Convex Extensions of Partially Ordered Rings

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This series of lectures is about algebraic aspects of real algebraic geometry. Partially ordered rings (*porings*) occur very naturally in real algebraic geometry, as well as in other branches of mathematics. There is a rich classical theory of porings (the beginnings dating back to the 1930s and 1940s). To a large extent it was inspired by the study of rings of continuous functions in analysis and topology. I will mention rings of continuous functions from time to time, but will not attempt an in-depth discussion.

With the rise of real algebraic geometry in the last 25 years there also arose a need for algebraic methods that are specifically adapted to *real* algebraic geometry, as opposed to *general* algebraic geometry. For a long time commutative algebra has been the answer to a similar need in general algebraic geometry. For real algebraic geometry, an important part of the answer is the theory of porings. However, the classical theory of porings frequently does not address the questions from real algebraic geometry. Therefore it is necessary to develop a more comprehensive theory of porings that includes rings of continuous functions, but also deals with phenomena occurring in real geometry. This is a major task that is not nearly finished at the present time.

First I give the definition of porings:

Definition 1
A *partially ordered ring* (*poring*) is a ring A (all rings are commutative with 1) together with a subset $P \subseteq A$ that satisfies the following conditions

- $P + P \subseteq P$;
- $P \cdot P \subseteq P$;
- $A^2 \subseteq P$;
- $P \cap -P = \{0\}$.

The set P is called the *positive cone* of A. **Ω**

This presentation of a partial order differs from the most intuitive one. The basic idea is that a partial order encodes the concept of "size" or "precedence" for the elements of a set, group, ring or other structure – given any two elements *a* and *b*, it may happen that *a* is larger than *b*, or that *b* is larger than *a*, or that the two are incomparable. A binary relation is the most natural way to express such relationships between elements. Of course, the relation must satisfy certain axioms to capture the idea of comparison by size. If it does so then it is called an *order relation*. In the case of algebraic structures, the order relation also has some monotonicity properties. The definition above looks differently, it does not talk about a binary relation. However, the positive cone of a partially ordered ring can be used to define a binary relation:

$$a \leq b \text{ if and only if } b - a \in P.$$

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One checks easily that the relation defined in this way has the following properties:

- $\forall a \in A : a \leq a$
- $\forall a,b \in A : (a \leq b \;\&\; b \leq a \Rightarrow a = b)$
- $\forall a,b,c \in A : (a \leq b \;\&\; b \leq c \Rightarrow a \leq c)$
- $\forall a,b,c \in A : (a \leq b \Rightarrow a + c \leq b + c)$
- $\forall a,b,c \in A : (a \leq b \;\&\; 0 < c \Rightarrow a \cdot c \leq b \cdot c)$
- $\forall a \in A : 0 \leq a^2$

The first three properties express that the relation is an order relation; the fourth and fifth properties say that addition and multiplication by certain elements preserve the order relation, thus establishing a connection between the order relation and the algebraic operations of the ring; the last property makes sure that many elements of the ring can be compared with each other and thus gives substance to the order relation.

Conversely, for any binary relation "≤" with these properties one defines the set $P = \{a \in A \mid 0 \leq a\}$. Then A is a partially ordered ring with positive cone P. The correspondence between partial order relations and positive cones is bijective; hence positive cones will be identified with the corresponding order relations.

There are a many variations of the definition of a poring. Some of them make a very considerable difference. But in this series of lectures only the one given above will be used, and, to avoid confusion, alternative definitions will not be discussed.

In this lecture I make one more general assumption: All porings are reduced, i.e., there are no nilpotent elements. This hypothesis has far-reaching consequences. It is known that general porings can be much more complicated than reduced ones. This is true in particular with respect to category-theoretic properties. In fact, a considerable part of the theory of porings has been developed only for reduced porings so far. If I do not explicitly say otherwise, porings are always reduced.

Here are some examples of porings that are known to every mathematician:

Example 2
Some of the most basic rings of numbers are *totally* ordered (i.e., the axioms of porings hold and, in addition, any two elements can be compared), e.g., the integers, the rational numbers and the real numbers. The integers can be considered as a subring of every poring. **Ω**

Example 3
If X is a set and (C,T) is a poring then the ring $A = C^X$ of all maps from X to C is partially ordered by pointwise comparison of maps, i.e., $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$. There are many applications in which the poring (C,T) is even totally ordered. **Ω**

Example 4
If (B,Q) is a poring and if $A \subseteq B$ is a subring then $P = A \cap Q$ is the positive cone for a partial order of A. In this case (A,P) is a *sub-poring* of (B,Q). **Ω**

Example 5
The previous two examples can be combined with each other: Let X be a topological space. The ring $A = \mathbb{R}^X$ is partially ordered. The set $\mathrm{C}(X,\mathbb{R})$ of continuous functions is a subring of

A, hence it is a poring through pointwise comparison of functions.

If one speaks about rings of continuous functions, then one always means this poring. Rings of continuous functions are important invariants of topological spaces. Topologists have studied them intensively for a long time. Rings of continuous functions were one of the main motivations for the development of a general theory of porings.

In this lecture it is always assumed that topological spaces are *completely regular*. The reason is that rings of continuous functions are not suited to the study of more general classes of spaces: If X is any topological space then there is a natural continuous map $X \to \mathbb{R}^{\mathrm{C}(X,\mathbb{R})} : x \to \hat{x}$, where $\hat{x} : \mathrm{C}(X,\mathbb{R}) \to \mathbb{R}$ is evaluation at x. If $X' \subseteq \mathbb{R}^{\mathrm{C}(X,\mathbb{R})}$ is the image of this map with the subspace topology then the natural map $\mathrm{C}(X',\mathbb{R}) \to \mathrm{C}(X,\mathbb{R})$ is an isomorphism, i.e., the rings of continuous functions cannot distinguish between X and X'. The space X' is completely regular. **Ω**

A ring is said to be *real* if $\sum_{i=1}^{r} a_i^2 = 0$ always implies $a_1 = \ldots = a_r = 0$. If A is a real ring then $\sum A^2$, the set of sums of squares in A, is the positive cone of a partial order. (This can be checked easily.) Many real rings carry various different partial orders. The smallest one (i.e., the one with the smallest positive cone) is always formed by the sums of squares.

Example 6

A field can be partially ordered if and only if it can be totally ordered, if and only if it is real. Zorn's Lemma shows that the positive cone of a partial order of a field is contained in a maximal one. The maximal partial orders are, in fact, total orders. A partial order of a field is the intersection of all the total orders that contain it. Real fields have characteristic 0.

Suppose that (K, K^+) is a totally ordered field. Then an algebraic extension of K is not necessarily real, hence not every algebraic extension can be ordered, e.g., the algebraic closure can never be ordered. However, there are maximal algebraic extensions that are real; these fields are called *real closed*. Real closed fields always carry a unique total order; the positive cone consists of the squares. Every totally ordered field has an algebraic extension that is real closed and whose total order extends the given total order of K. This real closed extension is unique in a very strong sense; it is called the *real closure* of (K, K^+). **Ω**

Example 7

If A is a real ring and $B = A\left[X_i \,\middle|\, i \in I\right]$ is a polynomial ring then B is also real. Since totally ordered fields are real, all polynomial rings with coefficients in $\mathbb{R}$ are real, hence are partially ordered by the sums of squares. It is important to note that these rings also carry other partial orders (e.g., see Example 9 and Example 10 below). Polynomial rings over the real numbers are among the most important rings in real algebraic geometry. **Ω**

Example 8

If A is a real ring and if X is a set then the ring A^X is a real ring. Subrings of real rings are always real. Thus, rings of continuous functions are real rings. In fact, in rings of continuous functions every sum of squares is even a square, and the pointwise partial order coincides with the partial order given by the (sums of) squares. **Ω**

Example 9
In the previous example, let $X = \mathbb{R}^n$ for some $1 \le n \in \mathbf{N}$. Then a polynomial is positive with respect to the pointwise partial if and only if it is *positive semi-definite*. If $n = 1$ then this partial order coincides with the sums of squares. If $n \ge 2$ then the two partial orders do not coincide. Hilbert (in 1888) was the first to prove this fact; the first explicit example of a positive semi-definite polynomial that is not a sum of squares was given by Motzkin in 1967:

$$X^4 \cdot Y^2 + X^2 \cdot Y^4 + 1 - 3 \cdot X^2 \cdot Y^2 .$$

Today many other examples are known. **Ω**

Example 10
If (A,P) is a partially ordered ring and if $M \subseteq A$ is any subset then the set

$$P[M] = \left\{ \sum_{i=1}^{r} p_i \cdot \prod_{j=1}^{s_i} a_{ij} \,\middle|\, r, s_i \in \mathbf{N} \;\&\; p_i \in P \;\&\; a_{ij} \in M \right\}$$

satisfies all conditions of a partial order except possibly the last one, i.e., it can happen that $P[M] \cap -P[M] \ne \{0\}$. Thus $P[M]$ is a partial order if and only if the condition $P[M] \cap -P[M] = \{0\}$ is satisfied. If $P[M]$ is a partial order, then it is called the *partial order generated by M over P*.

Let $A = \mathbb{R}[X_1, \ldots, X_n]$ be partially ordered by the sums of squares and let $f_1, \ldots, f_k \in A$, set $X = \{ x \in \mathbb{R}^n \mid f_1(x) \ge 0 \;\&\ldots\&\; f_k(x) \ge 0 \}$. Then $Q = \sum A^2 [f_1, \ldots, f_k]$ is a partial order of *A* if the set *X* has nonempty interior. **Ω**

The maps that are used to relate different porings to each other are the *monotonic* or *order preserving homomorphisms*: If (A,P) and (B,Q) are porings, a homomorphism $f : A \to B$ is *order preserving* if $f(P) \subseteq Q$.

