state lies and gives the center $\bar{x}$ of the subregion to the encoder. With the help of this information the encoder constructs a value s from a set of N^n different symbols and sends this information to the decoder. Using s the decoder reconstructs the center $\bar{x}$ of the subregion in which x is known to lie and uses this information to close the loop.

Now both the encoder and the decoder let the center $\bar{x}$ of the subregion follow the closed loop dynamics for some time τ. Because of the dynamics the error between the estimate $\bar{x}$ and the state x can grow by a certain factor. If we let the

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subregion grow by the same amount (the augmented region is the dashed box in Figure 2), we are sure that the state at time τ is still within the subregion. Then this subregion becomes the new quantization region. Now we are in the same situation as we started, namely to know a hypercube in which the state lies and are able to repeat the same steps. If the quotient between the growth of the quantization region and the reduction of the error due to N is smaller than 1, the quantization converges to 0.

For this scheme to work properly it is crucial that the encoder and the decoder agree on how large the quantization region is, into how many subregions it is divided and where the center of the quantization region is.

To achieve the last property we need encoder and decoder close their loops with the same signal. This is easily achieved, if there is no delay in the channel, because the encoded state is available for encoder and decoder at the same time. The case of fixed delay has been examined in [8]. In order to cope with time varying delays we propose to send time information along with the encoded state. The encoder has to send the time information so the decoder knows the time when the state was encoded. As soon as the new encoded state is available to the decoder, it changes the control action. This time information has to be known by the encoder to copy the behavior of the state dynamics of the decoder.

A sketch of the time evolution of the involved trajectories can be found in Figure 3 where $\bar{x}$ is the state of the encoder and $\bar{x}_d$ is the trajectory which will be used to close the loop.

We proceed as follows. In the ensuing Section II we collect the necessary notation and definition. The problem statement is found in Section II.B.

In Section III we give a detailed description of the quantization scheme and of the corresponding dynamics of encoder and decoder. It is important to note that both encoder and decoder have identical internal models of the system. The important idea is to ensure that at certain time instances encoder and decoder are certain to have the same information about the state of their respective internal models.

In Section IV we prove that with the encoding-decoding-scheme introduced in Section III it is possible to achieve asymptotic stability. We conclude our paper with some remarks in Section V.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Preliminaries

We use the following definitions. The symbol $|x| = \max\{|x_i| | 1 \leq i \leq n\}$ denotes the maximum norm on $\mathbb{R}^n$. The floor function $\lfloor\cdot\rfloor : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \lfloor x \rfloor$ is componentwise the biggest integer smaller or equal than x_i, $1 \leq i \leq n$. Similarly the ceiling function $\lceil\cdot\rceil : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \lceil x \rceil$ is componentwise the smallest integer bigger or equal than x_i, $1 \leq i \leq n$. We introduce $r(t^-) := \lim_{t \nearrow t^-} r(t)$, if the limit exists. If a continuous function $\alpha : [0,\infty) \to [0,\infty)$ is strictly increasing and $\alpha(0) = 0$ then it is said to be of class $\mathcal{K}$. If α is also unbounded, we say it is of class $\mathcal{K}_\infty$. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r,t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

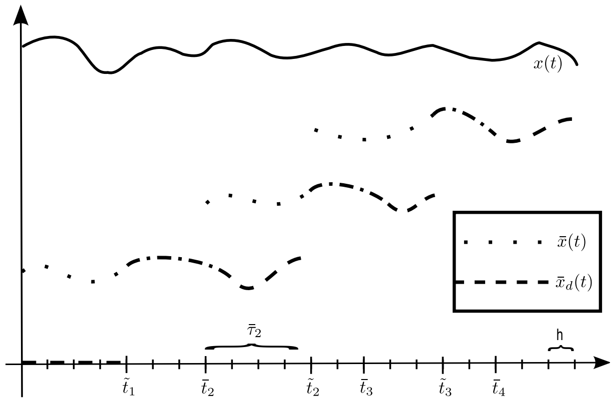

Fig. 3. Sketch of the time evolution of the trajectories involved

Definition 1: System (1) is called *input to state stable* (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every essentially bounded and measurable input u and any initial state $x(0)$ the solution exists for all $t > 0$ and satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|_\infty), \quad \forall t > 0. \tag{2}$$

The concept of ISS was introduced by Sontag [10] in the late 80s and has been adopted by the control community as a beneficial tool in analyzing control systems with disturbances (the interested reader is referred to [11]).

