# Ensemble Feedback Methods for Families of Linear Systems

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## 1. EXTENDED ABSTRACT

Ensemble control is a rather new research area of control theory which is concerned with a whole parameter-dependent "family of systems" (= ensemble) instead of a single one. Here, the major challenge is to achieve classical control task simultaneously, i.e. for the entire ensemble via controls that are *independent* of the system parameter. Thus the topic of ensemble control is located at the crossroad of finite- and infinite-dimensional control theory, operator theory and approximation theory. From a mathematical point of view, this intimate interplay of various disciplines is expected to lead to deep results with impact far beyond control theory.

The problem of simultaneous stabilization of parameterized families of linear systems falls into this setting. In this context, parameter-dependent pole-shifting has been addressed in the 1980s and 1990s. We note that the contributions used rather different methods. On the one hand, in Hautus and Sontag (1986); Sontag (1985); Sontag and Wang (1990) the algebraic theory of systems over rings is used. On the other hand, for the simultaneous stabilization problem a frequency domain approach using function theoretic methods was proposed by various authors. For details and more references we refer to the comprehensive monograph Blondel (1994). Moreover, more recent contributions are Guth et al. (2023); Ryan (2014). In this work, we will use tools from functional analysis and approximation theory to study the possibilities and limits of ensemble feedback methods for families of one-parameter families of linear systems.

### *1.1 Setting and Notation*

We consider parameterized system, where the parameter space $\mathbf{P} \subset \mathbb{C}$. It is assumed that $\mathbf{P}$ is compact with empty interior and that $\mathbb{C}\backslash\mathbf{P}$ is connected. The parameter-dependent systems under consideration are of the form

$$\frac{\partial}{\partial t}x(t,\theta) = A(\theta)x(t,\theta) + B(\theta)u(t) \tag{1}$$

or in discrete-time

$$x(t+1,\theta) = A(\theta)x(t,\theta) + B(\theta)u(t), \tag{2}$$

where $x(0,\theta) = x_0(\theta) \in \mathbb{C}^n$ denotes the initial conditions and the matrices $A(\theta) \in \mathbb{C}^{n\times n}$ depend continuously on the parameter $\theta$. We consider the families of states $\{x(\theta) \mid \theta \in \mathbf{P}\}$ in (1) as functions from the parameter space $\mathbf{P}$ to $\mathbb{C}^n$ located in a suitable separable Banach space. The formal definition of the state space is as follows. Let $X(\mathbf{P})$ denote an arbitrary separable Banach space of functions defined on $\mathbf{P}$ with values in $\mathbb{C}$ and let $X_{n,m}(\mathbf{P})$ consist of all $(n \times m)$-matrices with entries in $X(\mathbf{P})$. Furthermore, set $X_n(\mathbf{P}) := X_{n,1}(\mathbf{P})$. Thus $X_n(\mathbf{P})$ is simply the $n$-fold Cartesian product of $X(\mathbf{P})$ and therefore again a Banach space. Unless stated otherwise, we assume that $X_n(\mathbf{P})$ is equipped with the maximum norm of the entrywise norms, i.e. $\|x\|_{X_n(\mathbf{P})} := \max_{1\le i\le n} \|x_i\|_{X(\mathbf{P})}$. In this paper, despite of treating the general case, we consider the Banach space of continuous functions. We note that for other cases, e.g. the space of integrable functions $X_n(\mathbf{P}) := L^q_n(\mathbf{P})$ the corresponding $L^q$-construction might be a better choice. Moreover, we assume that the matrices $B(\theta) \in \mathbb{C}^{n\times m}$ to lie in $X_{n,m}(\mathbf{P})$.

In the following, we denote the space of continuous functions $f : \mathbf{P} \to \mathbb{C}$ by $C(\mathbf{P})$ and then $C_n(\mathbf{P})$ respectively $C_{n,m}(\mathbf{P})$ denote the spaces of $n$-vectors, $n \times m$-matrices, whose entries are continuous functions on $\mathbf{P}$.

