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The Joint Spectral Radius of Principal Submatrices

J. Epperlein and F. Wirth

Faculty of Computer Science and Mathematics, University of Passau, Innstraße 33, 94032 Passau, Germany, (e-mail: jeremias.epperlein,fabian.(lastname)@uni-passau.de).

Abstract: For compact sets of complex matrices it is shown that there always exists a similarity transformation such that in the transformed set all entries of all matrices are bounded in absolute value by the joint spectral radius. The key tool for this is that every extremal norm of a matrix set has an Auerbach basis. The result implies in particular that all diagonal entries, or equivalently all one-dimensional principal submatrices, are upper bounded by the joint spectral radius. It is shown that the corresponding statement for higher dimensional principal submatrices is false. More precisely, there are finite matrix sets, such that for all points in the similarity orbit the joint spectral radii of all higher dimensional principal submatrices are strictly larger than the joint spectral radius of the original matrix set.

Keywords: Joint spectral radius, linear discrete-time system, Auerbach basis, switched systems, stability theory.

1. INTRODUCTION

The joint spectral radius of a compact set of matrices describes the maximal exponential growth rate of products of matrices from this set. Interest in this quantity has been driven by applications in wavelet theory, Daubechies and Lagarias (1992), Lagarias and Wang (1995). Also for switched systems, it characterizes the exponential stability of a discrete-time linear switched system with arbitrary switching, Shorten et al. (2007). For a background on the joint spectral radius see Jungers (2009).

It was already noticed in the original paper by Rota and Strang (1960) that norms are instrumental in the analysis of the joint spectral radius, and this has been reconfirmed, e.g. in Elsner (1995), Barabanov (1988), Wirth (2002), Morris (2010), Bochi and Laskawiec (2023).

In this paper we study the relation of the entries the matrices in a given matrix sets and the joint spectral radius. Of course, the trace of all matrices may be 0 and the joint spectral radius be arbitrarily large. Similarly, there are bounded matrix sets with joint spectral radius 1 and arbitrarily large entries. So at first glance there is no relation. The problem is whether it is possible to find a representation of a given matrix set in its similarity orbit so that the entries of all matrices in the transformed set are bounded by the joint spectral radius. For matrix sets with positive joint spectral radius this is possible and the idea of proof is to show that it is always possible to transform the matrix set in such a way that there is an extremal norm sandwiched between the 1- and the ∞-norm.

The results relate to an approximation theorem due to Ando and Shih (1998) where it was shown to which extent quadratic norms can be used to approximate the joint spectral radius. In general, many of the numerical procedures available for the computation of the joint spectral radius can be interpreted in terms of extremal norms, see e.g. Guglielmi and Zennaro (2008), Guglielmi and Protasov (2013)

To place our problem in the context of the normal form problem, note that the general linear group acts on the space of compact matrix sets and the joint spectral radius is an invariant of this action. The normal form problem is to find canonical representatives of the orbits of the action, but this problem seems as yet elusive, even for finite matrix sets. We may interpret the result as the identification of an interesting subset of the similarity orbit, in which canonical forms could be looked for.

Our first result is that if the joint spectral radius is strictly positive, then the similarity orbit of the matrix set contains an instance in which all entries of all matrices are less or equal to the joint spectral radius in absolute value. This result is obtained in Section 3. Our main tool for this are extremal norms and their associated Auerbach bases. Auerbach bases are a tool from the theory of the geometry of finite-dimensional Banach spaces. They are given by biorthogonal sequences with unit norm in the norm, respectively the dual norm. Their existence was first shown by Auerbach with independent proofs by Day (1947) and Taylor (1947). More recently, Weber and Wojciechowski (2017) have shown that for $d > 2$ there exist at least $(d-1)d/2 + 1$ such bases, ignoring permutations of the basis vectors or multiplication by scalars of modulus 1. As it turns out, Auerbach bases of extremal norms can be used to define a similarity transformation that yields the desired rescaling of the matrix set.