In the next section we formulate the problem precisely.

B. Problem statement

The control system is of the form (1) where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and Lipschitz in the first component, i.e., for all $\eta > 0$ there exists an $L(\eta) \in \mathbb{R}$ such that

$$|f(x,u) - f(y,u)| \leq L(\eta)|x - y|, \; \forall x, y \in \mathbb{R}^n, |u| \leq \eta. \tag{3}$$

The results of this paper can be extended to the case of local Lipschitz continuity of f in x. To ease presentation we assume a global condition. This is no major restriction.

Assumption 1: There exists a smooth $k : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto k(x)$ with $k(0) = 0$ such that

$$\dot{x} = f(x, k(x + e_d)) \tag{4}$$

is ISS with respect to the measurement error $e_d(t)$. Note that this is equivalent to the existence of functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ so that the solutions of (4) satisfies

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma\left(\sup_{s \in [t_0, t]} |e_d(s)|\right) \quad \forall t \geq t_0. \tag{5}$$

Note that under Assumption 1 the unperturbed ($e_d \equiv 0$) system (4) has an asymptotically stable equilibrium $x^* = 0$.

Now we want to introduce the communication channel and its properties. In our approach we consider TCP like packet based transmissions over a noiseless, errorfree channel with delay and packet loss. The delay of packets from encoder to decoder is denoted by $\bar{\tau}$ and from decoder to encoder by $\tilde{\tau}$.

Assumption 2: For the communication channel the following should hold:

1) The delays $\bar{\tau}, \tilde{\tau}$ are bounded (i.e., $\exists h \in \mathbb{R}, \rho \in \mathbb{N}$ s.t. $h =: \tau_{min} \leq \bar{\tau}, \tilde{\tau} \leq \tau_{max} := \rho h$)
2) The number of consecutive packet losses δ is bounded (i.e., $\exists \delta_{max}$ s.t. $\delta \leq \delta_{max}$)
3) Only packets sent from the encoder to the decoder are lost
4) The channel is able to transmit packets containing a value from a set of N_{max}^n (N_{max} odd) discrete values and a single positive integer not greater than $j_{max} \in \mathbb{N}$

Remark 1: The time a packet travels through the channel can be greater than τ_{max}. If this is the case it is considered lost to ease the analysis. In this sense τ_{max} may be regarded as a design parameter for the trade-off between delay and packet loss.

Remark 2: The assumption of reliable transmission between decoder and encoder is quite strong. However the ack's are much smaller than the packets travelling from the encoder to the decoder. Therefore the decoder could send many ack's to ensure the encoder receives one of them within τ_{max} units of time.

Remark 3: For simplicity we require N_{max} and in general all values of N to be odd. This ensures that the center of the quantization region is in the interior of one of the subregions.

Remark 4: If binary encoding is used, item 4 of Assumption 2 states that the communication channel must be able to transmit $\log_2(N_{max}^n + j_{max})$ bits within τ_{max} units of time.

