Numerical verification of local input-to-state stability for large netwoks

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***Abstract*— We consider networks of locally input-to-state stable (LISS) systems. Under a small gain condition the entire network is again LISS. An efficient numerical test to check the small gain condition is presented in this paper. An example from applications serves as a demonstration for quantitative results.**

I. INTRODUCTION

This paper studies local stability properties of a system

$$\dot{x} = f(x, u) \tag{1}$$

that can be viewed as a composite of subsystems

$$\dot{x}_i = f_i(x_1, \ldots, x_n, u_i), \quad i = 1, \ldots, n, \tag{2}$$

where $x_i \in \mathbb{R}^{N_i}$, $u_i \in \mathbb{R}^{M_i}$, $f_i : \mathbb{R}^{\sum_j N_j + M_i} \to \mathbb{R}^{N_i}$, $i = 1, \ldots, n$, $x^T = (x_1^T, \ldots, x_n^T) \in \mathbb{R}^N$, $N = \sum_{i=1}^n N_i$, $u^T = (u_1^T, \ldots, u_n^T)$, $f(x,u)^T = (f_1(x,u_1)^T, \ldots, f_n(x,u_n)^T)$. To have existence and uniqueness of solutions of the subsystems on their own, each function f_i is assumed to be continuous and locally Lipschitz in x uniformly for u_i in compact sets.

Stability properties of such an interconnection have been studied in [1], [2], [3], [4], [5], [6]. A stability condition of a small gain type for the interconnection (1) of input to state stable (ISS) subsystems (2) was firstly derived in [1]. Some interpretations and consequences following from this small gain conditions can be found in [3], [5]. The construction of an ISS Lyapunov function for (1) was given in [2], [4]. These results generalize the known stability conditions derived in [9], [10].

It is not an easy task to check this generalized small gain condition in case of large scale interconnections. In this paper we develop a numerical procedure which allows to check this condition. Here we consider a local version of the stability property which seems to be more relevant for applications. Each subsystem (2) is assumed to be locally input-to-state stable (LISS). We present a local small gain condition (LSGC) for the stability of interconnection (1) and a numerical test to check this condition, i.e., to assure that the composite system is also LISS. This consists of two steps: By a fixed point algorithm and a convergence argument a region in the positive orthant is identified, where the gain matrix is strictly decreasing. A local version of the small gain theorem for general ISS networks then establishes LISS for the composite system. A region of stability can be explicitly stated. So far this estimate is still very conservative.

We organize this paper as follows: The next section introduces the necessary notions. Some auxiliary results and the problem statement is given in Section III. Section IV contains the main results of the paper. An illustrative example is considered in section V. Section VI concludes the paper and gives some remarks on the future directions of research.

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II. NOTATION AND DEFINITIONS

A. Local Input-to-State Stability (LISS)

Let $\mathbb{R}_+$ denote the interval $[0, \infty)$ and $\mathbb{R}_+^n$ be the positive orthant in $\mathbb{R}^n$. For any $a, b \in \mathbb{R}_+^n$ let $a < b \quad \Leftrightarrow \quad a_i < b_i,\ i = 1, \ldots, n$ and $a \leq b \quad \Leftrightarrow \quad a_i \leq b_i,\ i = 1, \ldots, n$. For $a, b \in \mathbb{R}_+^n$ let $[a, b] := \{s \in \mathbb{R}_+^n :\ a \leq s \leq b\}$ be a rectangular set in $\mathbb{R}_+^n$ and $]a, b[:= \{s \in \mathbb{R}_+^n :\ a < s < b\}$. Let $\|x\|$ denote the Euclidean norm of $x \in \mathbb{R}^n$. Before we move on to the stability concepts, we first recall the definition of comparison functions.

Definition 2.1: (i) A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded.

(ii) A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is non-increasing and tends to zero at infinity.

The concept of input-to-state stability (ISS) has been first introduced in [12]. Its local version was given by Sontag and Wang in [13].

Definition 2.2: System (1) is *locally input-to-state stable* (LISS), if there exists a $\rho > 0$, a $\gamma \in \mathcal{K}_\infty$, and a $\beta \in \mathcal{KL}$, such that for all $\|\xi\| \leq \rho$, $\|u\|_\infty \leq \rho$

$$\|x(t, \xi, u)\| \leq \beta(\|\xi\|, t) + \gamma(\|u\|_\infty) \quad \forall\, t \geq 0, \tag{LISS}$$

in this case γ is called gain.
If $\rho = \infty$, then system (1) is called *input-to-state stable* (ISS). It is known that ISS defined this way is equivalent to the existence of an ISS Lyapunov function. Here we give the definition of a LISS Lyapunov function:

Definition 2.3: A smooth function $V : \mathbb{R}^N \to \mathbb{R}_+$ is a *LISS Lyapunov function* of (1) if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}_\infty$, and a positive definite function α such that

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^N, \tag{3}$$

$$V(x) \geq \chi(|u|) \Longrightarrow \nabla V(x) \cdot f(x, u) \leq -\alpha(V(x)), \tag{4}$$

for all $\|\xi\| \leq \rho$, $\|u\|_\infty \leq \rho$. Function χ in then called Lyapunov gain. If $\rho = \infty$ then V is called an ISS Lyapunov function.