It might be hoped that this approach can be carried further and that also for higher dimensional submatrices, a similarity transformation can be found such that the joint spectral radius of these submatrices is bounded by that of the original matrix set. In Section 4 we show that this is in general not possible. There are finite matrix sets such that for the entire similiarity orbit all joint spectral radii corresponding to principal submatrices of order $2, ..., d-1$ are bounded away from the joint spectral radius of the given matrix set. We believe this to be a rather interesting and maybe surprising negative result that provides a further facet of the intricate theory of the joint spectral radius.

2. PRELIMINARIES

Let $\mathbb{N}$ be the set of natural numbers including 0. The real and complex field are denoted by $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}_{\geq 0} := [0, \infty)$. The Euclidean norm on $\mathbb{R}^d, \mathbb{C}^d$ is denoted by $\|\cdot\|_2$ and this also denotes the induced operator norm, i.e., the spectral norm on $\mathbb{R}^{d\times d}, \mathbb{C}^{d\times d}$. We write $\mathbb{K}$ if the field is either real or complex. The general linear group (of invertible matrices) in $\mathbb{K}^{d\times d}$ is denoted by $\mathbf{GL}_d(\mathbb{K})$. Where convenient, we use the abbreviating notation $\underline{d} = \{1, \dots, d\}$.

Let $d \geq 1$. For a bounded, nonempty set of matrices $\mathcal{M} \subset \mathbb{K}^{d\times d}$ we consider the set of arbitrary products of length t defined by

$$\mathcal{S}_t := \{A(t-1)\dots A(0) \; ; \; A(s) \in \mathcal{M}, s = 0, \dots, t-1\}.$$

The joint spectral radius of $\mathcal{M}$ is defined as

$$\rho(\mathcal{M}) := \lim_{t\to\infty} \sup\{\|S\| \; ; \; S \in \mathcal{S}_t\}^{1/t}. \tag{1}$$

It is known that taking the closure of $\mathcal{M}$ does not change the value of the joint spectral radius. Thus we will assume that $\mathcal{M}$ is compact from now on.

A helpful formulation of the joint spectral radius is in terms of operator norms. Given a norm $\|\cdot\|$ on $\mathbb{K}^d$ and its induced operator norm also denoted by $\|\cdot\|$, we define

$$\|\mathcal{M}\| := \max\{\|A\|; A \in \mathcal{M}\}.$$

Then it is known, Rota and Strang (1960), that

$$\rho(\mathcal{M}) = \inf\{\|\mathcal{M}\|; \|\cdot\| \text{ is an operator norm}\}. \tag{2}$$

A norm $\|\cdot\|$ on $\mathbb{K}^d$ is called extremal for $\mathcal{M}$, if $\rho(\mathcal{M}) = \|\mathcal{M}\|$. An extremal norm is called a Barabanov norm, if in addition for every $x \in \mathbb{K}^d$ there exists an $A \in \mathcal{M}$ such that

$$\|Ax\| = \rho(\mathcal{M})\|x\|.$$

A sufficient condition for the existence of Barabanov norms is that the set $\mathcal{M}$ is irreducible, i.e. only the trivial subspaces $\{0\}$ and $\mathbb{K}^d$ are invariant under all $A \in \mathcal{M}$, Barabanov (1988); Wirth (2002).

3. AUERBACH BASES AND BOUNDS ON THE ENTRIES OF THE MATRICES

In this section we investigate how the entries of a given compact set of matrices may be uniformly smaller than the joint spectral radius. We need some preliminary statements for this.

Given a norm v on $\mathbb{K}^d$, the dual norm is defined by

$$v^*(y) := \max\{|\langle y, x\rangle| \mid v(x) \leq 1\}, \quad y \in \mathbb{K}^d. \tag{3}$$

A vector $y \in \mathbb{K}^n$ is called a dual vector of $x \in \mathbb{K}^d$, if

$$\langle x, y\rangle = v(x)v^*(y). \tag{4}$$

We will call x *self-dual*, if x is a dual vector to itself.