For fixed $A \in C_{n,n}(\mathbf{P})$ we always assume that the induced multiplication operator

$$\mathcal{A} : X_n(\mathbf{P}) \to X_n(\mathbf{P}), \quad \mathcal{A}f(\theta) := A(\theta)f(\theta) \tag{3}$$

is well-defined, linear and bounded. Also, for any $B \in X_{n,m}(\mathbf{P})$ the input operator

$$\mathcal{B} : \mathbb{C}^m \to X_n(\mathbf{P}), \quad (\mathcal{B}v)(\theta) := B(\theta)v$$

is well-defined, linear and bounded as it is a multiplication operator with finite dimensional domain $\mathbb{C}^m$. In terms of these matrix multiplication operators, the dynamic equations (1) and (2) are equivalent to the (infinite dimensional) linear control systems

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad x(0) = x_0 \in X_n(\mathbf{P}) \tag{4}$$

and correspondingly

$$x(t+1) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad x(0) = x_0 \in X_n(\mathbf{P}). \tag{5}$$

To handle the continuous and discrete-time case at the same time we take $x_0 = 0$ and we will denote for $T \ge 0$ the set of inputs by $U(T)$. That is, in continuous-time one has $U(T) := L^1([0,T],\mathbb{C}^m)$ and in discrete-time $U(T) := (\mathbb{C}^m)^{T+1}$ From now on $T \ge 0$ is either a nonnegative real or a natural number depending on the system under consideration. Moreover, we denote the solutions to (1) and (2) by $\varphi\big(T,u,x_0\big)(\theta) := \varphi\big(T,u,x_0(\theta),\theta\big)$, i.e. in the continuous-time case it holds

$$\varphi\big(T,u,0\big)(\theta) = \int_0^T \mathrm{e}^{(T-\tau)A(\theta)}B(\theta)u(\tau)\,\mathrm{d}\tau$$

and in discrete-time we have

$$\varphi\big(T,u,0\big)(\theta) = \sum_{k=0}^{T-1} A(\theta)^{T-1-k}B(\theta)u(k).$$

A central notion of this paper is the following version of reachability. We refer to standard literature for the definition of the notion of reachability for finite-dimensional linear systems, e.g. (Sontag, 1998, Chapter 3).

*Definition 1.* A pair $(A,B) \in C_{n,n}(\mathbf{P}) \times X_{n,m}(\mathbf{P})$ is called

(i) *pointwise reachable*, if for all $\theta \in \mathbf{P}$ the pair $(A(\theta),B(\theta)) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ is reachable.
(ii) *ensemble reachable* with respect to $X_n(\mathbf{P})$, if for all $f \in X_n(\mathbf{P})$ and $\varepsilon > 0$ there exist $T \geq 0$ and an input $u \in U(T)$ such that
$$\|\varphi(T,u,0) - f\|_{X_n(\mathbf{P})} < \varepsilon.$$
(iii) *uniformly ensemble reachable* if it is ensemble reachable with respect to $C_n(\mathbf{P})$.

It is shown in (Triggiani, 1975a, Theorem 3.1.1), that a pair $(A,B) \in C_{n,n}(\mathbf{P}) \times X_{n,m}(\mathbf{P})$ is ensemble reachable with respect to $X_n(\mathbf{P})$ if and only if

$$\operatorname{span}\big\{\operatorname{Im}\mathcal{A}^k\mathcal{B}\,|\,k=0,1,2,3,...\big\}$$

is dense in $X_n(\mathbf{P})$, where Im denotes the image.

## 2. ENSEMBLE FEEDBACK FOR FAMILIES ON BANACH SPACES

The goal of this paper is to start/continue with a systematic analysis of feedback methods for this class of systems. Recall that a central point in ensemble reachability is that the input $u$ does not depend on the parameters and serves as a simultaneous input applied for all parameters. Consequently, we shall consider *ensemble feedback operators* given by bounded linear operators

$$\mathcal{F} : X_n(\mathbf{P}) \to \mathbb{C}^m. \tag{6}$$

We emphasize here that, because $\mathcal{F}$ has finite dimensional range, ensemble feedback operators are automatically compact, (Conway, 1990, p. 174). Natural choices might be given by the integral operator

$$\mathcal{F}f = \int_{\mathbf{P}} K(\theta)f(\theta)\,\mathrm{d}\theta, \qquad K \in C_{m,n}(\mathbf{P}),$$

or the weighted average operator

$$\mathcal{F}f = \sum_{k=1}^{N} K_k f(\theta_k), \qquad K_k \in \mathbb{C}^{m\times n}.$$

Note that, the latter serve as examples and the subsequent analysis is not limited to these choices.

The contributions of this paper are twofold. First, we treat ensemble defined on arbitrary separable Banach spaces and we will explore classical system-theoretic properties for ensemble feedback operators.