The porings together with the order preserving homomorphisms form *the category of porings*, **POR**. This is a very large category, all rings of continuous functions are among its objects, as well as the porings that arise in real algebraic geometry, e.g., partially ordered polynomial rings. The category is closed under many constructions. It is complete and co-complete. In this respect it behaves much more favorably than either the class of rings of continuous functions or the class of partially ordered polynomial rings.

To understand the advantage of using a category of porings in which many ring-theoretic constructions are possible, suppose that you study some question about rings of continuous functions. Your arguments may require the use a ring theoretic construction. Very often such constructions lead to rings that are porings, but not rings of continuous functions. General porings, even if they are not themselves rings of continuous functions, can contribute to a better understanding of rings of continuous functions. Therefore it is desirable to work in a more flexible category that contains all rings of continuous functions and admits as many ring theoretic constructions as possible. The literature contains several approaches, e.g., one may study *f-rings*, a special class of lattice-ordered rings. However, from the point of view of real algebraic geometry, this is a bad choice – the porings that arise in real algebraic geometry are rarely *f*-rings. Therefore, the category **POR** is more appropriate if one wants to develop a theory that also includes the porings from real algebraic geometry.

Completeness of **POR** implies the existence of a terminal object, which is the zero ring, as one checks easily. Similarly, co-completeness of **POR** implies the existence of an initial object, which is the totally ordered ring of integers.

One should note that well-known category-theoretic constructions, if they are performed in the category **POR**, can lead to results that are quite different from what the underlying rings yield in the category of rings. Here is an example with fibre sums:

Example 11
In the category of rings the fibre sum, i.e., the tensor product, of two copies of the field $\mathbb{Q}(\sqrt{2})$ over itself is $\mathbb{Q}(\sqrt{2})$. Now consider $\mathbb{Q}(\sqrt{2})$ with

- the total order T_1, determined by $\sqrt{2} > 0$,
- the total order T_2, determined by $\sqrt{2} < 0$, and
- the partial order $P = T_1 \cap T_2$.

By co-completeness of the category **POR**, the fibre sum of the porings $(\mathbb{Q}(\sqrt{2}), T_1)$ and $(\mathbb{Q}(\sqrt{2}), T_2)$ over $(\mathbb{Q}(\sqrt{2}), P)$ exists, say

$$\begin{array}{ccc} (\mathbb{Q}(\sqrt{2}), P) & \longrightarrow & (\mathbb{Q}(\sqrt{2}), T_1) \\ \downarrow & & \downarrow \alpha_1 \\ (\mathbb{Q}(\sqrt{2}), T_2) & \xrightarrow{\alpha_2} & (A, Q) \end{array}$$

Looking only at the underlying rings, there is a canonical homomorphism $f : \mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}(\sqrt{2})} \mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\sqrt{2}) \to A$, which must be surjective since otherwise the ring A can be replaced by the image of *f*. The maps α_1 and α_2 coincide on the underlying rings. Now taking into account that both maps are order preserving one gets $\alpha_1(\sqrt{2}) \in Q$ and $\alpha_2(\sqrt{2}) \in -Q$, which is possible only if A is the zero ring. **Ω**

Construction – rings of quotients
Suppose that (A, P) is a poring and that $S \subseteq A$ is a multiplicative subset. Let $i_S : A \to A_S$ be the canonical homomorphism into the ring of quotients. The subset

$$P_S = \left\{ \frac{a}{s} \middle| \exists t \in S : a \cdot s \cdot t^2 \in P \right\} \subseteq A_S$$

is a partial order: The conditions $P_S + P_S \subseteq P_S$, $P_S \cdot P_S \subseteq P_S$, and $A_S^2 \subseteq P_S$ are trivially satisfied. To see that $P_S \cap -P_S = \{0\}$, suppose that $\frac{a}{s} \in P_S \cap -P_S$, say $a \cdot s \cdot t^2 \in P$ and $-a \cdot s \cdot u^2 \in P$. Then also $a \cdot s \cdot t^2 \cdot u^2 \in P$ and $-a \cdot s \cdot t^2 \cdot u^2 \in P$. This implies $a \cdot s \cdot t^2 \cdot u^2 = 0$, and it follows that $\frac{a}{s} = 0$, which proves the claim.

The canonical homomorphism $i_S : A \to A_S$ is a homomorphism of porings. The poring (A_S, P_S) is called the *quotient poring of* (A,P) *with denominators in S*. The quotient poring has the same universal mapping property as quotient rings (without partial order): If $f : (A,P) \to (B,Q)$ is a map of porings with $f(S) \subseteq B^\times$ then there is a unique map $f_S : (A_S, P_S) \to (B,Q)$ such that $f = f_S \circ i_S$.

If $S \subseteq A$ is multiplicative, then so is $S^2 = \{ s^2 \mid s \in S \}$. The canonical map $(A_{S^2}, P_{S^2}) \to (A_S, P_S)$ is an isomorphism. Thus, one may always assume that the set of denominators of a quotient poring consists of positive elements.

The construction can be applied to a partially ordered domain (A,P), in particular. The quotient poring construction yields a partial order of the quotient field, say $(\mathrm{qf}(A), Q)$. By Zorn's Lemma there is a maximal partial order $\overline{Q}$ of $\mathrm{qf}(A)$ that contains Q. A simple argument shows that $\overline{Q}$ is a total order of the field. Thus, composing the canonical map from (A,P) to $(\mathrm{qf}(A), \overline{Q})$ with the map from $(\mathrm{qf}(A), \overline{Q})$ into its real closure one obtains an order preserving map into a real closed field.

Construction – factor rings and convex ideals

If $f : (A,P) \to (B,Q)$ is a homomorphism of porings then the kernel is a *convex* ideal. An ideal I is said to be *convex* if it has the following property: If $a, b \in P$ and if $a + b \in I$ then $a, b \in I$. The following statement is equivalent to the definition: If $a \le c \le b$ and if $a, b \in I$ then also $c \in I$. Note that kernels of poring homomorphisms are also radical ideals since the codomain is reduced.

Now suppose that $I \subseteq A$ is a convex radical ideal. The factor ring A/I is a reduced ring. Let $\pi : A \to A/I$ be the canonical homomorphism. Then $\pi(P) \subseteq A/I$ is the positive cone of a partial order: Once again, the first three conditions are clearly satisfied, only the last one needs to be checked. So, let $a + I \in \pi(P) \cap -\pi(P)$, say $a + I = p + I = -q + I$, $p, q \in P$. Then $p + q \in I$, which implies $p, q \in I$ (by convexity), and this means that $a + I = 0 + I$ in A/I. The poring $(A/I, \pi(P))$ is the *factor poring of* (A,P) *modulo I.*

The connection between the convex prime ideals of the poring (A,P) and those of a quotient poring (A_S, P_S), resp. those of a factor poring $(A/I, \pi(P))$, is the same as between the prime ideals of ring and the prime ideals of a quotient ring or, resp., a factor ring:

- If $S \subseteq A$ is a multiplicative subset then the bijective correspondence

$$\{ p \in \mathrm{Spec}(A) \mid p \cap S = \varnothing \} \to \mathrm{Spec}(A_S) : p \to p \cdot A_S,$$
$$\mathrm{Spec}(A_S) \to \{ p \in \mathrm{Spec}(A) \mid p \cap S = \varnothing \} : q \to i_S^{-1}(q)$$

  restricts to a bijection between the sets of convex prime ideals.

- If $I \subseteq A$ is a convex radical ideal then the bijective correspondence

$$\left\{ p \in \operatorname{Spec}(A) \middle| \, p \supseteq I \right\} \to \operatorname{Spec}(A/I) : p \to \pi_I(p),$$
$$\operatorname{Spec}(A/I) \to \left\{ p \in \operatorname{Spec}(A) \middle| \, p \supseteq I \right\} : q \to \pi_I^{-1}(q)$$

restricts to a bijection between the sets of convex prime ideals.

It is also easy to check that the convex radical ideals of A/I correspond bijectively to the convex radical ideals of A that contain I.

If I is a convex radical ideal then it is the intersection of all prime ideals that are minimal among those containing it. In fact, these prime ideals are also convex: To prove this assertion, first one forms the partially ordered factor ring A/I. Now the claim is the same as to say that the minimal prime ideals of this poring are convex. Thus, one may assume that $I = (0)$, and it suffices to show that every minimal prime ideal in a poring is convex.

Suppose that p is one of the minimal prime ideals of A. Then the quotient ring A_p is a partially ordered field, and the canonical homomorphism $A \to A_p$ is order preserving, hence the kernel, which is p, is convex. There is an order preserving homomorphism from the partially ordered field A_p into a real closed field (see Example 6). Thus, the minimal prime ideal p is also the kernel of a homomorphism into a real closed field.

It has just been shown that there exist many homomorphisms from porings to totally ordered fields, even to real closed fields; every convex prime ideal, in particular every minimal prime ideal is the kernel of such a homomorphism. These homomorphisms can be used to introduce the *real spectrum* of a poring. The real spectrum is a contravariant functor from the category **POR** to the category **TOP** of topological spaces. This is one of the most important and useful constructions in real algebraic geometry. It was first proposed by Coste and Roy, however without reference to partial orders; the adaptation to porings is a completely trivial matter.

Real spectra of porings are not arbitrary topological spaces, rather, they belong to the class of *spectral spaces*. This is the class of spaces that are homeomorphic to prime spectra of commutative rings. Spectral spaces were first introduced and studied by Hochster. In these lectures it is assumed that the prime spectra of rings are familiar to the audience.