Definition 1. Let v be a norm on $\mathbb{K}^d$. A set of pairs $(x_i, y_i)_{i=1}^d$ in $\mathbb{K}^d \times \mathbb{K}^d$ is called a biorthogonal basis of $\mathbb{K}^d$, if $\langle x_i, y_j\rangle = \delta_{ij}$. If, in addition, $v(x_i) = v^*(y_i) = 1$, $i \in \underline{d}$, then $(x_i, y_i)_{i=1}^d$ is an Auerbach or biorthonormal basis.

It was shown by Auerbach that every finite dimensional normed real or complex space has a biorthonormal basis. We note the following small lemma, the proof is omitted.

Lemma 2. Let v be a norm on $\mathbb{K}^d$ with Auerbach basis $(x_i, y_i)_{i=1}^d$. Let $T \in \mathbb{K}^{d\times d}$ with columns x_i, $i = 1, \dots, d$. Then $v_T(\cdot) := v(T\cdot)$ is a norm for which the standard basis vectors form an Auerbach basis $(e_i, e_i)_{i=1}^d$. In particular,

$$\|\cdot\|_\infty \leq v_T(\cdot) \leq \|\cdot\|_1. \tag{5}$$

Theorem 3. Let $\mathcal{M} \subset \mathbb{K}^{d\times d}$ be compact and nonempty. If $\rho(\mathcal{M}) > 0$, then there exists a $T \in \mathbf{GL}_d(\mathbb{K})$ such that

$$\rho(\mathcal{M}) \geq \max\{|a_{ij}| \; ; \; A = (a_{\mu\nu})_{\mu,\nu=1}^d \in T^{-1}\mathcal{M}T, i, j \in \underline{d}\}.$$

Proof. We first consider the case that $\mathcal{M}$ is irreducible. Then we may choose an extremal norm v for $\mathcal{M}$. Let $(x_i, y_i)_{i=1}^d$ be an Auerbach basis for v and let T be the invertible matrix from Lemma 2 such that the norm v_T has an Auerbach basis $(e_i, e_i)_{i=1}^d$. Note that v_T is extremal for $T^{-1}\mathcal{M}T$ because for all $A \in \mathcal{M}$ and the corresponding induced norm we have

$$v_T(T^{-1}AT) = \max_{x\neq 0} \frac{v_T(T^{-1}ATx)}{v_T(x)} = \max_{x\neq 0} \frac{v(ATx)}{v(Tx)} = v(A) \leq \rho(\mathcal{M}) = \rho(T^{-1}\mathcal{M}T).$$

Using (5) we get for arbitrary $A \in T^{-1}\mathcal{M}T$ and $j = 1, \dots, d$ that

$$\rho(\mathcal{M}) = \rho(T^{-1}\mathcal{M}T) \geq v_T(Ae_j) \geq \|Ae_j\|_\infty = \max_{i=1,\dots,d} |a_{ij}|.$$

As j was arbitrary, this shows the assertion.

For the remainder of the proof, note that if $\mathcal{M}$ is reducible, then a similarity transformation brings all matrices $A \in \mathcal{M}$ into upper block triangular form and the previous argument may be applied on the diagonal blocks as they are irreducible or 0. The entries in the off-diagonal blocks can then be made small by similarity scaling. We leave the details to the reader. □

Note that the previous Theorem 3 has no chance of being true in the case $\rho(\mathcal{M}) = 0$, as this condition only means that all matrices in $\mathcal{M}$ are nilpotent.

In addition, we point out that the same argument applies to all $S \in \mathcal{S}_t$, as v_T is an extremal norm. Thus

$$\rho(\mathcal{M}) \geq \sup\{|a_{ij}|^{1/t} \; ; \; A \in \mathcal{S}_t(T^{-1}\mathcal{M}T), i, j \in \underline{d}, t \geq 1\}. \tag{6}$$

The statement of (6) can be strengthened. Indeed, Theorem 3 may be used to obtain the following characterization of the joint spectral radius. This builds on a characterization of the joint spectral radius in terms of trace due to Chen and Zhou (2000).