III. Quantization and Encoding

As sketched in the introduction we have to send time information along with the encoded state. It is not reasonable to assume that it is possible to transmit one part of the information quantized and the other not. Therefore we have to quantize the timeline as well. The encoder or the decoder use the last time instance both agree upon to encode the time elapsed since then. Hence the quantizer we propose for the timeline takes the following form:

$$\nu(t, t_0) := \underbrace{\left\lceil \frac{t - t_0}{h} \right\rceil}_{:= \tilde{\bar{j}}_k} h + t_0. \qquad (6)$$

Here h is the coarseness of the quantizer, as defined in Assumption 2.1, and $\bar{j}_k$ and $\tilde{j}_k$ are the values which will be transmitted over the communication channel from encoder to decoder or back. Note that k is a counter, which is increased by one each time the encoder sends information to the decoder. The values of $\bar{j}_k$ and $\tilde{j}_k$ are used to reconstruct the corresponding time instances. Let $\bar{t}_k$ be the time instances when the encoder sends the kth packet and $\bar{\tau}_k$ is the delay of the kth packet. On the other hand $\tilde{t}_k$ is the time when the decoder sends an ack and $\tilde{\tau}_k$ is the delay of the kth ack. It is important to note that if the encoder or the decoder receive a packet (e.g., at $t = \tilde{t}_{k-1} + \tilde{\tau}_{k-1}$), they wait until the next quantized time step ($\bar{t}_k = \nu(t, \tilde{t}_{k-1})$) before they send a new packet or take any action. We have the encoder and the decoder agree on these time instances. To achieve this we make the following assumption.

Assumption 3: Clocks of encoder and decoder are synchronized and the time $\bar{t}_1 = 0$ when the encoder sends the first packet is known by the encoder and the decoder.

Remark 5: Assumption 3 is necessary in order to be able to transmit durations as opposed to time instances.

Assumption 4: Both the encoder and the decoder know the same bound of the initial state of the system (i.e., encoder and decoder know the same $X \in \mathbb{R}$ s.t. $|x(0)| \leq X$).

By Assumptions 1 and 4 there is a bounded region in which the trajectory remains for all positive times. As k is continuous the input is bounded as well. Hence the Lipschitz constant from (3) may be chosen as $L := L(\eta)$ with $\eta = \max_{|x| \leq E}(|k(x)|)$ with E as defined in Section IV

The table below illustrates when encoder or decoder send respectively receive information.

Time	Encoder		Decoder	
	sends	receives	sends	receives
$\bar{t}_k$	s, $\bar{j}_k$			
$\bar{t}_k + \bar{\tau}_k$				s, $\bar{j}_k$
$\tilde{t}_k$			ack, $\tilde{j}_k$	
$\tilde{t}_k + \tilde{\tau}_k$		ack, $\tilde{j}_k$		

Where the symbol s carries the state information from time $\bar{t}_k$, ack is the acknowledgment and $\bar{j}_k$ and $\tilde{j}_k$ carry the time information. Let us now summarize the variables we need for the encoder and the decoder.

$\hat{x}$	center of the quantization region
$\bar{x}$	center of the subregion in which the state of the system lies
$\bar{x}_d$	trajectory which is used to close the loop
ℓ	length of the quantization region
N	number of partitions per dimension
λ	design parameter which controls the speed of convergence of the quantization error ($0 < \lambda < 1$)
φ_d, φ_e	functions describing the quantization (see (35) respectively (34))
μ	function that assigns values to N (see (37))
t_k^*	last time the encoder successfully transmitted information to the decoder
δ	number of consecutive packets lost since the last successful transmission

The initial states for the encoder and the decoder are:

$$k = 1, \quad \tilde{\tau}_0 = \tilde{t}_0 = 0, \quad \bar{t}_0 = 0, \quad \bar{t}_1 = 0 \text{ and } t_1^* = 0$$
$$\delta(0) = 0 \quad \bar{x}(0^-) = 0, \quad \ell(0^-) = 2X \text{ and } \bar{x}_d(0) = 0.$$

In the following equations for the encoder we have to distinguish between three different cases. A detailed description of the meaning of the particular equations is found later on.