The real spectrum functor is denoted by Sper. It will be defined in three steps:

- The underlying set of $\operatorname{Sper}(A,P)$ for a poring (A,P);
- the topology of $\operatorname{Sper}(A,P)$;
- the map $\operatorname{Sper}(f): \operatorname{Sper}(B,Q) \to \operatorname{Sper}(A,P)$ for a map $f:(A,P) \to (B,Q)$ of porings.

First, the underlying set of $\operatorname{Sper}(A,P)$ will be explained: Consider all homomorphisms from (A,P) to real closed fields, $f:(A,P) \to \left(R,R^2\right)$. With each such homomorphism one associates the subset $f^{-1}\left(R^2\right) \subseteq A$, which is called the *prime cone associated with f.*

Without reference to homomorphisms into real closed fields, a subset $\alpha \subseteq A$ is called a *prime cone* if it satisfies the following list of conditions:

- $\alpha + \alpha \subseteq \alpha$;
- $\alpha \cdot \alpha \subseteq \alpha$;
- $P \subseteq \alpha$;
- $\alpha \cup -\alpha = A$;

- $\alpha \cap -\alpha$ is a prime ideal, the *support* of α.

One sees easily that $f^{-1}\left(R^2\right)$ is a prime cone in this sense. The support of the prime cone $f^{-1}\left(R^2\right)$ is exactly the kernel of *f*.

Given the prime cone α, one can define a homomorphism into a real closed field in a rather natural way: The support $\alpha \cap -\alpha$ is a prime ideal, hence $A/\alpha \cap -\alpha$ is a domain, let $\pi_\alpha : A \to A/\alpha \cap -\alpha$ be the canonical map. It is an immediate consequence of the definition that $\pi_\alpha(\alpha)$ is closed under addition and multiplication, that $\left(A/\alpha \cap -\alpha\right)^2 \subseteq \pi_\alpha(\alpha)$, $\pi_\alpha(\alpha) \cap -\pi_\alpha(\alpha) = \{0\}$ and $\pi_\alpha(\alpha) \cup -\pi_\alpha(\alpha) = A/\alpha \cap -\alpha$, i.e., $\pi_\alpha(\alpha)$ is a total order of $A/\alpha \cap -\alpha$. The poring $\left(A/\alpha \cap -\alpha, \pi_\alpha(\alpha)\right)$ is denoted by A/α. Because of $P \subseteq \alpha$, the canonical map $\pi_\alpha : (A;P) \to A/\alpha$ is order preserving. Passing from the totally ordered domain A/α to its quotient field, the total order of the ring extends uniquely to a total order of the field. The totally ordered quotient field is denoted by $\kappa(\alpha)$, the inclusion is denoted by $i_\alpha : A/\alpha \to \kappa(\alpha)$. Finally, $\kappa(\alpha)$ has a unique real closure $\rho(\alpha)$ with inclusion map $j_\alpha : \kappa(\alpha) \to \rho(\alpha)$. The composition $\rho_\alpha = j_\alpha \circ i_\alpha \circ \pi_\alpha : (A,P) \to \rho(\alpha)$ is the desired homomorphism into a real closed field. Note that $\alpha = \rho_\alpha^{-1}\left(\rho(\alpha)^2\right)$, i.e., α is the prime cone associated with ρ_α.

Note that the definition of the map ρ_α is essentially unique since the formation of the totally ordered factor ring A/α and of the quotient field $\kappa(\alpha)$ are unique, whereas the formation of the real closure $\rho(\alpha)$ of $\kappa(\alpha)$ is unique up to isomorphism.

Two maps into real closed fields are said to be *equivalent* if the associated prime cones agree. Thus, the prime cones of a poring correspond bijectively to the equivalence classes of maps into real closed fields. For each prime cone α one picks once and for all the map ρ_α (that has just been defined) as representative of the equivalence class. The field $\rho(\alpha)$ is called the *real closed residue field* at α.

The *set* $\mathrm{Sper}(A,P)$ consists of the prime cones of (A,P). The maps ρ_α can be combined to define the map $\rho_{(A,P)} : (A,P) \to \prod_{\alpha \in \mathrm{Sper}(A,P)} \rho(\alpha) : a \to \left(\rho_\alpha(a)\right)_\alpha$. (The partial order on the product of real closed fields is defined componentwise, of course.) In the discussion of factor porings it has been shown that every minimal prime ideal is convex and is the kernel of some homomorphism into a real closed field, hence is the support of a point in the real spectrum. Thus, the homomorphism $\rho_{(A,P)}$ is injective. The elements of *A* are viewed as functions from the set $\mathrm{Sper}(A,P)$ into the family $\left(\rho(\alpha)\right)_{\alpha \in \mathrm{Sper}(A,P)}$ of real closed fields. The value of $a \in A$ at the point $\alpha \in \mathrm{Sper}(A,P)$ is $\rho_\alpha(a)$. Viewing the elements of *A* as functions in this way, I will frequently use the notation $a(\alpha)$ for the value of *a* at α, i.e., $a(\alpha) = \rho_\alpha(a)$.

The real spectrum is considered as a geometric object, the ring *A* as a ring of functions defined on the geometric object. In this way $\mathrm{Sper}(A,P)$ is a far-reaching generalization of the affine space $\mathbb{R}^n$, and *A* is a generalization of the polynomial ring $\mathbb{R}\left[X_1,\ldots,X_n\right]$.

Although the values of the "functions" $a \in A$ at the different points of the real spectrum belong to different fields, the values have a definite sign at every point, either $+1$ or 0 or -1.

The euclidean topology of $\mathbb{R}^n$ is generated by the sets of positivity of polynomial functions, $P(F)=\{\, x\in\mathbb{R}^n \mid F(x)>0 \,\}$ with F ranging in $\mathbb{R}[X_1,\dots,X_n]$. Viewing ring elements as functions on the real spectrum, one proceeds analogously and defines a topology on $\operatorname{Sper}(A,P)$ through the subbasis consisting of the sets of positivity in the real spectrum:

$$P(a)=\{\, \alpha\in\operatorname{Sper}(A,P) \mid a(\alpha)>0 \,\},\ a\in A.$$

There is also another important topology on the real spectrum, which will be described below. To distinguish the topologies, the present one is referred to as the *spectral topology*.

The definition of the topological space $\operatorname{Sper}(A,P)$ is complete now. However, spectral spaces always carry some additional structure – *constructible subsets*, the *constructible topology* and the *specialization relation* – which can be defined via the spectral topology. For real spectra these concepts will now be explained as explicitly as possible.

First of all, there are *constructible subsets* – they are the elements of the Boolean algebra of subsets generated by the sets $P(a)$. The constructible sets generate a topology on the set $\operatorname{Sper}(A,P)$, which is called the *constructible topology*. In fact, the constructible sets form a basis of the constructible topology since the class of constructible sets is closed under finite intersections. It is clear from the definition that the constructible topology is finer than the spectral topology. It is frequently necessary to work with both topologies, in particular to consider the closure of a subset $X\subseteq\operatorname{Sper}(A,P)$ with respect to both topologies; $\overline{X}$ denotes the closure with respect to the spectral topology, $\widetilde{X}$ denotes the closure with respect to the constructible topology.

The constructible topology is always *Boolean*, i.e., totally disconnected and compact. Therefore the spectral topology is quasi-compact. Usually the spectral topology is not Hausdorff. There is a partial order, called *specialization*, on $\operatorname{Sper}(A,P)$ that can be defined through the topology: $\alpha\le\beta$ if and only if $\beta\in\overline{\{\alpha\}}$. If this is the case, then β is called a *specialization* of α, and α is a *generalization* of β.

The topological description of the specialization relation is rather abstract. Everybody who works with real spectra must be aware of it. But there are other more easily accessible descriptions:

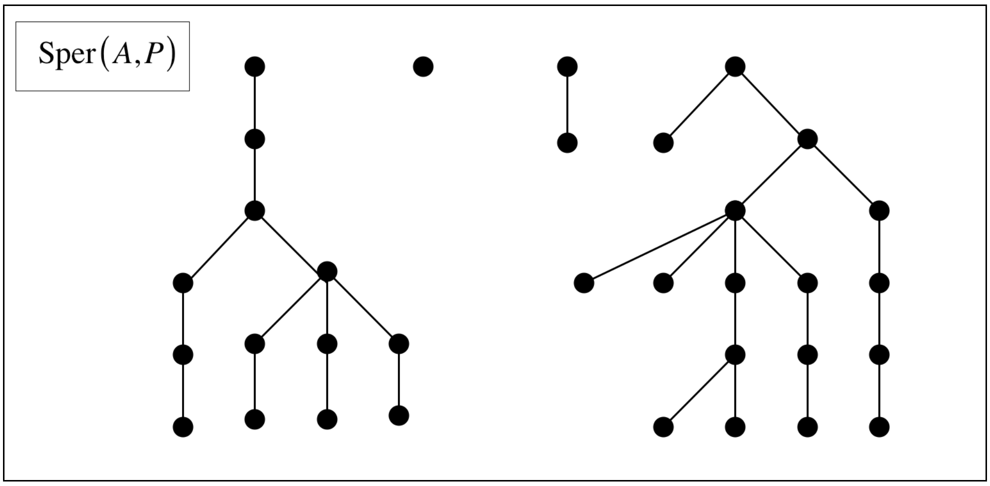
- $\alpha\le\beta$ if and only if $\alpha\subseteq\beta$; or:
- $\alpha\le\beta$ if and only if $\alpha\cap-\alpha\subseteq\beta\cap-\beta$ and the canonical *ring* homomorphism $A/\alpha\to A/\beta$ is also order preserving.