Proposition 4. Let $\emptyset \neq \mathcal{M} \subset \mathbb{K}^{d\times d}$ be compact. Then

$$\rho(\mathcal{M}) = \min_{T\in\mathbf{GL}_d(\mathbb{K})} \sup_{t\geq 1} \max\{|\langle e_i, Se_i\rangle|^{1/t} \mid i \in \underline{d}, S \in T\mathcal{S}_tT^{-1}\}.$$

Proof. If $\rho(\mathcal{M}) = 0$, then it is known that there is a similarity transformation that brings all $A \in \mathcal{M}$ simultaneously in upper triangular form with zero diagonal. The statement is then evident.

Assume $\rho(\mathcal{M}) > 0$. Then we may apply Theorem 3 and with the T constructed therein we see that for all $S \in T\mathcal{S}_t T^{-1}$ we have (as (e_i, e_i) is a dual pair for v_T) that

$$\rho(\mathcal{M})^t \geq v_T(Se_i) \geq |\langle e_i, Se_i\rangle|, \quad i = 1, \dots, d.$$

This shows $\geq$.

For the converse recall that by Chen and Zhou (2000) we know that

$$\rho(\mathcal{M}) = \limsup_{t\to\infty} \max_{S\in\mathcal{S}_t} |\operatorname{trace} S|^{1/t}. \tag{7}$$

Let $T \in \mathbf{GL}_d(\mathbb{K})$ be arbitrary and recall that $\rho(\mathcal{M}) = \rho(T\mathcal{M}T^{-1})$. If there exists a $c > 0$ such that

$$\rho(\mathcal{M}) - c \geq \sup_{t\geq 1} \max \left\{ |\langle e_i, Se_i\rangle|^{1/t} \ ; \ i \in \underline{d}, S \in T\mathcal{S}_t T^{-1} \right\},$$

then

$$\rho(\mathcal{M}) - c \geq \sup_{t\geq 1} \left\{ \left| \frac{1}{d} \operatorname{trace} S \right|^{1/t} \ ; \ S \in T\mathcal{S}_t T^{-1} \right\}.$$

As $(1/d)^{1/t} \to 1$ this yields a contradiction to (7). Note that we have also shown that the minimum in the claim is attained, as the T provided by Theorem 3 is a minimizer. The proof is complete. □

4. HIGHER DIMENSIONAL PRINCIPAL SUBMATRICES

It seems natural to ask for a generalization of the results of the previous section to larger submatrices. The question is then whether we can use a similarity transformation to ensure that the joint spectral radius of all principal submatrices is bounded by the joint spectral radius of the whole matrix set. The aim of this section is to show that this is in general only possible up to a multiplicative constant and that this question has intricate connections to convex geometry and the local theory of Banach spaces.

We begin by fixing some necessary notation. Let $\emptyset \neq J \subset \{1, \dots, d\}$. For $A \in \mathbb{K}^{d\times d}$ we denote by $A_{J,J}$ the principal submatrix of A obtained by deleting all rows and columns with index not in J. Similarly, for a set of matrices $\mathcal{M} \subset \mathbb{K}^{d\times d}$ we define $\mathcal{M}_{J,J} := \{A_{J,J} \ ; \ A \in \mathcal{M}\}$.

We start with a negative result which shows that the direct generalization of Theorem 3 to larger submatrices is false.

Theorem 5. For every $d \geq 3$ there is a finite set $\mathcal{M} \subset \mathbb{R}^{d\times d}$ with $\rho(\mathcal{M}) = 1$ and a constant $C > 1$ such that for all $T \in \mathbf{Gl}_d(\mathbb{R})$ and for every index set J with cardinality in $\{2, \dots, d-1\}$ we have $\rho((T^{-1}\mathcal{M}T)_{J,J}) \geq C$.

We omit the proof of this result for reasons of space. Nonetheless, some comments are in order. The key ingredients for this result is the following lemma and a result from Banach space theory by Bosznay and Garay (1986).

Lemma 6. Let ν be a norm on $\mathbb{R}^d$ whose unit ball is a polytope K. For every vertex v and face F of K there is a rank one matrix A with operator norm $\nu(A) \leq 1$ which maps F to v.

