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Characterizations of input-to-state stability for time-delay systems ⋆

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Abstract: For nonlinear time-delay systems, characterizations of input-to-state stability (ISS) are investigated. While general ISS superposition theorems for infinite-dimensional systems can be applied in this context, the criteria provided by such theorems are unnecessarily demanding. It is shown that for time-delay systems, relaxed characterizations can be obtained. While recovering some ISS characterizations known for ordinary differential equations, we also highlight specific obstructions posed by time-delay systems. In particular, the boundedness of finite-time reachability sets becomes a central property in this context. As it turns out, this assumption cannot be relaxed in a meaningful sense. With this assumption, however, several uniformity properties may be derived for time-delay systems.

Keywords: Delay systems, stability of nonlinear systems, input-to-state stability, superposition theorems, reachability sets

1. INTRODUCTION

During the last decade, significant progress has been made both in the understanding of input-to-state stability (ISS) for general infinite-dimensional systems (see Mironchenko and Prieur (2020) for an overview), and of the peculiarities of ISS for time-delay systems (see Chaillet et al. (2023) for a survey).

One of the important questions in the theory of ISS is in which way it may be characterized by other dynamic properties of a given system. An important point in this respect is the characterization through Lyapunov functions, see, e.g., Sontag and Wang (1995). Another central result is an ISS superposition theorem Sontag and Wang (1996), which states in particular that ISS is equivalent to the various combinations of other dynamic properties of the system. One such characterization states that ISS is equivalent to local stability and the limit property. The latter notion essentially imposes that, in response to a bounded input, all solutions eventually visit a neighborhood of the origin whose size is "proportional" to the amplitude of the applied input. The results of Sontag and Wang (1996) have been (partially) extended in Mironchenko and Wirth (2018) to abstract infinite-dimensional control systems, including evolution PDEs, differential equations in Banach spaces, switched systems, etc. In that general class, the infinite-dimensional result is tight, but for the specific class of time-delay systems, it has been open for some time how to obtain minimal characterizations.

The possibility of establishing ISS by proving a number of apparently weaker properties has proved very useful for the development of other fundamental results. This technique has been employed with much success in areas such as ISS small-gain theorems for ODEs Dashkovskiy et al. (2007), hybrid systems Cai and Teel (2009); Dashkovskiy and Kosmykov (2013), or, in the context of time-delay systems, for Lyapunov-Razumikhin Teel (1998); Dashkovskiy et al. (2012) and Lyapunov-Krasovskii Ito et al. (2010, 2013) approaches and non-coercive Lyapunov functions theory Mironchenko and Wirth (2019); Jacob et al. (2020), to name a few examples.

Whereas there are several criteria for ISS of time-delay systems in terms of Lyapunov functions Karafyllis (2006), Qiao and Guang-Da (2010), obtaining tight ISS superposition theorems for time-delay systems remained an open problem. Several difficulties appear on this way:

(i) noncompactness of closed bounded balls in infinite-dimensional normed linear spaces, which prevents the extended use of finite-dimensional arguments in many places.
(ii) the convergence rate for solutions of globally asymptotically stable infinite-dimensional systems is not necessarily uniform, as was shown for nonlinear time-delay systems in Chaillet et al. (2024).
(iii) reachability sets of delay systems may be unbounded even for nonuniformly globally asymptotically stable nonlinear systems, as demonstrated in Mancilla-Aguilar and Haimovich (2023).

This makes time-delay systems strikingly different from nonlinear ODE systems, where all these problems do not arise.

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In this paper, *we provide tight ISS superposition theorems for time-delay systems. That is, we give equivalent characterizations of ISS, which cannot be essentially weakened in the sense that the required properties are independent of each other*. More precisely, we show that for delay systems which have bounded finite-time reachability sets (are BRS), the limit property is uniform over bounded sets of initial states and inputs. This result allows for a substantial simplification of the superposition theorem when compared to the general infinite-dimensional case. On the other hand, it is an essential requirement to assume that the time-delay system is BRS. This is in strong contrast to the ODE case where boundedness of reachability sets is a consequence of forward completeness of the system. We highlight the key role played by the BRS property by referring to recent counter-examples from the literature Mancilla-Aguilar and Haimovich (2023); Chaillet et al. (2024).