A related and strictly weaker stability concept is that of local stability:

Definition 2.4: System (1) is *locally stable* (LS), if there exist $\rho > 0$, $\sigma, \gamma \in \mathcal{K}_\infty$, such that for all $\|\xi\| \le \rho$, $\|u\|_\infty \le \rho$

$$\|x(\cdot, \xi, u)\|_\infty \le \sigma(\|\xi\|) + \gamma(\|u\|_\infty). \qquad \text{(LS)}$$

Also related is the concept of asymptotic gains.

Definition 2.5: System (1) has the *local asymptotic gain property* (LAG), if there exist $\rho > 0$, $\gamma \in \mathcal{K}_\infty$, such that for all $\|\xi\| \le \rho$, $\|u\|_\infty \le \rho$

$$\limsup_{t\to\infty} \|x(t, \xi, u)\| \le \gamma(\|u\|_\infty). \qquad \text{(LAG)}$$

Note that inequality (LAG) is equivalent to

$$\limsup_{t\to\infty} \|x(t, \xi, u)\| \le \gamma(\text{ess.} \limsup_{t\to\infty} \|u\|). \qquad \text{(LAG}')$$

B. Gain Matrices

There many nonlinear gains in case of an interconnected system (2) which can be combined in a matrix in the following way. Subsystem i is LISS, provided there exist $\rho_i > 0$, $\gamma_{ij}, \gamma_i \in \mathcal{K}_\infty$, and a $\beta_i \in \mathcal{KL}$, such that for all $\|\xi_i\| \le \rho_i$, $\|u_i\|_\infty \le \rho_i$

$$\begin{aligned} \|x_i(t, \xi_i, x_j : j \ne i, u_i)\| &\le \beta_i(\|\xi_i\|, t) \\ &+ \sum_{j\ne i} \gamma_{ij}(\|x_j\|_\infty) + \gamma_i(\|u_i\|_\infty) \quad \forall\, t \ge 0. \end{aligned} \qquad (5)$$

Remark 2.6: Note that instead of (5) we could also write

$$\begin{aligned} \|x_i(t, \xi_i, x_j : j \ne i, u_i)\| &\le \max\{\beta_i(\|\xi_i\|, t), \\ &\max_{j\ne i} \gamma_{ij}(\|x_j\|_\infty),\, \gamma_i(\|u_i\|_\infty)\} \quad \forall\, t \ge 0, \end{aligned} \qquad (6)$$

which is qualitatively equivalent. Of course the gains in (5) and (6) are in general different.

If all n subsystems are LISS then these estimates give rise to a *gain matrix*

$$\Gamma = (\gamma_{ij})_{i,j=1}^n, \text{ with } \gamma_{ij} \in \mathcal{K}_\infty \text{ or } \gamma_{ij} \equiv 0, \qquad (7)$$

where we use the convention $\gamma_{ii} \equiv 0$ for $i = 1, \dots, n$.

C. The Small Gain Condition

The gain matrix Γ defines a monotone operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by $\Gamma(s)_i := \sum_j \gamma_{ij}(s_j)$ for $s \in \mathbb{R}^n_+$.

Remark 2.7: If we used the notation (6) instead of (5), then we would define $\Gamma(s)_i := \max_j \gamma_{ij}(s_j)$.

An operator $A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is called monotone, if $r \le s$ implies $A(r) \le A(s)$. By construction, Γ is monotone.

We say that Γ satisfies the *local small gain condition* on $[0, w^*]$, provided that

$$\Gamma(w^*) < w^* \text{ and } \Gamma(s) \not\ge s,\ \forall\, s \in [0, w^*],\ s \ne 0. \quad \text{(LSGC)}$$

Here $s \not\ge r$ for $r, s \in \mathbb{R}^n_+$ means that there is at least one component i where $s_i < r_i$ holds.

The global small gain condition assuring the ISS property for an interconnection of ISS subsystems was derived in [1]. An alternative proof has been given in [5]. We quote the following result from these papers.