For a finite dimensional normed space Y and a linear subspace $X \subset Y$, the (relative) *projection constant* $\lambda(X, Y)$ is defined as the minimal operator norm that a projection from Y to X can have.

Theorem 7. (Bosznay and Garay (1986)). For every $d \geq 3$ there are a constant $C > 1$ and a d-dimensional real normed space with polytopal unit ball such that every k-dimensional subspace with $2 \leq k \leq d-1$ has projection constant at least C.

An explicit example of such a polytopal normed space in dimension 3 can be constructed by chopping off the vertices of a regular dodecahedron in a specific manner, see (Singer, 1970, Chapter II, Theorem 1.1, p. 217).

We can nevertheless obtain for example the following upper bounds for the joint spectral radius of submatrices.

Theorem 8. Let $\mathcal{M} \subset \mathbb{K}^{d\times d}$ be a nonempty, compact set of matrices. There is a $T \in \mathbf{GL}_d(\mathbb{K})$ such that $\rho((T^{-1}\mathcal{M}T)_{J,J}) \leq |J|\,\rho(\mathcal{M})$ for every non-empty index set $J \subset \{1, \dots, d\}$.

Proof. If $\rho(\mathcal{M}) = 0$, then all matrices in $\mathcal{M}$ can be brought simultaneously into strict upper triangular form and the result follows. On $\mathbb{K}^{k\times k}$ consider the induced $\|\cdot\|_1$ matrix norm, i.e. the maximal column sum norm. By Theorem 3 there is a similarity transformation T such $\|T^{-1}AT\|_1 \leq d\rho(\mathcal{M})$ for all $A \in \mathcal{M}$. Let $J \subset \{1, \dots, d\}$ be a non-empty set of indices. Then $\|(T^{-1}AT)_{J,J}\|_1 \leq |J|\,\rho(\mathcal{M})$ for all $A \in \mathcal{M}$, hence $\rho((T^{-1}AT)_{J,J}) \leq |J|\,\rho(\mathcal{M})$. □

Using John's ellipsoids (see the reprint of the original paper from 1948 in John (2013)) we can improve this theorem for large submatrices (where $|J| > \sqrt{d}$).

Theorem 9. Let $\mathcal{M} \subset \mathbb{R}^{d\times d}$ be a nonempty, compact set of matrices. There is an invertible matrix $T \in \mathbb{R}^{d\times d}$ such that $\rho((T^{-1}\mathcal{M}T)_{J,J}) \leq \sqrt{d}\rho(\mathcal{M})$ for every non-empty index set $J \subset \{1, \dots, d\}$.

Proof. The case $\rho(\mathcal{M}) = 0$ is clear. For $\rho(\mathcal{M}) > 0$ assume $\mathcal{M}$ is irreducible, the reducible case then follows by induction. Assume $\rho(\mathcal{M}) = 1$. Then there exists an extremal norm ν and by applying a similarity transformation we may assume that the unit ball of ν is in John position, i.e. the Euclidean unit ball is the ellipsoid of maximal volume inscribed in the unit ball of ν. By John's theorem, see e.g. (Ball, 1997, p. 13) this implies $\frac{1}{\sqrt{d}}\|x\|_2 \leq \nu(x) \leq \|x\|_2$. Let $J \subset \{1, \dots, d\}$ be any index set and set $V := \operatorname{span}\{e_i \ ; \ i \in J\}$. Let P be the orthogonal projection of $\mathbb{R}^d$ onto V. Let $\tilde{\nu}$ be the norm on V induced by ν. Using the basis $\{e_i \ ; \ i \in J\}$ of V we get an isomorphism between $\mathbb{R}^{|J|}$ and V. Under this isomorphism we have $\tilde{\nu}(A_{J,J}(x)) = \nu(P(Ax))$ for every $x \in V$. Let $x \in V$ with $\tilde{\nu}(x) \leq 1$. Since ν is extremal, we get $\nu(Ax) = 1$. Using the assumption that the unit ball of ν is in John position, we obtain $\|Ax\|_2 \leq \sqrt{d}$ and hence $\nu(P(Ax)) \leq \|P(Ax)\|_2 \leq \|P\|_2 \|Ax\|_2 \leq \sqrt{d}$. Since x was an arbitrary point in the unit ball of $\tilde{\nu}$ we obtain $\tilde{\nu}(A_{J,J}) \leq \sqrt{d}$ for all $A \in \mathcal{M}$ and thus $\rho(\mathcal{M}_{J,J}) \leq \sqrt{d}$. □