In Theorem 1, we prove our main result showing that ISS is equivalent to a combination of the limit property, boundedness of reachability sets, and uniform local stability.

For systems without inputs, this result recovers a recent characterization of uniform global asymptotic stability of a time-delay system as a combination of (non-uniform) global asymptotic stability with the boundedness of reachability sets, (Karafyllis et al., 2022, Theorem 1).

The problem of getting the ISS characterizations for delay systems was stated and discussed in the conference paper Mironchenko and Wirth (2017). In particular, in Mironchenko and Wirth (2017), some technical lemmas have been derived that can be useful on this way. Here, we give a decisive answer to the problem posed in Mironchenko and Wirth (2017).

We proceed as follows. In Section 2, we define the general class of nonlinear time-delay systems with bounded delays that are considered in this paper. Our general assumptions are such that standard hypotheses of solution theory are satisfied. In Subsection 2.1, we introduce the various stability, attractivity, and limit properties that are required for our characterizations. Section 3 contains the main result, which lists a number of statements that are equivalent to input-to-state stability. We conclude in Section 4.

2. PRELIMINARIES

We consider retarded differential equations of the form

$$\dot{x}(t) = f(x_t, u(t)), \tag{1}$$

where $x_t \in X := C([-\theta,0],\mathbb{R}^n)$ for some $n \in \mathbb{N} \setminus \{0\}$, and $\theta > 0$ is the fixed maximal time-delay involved in the dynamics. $u \in \mathscr{U}$ denotes an exogenous input. For system (1), we use the following assumption concerning the vector field f.

Assumption 1. The vector field $f : X \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous, jointly in both variables, satisfies $f(0,0) = 0$, and is Lipschitz continuous in its first argument on bounded subsets of X and $\mathbb{R}^m$, uniformly with respect to the second argument, i.e., for all $r > 0$, there exists $\ell(r) > 0$ such that for all $x_1, x_2 \in B_X(r,0)$ and all $v \in \mathbb{R}^m$ with $|v| \le r$, it holds that

$$|f(x_1,v) - f(x_2,v)| \le \ell(r)\|x_1 - x_2\|. \tag{2}$$

By (Chaillet et al., 2023, Theorem 2), Assumption 1 implies that, for any initial condition $x_0 \in X$ and any input $u \in \mathscr{U}$, there exists a unique maximal solution of (1), defined on some time interval $[0, T^{\max}_{x_0,u})$ with $T^{\max}_{x_0,u} \in (0,+\infty]$. This solution is denoted by $x(\cdot;x_0,u)$. Given $t \in [0, T^{\max}_{x_0,u})$, $x_t(x_0,u) \in X$ then refers to its history function at time t and it holds that $\|x_t(x_0,u)\| = \max_{s\in[-\theta,0]} |x(t+s;x_0,u)|$. The system is called forward complete if for all x_0, u we have $T^{\max}_{x_0,u} = \infty$, i.e., solutions may be continued for all positive times.

If we denote the flow induced by system (1) by $\phi : (t, x_0, u(\cdot)) \mapsto x_t(x_0, u(\cdot))$, then the triple $(X, \mathscr{U}, \phi)$ defines an abstract control system in the sense of Mironchenko and Wirth (2018), which satisfies the boundedness-implies-continuation (BIC) property, i.e., every maximal bounded solution is defined on $\mathbb{R}_+$ (Chaillet et al., 2023, Theorem 2).

2.1 Stability Concepts

The objective of this paper is to provide links between different stability and robustness notions. These notions can be seen as natural extensions to time-delay systems of the properties addressed in Sontag and Wang (1996). We start with stability notions for systems with inputs.