A constructible set is closed (or open) if and only if it contains every specialization (resp., every generalization) of each of its points.

The space of closed points in $\operatorname{Sper}(A,P)$ is denoted by $\operatorname{Spermax}(A,P)$. This is a Hausdorff space. The minimal prime cones are the points that do not have any proper generalization; they are called *generic points*. The generic points also form a subspace of the spectrum, which, however, is not compact in general.

It is a very important property of real spectra (prime spectra usually do not have this property) that the specializations of any point α, i.e., the elements of $\overline{\{\alpha\}}$, form a chain: If $\alpha\subseteq\beta,\gamma$ then either $\beta\subseteq\gamma$ or $\gamma\subseteq\beta$. In particular, there is a unique maximal specialization of α, namely the union of all the prime cones that contain α is itself a prime cone. Thus there is a *specialization map* $\text{spec}:\text{Sper}(A,P)\to\text{Spermax}(A,P)$ that sends a prime cone to its unique maximal specialization. The map spec is continuous, and the space $\text{Spermax}(A,P)$ is compact (recall that the space is Hausdorff).

The specialization relation can be used to visualize the real spectrum in a graph-theoretic way. The underlying set of the real spectrum $\text{Sper}(A,P)$ becomes a graph if one defines the edges to be the two-element sets $\{\alpha,\beta\}$ that consist of two comparable prime cones. The connected components of the graph are the sets of generalizations of the maximal prime cones. Each connected component is a *tree*, the entire spectrum is a *forest*. The following picture shows a few trees of the forest. Large prime cones are at the top of the diagram, small ones are at the bottom.



The roots of the trees are the maximal prime cones, the leaves are the generic points of the spectrum. A point of a tree is called a *branching point* if it is the smallest common specialization of two incomparable prime cones. (If there is a common specialization at all then there always exists a first one.)

The next example contains a first discussion of the real spectrum of a ring of continuous functions.

Example 12

The ring $\mathrm{C}(X,\mathbb{R})$ of continuous functions associated with a completely regular topological space is a poring. The support map is a homeomorphism in this case. Therefore the real spectrum can be identified with the prime spectrum.

Every point $x\in X$ defines the evaluation map $\hat{x}:\mathrm{C}(X,\mathbb{R})\to\mathbb{R}$, hence a point $e_x\in\text{Sper}(\mathrm{C}(X,\mathbb{R}))$. The prime cone e_x is maximal since its support is a maximal ideal. Clearly, the map $e:X\to\text{Sper}(\mathrm{C}(X,\mathbb{R})):x\to e_x$ is injective. One checks without difficulty that $\text{im}(e)\subseteq\text{Sper}(\mathrm{C}(X,\mathbb{R}))$ is a dense subspace, hence also a dense subspace of $\text{Spermax}(\mathrm{C}(X,\mathbb{R}))$, and that *e* is a homeomorphism onto its image.

The closure of $\text{im}(e)$ in $\text{Sper}(\mathrm{C}(X,\mathbb{R}))$ *with respect to the constructible topology* is the subspace of the so-called *prime z-cones* of $\mathrm{C}(X,\mathbb{R})$; this subspace is denoted by z-$\mathrm{Sper}(\mathrm{C}(X,\mathbb{R}))$. A prime cone $\alpha \subseteq \mathrm{C}(X,\mathbb{R})$ is a *prime z-cone* if the following condition is satisfied: If $f \in \mathrm{C}(X,\mathbb{R})$, if $g \in \alpha$ and if $\{x \in X \mid g(x) \geq 0\} \subseteq \{x \in X \mid f(x) \geq 0\}$ then also $f \in \alpha$. Usually not every prime cone is a prime z-cone. The importance of prime z-cones lies in the fact that they can be described "geometrically", i.e., the decision whether or not a function f belongs to a prime z-cone only depends on its set of nonnegativity, $\{x \in X \mid f(x) \geq 0\}$. Maximal prime cones, minimal prime cones and branching points are always z-cones. **Ω**

To complete the definition of Sper as a contravariant functor on the category **POR**, it remains to describe the action of Sper on the **POR**-maps. So, let $f:(A,P) \to (B,Q)$ be an order preserving homomorphism. If $\beta \in \mathrm{Sper}(B,Q)$ is a prime cone then $\alpha = f^{-1}(\beta)$ is a prime cone of (A,P). Thus, there is a map of sets, $\mathrm{Sper}(f):\mathrm{Sper}(B,Q) \to \mathrm{Sper}(A,P):\beta \to f^{-1}(\beta)$. In fact, this map is continuous with respect to *both* the spectral topology and the constructible topology.

It is important to note that the map $f:(A,P) \to (B,Q)$ does not only define a map between the real spectra, but it also defines a map between the real closed residue fields at β and $\alpha = f^{-1}(\beta)$: First there is an order preserving homomorphism $\overline{f_\beta}: A/\alpha \to B/\beta$, which extends uniquely to a homomorphism between the totally ordered quotient fields, $\kappa_{f,\beta}:\kappa(\alpha) \to \kappa(\beta)$, and then, in a final step, to a homomorphism $\rho_{f,\beta}:\rho(\alpha) \to \rho(\beta)$ of the real closed residue fields.

One more property of the functorial maps $\mathrm{Sper}(f):\mathrm{Sper}(B,Q) \to \mathrm{Sper}(A,P)$ that is frequently useful is their *convexity*: Suppose that $\beta_1 \leq \beta_2$ in $\mathrm{Sper}(B,Q)$ and that there is some $\alpha \in \mathrm{Sper}(A,P)$ with $f^{-1}(\beta_1) \leq \alpha \leq f^{-1}(\beta_2)$. Then there exists a prime cone $\beta \in \mathrm{Sper}(B,Q)$, $\beta_1 \leq \beta \leq \beta_2$, such that $\alpha = f^{-1}(\beta)$.

Each prime cone $\alpha \in \mathrm{Sper}(A,P)$ has the support $\alpha \cap -\alpha$, which is a prime ideal. Thus, there is a map from the real spectrum to the prime spectrum, $\mathrm{supp}:\mathrm{Sper}(A,P) \to \mathrm{Spec}(A)$. The support map is continuous and preserves inclusion. Sometimes it is true that the support map is a homeomorphism. For example, rings of continuous functions have this property. (Therefore, in the study of rings of continuous functions it is not really necessary to consider the real spectrum.) On the other hand, for polynomial rings over the real numbers the support map is never a homeomorphism.

There are very substantial applications of the real spectrum in real algebra and geometry, notably in semi-algebraic geometry. Here I want to concentrate on its uses in connection with the category **POR**.

In every category the monomorphisms and the epimorphisms are particularly important classes of maps. A map $f:X \to Y$ is a *monomorphism* if the following holds: Any two maps $a,b:T \to X$ coincide if the compositions $f \circ a$ and $f \circ b$ coincide. Dually, a map $f:X \to Y$ is an *epimorphism* if: the two maps $g,h:Y \to Z$ coincide whenever the compositions $g \circ f$ and $h \circ f$ coincide.

In categories of algebraic structures the monomorphisms are frequently easy to describe – very often they are the injective homomorphisms. This is also true in the category **POR**. Epimorphisms are much harder; in many categories they do not have to be surjective. It is extremely useful to have a good description of epimorphisms. For the category of rings, for example, characterizations can be found in the notes of the Seminaire Samuel of 1967/68. Apparently, no characterization is known for reduced rings. However, in the category **POR** the following can be proved:

Theorem 13 (Schwartz, Madden)
The map $f:(A,P)\to(B,Q)$ in **POR** is an epimorphism if and only if the map $\operatorname{Sper}(f)$ is injective and, for every $\beta\in\operatorname{Sper}(B,Q)$, the map $\rho_{f,\beta}:\rho\left(f^{-1}(\beta)\right)\to\rho(\beta)$ of the real closed residue fields is an isomorphism. **Ω**

The result has many applications. One area of applications is the study of functorial extensions of porings. To give a first impression of these applications one such construction is explained briefly.

A direct product of totally ordered rings with the componentwise partial order is a poring. In fact, this partial order is even a lattice-order, i.e., for any two elements there exist both a supremum and an infimum. Every subring of such a direct product that is *also a sub-lattice* is called an *f-ring*. For example, rings of continuous functions are *f*-rings.

Compared with arbitrary porings, *f*-rings have many special properties, e.g., the support map from the real spectrum to the prime spectrum is a homeomorphism onto the image. There is a functorial way to associate a "smallest" *f*-ring with every poring. This smallest *f*-ring contains a large amount of information about the original poring. It can be used in the following way: An arbitrary poring may be difficult to understand and work with. In such a case it may help to pass to the associated *f*-ring, which is frequently easier to understand, to solve the problem for the *f*-ring and then to try to transfer the solution from the *f*-ring to the poring.

Here is the construction of the smallest *f*-ring containing a poring: Every poring (A,P) can be considered as a subring of a product of real closed fields using the map $\rho_{(A,P)}:(A,P)\to\prod_{\alpha\in\operatorname{Sper}(A,P)}\rho(\alpha)$ that was introduced before. It is easy to see that every intersection of sub-*f*-rings of $\prod_{\alpha\in\operatorname{Sper}(A,P)}\rho(\alpha)$ is a sub-*f*-ring as well. Thus there is a smallest sub-*f*-ring $\varphi(A,P)\subseteq\prod_{\alpha\in\operatorname{Sper}(A,P)}\rho(\alpha)$ that contains the image of $\rho_{(A,P)}$. Restricting the codomain of $\rho_{(A,P)}$ one obtains an injective map $\varphi_{(A,P)}:(A,P)\to\varphi(A,P)$, i.e., the construction enlarges the poring and yielding an *f*-ring.