Theorem 2.8 (global small-gain theorem for networks): Consider system (1) and suppose that each subsystem (2) is ISS, i.e., condition (5) holds for all $\xi_i \in \mathbb{R}^n_+,\ u_i \in L_\infty$, $i = 1, \dots, n$. Let Γ be given by (7). If there exists an $\alpha \in \mathcal{K}_\infty$, such that

$$(\Gamma \circ D)(s) \not\ge s, \qquad \forall s \in \mathbb{R}^n_+ \setminus \{0\}, \qquad (8)$$

with $D = \text{diag}_n(\text{id} + \alpha)$ then the system (1) is ISS from u to x.

Furthermore it is known that under the same small gain condition stated for Lyapunov gains of subsystems (2) an ISS Lyapunov function for (1) can be explicitly constructed as a combination of the ISS Lyapunov functions of subsystems, see [4]:

If we replace (4) for each subsystem (2) by

$$\begin{aligned} V_i(x_i) &\ge \chi(\|u\|) + \sum_{j\ne i} \gamma_{ij}(V_j(x_j)) \\ &\Longrightarrow \nabla V_i(x_i) \cdot f_i(x, u) \le -\alpha(V_i(x_i)), \end{aligned} \qquad (9)$$

for $\gamma_{ij} \in \mathcal{K}_\infty$ or $\gamma_{ij} \equiv 0$, and define Γ just as before, then the following theorem holds.

Theorem 2.9: Let each subsystem (2) have an ISS Lyapunov function V_i, i.e., V_i satisfies $V_i(0) = 0$, is radially unbounded, proper, and locally Lipschitz, such that (9) holds for $i = 1, \dots, n$. Let Γ be given by (7) but with γ_{ij} the Lyapunov gains. Assume that Γ is irreducible. If there exists an $\alpha \in \mathcal{K}_\infty$, such that

$$(\Gamma \circ D)(s) \not\ge s, \qquad \forall s \in \mathbb{R}^n_+ \setminus \{0\}, \qquad (10)$$

with $D = \text{diag}_n(\text{id} + \alpha)$ then

$$V(x) = \max_i \sigma_i^{-1}(V_i(x_i))$$

is an ISS Lyapunov function for the system (1), where $\sigma : \mathbb{R}_+ \to \mathbb{R}^n_+$ is a function satisfying $(\Gamma \circ D)(\sigma(t)) < \sigma(t),\ \forall t > 0$, such that each component function is of class $\mathcal{K}_\infty$.

Note that $V(x)$ in this case is not smooth but Lipschitz continuous. A local version of the function σ can be constructed explicitly as we will show below.

In this paper we give a local result similar to Theorem 2.8. The following lemmas will be used to obtain the main result.

III. Auxiliary lemmas and problem statement

Lemma 3.1: Let Γ be a gain matrix as in (7). For any $w^* \in \mathbb{R}^n_+$ consider the trajectory $\{w(k)\}$ of the discrete monotone system $w(k+1) = \Gamma(w(k)),\ k = 0, 1, 2 \dots$ with $w(0) = w^*$. If $w(k) \to 0$ for $k \to \infty$ then Γ satisfies the small gain condition (LSGC) on $[0, w^*]$.

Proof: Suppose this is not true, i.e., there exists a point $0 \ne v \in [0, w^*]$ with

$$\Gamma(v) \ge v \qquad (11)$$

and $v \ne 0$. Since Γ is monotone, so is Γ^k, i.e., the k-times application of Γ. Hence (11) implies $\Gamma^k(v) \ge v \ge 0$, so $\Gamma^k(v)$ does not tend to zero as k approaches infinity. But $v \le w^*$ implies $\Gamma^k(v) \le \Gamma^k(w^*) = w(k)$, which is assumed to tend to zero. A contradiction. Hence there cannot exist such a v and (LSGC) holds on all of $[0, w^*]$. ∎

Lemma 3.2: Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ satisfy (LSGC) on $[0, w^*]$, such that Γ has no zero rows. Then for all $w_1 \in]\Gamma(w^*), w^*[$ there exists an $\alpha \in \mathcal{K}_\infty$, such that for $D = \operatorname{diag}_n(\mathrm{id} + \alpha)$

$$D \circ \Gamma(s) \ngeq s, \ \forall s \in [0, w_1], \ s \neq 0.$$

Sketch of Proof. 1. Using $\Gamma(w^*) < w^*$ and monotonicity of Γ one can easily show that $\Gamma^k(w^*) \to 0$, strictly decreasing.

2. Note that $\Gamma^{k+2}(w^*) < \Gamma^{k+1}(w_1) < \Gamma^{k+1}(w^*) < \Gamma^k(w_1)$, k=0, 2, ...

3. We have then $\Gamma^k(w_1) \to 0$, strictly decreasing.

4. On a discrete set define a diagonal operator $\tilde{D}$ inductively by $w_2 = \Gamma(w^*) = \tilde{D} \circ \Gamma(w_1)$, $\tilde{D} \circ \Gamma(w_2) = \Gamma(w_1) = w_3$, and so on, $\tilde{D} \circ \Gamma(w_i) = w_{i+1}$.