Definition 2.1. System (1) is called

- *uniformly locally stable (ULS)* if there exist $\sigma, \gamma \in \mathscr{K}_\infty$ and $r > 0$ such that for all $x_0 \in X$ with $\|x_0\| \le r$ and all $u \in \mathscr{U}$ with $\|u\| \le r$, its solution satisfies
$$\|x_t(x_0,u)\| \le \sigma(\|x_0\|) + \gamma(\|u\|) \quad \forall t \ge 0. \tag{3}$$
- *uniformly globally stable (UGS)*, if there exist $\sigma, \gamma \in \mathscr{K}_\infty$ such that the estimate (3) holds for all $x_0 \in X$ and all $u \in \mathscr{U}$.
- *uniformly locally stable for zero input (0-ULS)*, if there exist $\sigma \in \mathscr{K}_\infty$ and $r > 0$ such that, for all $x_0 \in X$ with $\|x_0\| \le r$,
$$\|x_t(x_0,0)\| \le \sigma(\|x_0\|) \quad \forall t \ge 0. \tag{4}$$

Next, we define (uniform) attractivity for systems with inputs.

Definition 2.2. System (1) has

- an *asymptotic gain (AG)* if there exists $\gamma \in \mathscr{K}_\infty \cup \{0\}$ such that, for all $\varepsilon > 0$, all $x_0 \in X$ and all $u \in \mathscr{U}$, there exists $\tau = \tau(\varepsilon, x_0, u) < \infty$ such that
$$\|x_t(x_0,u)\| \le \varepsilon + \gamma(\|u\|) \quad \forall t \ge \tau(\varepsilon, x_0, u). \tag{5}$$
- a *strong asymptotic gain (SAG)* if for (5), there exist $\tau(\varepsilon, x_0, r) < \infty$, $r \ge 0$, such that $\tau = \tau(\varepsilon, x_0, u) \le \tau(\varepsilon, x_0, r)$ for all $\varepsilon > 0, x_0 \in X, u \in \mathscr{U}$ with $\|u\| \le r$.
- a *uniform asymptotic gain (UAG)* if there exists $\gamma \in \mathscr{K}_\infty \cup \{0\}$ such that, for all $\varepsilon, r > 0$, there is a $\tau = \tau(\varepsilon, r) < \infty$ such that, for all $x_0 \in X$ with $\|x_0\| \le r$ and all $u \in \mathscr{U}$ with $\|u\| \le r$,
$$\|x_t(x_0,u)\| \le \varepsilon + \gamma(\|u\|) \quad \forall t \ge \tau(\varepsilon, r). \tag{6}$$

Both AG and UAG impose that solutions are eventually in a neighborhood of the origin, with a size is depending continuously on the magnitude of the input. For UAG it is required that this attractivity is uniform over bound sets of initial states and inputs. While the inequalities (5) and (6) seem to suggest the same estimate, the point is that in the uniform case the time beyond which the estimate is valid cannot depend on the initial condition, but only on its norm. We also point out that neither of the properties is local in nature but refers to globally (in the state space) valid attractivity properties. It is less demanding to require that solutions eventually reach this neighborhood, this leads to different versions of the limit property, which are again global properties in the state space.

Definition 2.3. We say that (1) has the

- *limit property (LIM)*, if there exists $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that, for all $\varepsilon > 0$, all $x_0 \in X$ and all $u \in \mathcal{U}$, there is a $t = t(\varepsilon, x_0, u) \geq 0$ such that
$$\|x_t(x_0,u)\| \leq \varepsilon + \gamma(\|u\|).$$
- *strong limit property (SLIM)*, if there exists $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $\varepsilon, r > 0$ and all $x_0 \in X$, there is a $\tau = \tau(\varepsilon, x_0, r) \geq 0$ such that, for all $u \in \mathcal{U}$ with $\|u\| \leq r$, there exists $t \in [0, \tau]$ such that
$$\|x_t(x_0,u)\| \leq \varepsilon + \gamma(\|u\|). \tag{7}$$
- *uniform limit property (ULIM)*, if there exists $\gamma \in \mathcal{K}_\infty \cup \{0\}$ so that, for all $\varepsilon, r > 0$, there exists a $\tau = \tau(\varepsilon, r) \geq 0$ such that, for all $x_0 \in X$ with $\|x_0\| \leq r$ and all $u \in \mathcal{U}$ with $\|u\| \leq r$, there is a $t \in [0, \tau]$ such that
$$\|x_t(x_0,u)\| \leq \varepsilon + \gamma(\|u\|).$$