Let $X_n(\mathbf{P})$ denote a separable Banach space of functions from the parameter space $\mathbf{P}$ to $\mathbb{C}^n$. Also We assume that the multiplication operators $\mathcal{A}\colon X_n(\mathbf{P}) \to X_n(\mathbf{P})$ and $\mathcal{B}\colon \mathbb{C}^m \to X_n(\mathbf{P})$ are bounded linear. Then, using an bounded linear ensemble feedback operator $\mathcal{F}\colon X_n(\mathbf{P}) \to C^m$, the overall systems can be written as follows

$$\dot{x}(t) = (\mathcal{A} + \mathcal{B}\,\mathcal{F})\,x(t) + \mathcal{B}u(t) \tag{7}$$

The first result is obtained by using arguments of (Hinrichsen and Pritchard, to appear, Chapter 8).

*Theorem 1.* Ensemble reachability is invariant under ensemble feedback.

The proof will appear in a forthcoming publication.

We will now study the stabilization problem for the class of infinite dimensional systems given by (4) and (5). We start by considering the stability properties of the uncontrolled system

$$\begin{aligned}\dot{x}(t) &= \mathcal{A}x(t)\\ x(0) &= x_0.\end{aligned} \tag{8}$$

The multiplication operator $\mathcal{A}$ generates the semigroup

$$T(t)\colon X_n(\mathbf{P}) \to X_n(\mathbf{P}), \quad T(t)f(\theta) := e^{tA(\theta)}f(\theta). \tag{9}$$

From (Engel and Nagel, 2000, Ch I Sec. 4 a,Ch II Sec. 2 b) it follows that $T(t)$ is uniformly continuous. Recalling that the spectrum of a bounded linear operator $\mathcal{L}$ is defined as

$$\sigma(\mathcal{L}) = \{\lambda \in \mathbb{C} \mid \lambda - \mathcal{L} \quad \text{is not a bijection}\},$$

we note the spectrum of the matrix multiplication operator $\mathcal{A}$ is given by

$$\sigma(\mathcal{A}) = \bigcup_{\theta\in\mathbf{P}} \sigma(A(\theta)), \tag{10}$$

where $\sigma(A(\theta))$ denotes the set of eigenvalues of the matrix $A(\theta)$, cf. (Hardt and Wagenführer, 1996, Example 2.7 2). Moreover, from (Engel and Nagel, 2000, Ch. I, Sec. 3) we deduce the following characterization of stability for system (8).

*Proposition 2.* Let $A \in C_{n,n}(\mathbf{P})$ and let $\mathcal{A}$ denote the corresponding matrix multiplication operator defined on a separable Banach space $X_n(\mathbf{P})$. Then, for the linear system (8) the following statements are equivalent.

(1) The origin is uniformly asymptotically stable, i.e. one has
$$\lim_{t\to\infty} \|T(t)\| = 0.$$
(2) The origin is exponentially stable, i.e. there are constants $M \geq 1$ and $\gamma > 0$ such that $\|T(t)\| \leq Me^{-\gamma t}$.
(3) The spectrum of $\mathcal{A}$ is contained in the open left half plane, i.e.
$$\sigma(\mathcal{A}) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}.$$

Note that (Engel and Nagel, 2000, Exercise 3.4.8) gives an example of a multiplication operator on $C(\mathbb{R},\mathbb{C})$ which generates an asymptotically stable semigroup in the sense that for all $x_0$

$$\lim_{t\to\infty} \|T(t)x_0\| = 0,$$

but the semigroup is not uniformly asymptotically stable.

From this characterization we turn to the stabilization problem. We start with a definition of stabilizability, cf. Pritchard and Zabczyk (1981); Triggiani (1975b).

*Definition 2.* A pair $(A, B) \in C_{n,n}(\mathbf{P}) \times X_{n,m}(\mathbf{P})$ is called (ensemble) stabilizable if there is exists an ensemble feedback operator $\mathcal{F}\colon X_n(\mathbf{P}) \to \mathbb{C}^m$ such that the closed-loop system defined by $\mathcal{A} + \mathcal{B}\mathcal{F}$ is exponentially stable.

*Theorem 3.* Suppose the pair $(A, B) \in C_{n,n}(\mathbf{P}) \times X_{n,m}(\mathbf{P})$ is not exponentially stable. Then, it cannot be exponentially stabilized by any ensemble feedback, i.e. for any ensemble feedback operator $\mathcal{F}\colon X_n(\mathbf{P}) \to \mathbb{C}^m$ the closed-loop system $\mathcal{A} - \mathcal{B}\mathcal{F}$ is not exponentially stable.