The construction is functorial and has the following universal property: Every homomorphism $f:(A,P)\to(B,Q)$ into an *f*-ring has a unique extension $\overline{f}:\varphi(A,P)\to(B,Q)$ to a homomorphism between *f*-rings. Thus the maps with domain (A,P) are very closely related to the maps with domain $\varphi(A,P)$. The maps $\varphi_{(A,P)}$ are epimorphisms of porings, which is one of their most useful properties. Therefore it is extremely important in this context that the epimorphisms in **POR** have a characterization that can be applied easily in many concrete situations. It is also noteworthy that the functorial map $\operatorname{Sper}\left(\varphi_{(A,P)}\right)$ is always a homeomorphism.

There are a large number of other similar functorial constructions in the category **POR**. Schwartz and Madden started a systematic study of such constructions.

The notion of a convex subring of a poring will be introduced now. Convex subrings occur naturally and frequently in the study of porings. They have many applications, in real algebraic geometry, in real algebra and in topology.

Definition 14

Suppose that (B,Q) is a poring and that $A \subseteq B$ is a subring. Then A is *convex* if, for all $a \in A$ and all $b \in B$, the inequality $-a \le b \le a$ implies $b \in A$. **Ω**

A priori the convex subring $A \subseteq (B,Q)$ does not have to carry a partial order. But it can always be endowed with the restriction of the partial order of (B,Q).

It is easy to find convex subrings of a given poring by forming *convex hulls*: If $A \subseteq (B,Q)$ is any subring then $\mathrm{Conv}(B/A) = \{ b \in B \mid \exists a \in A : -a \le b \le a \}$ is a convex subring, the *convex hull* of A in (B,Q). Clearly, $\mathrm{Conv}(B/A)$ is the smallest convex subring of (B,Q) that contains A. Among all convex subrings there is a smallest one, namely $\mathrm{Conv}(B/\mathbb{Z})$. This subring will be called the *ring of bounded elements*; in the literature it is also referred to as the *real holomorphy ring* of (B,Q).

Example 15

If (K,T) is a totally ordered field and $A \subseteq K$ is a subring then the convex hull $\mathrm{Conv}(K/A)$ is a valuation ring. Valuation rings also occur in other contexts, e.g., number theory. The convex valuation rings arising in real algebra are rarely Noetherian, which contrasts sharply with the situation in number theory, where many valuation rings are Noetherian.

The formation of convex hulls in totally ordered fields is a very natural construction. Therefore non-Noetherian valuation rings play a considerable role in the theory of porings. **Ω**

Example 16

The ring of bounded elements in a ring $\mathrm{C}(X,\mathbb{R})$ of continuous functions is denoted by $\mathrm{C}^*(X,\mathbb{R})$. This ring is another important invariant of the topological space *X*; it has been used and studied intensively by topologists.

The *Stone-Cech compactification* of a completely regular space is a topological construction. It is a compact space, usually denoted by βX, that contains X as a dense subspace and has a universal property: If $f : X \to Y$ is any continuous map into a compact space then there is a unique extension $\overline{f} : \beta X \to Y$. (Note that this is the same kind of universal property as the one of the functorial *f*-ring extension of porings described earlier.)

The Stone-Cech compactification can be constructed in various different ways. Two of them will be described here: Suppose that $A \subseteq \mathrm{C}(X,\mathbb{R})$ is a subring that contains $\mathbb{R}$ and separates the points of *X*, i.e., for each pair of distinct points $x, y \in X$ there is some $a \in A$ with $a(x) \ne a(y)$. The evaluation maps $\hat{x} : A \subseteq \mathrm{C}(X,\mathbb{R}) \to \mathbb{R}$ at the points of *X* are surjective ring homomorphisms into a real closed field, hence they define maximal points $e_A(x)$ in the real spectrum of A. This defines a map $e_A : X \to \mathrm{Spermax}(A) : x \to e_x$. The map e_A is injective because A separates the points of X. In fact, e_A is a homeomorphism onto a dense subspace of the compact space $\mathrm{Spermax}(A)$, i.e., $\mathrm{Spermax}(A)$ is a compactification of X. Choosing A to be $\mathrm{C}(X,\mathbb{R})$ or $\mathrm{C}^*(X,\mathbb{R})$, one obtains the Stone-Cech compactification in both cases.

The fact that the same construction applied to both rings yields the same compactification looks like a marvelous coincidence. However, there is a deeper reason behind this apparent coincidence, which will be explained later in this lecture. **Ω**

Important questions about convex extensions of porings are:

- If $A \subseteq (B,Q)$ is a convex subring, how close is the connection between the algebraic properties of the two rings or between their prime spectra or between their real spectra?
- If (A,P) is a poring, do there exist extensions $(A,P) \subseteq (B,Q)$ that are convex? How can they be determined?

In the study of convex extensions there is a sharp dichotomy between two cases, depending on whether or not the rings involved satisfy the *bounded inversion property*:

Definition 17
A poring (A,P) has *bounded inversion* if the set $1+P = \{1+p \mid p \in P\}$ consists of units. **Ω**

Partially ordered fields and rings of continuous functions have the bounded inversion property, polynomial rings over the real numbers – regardless of the partial order – do not have bounded inversion. More generally, if A is a reduced ring that can be partially ordered then the polynomial ring $A[X]$ can also be partially ordered. Choosing any partial order P of the polynomial ring, the poring $(A[X],P)$ cannot have bounded inversion since $A[X]^\times = A^\times$.

If the poring (A,P) has bounded inversion and if Q is a weakening of the partial order P, i.e., $Q \subseteq P$, then (A,Q) has bounded inversion as well.

The notion of bounded inversion was first introduced by Henriksen, Isbell and Johnson in the study of lattice-ordered algebras. However it is important also in the context of general porings. A very useful characterization is:

Proposition 18 (Knebusch, Zhang)
The poring (A,P) has bounded inversion if and only if all maximal ideals are convex. **Ω**

It follows immediately from this proposition, or from the definition, that factor porings of porings with bounded inversion also have bounded inversion. The corresponding statement about quotient porings is false. (If a poring with bounded inversion has a prime ideal that is not convex then the partially ordered localization at this prime ideal does not have bounded inversion.)

Rings with bounded inversion always have "many" bounded elements: Suppose that (A,P) has bounded inversion and that $0 < a \in A$. Then $1+a \in A^\times$, and $0 < \frac{1}{1+a} = \frac{1+a}{(1+a)^2} < 1$.

Also, $1+a^2 \in A^\times$, and $0<\frac{a}{1+a^2}<1$. Thus, every principle ideal (a) has a bounded generator, namely $\frac{a}{1+a^2}$.

Without bounded inversion the situation can be entirely different:

Example 19
In the polynomial ring $\mathbb{R}[X]$, partially ordered by the sums of squares, the only bounded elements are the constant polynomials. Thus, only the trivial ideals are generated by bounded elements. **Ω**

Given a subring $A \subseteq (B,Q)$ of a poring, there are two related conditions involving convexity – the subring may be convex or it may contain the subring $\mathrm{Conv}(B/\mathbb{Z})$ of bounded elements. In general these are different conditions: A convex subring always contains the bounded elements, but usually the converse is not true. However:

Proposition 20
If (B,Q) has bounded inversion then the subring A is convex if and only if it contains the subring of bounded elements.

Proof
It has been noted already that one implication is always true (without any assumption about bounded inversion). Conversely, suppose that A contains all bounded elements. If $-a \le b \le a$ with $a \in A$ and $b \in B$ then $-1<-\frac{a}{1+a^2} \le \frac{b}{1+a^2} \le \frac{a}{1+a^2}<1$ shows that $\frac{b}{1+a^2}$ is bounded, hence $\frac{b}{1+a^2} \in A$, and $b \in A$. **Ω**

Example 21
Let $A=\mathbb{R}[X,Y]$ with the total order in which $1 \ll X$ and Y is positive and infinitely large compared with all powers of X. The subring of bounded elements is the field of real numbers. The polynomial ring $\mathbb{R}[Y]$ contains the bounded elements, but is not convex. **Ω**

Assuming that $A \subseteq (B,Q)$ is convex and that A is equipped with the partial order $P=A \cap Q$ one may ask whether bounded inversion for one of the two porings implies bounded inversion for the other one.

Proposition 22
If $A \subseteq (B,Q)$ is convex, if A is carries the partial order $P=A \cap Q$ and if (B,Q) has bounded inversion then (A,P) has bounded inversion. **Ω**

Example 23
Let (A,P) be a poring with bounded inversion, let $B=A[X]$ be the polynomial ring in one variable and let Q be the positive cone generated by P and $\sum B^2$. Then $A \subseteq (B,Q)$ is

convex, and (B,Q) does not have bounded inversion (which is easy and has already been mentioned).