It clearly holds that ULIM $\Rightarrow$ SLIM $\Rightarrow$ LIM. A further important property in our characterization is that finite-time reachability sets are bounded.

Definition 2.4. We say that (1) has *bounded reachability sets (is BRS)*, if for all $r > 0$ and all $\tau > 0$, it holds that
$$\sup\left\{\|x_t(x_0,u)\| : \|x_0\| \leq r,\ \|u\| \leq r,\ t \in [0,\tau]\right\} < \infty.$$

Recently, an important example has been provided in Mancilla-Aguilar and Haimovich (2023) which shows that forward completeness does not ensure BRS [1]. This contrasts the finite-dimensional case for which the implication holds, see Lin et al. (1996). The class of time-delay system does provide a continuity property of the flow at the equilibrium position $(x^*, u^*) = (0,0)$. This is the content of the next statement.

Definition 2.5. We say that (1) is *continuous at the equilibrium (CEP)* if, for every $\varepsilon, h > 0$, there exists a $\delta = \delta(\varepsilon, h) > 0$, so that its solutions satisfy
$$t \in [0,h],\ \|x_0\| \leq \delta,\ \|u\| \leq \delta \quad \Rightarrow \quad \|x_t(x_0,u)\| \leq \varepsilon. \tag{8}$$

As we will shortly recall, continuity at the equilibrium point was an important point for general infinite-dimensional systems. In our case, however, we have the following result.

Lemma 2.6. Let Assumption 1 hold. If (1) is BRS, then it is CEP.

We omit the proof for reasons of space.

We finally recall the notion of input-to-state stability, originally introduced for finite-dimensional systems Sontag (1989, 2008); Mironchenko (2023) but also well documented in the context of time-delay systems Chaillet et al. (2023).

Definition 2.7. System (1) is called *input-to-state stable (ISS)*, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $x_0 \in X$ and all $u \in \mathcal{U}$, it holds that
$$\|x_t(x_0,u)\| \leq \beta(\|x_0\|, t) + \gamma(\|u\|), \quad \forall t \geq 0. \tag{9}$$

Although not stated explicitly in the acronym, the ISS property is uniform both in the initial state and the input.

3. ISS SUPERPOSITION THEOREM FOR TIME-DELAY SYSTEMS

The numerous characterizations of ISS proposed in Sontag and Wang (1996) provide a superposition theorem for ISS in a finite-dimensional context, in the sense that ISS is established based on the combination of internal stability properties and attractivity-like notions for systems with exogenous inputs. In Mironchenko and Wirth (2018), these characterizations have been partially extended to abstract control systems, including ODEs, time-delay systems, evolution partial differential equations, and broad classes of evolution equations in Banach spaces. The following was established:
$$\text{ISS} \Leftrightarrow \text{ULIM} \wedge \text{ULS} \wedge \text{BRS} \Leftrightarrow \text{UAG} \wedge \text{CEP} \wedge \text{BRS}.$$

In this general class of control systems, this theorem is tight. However, when focusing on time-delay systems, we can use the special structure of these systems to achieve a stronger ISS characterization.

Before stating our main result, we need to establish the equivalence of LIM and ULIM for systems that are BRS. This observation is based upon (and extends) the corresponding result for systems without inputs shown in (Karafyllis et al., 2022, Theorem 7).