5. $\tilde{D}$ is defined on set of strictly decreasing points, consider the componentwise slopes of $\tilde{D}$ between these points $\tilde{\alpha}_i = \min_j\{(\tilde{D} - \mathrm{id})(\Gamma(w_i))_j / \Gamma(w_i)_j\}$ as a function of these points $\Gamma(w_i)$, associate this to $r_i := \|\Gamma(w_i)\|_1$. So $\{r_i\}$ is null sequence, hence $\tilde{\alpha}_i$ is function of r_i.

6. Let $\tilde{\alpha}_i$ be bounded from below by some strictly decreasing null sequence α_i.

7. Define by linear interpolation $D(s) = \operatorname{diag}((\alpha_i - \alpha_{i+1}) \cdot \frac{\|s\|_1 - r_{i+1}}{r_i - r_{i+1}} + (1 + \alpha_{i+1})$ for $\|s\|_1 \in]r_{i+1}, r_i]$.

Then clearly $D \circ \Gamma(w_1) < w_1$, $D \in \operatorname{diag}(\mathrm{id} + \mathcal{K}_\infty)$, and $(D \circ \Gamma)^k(w_1) \to 0$. Hence $(D \circ \Gamma)(s) \ngeq s$ for all $s \in [0, w_1], s \neq 0$ by Lemma 3.1 applied to $D \circ \Gamma$. ■

Note that by continuity of Γ for w^* such that $\Gamma(w^*) < w^*$ we also have a diagonal operator D such that $D \circ \Gamma(w^*) < w^*$.

A. Problem Statement

We will consider the following questions. Suppose we have a network of n interconnected LISS systems like (2), each satisfying (5), $i = 1, \ldots, n$. Under what conditions does the composite system (1) satisfy (LISS)? How can we check this condition numerically by just looking at Γ? Can we estimate the stability region in (LISS), i.e., determine estimates for ρ numerically?

In the next section we will show that (LSGC) is sufficient for LISS of (1) and show how this condition can be checked. We will see how the stability region can be estimated.

IV. MAIN RESULTS

A. The Numerical Test

The procedure to check the small gain condition explicitly for a given $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ consists of two steps.

a) Step 1: Given a radius $r > 0$, find a $w \in \mathbb{R}^n_+$, $\|w\|_1 := \sum w_k = r$, such that $\Gamma(w) \leq w$.

In [8] two algorithms are proposed to find distinguished points in simplices. In that paper, usually fixed points are under consideration, but the method is more general and suitable for our purposes. For reasons of space, we just indicate how the method is applicable to our setting:

The set $S := S_r := \{s \in \mathbb{R}^n_+ : \|s\|_1 = r\}$ defines an $(n-1)$-simplex. Let function $l : S \to \{1, \ldots, n\}$ be defined by $l(s) = l_{i(s)}$, where

$$i(s) = \arg\min_i \{s_i > 0 \text{ and } \Gamma(s)_i \leq s_i\},$$

defines a proper labeling, see [8] for the definition and details, where l_i is a n-vector of zeros, with $+1$ at position i and -1 at position n for $i = 1$ and at position $i-1$ otherwise. Note that for actual computations it is sufficient to consider $l(s) = i(s)$.

The algorithm described in §6 and §7 in [8] then finds a point $w \in \mathbb{R}^n_+$, $\|w\|_1 = r$, with $\Gamma(w) \leq w$, provided such a point exits. An even faster but more involved algorithm is given in §8 of that paper.

b) Step 2: Given a point $w \in \mathbb{R}^n_+$, $\|w\|_1 = r$, with $\Gamma(w) \leq w$ then, provided that (LSGC) can be satisfied at all, in an arbitrary small neighborhood $U(w)$ of w, there exists a w^* such that $\Gamma(w^*) < w^*$, see interpretations of the small gain condition in [1].

Now consider the trajectory $\{w(k)\}$ with $w(0) = w^*$ of the discrete monotone system $w(k+1) = \Gamma(w(k))$. If it tends to zero, then this implies that on the set $[0, w^*]$ the local small gain condition (LSGC) is satisfied, see Lemma 3.1.

B. A Local Small Gain Theorem

In this section we state a small gain theorem for LISS systems in the spirit of [1], [5].

The first step is a local version of the main ingredient used to prove the other small gain theorems.