(i) If the encoder receives an ack (i.e., $t = \tilde{t}_{k-1} + \tilde{\tau}_{k-1}$):

$$\tilde{t}_{k-1} = \bar{t}_{k-1} + \tilde{j}_{k-1}h \quad (7)$$

$$\bar{t}_k = \nu(t, \tilde{t}_{k-1}) \quad (8)$$

$$\ell(\bar{t}_k^-) = \ell(\bar{t}_{k-1})e^{L(\bar{t}_k - \bar{t}_{k-1})} \quad (9)$$

$$\ell(\bar{t}_k) = \lambda\ell(\bar{t}_{k-1}) \quad (10)$$

$$N(\bar{t}_k) = \mu(\bar{t}_k, \bar{t}_{k-1}, \lambda) \quad (11)$$

$$\bar{x}(\bar{t}_k^-) = \bar{x}(\bar{t}_{k-1}) + \int_{\bar{t}_{k-1}}^{\tilde{t}_{k-1}} f(\bar{x}(s), k(\hat{x}(s)))ds + \int_{\tilde{t}_{k-1}}^{\bar{t}_k} f(\bar{x}(s), k(\bar{x}(s)))ds \quad (12)$$

$$\bar{x}(\bar{t}_k) = \varphi_e(\bar{x}(\bar{t}_k^-), x(\bar{t}_k), \ell(\bar{t}_k^-), N(\bar{t}_k)) \quad (13)$$

$$\hat{x}(\bar{t}_k) = \bar{x}(\bar{t}_k^-) \quad (14)$$

$$t_k^* = \bar{t}_{k-1}\,, \quad \tilde{t}_k^* = \tilde{t}_{k-1} \quad (15)$$

$$\delta(\bar{t}_k) = 0 \quad (16)$$

If the encoder receives an ack (case (i)), it reconstructs the time the decoder sent the ack (7), calculates the next quantized time (8) and updates $\bar{x}(\bar{t}_k^-)$ accordingly (12). Both integrals are needed because the control action changes between $\bar{t}_{k-1}$ and $\bar{t}_k$. The value $\bar{x}(\bar{t}_k^-)$ becomes the new center of the quantization region (14). As the estimation error grows, the length ℓ of the quantization region has to be augmented (9). The quantization region is then divided into N^n subregions. The jump from the center of the whole quantization region to the center of the subregion where the state lies is described by (13) respectively (34). Equation (11) ensures that N is large enough so that the new quantization region is by a factor of λ smaller than the old (10). The value t_k^* introduced in (15) is always the quantized time instant after the encoder received an ack. Equation (16) sets δ to zero to indicate that no packet loss occurred.

(ii) If packet loss occurred (i.e., $t = t_k^* + 2(\delta(t) + 1)\tau_{max}$):

$$\bar{t}_k = \bar{t}_{k-1} + 2\tau_{max} \quad (17)$$

$$\ell(\bar{t}_k^-) = \ell(\bar{t}_k) = \ell(\bar{t}_{k-1})e^{L(\bar{t}_k - \bar{t}_{k-1})} \quad (18)$$

$$N(\bar{t}_k) = \mu(\bar{t}_k, t_k^*, \lambda) \quad (19)$$

$$\bar{x}(\bar{t}_k^-) = \hat{x}(\bar{t}_k^-) \quad (20)$$

$$\bar{x}(\bar{t}_k) = \varphi_e(\bar{x}(\bar{t}_k^-), x(\bar{t}_k), \ell(\bar{t}_k^-), N(\bar{t}_k)) \quad (21)$$

$$t_k^* = t_{k-1}^*\,, \quad \tilde{t}_k^* = \tilde{t}_{k-1}^* \quad (22)$$

$$\delta(\bar{t}_k) = \delta(\bar{t}_k^-) + 1 \quad (23)$$

If $2\tau_{max}$ units of time elapse without receiving an ack (case (ii)), the encoder updates the length of the quantization region, the number N and the center of the quantization region but it cancels the jump made by the encoder the last time step by using the old value (20) to encode the state. The counter of packet losses is increased by one due to (23).

In both cases (i) and (ii) the encoder sends the quantization information s and the time information $\bar{j}_k = \lceil (t - \tilde{t}_{k-1}^*)/h \rceil$.