To prove convexity, suppose that $-a \le \sum_{i=0}^{r} \alpha_i \cdot X^i \le a$. Assume that $0 < r$ and $\alpha_r \ne 0$. Then there exists a minimal prime ideal $q \subseteq A$ that does not contain α_r, and there is an order preserving homomorphism $f : A \to \rho$ with kernel q into a real closed field. It follows that $f(\alpha_r) \ne 0$, and the polynomial $\sum_{i=0}^{r} f(\alpha_i) \cdot X^i \in \rho[X]$ is not constant. The homomorphism $F : B \to \rho[X]$ defined by $F|_A = f$ and $F(X) = X$ is order preserving, where $\rho[X]$ is partially ordered by the sums of squares (which is the same as the partial order defined by the positive semi-definite polynomials). The inequality $-a \le \sum_{i=0}^{r} \alpha_i \cdot X^i \le a$ yields $-f(a) \le \sum_{i=0}^{r} f(\alpha_i) \cdot X^i \le f(a)$. Thus the values of the non-constant polynomial $\sum_{i=0}^{r} f(\alpha_i) \cdot X^i \in \rho[X]$ are bounded. This is impossible, and the subring A is convex, as claimed. **Ω**

Every poring can be extended to a poring with bounded inversion: The subset $1+P$ of the poring (A,P) is clearly multiplicative, consists of positive elements and does not contain zero-divisors (since the minimal prime ideals are convex). The quotient ring (A_{1+P}, P_{1+P}) has bounded inversion and contains (A,P) as a subring.

The importance of bounded inversion in connection with convex subrings is evident from the following

Theorem 24 (Knebusch, Zhang)
Suppose that the poring (B,Q) and its subring (A,P) (with $P = A \cap Q$) both have bounded inversion. Then A is convex in B if and only if $A \subseteq B$ is a Prüfer extension, i.e.: given any intermediate ring $A \subseteq C \subseteq B$ the extension $A \subseteq C$ is an epimorphism in the category of rings. Moreover, if the equivalent conditions hold, then $B = A_S$, where $S = \left\{ s \in A \,\middle|\, 0 < s \;\&\; s \in B^{\times} \right\}$.
Ω

This result establishes very strong algebraic connections between a poring and a convex subring. The next question is how the real spectra of the two rings are related. It is clear that the canonical map $\mathrm{Sper}(B,Q) \to \mathrm{Sper}(A,P)$ is a homeomorphism onto the image and that the image is a generically closed subset of $\mathrm{Sper}(A,P)$. Thus, $\mathrm{Sper}(B,Q)$ can be identified with this subspace of $\mathrm{Sper}(A,P)$. The proof of a result of Zhang in connection with results by Knebusch and Zhang show that the subspace contains all minimal prime cones in $\mathrm{Sper}(A,P)$. Even more can be said with an additional hypothesis.

Proposition 25
Suppose that (B,Q) is a poring with bounded inversion and that (A,P), $P = A \cap Q$, is a convex subring. Assume that the support map of B restricts to an injective map $\mathrm{Spermax}(B,Q) \to \mathrm{Spec}(B)$. Then the following holds: If $\beta,\gamma \in \mathrm{Sper}(B,Q)$ have a common specialization $\alpha \in \mathrm{Sper}(A,P)$ then there is a common specialization in $\mathrm{Sper}(B,Q)$.

Proof
Assume by way of contradiction that β and γ do not have a common specialization in $\mathrm{Sper}(B,Q)$. Then the maximal specializations in $\mathrm{Sper}(B,Q)$ are distinct, and both specialize to α. Thus, one may assume that β and γ are distinct points of $\mathrm{Spermax}(B,Q)$. The supports of α, β and γ are denoted by *r*, *p* and *q*. The hypothesis implies that $p \neq q$; without loss of generality one may assume that *q* is not contained in *p*. The subset $\delta = \beta + q = \{x + y \mid x \in \beta \ \&\ y \in q\} \subseteq B$ has the following properties:

- $\delta + \delta \subseteq \delta$;
- $\delta \cdot \delta \subseteq \delta$;
- $Q \subseteq \delta$;
- $\delta \cup -\delta = B$;
- $\delta \cap -\delta$ is an ideal of *B* that contains *p* properly.

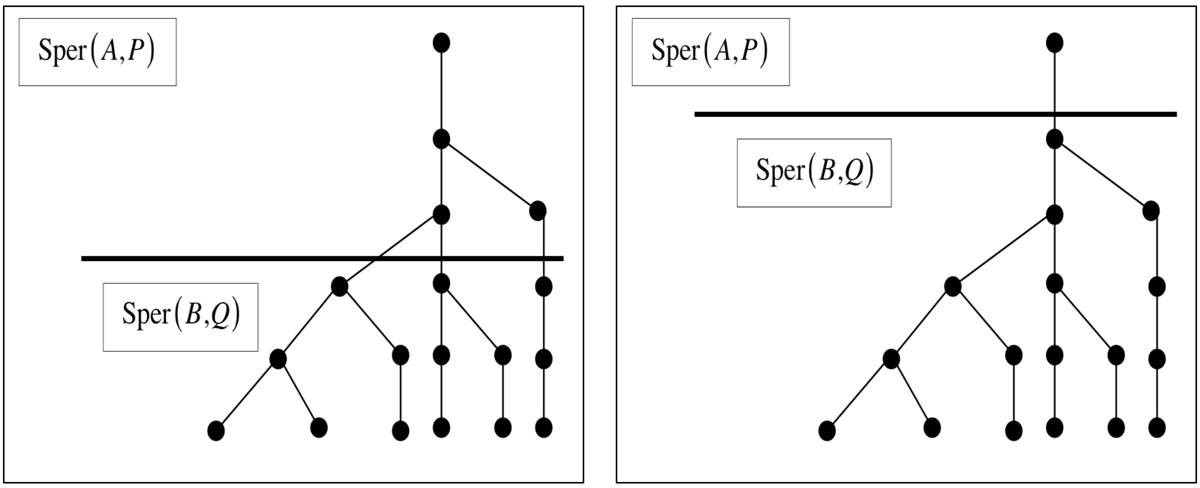
One checks that $\delta \cap -\delta/p$ is convex in the totally ordered domain B/p (with the total order β/p). Since β is a maximal point of the real spectrum there are no non-trivial convex ideals in B/p, hence $\delta \cap -\delta = B$. Thus $-1 \in \delta \cap -\delta$, and $-1 = x + y$, $x \in \beta$, $y \in q$. The convexity of *A* implies $\frac{1}{1+y^2}, \frac{y}{1+y^2} \in A$, hence the identity $-\frac{1}{1+y^2} = \frac{x}{1+y^2} + \frac{y}{1+y^2}$ shows that also $\frac{x}{1+y^2} \in A$. Moreover, $\frac{x}{1+y^2} \in \beta$ and $\frac{y}{1+y^2} \in q$. The restrictions of β and γ to *A* are denoted by β' and γ', the supports by p' and q'; both are contained in *r*. The homomorphisms $A/p' \to A/r$ and $A/q' \to A/r$ are both order preserving. In B/q one has $1 + y^2 + q = 1 + q$, hence $\frac{1}{1+y^2} + q' = 1 + q'$ holds in A/q', which yields $-\frac{1}{1+y^2} + r = -1 + r$ in A/r. On the other hand, the identity $-\frac{1}{1+y^2} + p' = \frac{x}{1+y^2} + \frac{y}{1+y^2} + p'$ in A/p' is mapped to $-\frac{1}{1+y^2} + r = \frac{x}{1+y^2} + \frac{y}{1+y^2} + r$ in A/r. It follows from $-\frac{1}{1+y^2} + r = -1 + r$, $\frac{y}{1+y^2} + r = 0 + r$ and $\frac{x}{1+y^2} \in \beta'$ that $-1 + r = \frac{x}{1+y^2} + r \in \alpha/r$, which implies $-1 \in \alpha$, a contradiction. **Ω**

The result can be applied whenever the larger of the two rings has bounded inversion and its support map is a homeomorphism onto its image. Note that bounded inversion holds automatically if the support map is a homeomorphism. This applies to all rings of continuous functions. For *f*-rings with bounded inversion the support map is a homeomorphism onto its image, but it needs not be surjective. It also happens that the support map is injective on

$\mathrm{Spermax}(B,Q)$, but does not map $\mathrm{Spermax}(B,Q)$ homeomorphically onto its image:

Example 26
Suppose that B is a field with two total orders T_1 and T_2. Assume that there is a T_1-convex valuation ring $V_1 \subseteq B$ that has rank 1 and is not T_2-convex; let M_1 be its maximal ideal. (E.g., one may define $B = \mathbb{R}[X]$, the total order T_1 by the condition $0 < X \ll 1$ and the total order T_2 by the condition $0 < X - 1 \ll 1$. Then the natural valuation ring of (B,T_1) is $V_1 = \mathbb{R}[X]_{(X)}$ with maximal ideal $M_1 = X \cdot V_1$.) The poring $(V_1, V_1 \cap T_1 \cap T_2)$ has bounded inversion. Its real spectrum has three points: T_1, T_2 and $T_1 + M_1$. The two points T_2 and $T_1 + M_1$ are maximal in $\mathrm{Sper}(V_1, V_1 \cap T_1 \cap T_2)$, whereas $T_1 \subset T_1 + M_1$. The supports of T_2 and $T_1 + M_1$ are $(0) \subset M_1$, which shows that the support map is injective on the maximal real spectrum, but is not a homeomorphism onto the image. **Ω**

The Proposition and the considerations preceding it yield the following intuitive picture of the relationship between the real spectra of (B,Q) and its convex subring (A,P): As pointed out before, in graph-theoretical terms, the real spectrum of (A,P) is a forest; the trees are the generalizations of the maximal points. Looking only at one single tree, the minimal points belong to $\mathrm{Sper}(B,Q)$, and if $\mathrm{Sper}(B,Q)$ contains a prime cone β then it contains all its generalizations. The Proposition says that all branching points of the tree must be contained in $\mathrm{Sper}(B,Q)$.