Lemma 4.1: Let $w^* \in \mathbb{R}^n_+$, $w^* > 0$. Let Γ satisfy (LSGC) on $[0, w^*]$. Then there exists a $\varphi \in \mathcal{K}_\infty$, such that for all $w \in \mathbb{R}^n_+$, $w \leq w^*$ and all $v \in \mathbb{R}^n_+$ we have

$$(\mathrm{id} - \Gamma)(w) \leq v \implies \|w\| \leq \varphi(\|v\|).$$

The proof is essentially the same as in [5, Lemma 13]. The other important ingredient is an operator $D = \operatorname{diag}(\mathrm{id} + \alpha) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ for some $\alpha \in \mathcal{K}_\infty$.

Theorem 4.2: Let all subsystems (2), $i = 1, \ldots, n$, satisfy (5). Suppose Γ satisfies (LSGC). Then there exists a $\rho > 0$, a $\beta \in \mathcal{KL}$, and a $\gamma \in \mathcal{K}_\infty$, such that system (1) satisfies (LISS).

The proof divides into the following steps: First we establish that system (1) satisfies (LS) and (LAG), then we construct β for the (LISS) estimate. But beforehand we have to establish existence of solutions for all times and find ρ.

Proof: Throughout the proof let $w^* > 0$ be given by (LSGC), φ be given by Lemma 4.1. Let $x \in \mathbb{R}^N$, $N := \sum_{i=1}^n N_i$ as in (1) be split into parts as in (2). We use the notation $\|\cdot\|_c$ to denote componentwise norm, i.e., for $x \in \mathbb{R}^N$ this yields $\|x\|_c = (\|x_1\|, \ldots, \|x_n\|)^T$, where $\|x_i\|$ denotes the corresponding norm on $\mathbb{R}^{N_i}$, $i = 1, \ldots, n$.

Let $C > 0$ be the minimal constant, such that with $\|x\|_c \leq v \in \mathbb{R}^n_+$ also $\|x\| \leq C \cdot \|v\|$ holds for all $x \in \mathbb{R}^N$ and all $v \in \mathbb{R}^n_+$.

In the following a vector notation will be extremely useful: For now let $\beta(s,t) := (\beta_i(s_i, t), i = 1, \ldots, n)^T$ and $G(\|u\|_{c,\infty}) := (\gamma_i(\|u_i\|_\infty), i = 1, \ldots, n)^T$.

STEP 0 – EXISTENCE AND ρ Let $\varepsilon := \min_j w^*_j > 0$, so that $\|x\| < \varepsilon$ implies $\|x\|_c < w^*$. Let $\delta = \varphi^{-1}(\varepsilon/2\cdot C)$. For actual computations one could choose special norms to get $C \approx 1$ and instead of ${\varepsilon}/{2 \cdot C}$ one could use something just below ε, to get less conservative numerical estimates.

Let $\xi \in \mathbb{R}^{\sum_{i=1}^n N_i}$ be such that $\|\beta(\|\xi\|_c, 0)\| < \delta/2$ and $\|\xi\| < \varepsilon$. Let $u \in L^\infty(\mathbb{R}_+; \mathbb{R}^{\sum_i M_i})$ as in (1) such that $\|G(\|u\|_{c,\infty})\| < \delta/2$.

Define $T^* := \min\{t \geq 0 | \|x(t, \xi, u)\| \geq \varepsilon\}$. Clearly $\|x(\cdot, \xi, u)\|_{[0,T^*]} \leq \varepsilon$ and hence $\|x(\cdot, \xi, u)\|_{c,[0,T^*]} \leq w^*$. So we may apply Lemma 4.1 to the following inequality following from (5):

$$(\mathrm{id} - \Gamma)(\|x(\cdot, \xi, u)\|_{c,[0,T^*]}) \leq \beta(\|\xi\|_c, 0) + G(\|u\|_{c,\infty}).$$

Hence

$$\begin{aligned}\|x(\cdot, \xi, u)\|_{[0,T^*]} &\leq C \cdot \varphi(\|\beta(\|\xi\|_c, 0)\| + \|G(\|u\|_{c,\infty})\|) \\ &< C \cdot \varphi(\delta/2 + \delta/2) \\ &= C \cdot \varphi(\delta) \\ &= C \cdot {\varepsilon}/{2 \cdot C} = \varepsilon/2.\end{aligned}$$

This implies that there is no such minimal time T^*, such that the norm of the trajectory $x(\cdot, \xi, u)$ leaves the ε-ball around the origin. Hence this trajectory stays in that ball for all times.

Now let $\rho < \sup\{\|s\| : s \in \mathbb{R}^n, s \leq w^*, \|G(s)\| < \delta/2, \|\beta(s, 0)\| < \delta/2\}$.

Note that if LISS was defined as in (6), then in the above estimates we would use a maximum instead of a sum.