(iii) Otherwise:

$$\dot{\hat{x}}(t) = f(\hat{x}(t), k(\hat{x}(t))) \quad (24)$$

Equation (24) makes sure that the value the decoder uses to close the loop on the interval $[\bar{t}_k, \tilde{t}_k)$ is also known by the encoder.

Remark 6: Note that the above equations are valid for $k \geq 2$. The very first time ($k = 1$) the encoder sends information only the equations (11),(13),(14), (15) and (16) are used. If the packet sent at time $\bar{t}_1$ is not lost, the decoder does not use equation (27) and (28) until it receives the next packet. The equations for the decoder are:

(i) If the decoder received a packet (i.e. $t = \bar{t}_k + \bar{\tau}_k$):

$$\bar{t}_k = \tilde{t}_{k-1} + \bar{j}_k h \quad (25)$$

$$\tilde{t}_k = \nu(t, \bar{t}_k) \quad (26)$$

$$\ell(\bar{t}_k^-) = \ell(t_k^*)e^{L(\bar{t}_k - t_k^*)} \quad (27)$$

$$\ell(\bar{t}_k) = \lambda\ell(\bar{t}_k^*) \quad (28)$$

$$N(\bar{t}_k) = \mu(\bar{t}_k, t_k^*, \lambda) \quad (29)$$

$$\bar{x}(\bar{t}_k^-) = \bar{x}_d(\tilde{t}_{k-1}) + \int_{\tilde{t}_{k-1}}^{\bar{t}_k} f(\bar{x}(s), k(\bar{x}(s)))ds \quad (30)$$

$$\bar{x}(\bar{t}_k) = \varphi_d(\bar{x}(\bar{t}_k^-), s, \ell(\bar{t}_k^-), N(\bar{t}_k)) \quad (31)$$

$$\bar{x}_d(\tilde{t}_k) = \bar{x}(\bar{t}_k) + \int_{\bar{t}_k}^{\tilde{t}_k} f(\bar{x}(s), k(\bar{x}_d(s)))ds \quad (32)$$

(ii) Otherwise

$$\dot{\bar{x}}_d(t) = f(\bar{x}_d(t), k(\bar{x}_d(t))) \quad (33)$$

Equations (27)-(31) ensure that encoder and decoder agree on the value of $\bar{x}(\bar{t}_k)$ at time $\tilde{t}_k$ while (32) calculates the trajectory forward in time to compensate the delay between encoder and decoder. By Assumption 2.3 no ack's are lost. Hence encoder and decoder agree on the value of t_k^*.

The function φ describes the quantization

$$\varphi_e(\bar{x}, x, N, \ell) = \bar{x} + \underbrace{\left\lfloor \frac{N}{\ell}(x - \bar{x}) + \frac{1}{2} \right\rfloor}_{=s} \frac{\ell}{N} \quad (34)$$

and

$$\varphi_d(\bar{x}, s, N, \ell) = \bar{x} + s\frac{\ell}{N}. \quad (35)$$

Remark 7: If at time $\bar{t}_k$ the state lies within the quantization region, the error between the state and the estimate shrinks by N because of the jump from the center of the region to the center of a subregion. Hence

$$|x(\bar{t}_k) - \bar{x}(\bar{t}_k^-)| \leq \frac{\ell}{2} \Rightarrow |x(\bar{t}_k) - \bar{x}(\bar{t}_k)| \leq \frac{\ell}{2N} \quad (36)$$

holds, which can be seen from (34) and (13).

The function μ takes the following form

$$\mu(\bar{t}_k, \bar{t}_{k-1}, \lambda) := 2\left\lceil \frac{e^{L(\bar{t}_k - \bar{t}_{k-1})}}{2\lambda} \right\rceil + 1. \quad (37)$$

The reason for this particular choice of μ is that we need N (11) to be an integer large enough to compensate for the growth of ℓ (9). The value N has to be odd otherwise the

quantizer (34) would take a different form.
The evolution of the closed loop system is thus given by

$$\dot{x}(t) = f(x(t), k(\bar{x}_d(t)))\,. \qquad (38)$$

We now give conditions under which this scheme results in stabilization.