5. AN EXAMPLE

The following example is based on (Guglielmi and Protasov, 2023, Example 3.1). Consider $\mathcal{M} = \{A_1, A_2\}$ with

$$A_1 := \begin{pmatrix} 6 & -4 \\ 7 & -4 \end{pmatrix}, \quad A_2 := \begin{pmatrix} -4 & 4 \\ -5 & 4 \end{pmatrix}.$$

The joint spectral radius is $\rho(\mathcal{M}) = (48 + 16\sqrt{5})^{1/5} \approx 2.4245$ and a spectrum maximizing product is given by $B := A_1 A_2 A_1^2 A_2$. Set $\tilde{A}_i := \frac{1}{\rho(\mathcal{M})} A_i$, $i = 1, 2$. Then $v_1 := (1, \frac{3+\sqrt{5}}{4})^\top$ is a right eigenvector of B for the leading eigenvalue $\rho(\mathcal{M})^5$. Define $v_2 := \tilde{A}_2 v_1$, $v_3 := \tilde{A}_1 v_2$, $v_4 := \tilde{A}_1 v_3$, $v_5 := \tilde{A}_2 v_4$, $v_6 := \tilde{A}_2 v_3$. The unit ball K of an extremal norm for $\mathcal{M}$ is given by the convex hull of $\{v_1, \ldots, v_6, -v_1, \ldots, -v_6\}$. An Auerbach basis is given by (v_3, v_6) together with its dual basis, see Figure 1.

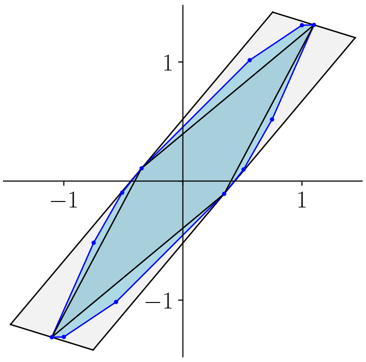

Fig. 1. The unit ball K of an extremal norm

Set $T := (v_3\ v_6)$. Then $T^{-1} A_i T$, $i = 1, 2$ are given approximately by

$$\begin{pmatrix} 1.7454 & 2.1998 \\ -1.6163 & 0.2546 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1.6498 \\ 2.4245 & 0 \end{pmatrix}.$$

The absolute values of all entries of these two transformed matrices are now bounded by $\rho(\mathcal{M})$. See Figure 2 for

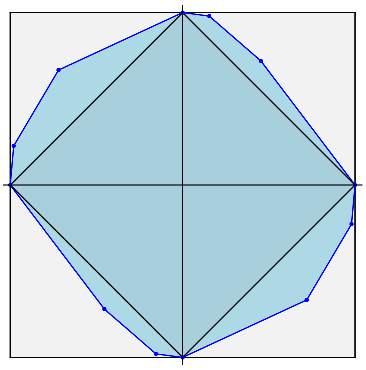

Fig. 2. The transformed unit ball $T^{-1}K$.

an illustration of the transformed unit ball $T^{-1}K$ which clearly lies between the unit ball of the 1- and the ∞-norm.

6. CONCLUSIONS

Using the geometric theory of finite-dimensional Banach spaces, properties of the joint spectral radius for principal submatrices of matrix sets have been obtained. While in the 1-dimensional case the joint spectral radius provides an upper bound, this does not hold in the higher dimensional case. The question of precise bounds for these results is currently under investigation.

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