Not admissible *admissible*

It is now possible to understand the fact that the two, a priori different, compactifications of a completely regular space X (constructed via the maximal real spectrum of the ring $\mathrm{C}(X,\mathbb{R})$ of all continuous functions or via the maximal real spectrum of the ring $\mathrm{C}^*(X,\mathbb{R})$ of bounded continuous functions) both lead to the Stone-Cech compactification of X: The map

$$\operatorname{Spermax}(\mathrm{C}(X,\mathbb{R})) \subseteq \operatorname{Sper}(\mathrm{C}(X,\mathbb{R})) \subseteq \operatorname{Sper}(\mathrm{C}^*(X,\mathbb{R})) \xrightarrow{\text{spec}} \operatorname{Spermax}(\mathrm{C}^*(X,\mathbb{R}))$$

is a continuous map of compact spaces. It is injective by the Proposition, surjective since X can be considered as a dense subspace both of $\operatorname{Spermax}(\mathrm{C}^*(X,\mathbb{R}))$ and $\operatorname{Spermax}(\mathrm{C}(X,\mathbb{R}))$. A continuous bijective map of compact spaces is a homeomorphism, hence $\operatorname{Spermax}(\mathrm{C}(X,\mathbb{R})) \cong \operatorname{Spermax}(\mathrm{C}^*(X,\mathbb{R}))$, and the two compactifications coincide.

The Proposition can also help to recognize whether or not a given poring has a convex extension: Suppose that

- (A,P) is a poring with bounded inversion,
- the support map of (A,P) is injective, and that
- in $\operatorname{Spermax}(A,P)$ there is a subspace Z that is dense *with respect to the constructible topology* and consists only of branching points and minimal points of $\operatorname{Sper}(A,P)$.

Assume that $(A,P) \subseteq (B,Q)$ is a convex extension and that (B,Q) has bounded inversion. Then the Theorem of Knebusch and Zhang shows that B is a quotient ring of A, say $B = A_S$, where S is a multiplicative subset of positive elements of A. In particular, the support map of (B,Q) is also injective. Given any element $s \in S \setminus A^\times$,

$$\operatorname{Sper}(B,Q) \subseteq P(s) = \left\{ \alpha \in \operatorname{Sper}(A,P) \mid s(\alpha) > 0 \right\} \subset \operatorname{Sper}(A,P).$$

The constructible set $V(s) = \left\{ \alpha \in \operatorname{Sper}(A,P) \middle| s(\alpha) = 0 \right\}$ is non-empty, hence contains some element of Z. Since $\operatorname{Sper}(B,Q)$ contains all minimal prime cones of (A,P) it follows that $Z \cap V(s)$ consists of branching points. There are incomparable points $\alpha, \beta \in \operatorname{Sper}(A,P)$ that have a point $\gamma \in Z \cap V(s)$ as their first common specialization. On may assume that α and β are minimal prime cones, which implies that they belong to $\operatorname{Sper}(B,Q)$. Since there is a common specialization in $\operatorname{Sper}(A,P)$, namely γ, there must also be a common specialization in $\operatorname{Sper}(B,Q)$ (by the Proposition). However, this contradicts the choice of α, β and γ. The contradiction shows that $S \subseteq A^\times$, hence $A = B$. Thus, (A,P) does not have any proper convex extension with bounded inversion.

These considerations apply to rings of continuous functions: Assume that X is a completely regular space and that every point has a countable neighborhood basis (i.e., the space satisfies the first countability axiom). It is claimed that $\mathrm{C}(X,\mathbb{R})$ does not have any proper convex extension with bounded inversion.

The subset $\left\{ e_x \mid x \in X \right\} \subseteq \operatorname{Spermax}(\mathrm{C}(X,\mathbb{R}))$ is dense in the constructible topology. Since $\mathrm{C}(X,\mathbb{R})$ has bounded inversion and the support map is bijective, it suffices to show that every prime cone e_x is either minimal or is a branching point of $\operatorname{Sper}(\mathrm{C}(X,\mathbb{R}))$. If the point x is isolated then the prime cone e_x is clearly minimal. So, assume that x is not isolated. Then there is a neighborhood basis for x that consists of a properly decreasing sequence $(V_n)_{n\in\mathbb{N}}$ of open sets with $\{x\} = \bigcap_{n\in\mathbb{N}} V_n$. For each $1 \le n \in \mathbb{N}$, pick a point $x_n \in V_{n-1} \setminus V_n$. The map

$\omega:\left\{\frac{1}{n}\mid 1\le n\in\mathbb{N}\right\}\cup\{0\}\to X$, $\frac{1}{n}\to x_n$, $0\to x$, is a homeomorphism onto a closed subset of *X*. Since $\{x_n\mid n\in\mathbb{N}\}\cup\{x\}\subseteq X$ is a compact subset in a completely regular space, the restriction map

$$\mathrm{C}(X,\mathbb{R})\to\mathrm{C}\left(\{x_n\mid n\in\mathbb{N}\}\cup\{x\},\mathbb{R}\right)\cong\mathrm{C}\left(\left\{\frac{1}{n}\mid 1\le n\in\mathbb{N}\right\}\cup\{0\},\mathbb{R}\right)$$

is surjective. Therefore $\mathrm{Sper}\left(\mathrm{C}\left(\left\{\frac{1}{n}\mid 1\le n\in\mathbb{N}\right\}\cup\{0\},\mathbb{R}\right)\right)$ can be identified with a closed subspace of $\mathrm{Sper}(\mathrm{C}(X,\mathbb{R}))$, and it suffices to show that $e_0\in\mathrm{Sper}\left(\mathrm{C}\left(\left\{\frac{1}{n}\mid 1\le n\in\mathbb{N}\right\}\cup\{0\},\mathbb{R}\right)\right)$ is a branching point.

Let $D_1=\left\{\frac{1}{n}\mid 1\le n\in\mathbb{N}\text{ odd}\right\}$, $D_2=\left\{\frac{1}{n}\mid 1\le n\in\mathbb{N}\text{ even}\right\}$. Let $\mathfrak{U}_1$ and $\mathfrak{U}_2$ be free ultrafilters on D_1 and D_2. Then $\mathfrak{P}_1=\left\{\bar{U}\mid U\in\mathfrak{U}_1\right\}$ and $\mathfrak{P}_2=\left\{\bar{U}\mid U\in\mathfrak{U}_2\right\}$ are prime filters of closed sets on $\left\{\frac{1}{n}\mid 1\le n\in\mathbb{N}\right\}\cup\{0\}$. They are incomparable, and the unique prime filter that contains them both is the fixed ultrafilter belonging to the point 0. Now

$$e_1=\left\{f\in\mathrm{C}\left(\left\{\frac{1}{n}\mid 1\le n\in\mathbb{N}\right\}\cup\{0\},\mathbb{R}\right)\Big|\ \exists\bar{U}\in\mathfrak{P}_1\ \forall u\in\bar{U}:f(u)\ge 0\right\},$$
$$e_2=\left\{f\in\mathrm{C}\left(\left\{\frac{1}{n}\mid 1\le n\in\mathbb{N}\right\}\cup\{0\},\mathbb{R}\right)\Big|\ \exists\bar{U}\in\mathfrak{P}_2\ \forall u\in\bar{U}:f(u)\ge 0\right\}$$

are incomparable prime cones, and e_0 is the first (and only) common specialization. Thus, e_0 is a branching point.

As a consequence, no metric spaces has a convex extension with bounded inversion.

Finally, I turn to convex extensions of porings without bounded inversion. This is an important topic because numerous porings that arise in real algebraic geometry do not have bounded inversion, and therefore the preceding considerations are not applicable. It is clear (and has been mentioned) that polynomial rings over the real numbers, for example, do not have bounded inversion. In fact, no finitely generated $\mathbb{R}$-algebra of positive transcendence degree has bounded inversion. This can be seen as follows:

Assume that (A,P) is a partially ordered finitely generated $\mathbb{R}$-algebra of positive transcendence degree with bounded inversion. Thus, all maximal ideals of *A* are convex. The minimal prime ideals $p_1,\ldots,p_r$ are convex, and the maximal ideals of A/p_i are convex with respect to the partial order $P+p_i/p_i$. Thus, each A/p_i has bounded inversion as well. Among the A/p_i there is one with positive transcendence degree. Therefore one may assume that *A* is a domain.

Noether's Normalization Theorem says that A is an integral extension of a polynomial ring $\mathbb{R}[X_1,\ldots,X_k]$, where $1 \le k = \mathrm{trdeg}_{\mathbb{R}} A$. The Going-up Theorem holds for the integral extension $\mathbb{R}[X_1,\ldots,X_k] \subseteq A$, hence every maximal ideal $\mathfrak{m} \subseteq \mathbb{R}[X_1,\ldots,X_k]$ extends to a maximal ideal $\mathfrak{M} \subseteq A$. If $\mathbb{R}[X_1,\ldots,X_k]/\mathfrak{m} \cong \mathbb{C}$ (and there are such maximal ideals) then also $A/\mathfrak{M} \cong \mathbb{C}$, the maximal ideal $\mathfrak{M} \subseteq A$ is not convex with respect to any partial order, and A cannot have bounded inversion, a contradiction.

There is a stronger variant of the above result: Suppose that A is a finitely generated $\mathbb{R}$-algebra of positive transcendence degree. Let $B \subseteq A$ be a sub-algebra, also of positive transcendence degree, not necessarily finitely generated. Then B cannot have bounded inversion with respect to any partial order.

Exactly as above, the problem can be reduced to the case that B is a domain.