IV. MAIN RESULT

Here we show that with the encoder/decoder scheme introduced in the previous section, asymptotic stability can be achieved if Assumptions 1-4 hold and a bandwidth condition is satisfied.

Theorem 1: Consider system (1) with encoder/decoder scheme described in (7)-(33) and let Assumptions 1-4 hold. If

$$N_{max} > e^{2L\tau_{max}\delta_{max}} + 1 \qquad (39)$$

and

$$j_{max} h \geq 2\tau_{max}\delta_{max}\,, \qquad (40)$$

then $u = k(\bar{x}_d)$, where $\bar{x}_d$ is generated by the decoder (33), asymptotically stabilizes the equilibrium $x^* = 0$ of (1).

Remark 8: Condition (40) states how small the coarseness h can be chosen without violating the bandwidth constraints due to j_{max}. Although (40) can be satisfied by choosing h large, a small value of h can possibly save bandwidth.
It follows readily from (6) and Assumption 2 that $\bar{j}_k, \tilde{j}_k \leq j_{max}\,, \forall k \in \mathbb{N}$ whenever (40) holds.
We prove Theorem 1 as follows: By Lemma 1 all systems close their loop using the same signal. This is used to bound the error between state and encoder estimate in Lemma 2. As an easy consequence a bound on the error on the decoder side can be obtained (Corollary 1).
The evolution of $\bar{x}(t)$ is governed by the following equation.

$$\dot{\bar{x}}(t) = \begin{cases} f(\bar{x}(t), k(\hat{x}(t))) & t \in [\bar{t}_k, \tilde{t}_k) \\ f(\bar{x}(t), k(\bar{x}(t))) & t \in [\tilde{t}_k, \bar{t}_{k+1}) \end{cases}. \qquad (41)$$

The next lemma shows on which time intervals certain signals coincide.

Lemma 1: Consider encoder/decoder scheme described in (7)-(33). If Assumptions 1-4 hold, then for all $t \in [\bar{t}_k, \tilde{t}_k)$

$$\bar{x}_d(t) = \hat{x}(t)$$

and for all $t \in [\tilde{t}_k, \bar{t}_{k+1})$ we have $\bar{x}_d(t) = \bar{x}(t)$.

Proof: We first treat the case of no packet loss. Because of the initial condition of the encoder and the decoder and (14) it holds that $\bar{x}_d(0) = \hat{x}(0) = 0$. Using $\bar{t}_1 = 0$ and equations (24) and (33) we obtain

$$\bar{x}_d(t) = \hat{x}(t) \quad \forall t \in [\bar{t}_1, \tilde{t}_1). \qquad (42)$$

At time $\tilde{t}_1$ the value of $\bar{x}(\bar{t}_1)$ becomes available to the decoder (31). By (32), (41) and (42) we have

$$\bar{x}_d(\tilde{t}_1) = \bar{x}(\tilde{t}_1).$$

Since both trajectories follow the same dynamics on the interval $[\tilde{t}_k, \bar{t}_{k+1})$ by (33) and (41) we get

$$\bar{x}_d(t) = \bar{x}(t)\,, \quad \forall t \in [\tilde{t}_1, \bar{t}_2).$$

Due to the continuity of $\bar{x}_d$ at $\bar{t}_k$ and (14)

$$\bar{x}_d(\bar{t}_2) = \hat{x}(\bar{t}_2)$$

holds and we can repeat the arguments inductively to conclude.