STEP 1 – ESTABLISHING LS For $\|u\|_\infty \leq \rho$ and $\|\xi\| < \rho$ we saw that

$$\|x(\cdot, \xi, u)\|_\infty \leq C \cdot \varphi(\|\beta(\|\xi\|_c, 0)\| + \|G(\|u\|_{c,\infty})\|)$$

which by the weak triangular inequality is

$$\leq C\varphi(2\|\beta(\|\xi\|_c, 0)\|) + C\varphi(2\|G(\|u\|_{c,\infty})\|) \tag{12}$$

and hence

$$\leq \sigma(\|\xi\|) + \gamma(\|u\|_\infty)$$

for some $\sigma, \gamma \in \mathcal{K}_\infty$. This establishes (LS).

STEP 2 – ESTABLISHING LAG$'$ For $u \in L^\infty$, $\|u\| \leq \rho$ and $\|\xi\| < \rho$ the LISS estimate for the subsystems gives us for $t > t_0 > 0$

$$\begin{aligned}\|x(t - t_0, x(t_0, \xi, u), u)\|_c &\leq \beta(\|x(t_0, \xi, u)\|_c, t - t_0) \\ &\quad + \Gamma(\|x(\cdot, x(t_0, \xi, u), u)\|_{c,[t_0,t]}) + G(\|u\|_{c,\infty}).\end{aligned}$$

For $t_0 = t/2$ this becomes

$$\begin{aligned}\|x(t/2, x(t/2, \xi, u), u)\|_c &\leq \beta(\|x(t/2, \xi, u)\|_c, t/2) \\ &\quad + \Gamma(\|x(\cdot, x(t/2, \xi, u), u)\|_{c,[t/2,t]}) + G(\|u\|_{c,\infty})\end{aligned}$$

and taking the $\limsup$ for $t \to \infty$, and by a result in [1], [5] we obtain

$$l(\xi, u) := \limsup_{t\to\infty} \|x(t/2, \xi, u)\|_c \leq \Gamma(l(\xi, u)) + G(\|u\|_{c,\infty}).$$

Since $l(\xi, u) \leq w^*$ we again may apply Lemma 4.1 to get $\|l(\xi, u)\| \leq C \cdot \varphi(\|G(\|u\|_{c,\infty})\|) \leq \gamma(\|u\|_\infty)$, which establishes (LAG$'$).

STEP 3 – CONSTRUCTING THE $\mathcal{KL}$-FUNCTION To this end let

$$\tilde{\beta}(s, t) := \sup_{\|u\|_\infty \leq \rho, \|\xi\| \leq s} (\|x(t, \xi, u)\| - \gamma(\|u\|_\infty))^+,$$

where a^+ denotes $\max\{a, 0\}$. By compactness of the set where the supremum is taken, the supremum is attained, finite, and for $t \to \infty$ the function $\tilde{\beta}$ tends to zero. It is clearly increasing in s and continuous and hence can be bounded above by a function β of class $\mathcal{KL}$, see for example [11]. With this β and γ from step 2 we obtain the LISS for (1). This completes the proof. ∎

V. AN EXAMPLE

Remark 5.1: In Definition 2.2 of LISS or in (5) for the case of several inputs we used the summation of β and gain function(s). An equivalent formulation can be given using a maximum of these terms, as in (6). The small gain conditions (10) and (LSGC) for ISS or LISS do not change in that case.

In the literature since [12] ISS using the $\beta + \gamma$-estimates has often been proved by showing first that there exists an ISS Lyapunov function. We use the same approach here. The gains we get using this approach are in our case also suitable for the $\max\{\beta, \gamma\}$-formulation of ISS.

The $\mathcal{KL}$-functions β_i, $i = 1, \ldots, n$, will not be given explicitly, but what is important to note is just that $\beta_i(s, 0) = s$ for all $s \in \mathbb{R}_+$.

A. A Network System

The following system on $\mathbb{R}^6_+$ is motivated by applications in logistics, see [7]. Although we establish that the network system is LISS, the estimates given by Lemmas 3.2 and 4.1 can be very conservative.

$$\dot{x}_1 = u - \frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} \tag{13}$$

$$\dot{x}_2 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_3, c_3x_3\} - \min\{b_2, c_2x_2\} \tag{14}$$

$$\dot{x}_3 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_2, c_2x_2\} - \min\{b_3, c_3x_3\} \tag{15}$$

$$\begin{aligned}\dot{x}_4 = &\frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_2, c_2x_2\} \\ &+ \min\{b_3, c_3x_3\} - \min\{b_4, c_4x_4\}\end{aligned} \tag{16}$$