Theorem 3.1. Assume that (1) is BRS and satisfies Assumption 1. If it is SLIM with a given gain $\gamma \in \mathcal{K}_\infty \cup \{0\}$, then it is ULIM with the same gain γ.

The proof is omitted for reasons of space. We stress that the result is false, if the assumption of BRS is dropped. By an example in Chaillet et al. (2024), there exists a time-delay system with trajectories in $\mathbb{R}^4$ with two point delays with unbounded reachability sets, which is SLIM but not ULIM. Recall that for (forward complete) ODE systems, it was already shown in Mironchenko and Wirth (2018) that LIM and ULIM are equivalent.

3.1 Superposition theorem for ISS

Our main result is the following characterization of ISS.

Theorem 1. For a time-delay system (1) satisfying Assumption 1, the following properties are equivalent:

i) ISS
ii) ULIM $\wedge$ UGS
iii) UAG $\wedge$ BRS
iv) SLIM $\wedge$ 0-ULS $\wedge$ BRS
v) SAG $\wedge$ 0-ULS $\wedge$ BRS

Proof.

i) $\Leftrightarrow$ ii). This was shown in Mironchenko and Wirth (2018) in a more general infinite-dimensional context.

i) $\Leftrightarrow$ iii). In Mironchenko and Wirth (2018), it was shown that ISS is equivalent to UAG $\wedge$ CEP $\wedge$ BRS. In view of Proposition 2.6, the CEP assumption can be removed, as it results from BRS.

i) $\Leftrightarrow$ iv). By Theorem 3.1, SLIM $\wedge$ BRS implies ULIM. Furthermore, ULIM $\wedge$ 0-ULS $\wedge$ BRS (applied for $u \equiv 0$) ensures that the system (1) is 0-UGAS (Mironchenko and Wirth, 2018, Proposition 14). By (Palumbo et al., 2013, Theorem 6), 0-UGAS delay systems (1) satisfying Assumption 1 are locally ISS (meaning satisfying the ISS estimate (9) for $\|x_0\|$ and $\|u\|$ small enough) and hence ULS. Overall, SLIM $\wedge$ 0-ULS $\wedge$ BRS is equivalent to ULIM $\wedge$ ULS $\wedge$ BRS, which is itself equivalent to ISS due to Mironchenko and Wirth (2018).

[1] Note, however, that for some alternative state spaces, forward completeness implies BRS, see Karafyllis et al. (2022); Brivadis et al. (2024)

i) $\Leftrightarrow$ v). Clearly, i) $\Rightarrow$ v). Moreover, since SAG $\Rightarrow$ SLIM, we have that v) $\Rightarrow$ iv) $\Rightarrow$ i). □

Remark 2. It is instructive to compare the statement of Theorem 1 to (some of) the results obtained in Sontag and Wang (1996). In this reference, the authors consider control systems of the form

$$\dot{x} = f(x,u),$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz continuous and $f(0,0) = 0$. Under these assumptions it is shown that we have the equivalence

$$\text{ISS} \Leftrightarrow \text{LIM} \wedge \text{ULS} \Leftrightarrow \text{UAG} \Leftrightarrow \text{AG} \wedge \text{ULS}.$$

(Note that what we call ULS here is precisely the property referred to as LS in Sontag and Wang (1996).) By comparing the statement, it becomes clear that the main missing ingredient to obtain a characterization of ISS for time-delay systems is the boundedness of reachability sets. By the result of Mancilla-Aguilar and Haimovich (2023), BRS is an independent property which is not implied by other easy properties. To obtain a complete picute however, the relation of LIM and SLIM for time-delay systems still needs to be clarified. For general infinite-dimensional systems further uniformity assumptions are required in any case.

4. CONCLUSION

We have shown that it is sufficient to extend the known criteria for ISS for ODE systems by BRS to obtain characterizations of ISS for time-delay systems. This significantly improves on general conditions for infinite-dimensional systems.

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