If a packet loss occurred ($\delta(\bar{t}_k) > 0$), then

$$\bar{x}_d(\bar{t}_k) = \bar{x}(\bar{t}_k) = \hat{x}(\bar{t}_k)$$

holds since (20) cancels the last jump made by the encoder. As all three trajectories follow the same dynamics we have

$$\bar{x}_d(t) = \bar{x}(t) = \hat{x}(t) \quad \forall t \in [\bar{t}_k, \bar{t}_{k+1}).$$

Hence the proof for the case of packet loss is similar to the previous arguments. ■

The introduction of δ_{max} in (39) guarantees that the number N_{max} is still big enough to cope with the larger time interval over which the length of the quantization region can grow.
By Assumption 2 the time between two successful transmissions from the encoder to the decoder is always smaller than $2\tau_{max}\delta_{max}$. Before we proceed with the next lemma, note that it follows from (37) and (11) respectively (29), (19) that

$$\lambda > \frac{e^{L(\bar{t}_k - \bar{t}_{k-1})}}{N(\bar{t}_k)}. \qquad (43)$$

Now define $e_e(t) := \bar{x}(t) - x(t)$ as the error between the estimated state and the state of the encoder system. To understand the ensuing lemma it is helpful to consider Figure 2 again. Note that at time $\bar{t}_k$ the value of N ((37) respectively (11)) compensates the growth of the error $e_e(t)$ on the interval $[\bar{t}_{k-1}, \bar{t}_k)$.

Lemma 2: Consider system (1) with encoder/decoder scheme described in (7)-(33). If Assumptions 1-4 hold, the error on the encoder side is bounded by

$$|e_e(\bar{t}_k)| \leq \lambda^k X\,, \quad \forall\, k \in \mathbb{N}\,.$$

Proof: Because of Assumption 4 and initialization of the encoder, the initial state $x(\bar{t}_1)$ is within the quantization region.

$$|0 - x(\bar{t}_1)| = |e_e(\bar{t}_1^-)| \leq X = \frac{\ell(\bar{t}_1^-)}{2}.$$

Hence we can use (36) to obtain

$$|e_e(\bar{t}_1)| \leq \frac{\ell(\bar{t}_1^-)}{2N(\bar{t}_1)} \leq \lambda X,$$

where the last conclusion follows from (43) and the fact that $\bar{t}_1 = \bar{t}_0$.
Let $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. The encoder error e_e satisfies for $t \in [\bar{t}_{k-1}, \bar{t}_k)$ according to (41) and (38):

$$|e_e(t)| = |\bar{x}(t) - x(t)| = |\bar{x}(\bar{t}_{k-1})$$

$$+ \int_{\bar{t}_{k-1}}^{\tilde{t}_{k-1} \wedge t} f(\bar{x}(s), k(\hat{x}(s)))ds + \int_{\tilde{t}_{k-1}}^{t \vee \tilde{t}_{k-1}} f(\bar{x}(s), k(\bar{x}(s)))ds$$

$$-x(\bar{t}_{k-1}) - \int_{\bar{t}_{k-1}}^{t} f(x(s), k(\bar{x}_d(s)))ds|.$$

We can split the last integral and collect the corresponding terms to get:

$$|e_e(t)| = |\bar{x}(\bar{t}_{k-1}) - x(\bar{t}_{k-1}) + \int_{\bar{t}_{k-1}}^{\tilde{t}_{k-1}\wedge t} (f(\bar{x}(s), k(\hat{x}(s))) - f(x(s), k(\bar{x}_d(s))))ds + \int_{\tilde{t}_{k-1}}^{t\vee\tilde{t}_{k-1}} (f(\bar{x}(s), k(\bar{x}(s))) - f(x(s), k(\bar{x}_d(s))))ds|.$$