The proof will appear in a forthcoming publication. The main argument is that the essential spectrum (in the sense of Kato) of a bounded linear operator is invariant under compact perturbations. To this end we recall relevant notions concerning the spectra of bounded operators. As in (Engel and Nagel, 2000, Ch. IV, 1.20) we call

$$\sigma_{\text{ess}}(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \lambda - \mathcal{A} \quad \text{is not a Fredholm operator}\}$$

the essential spectrum (in the sense of Kato) of $\mathcal{A}$. We note that there are various different definitions for the essential spectrum, which do not coincide in general. For a comprehensive summary of spectra of bounded operators and their relation, we refer to (Appell et al., 2004, Chapter 1) and the references therein. However, in our context it holds that $\sigma_{\text{ess}}(\mathcal{A}) = \sigma(\mathcal{A})$.

## 3. RELAXED ENSEMBLE FEEDBACK FOR CONTINUOUS FAMILIES

Motivated by the disappointing result of Theorem 3, we now intend to study a larger class of feedback operators. The second contribution of this paper is to show that feedback using multiplication operators can be used for stabilization.

For ensembles defined on the space of continuous functions, we will study *relaxed feedback operators* given by multiplication operators, i.e. feedback operators of the form

$$\mathcal{K} : X_n(\mathbf{P}) \to X_m(p), \quad \mathcal{K}x(\theta) = K(\theta)x(\theta), \tag{11}$$

where $K \in X_{m,n}(\mathbf{P})$. This class of feedback operators was recently considered in Schönlein (2021), where also the case $K(\theta) \equiv K \in \mathbb{R}^{m\times n}$ is considered for the controlled harmonic oscillator. For relaxed feedback operators in Theorem 7 provides new sufficient conditions so that a continuous ensemble becomes uniformly ensemble reachable by applying a mixture of open-loop inputs and relaxed feedback operators of the form

$$u(t, x(t, \theta)) = u(t) + K(\theta)x(t, \theta), \qquad K \in C_{m,n}(\mathbf{P}).$$

We note that this result improves Theorem 3 in Schönlein (2021), as Theorem 7 does not require the application of transformations in the state-space and the input space.

To overcome the limitations of ensemble feedback methods we consider in this section feedback operators of the form

$$\begin{aligned} &\mathcal{K} : X_n(\mathbf{P}) \to X_m(\mathbf{P}) \\ &\mathcal{K}f(\theta) = K(\theta)\, f(\theta), \qquad K \in C_{m,n}(\mathbf{P}). \end{aligned}$$

It is readily seen, that the range of these operators is infinite dimensional and one might expect to get results that are in line with finite-dimensional linear systems theory. More precisely, we are aiming for extension of (Schönlein, 2021, Theorem 3) in the sense that transformations in the state and input variables are not required.

A crucial step in the construction procedure is the following ensemble version of Heymann's Lemma. To state the first result, we start with a recap. For a pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, we consider the list (a permutation of columns the Kalman matrix of $(A, B)$)

$$b_1 \ Ab_1 \ \cdots A^{n-1}b_1 \ \cdots \ b_m \ Ab_m \cdots \ A^{n-1}b_m.$$

Then, we select from left to right the first linear independent columns

$$b_1, \dots, A^{h_1-1}b_1, \dots, b_m, \dots, A^{h_m-1}b_m.$$

The corresponding exponents $h_1, ..., h_m$ are called the *Hermite indices* of $(A, B)$, where $h_i := 0$ if $b_i$ is not selected, see also Fuhrmann and Helmke (2015). It may be shown that the columns thus selected are of the form

$$b_{l_1}, ..., A^{h_{l_1}-1}b_{l_1}, b_{l_2}, ..., A^{h_{l_2}-1}b_{l_2}, \dots, b_{l_\mu}, ..., A^{h_{l_\mu}-1}b_{l_\mu}$$

with $n = \sum_{j=1}^{\mu} h_{l_j}$. For brevity, we write

$$M_{A,B} =: \left(b_{l_1} \ \dots \ A^{h_{l_1}-1}b_{l_1} \ \dots \ b_{l_\mu} \ \dots \ A^{h_{l_\mu}-1}b_{l_\mu}\right).$$

The following generalizes to the ensemble case a lemma which is frequently used as a preparatory step in the proof of Heymann's lemma, Heymann (1968).