$$\dot{x}_5 = \frac{1}{2}\min\{b_4, c_4x_4\} - c_5x_5 \tag{17}$$

$$\dot{x}_6 = \frac{1}{2}\min\{b_4, c_4x_4\} - c_6x_6 \tag{18}$$

This system can be seen as a composition of 6 interconnected one dimensional systems Σ_i, $i = 1, \ldots, n$, each regarding only x_i as a state and x_j, $j \neq i$, as inputs. The associated graph of this network is depicted in figure 1. If the constants $a, b, b_2, b_3, b_4, c_2, c_3, c_4, c_5$, and c_6 are chosen properly, then each subsystem should be LISS: Choosing

$$\begin{aligned} a &= 1/4 & b &= 1/2 \\ b_2 &= 5 & b_3 &= 3 \\ b_4 &= 4 & c_2 &= 7/8 \\ c_3 &= 5/4 & c_4 &= 2/3 \\ c_5 &= 1 & c_6 &= 1 \end{aligned}$$

one can check that each subsystem is LISS using LISS Lyapunov functions $V_i(x_i) = x_i$ (see [4]) with the following gains:

$$\begin{aligned} \gamma_{12}(x_2) &= x_2^2/m^2 & \gamma_{13}(x_3) &= x_3^2/m^2 \\ \gamma_{21}(x_1) &= \sqrt{x_1}/p_2 & \gamma_{23}(x_3) &= x_3/q_2 \\ \gamma_{31}(x_1) &= \sqrt{x_1}/p_3 & \gamma_{32}(x_2) &= x_2/q_3 \\ \gamma_{41}(x_1) &= \sqrt{x_1}/p_4 & \gamma_{42}(x_2) &= x_2/r_{42} \\ \gamma_{43}(x_3) &= x_3/r_{43} & \gamma_{1u}(u) &= u^2/c_u^2 \end{aligned}$$

and

$$\begin{aligned} \gamma_{54}(x_4) &= 1/2 \min\{b_4/c_5, c_4/c_5 x_4\} + \varepsilon(x_4) \\ \gamma_{64}(x_4) &= 1/2 \min\{b_4/c_6, c_4/c_6 x_4\} + \varepsilon(x_4) \end{aligned}$$

Here ε denotes some arbitrary slowly growing $\mathcal{K}_\infty$ function and

$$\begin{aligned} m &= 9/15 & c_u &= 1/5 \\ p_2 &= 5 & p_3 &= 4 \\ p_4 &= 3/10 & & \\ q_2 &= 8/3 & q_3 &= 11/10 \\ r_{42} &= 16/70 & r_{43} &= 8/50. \end{aligned}$$

Namely the following holds true: For $i = 1$ we have

$$x_1 > \max_{j>1}\{\gamma_{1j}(x_j), \gamma_{1u}(u)\}$$

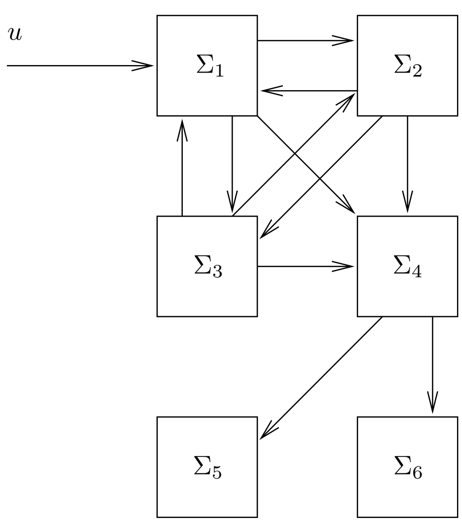

Fig. 1. The example in Section V consists of a strongly connected component and a cascade of subsystems. This translates into the block form of Γ.

implies

$$\dot{x}_1 < 0$$

and for $x_2 \le 9/10$, $x_3 \le 12/5$, and $x_4 \le 6$ it holds that

$$x_i > \max_{j \neq i} \gamma_{ij}(x_j)$$

implies

$$\dot{x}_i < 0$$

for $i = 2, \ldots, 6$.

The gain matrix Γ in this example looks like

$$\Gamma = \begin{bmatrix} 0 & \gamma_{12} & \gamma_{13} & 0 & 0 & 0 \\ \gamma_{21} & 0 & \gamma_{23} & 0 & 0 & 0 \\ \gamma_{31} & \gamma_{32} & 0 & 0 & 0 & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_{54} & 0 & 0 \\ 0 & 0 & 0 & \gamma_{64} & 0 & 0 \end{bmatrix}.$$

We see that Γ is lower block triangular with an upper left 3×3 irreducible block and a lower right nilpotent 3×3 block on the diagonal. We like to find a vector w^* satisfying $\Gamma(w^*) < w^*$. A few basic considerations lead us to the following vector

$$w^* = \begin{bmatrix} 2 \\ 0.3 \\ 0.4 \\ 5 \\ 2 \\ 2 \end{bmatrix}.$$

This vector will be the starting point of the numerical test described in Section IV-A. We could also use the algorithm proposed in [8] for some given radius r to find another w^*. Now we can apply Lemma 3.1 to see if the small gain condition is met on the set $[0, w^*] \setminus \{0\}$.