Now we can use (3) and Lemma 1 to arrive at:

$$|e_e(t)| \le |e_e(\bar{t}_{k-1})| + L\int_{\bar{t}_{k-1}}^{t} |e_e(s)|ds.$$

Because of the continuity of $e_e(t)$ on the interval $t \in [\bar{t}_{k-1}, \bar{t}_k)$ the Gronwall-lemma yields

$$|e_e(\bar{t}_k^-)| \le |e_e(\bar{t}_{k-1})|e^{L(\bar{t}_k - \bar{t}_{k-1})}.$$

If we assume that $|e_e(\bar{t}_{k-1})| \le \lambda^{k-1}X$ we conclude:

$$|e_e(\bar{t}_k^-)| \le \lambda^{k-1}Xe^{L(\bar{t}_k - \bar{t}_{k-1})}.$$

Hence we can use (9), (10), (36) and (43) to deduce:

$$|e_e(\bar{t}_k)| \le \frac{\ell(\bar{t}_k^-)}{2N(\bar{t}_k)} \le \lambda^k X,$$

which completes the proof. ∎

Now that we have established a bound on the error on the encoder side, we have to achieve a similar goal on the decoder side. To this end define the error on the decoder side $e_d(t) := \bar{x}_d(t) - x(t)$ and a constant $W := Xe^{L\tau_{max}}$.

Corollary 1: Consider system (1) with encoder/decoder scheme described in (7)-(33). Let Assumptions 1-4 hold, then the error on the decoder side is bounded by

$$|e_d(\tilde{t}_k)| \le \lambda^k W, \quad \forall k \in \mathbb{N}.$$

Proof: Using Lemma 1 we are able to conclude

$$|e_d(\tilde{t}_k)| = |e_e(\tilde{t}_k)|.$$

Using the Gronwall Lemma as in the proof of Lemma 2, the error evolves according to:

$$|e_d(\tilde{t}_k)| = |e_e(\tilde{t}_k)| \le |e_e(\bar{t}_k)|e^{L\bar{\tau}_k}.$$

Using Lemma 2 completes the proof because we obtain

$$|e_d(\tilde{t}_k)| \le \lambda^k Xe^{L\bar{\tau}_k} \le \lambda^k W.$$

∎

With the previous results we may prove Theorem 1.

Proof: (of Theorem 1) From (39) there exists $\lambda > 0$ with

$$1 > \lambda \ge \frac{e^{2L\tau_{max}\delta_{max}}}{N_{max} - 1} > 0.$$

It is easy to see from (37) and, (11), (19), (29) that this choice of λ guarantees that

$$N(\bar{t}_k) \le N_{max} \quad \forall k \in \mathbb{N},$$

as N_{max} is odd. Using Corollary 1 and the fact that $\lambda < 1$ it follows that

$$|e_d(\tilde{t}_k)| \underset{k\to\infty}{\to} 0.$$

From the bound of the initial value X and the maximal error on the decoder side W we obtain using (5) that

$$|x(t)| \le \beta(X, 0) + \gamma(W) =: E \quad \forall t \ge 0.$$

Using (5) again we get

$$|x(t)| \le \beta(E, t - t_0) + \gamma\left(\sup_{s\in[t_0,t]} |e_d(s)|\right) \quad \forall t \ge t_0.$$

As t_0 goes to infinity the right hand side converges to zero which shows the attractivity of $x^* = 0$. On the other hand we can interpret (5) as:

$$|x(t)| \le \beta(|x(0)|, 0) + \gamma(|\bar{x}_d(0) - x(0)|e^{2L\tau_{max}}).$$

Hence the right hand side is a $\mathcal{K}_\infty$ function depending only on $x(0)$ which together with the attractivity concludes the proof. ∎

V. CONCLUSION

The design of an encoder/decoder scheme has been presented, which despite quantization, delay and packet loss is able to achieve semi-global asymptotic stability if the closed loop system is ISS with respect to measurement errors and a bandwidth condition is satisfied. The construction is semi-global as we start with a bound for the initial state. In [5] an idea called “zooming-out” is presented to deal with this limitation. In [9], [7] the ISS assumption may be weakened. We expect that similar techniques can be used here. The scheme outlined in this work is sensitive to errors between the encoder state and the decoder state. Ideas to bound the mismatch between encoder and decoder initialization for a similar scheme can be found in [3].

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