*Lemma 4.* Let $(A, B) \in C_{n,n}(\mathbf{P}) \times C_{n,m}(\mathbf{P})$ be pointwise reachable and suppose that the Hermite indicies are constant. Then, there are $v_1, ..., v_n \in \mathbb{R}^m$ such that the vectors $x_1(\theta), ..., x_n(\theta)$ defined by

$$x_k(\theta) := A(\theta)x_{k-1}(\theta) + B(\theta)v_k(\theta), \qquad x_0(\theta) := 0$$

are linearly independent in $C_n(\mathbf{P})$.

*Corollary 5.* Let $(A, B) \in C_{n,n}(\mathbf{P}) \times C_{n,m}(\mathbf{P})$ be pointwise reachable and suppose that the Hermite indices are constant. Then, for the vectors $x_1(\theta), ..., x_n(\theta)$ constructed in Lemma 4 we have

$$\det\left(x_1 \ \dots \ x_n\right) = \det M_{A,B}.$$

Another result we will use in this work concerns sufficient conditions for uniform ensemble reachability that are verifiable just in terms of the matrices $A(\theta)$ and $B(\theta)$. The next result is a mild refinement of (Dirr and Schönlein, 2021, Corollary 4) in the sense that it puts weaker assumptions on the properties of the parameter space. We note that this is partially contained in Danhane et al. (2024).

*Proposition 6.* Let $\mathbf{P}$ be compact with empty interior such that $\mathbb{C} \setminus \mathbf{P}$ is connected. Then, a pair $(A, b) \in C_{n,n}(\mathbf{P}) \times C_n(\mathbf{P})$ is uniformly ensemble reachable if the following conditions are satisfied:

(a) $(A(\theta), b(\theta))$ is reachable for all $\theta \in \mathbf{P}$.
(b) For all distinct parameters $\theta, \theta' \in \mathbf{P}$, the spectra $\sigma\big(A(\theta)\big)$ and $\sigma\big(A(\theta')\big)$ are disjoint.
(c) For each $\theta \in \mathbf{P}$, the eigenvalues of $A(\theta)$ are simple.

A proof of this and the next statement will be provided in a forthcoming publication. Lemma 4 is essential for the verification of the following main result of this section.

*Theorem 7.* Suppose $(A, B) \in C_{n,n}(\mathbf{P}) \times C_{n,m}(\mathbf{P})$ is pointwise reachable and has constant Hermite indices. Then, there is a $K \in C_{m,n}(\mathbf{P})$ such that $(A-BK, B)$ is uniformly ensemble reachable.

Compared to (Schönlein, 2021, Theorem 3), Theorem 7 puts weaker assumptions on the matrix pair $(A, B)$. More precisely, the strong assumption that the Kronecker indices are constant is removed. In addition, for the parameter space $\mathbf{P}$ it is only assumed that $\mathbb{C} \setminus \mathbf{P}$ is connected (which is the case if $\mathbf{P}$ is a Jordan arc). In addition, the assertion of Theorem 7 is stronger than that of (Schönlein, 2021, Theorem 3) as the result shows that it is sufficient to use a continuous feedback matrix $F(\theta)$, whereas in (Schönlein, 2021, Theorem 3) a restricted feedback transformation is required, (i.e. additional similarity transformations on the input space $\mathbb{R}^m$ and the state space $\mathbb{R}^n$).

Using Theorem 7 together with Proposition 2 it is not hard to obtain the following result on exponential stabilization.

*Corollary 8.* Suppose $(A, B) \in C_{n,n}(\mathbf{P}) \times C_{n,m}(\mathbf{P})$ is pointwise reachable and has constant Hermite indices. Then, there is a relaxed ensemble feedback $\mathcal{K} : C_n(\mathbf{P}) \to C_m(\mathbf{P})$ such that $\mathcal{A} - \mathcal{B}\mathcal{K}$ is exponentially stable.

## 4. CONCLUSIONS

We have considered ensemble feedback for one-parameter families of linear systems and discussed different feedback methods. One the one hand, it is observed that feedback operators with finite-dimensional range cannot be used for stabilization. Consequently treated relaxed feedback methods by considering multiplication feedback operators. In this context we proved a version of Heymann's Lemma for continuous families of systems with constant Hermite indices. In future research we will investigate how this condition can be weakened.

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