The condition (LSGC) is satisfied by Lemma 3.1. By the method given in the proof of Lemma 3.2 we find an operator D. Figure 2 shows the computed sequence $\{\tilde{\alpha}\}_{k=1}^{k_0}$

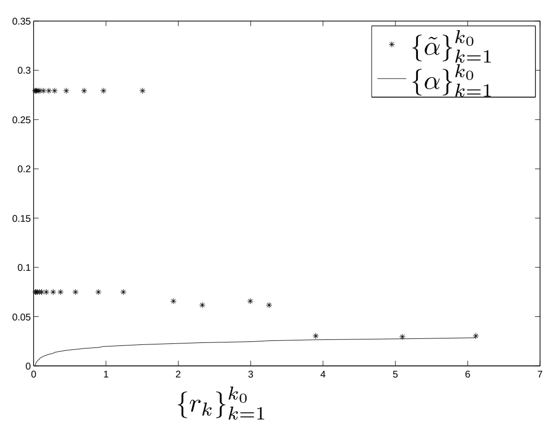

Fig. 2. Constructing the operator D: The sequences $\tilde{\alpha}$ and α plotted versus the sequence r up to some index k_0.

as function of $\{r_k\}_{k=1}^{k_0}$ and a bounding sequence $\{\alpha\}_{k=1}^{k_0}$ needed for the construction of D. Here we used the notation given in the proof of Lemma 3.2.

Next we are interested in the function φ of Lemma 4.1, as it gives us the gain from u to $(x_1,\dots,x_n)^T$ and φ^{-1} is involved in the computation of $\rho>0$, which gives us the region where LISS holds.

The function φ can be found by the method in the proof of Lemma 4.1, see [1] or [5] for details, in our case this estimate for φ is just

$$\varphi(s) = \|(D\circ(D-\mathrm{id})^{-1}\circ(\mathrm{id}+\Gamma))^4(s\cdot e)\|_\infty, \qquad (19)$$

where $s\in\mathbb{R}_+$ and $e=(1,\dots,1)^T$. Unfortunately this is not only not a sharp estimate but in fact quite conservative, since in the above mentioned proof at several steps very rough estimates are made. The function φ given by (19) is plotted in Figure 3. Now given φ, this allows us to choose $\delta^* < \varphi^{-1}(0.3)$, which leads to $\rho^* := \sup\{\|s\| : s\in\mathbb{R}^n, s\le w^*, \|G(s)\| < \delta^*/2, \|\beta(s,0)\| < \delta^*/2\}$. This ρ^* bounds the initial values $\|xi\|$, $\xi\in\mathbb{R}^6_+$, and inputs $\|u\|_\infty$, $u\in L^\infty(\mathbb{R}_+,\mathbb{R}_+)$. For actual applications this is too conservative, since already δ^* is very small. Better estimates for φ have to be found. Hence for $\|u\|_\infty\le\rho^*$ and $\|\xi\|\le\rho^*$ the system (13)—(18) satisfies (LISS) where γ can be taken from (12).

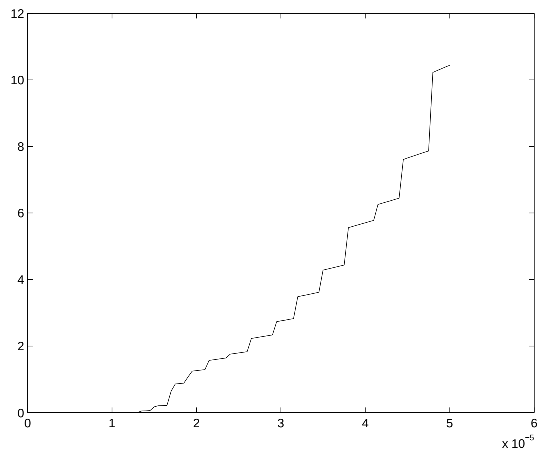

Fig. 3. The function φ constructed in the proof of Lemma 4.1 is too conservative for practical applications.

VI. CONCLUSIONS AND FUTURE WORK

A. Conclusions

We have presented an efficient tool for numerical verification of the stability properties of an interconnection of several locally input to state stable systems. The local small gain condition can be checked using this algorithm. An example illustrates how this method works. A drawback is the conservative estimate for the region, where the local stability property actually holds.

B. Future Work

Better estimates for φ in Lemma 4.1 should be possible and are needed for actual applications of this result. The next step is to extend this method on the systems satisfying practical stability property.

VII. ACKNOWLEDGMENTS

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