Once again, it suffices to prove that there is a maximal ideal $\mathfrak{M} \subseteq B$ with $B/\mathfrak{M} \cong \mathbb{C}$. First, pick a transcendence basis $t_1,\ldots,t_s$ for B over $\mathbb{R}$, and consider the extension $\mathbb{R}[t_1,\ldots,t_s] \subseteq A$ of Noetherian domains. Chevalley's Theorem says that the image of $\mathrm{Spec}(A)$ in $\mathrm{Spec}(\mathbb{R}[t_1,\ldots,t_s])$ is constructible, i.e., is a finite union of sets $U \cap C$, where $U \subseteq \mathrm{Spec}(\mathbb{R}[t_1,\ldots,t_s])$ is open and $C \subseteq \mathrm{Spec}(\mathbb{R}[t_1,\ldots,t_s])$ is closed. The 0-ideal belongs to the constructible set, hence $(0) \in U \cap C$ for some U and C. Since (0) is the unique generic point of $\mathrm{Spec}(\mathbb{R}[t_1,\ldots,t_s])$ it follows that $C = \mathrm{Spec}(\mathbb{R}[t_1,\ldots,t_s])$, i.e., $U \cap C = U$. The sets $D(a) = \{ p \in \mathrm{Spec}(\mathbb{R}[t_1,\ldots,t_s]) \mid a \notin p \}$, $a \in \mathbb{R}[t_1,\ldots,t_s]$, form a basis of the topology of $\mathrm{Spec}(\mathbb{R}[t_1,\ldots,t_s])$. Thus there is some $a \in \mathbb{R}[t_1,\ldots,t_s]$, $a \ne 0$, such that $D(a) \subseteq U$. There is some maximal ideal $\mathfrak{m} \in D(a)$ with $\mathbb{R}[t_1,\ldots,t_s]/\mathfrak{m} \cong \mathbb{C}$. By the choice of $\mathfrak{m}$ there exists a prime ideal $\mathfrak{p} \subseteq A$ with $\mathfrak{m} = \mathfrak{p} \cap \mathbb{R}[t_1,\ldots,t_s]$. Then $\mathfrak{M} = \mathfrak{p} \cap B$ is a prime ideal, and $\mathbb{R}[t_1,\ldots,t_s]/\mathfrak{m} \subseteq B/\mathfrak{M}$ is an algebraic extension (since $t_1,\ldots,t_s$ is a transcendence basis of B). This implies $B/\mathfrak{M} \cong \mathbb{C}$, and the proof is finished.

Partially ordered finitely generated $\mathbb{R}$-algebas and their convex subrings (frequently under the name "real holomorphy rings") have been used and studied by real algebraists. But the relationship between the real spectra or between the prime spectra of the rings involved is still mostly a riddle. The question whether a given partially ordered algebra has a proper convex extension has not been studied at all, so far. The methods of Knebusch and Zhang (using Prüfer extensions) are not applicable in this setting.

As a first step towards the study of convex extensions of partially ordered finitely generated $\mathbb{R}$-algebras it is reasonable to look for examples of such extensions. It has been pointed out repeatedly in these lectures that polynomial rings always yield convex extensions. It is not clear what other constructions can be used to provide examples. The idea that is pursued here is to look for quotient rings that are convex extensions of a given poring:

- Given a poring (A,P) and a multiplicative set S of positive non-zero divisors, when is it true that $(A,P) \subseteq (A_S,P_S)$ is a convex extension?
- Given a poring (A,P) and a multiplicative set S of positive non-zero divisors, when is it true that A contains the ring of bounded elements of (A_S,P_S)?

It has been shown that the two questions are equivalent if bounded inversion is assumed. An example has been given above that shows that the two questions are not equivalent without bounded inversion. The main point of the present discussion is precisely that bounded inversion is not assumed, hence one may expect different answers to these questions.

In this lecture the considerations must be limited to the most elementary facts. Thus, only special cases and very special examples will be presented.

The positive element s in the poring (A,P) is said to satisfy the *first convexity condition* if $0 \leq a \leq s$ implies $a \in (s)$. S. Larson studied conditions of this type (in particular, an n-th convexity condition for every natural number n) in connection with f-rings.

To start with, there are two easy observations about the first convexity condition:

Lemma 27
Suppose that A is a ring with two partial orders P and P', $P \subseteq P'$. If the element $s \in P$ has the first convexity condition with respect to P', then it has the first convexity property also with respect to P. **Ω**

Lemma 28
Given the poring (A,P), let $s,t \in P$ with $D(s) \cap \mathrm{Specmin}(A) \subseteq D(t) \cap \mathrm{Specmin}(A)$. If $t \cdot s$ satisfies the first convexity condition then s has the first convexity property as well. In particular, if A is a domain, if $s,t \in P \setminus \{0\}$ and if $s \cdot t$ has the first convexity property then both s and t have the first convexity property.

Proof
If $0 \leq a \leq s$ then $0 \leq t \cdot a \leq t \cdot s$ and the first convexity condition for $t \cdot s$ yields an element $c \in A$ such that $t \cdot a = c \cdot (t \cdot s)$. It is claimed that $a = c \cdot s$. It suffices to show that $a + q = c \cdot s + q$ for each minimal prime ideal $q \subseteq A$.

If $q \in V(s)$ then the inequality $0 \leq a \leq s$ and convexity of q imply $0 + q \leq a + q \leq s + q = 0 + q$, and the claim is clear. Now suppose that $q \in D(s)$, hence also $q \in D(t)$. Then the factor $b + q$ can be cancelled from the equality $(t+q)\cdot(a+q) = (t+q)\cdot(c \cdot s + q)$. **Ω**

The next result shows the importance of the first convexity condition in connection with convex extensions.

Proposition 29
Let (A,P) be a poring containing the rational numbers, let $s \in P$ be a non-zero divisor (so that A can be considered as a subring of the poring (A_s, P_s)).

(a) All bounded elements of (A_s, P_s) are contained in A if and only if all powers of s satisfy the first convexity condition.
(b) The ring A is convex in (A_s, P_s) if and only if, for all $n \in \mathbb{N}$, the principle ideal (s^n) is convex.

Proof

(a) First assume that A contains the bounded elements. If $0 \le a \le s^n$ then $0 \le \frac{a}{s^n} \le 1$ in A_s, and the assumption implies that $\frac{a}{s^n} \in A$, hence $a = \frac{a}{s^n} \cdot s^n \in (s^n)$.

On the other hand, suppose that $0 \le a \le s^n$ implies $a \in (s^n)$. It is claimed that $\frac{a}{s^n} \in A$ if $-k \le \frac{a}{s^n} \le k$ (with $a \in A$, $k, n \in \mathbb{N}$). Adding k to the inequality and dividing by $2 \cdot k$ one may assume that $0 \le \frac{a}{s^n} \le 1$. The definition of the partial order P_s implies that $0 \le a \cdot s^{2 \cdot r} \le s^{n+2 \cdot r}$ with a suitable $r \in \mathbb{N}$. The hypothesis yields an element $c \in A$ such that $a \cdot s^{2 \cdot r} = c \cdot s^{n+2 \cdot r}$. Since s is a non-zero divisor it follows that $a = c \cdot s^n \in (s^n)$.

(b) Suppose that A is convex, and let $0 \le a \le b \cdot s^n$ with $b \in A$. Then $0 \le \frac{a}{s^n} \le b$ in A_s, and it follows that $\frac{a}{s^n} \in A$, hence $a = \frac{a}{s^n} \cdot s^n \in (s^n)$.

Conversely, suppose that $0 \le a \le b \cdot s^n$, $b \in A$ and $n \in \mathbb{N}$, implies $a \in (s^n)$. If $0 \le \frac{a}{s^n} \le b$ in A_s then it follows that $0 \le a \cdot s^{2 \cdot r} \le b \cdot s^{n+2 \cdot r}$ with $r \in \mathbb{N}$. The hypothesis yields an element $c \in A$ such that $a \cdot s^{2 \cdot r} = c \cdot s^{n+2 \cdot r}$, and this implies $a = c \cdot s^n \in (s^n)$. **Ω**

The Proposition raises the question which elements in a concretely given poring satisfy the first convexity condition and which principal ideals are convex. If it is possible to give a complete characterization of these elements then one can recognize for which quotient porings (A_s, P_s) the ring of bounded elements is contained in A, or A is convex in (A_s, P_s).

Example 30

The polynomial ring $\mathbb{R}[T]$ in one variable carries many different partial orders. Important examples are cones of positive semi-definite polynomials: Let $K \subseteq \mathbb{R}$ be a one-dimensional semi-algebraic set, i.e., a finite union of intervals (open, closed, half-open, finite or infinite, degenerate or not) at least one of which is not reduced to a single point. Then

$$P_K = \{\, a \in \mathbb{R}[T] \mid \forall\, x \in K : a(x) \ge 0 \,\}$$

is a partial order. It is immediately clear that $P_K = P_{\overline{K}}$, i.e., one may assume that K is a closed semi-algebraic set. It will be necessary to distinguish three different classes of points of K:

- K_1 is the set of isolated points;
- K_2 is the set of boundary points;
- K_3 is the set of interior points.

The set of polynomials $s \in P_K$ with the first convexity condition will be determined.

Assume that $s \in P_K$ satisfies the first convexity condition. Let $\alpha_1 < \ldots < \alpha_r$ be the different roots of s in K with multiplicities $\mu_1,\ldots,\mu_r$. Then $s = F \cdot \prod_{i=1}^{r}\left(\varepsilon_i\left(T-\alpha_i\right)\right)^{\mu_i}$, where F is nowhere 0 on K and $\varepsilon_1,\ldots,\varepsilon_r \in \{+1,-1\}$. There is a polynomial F_1 that

- is nowhere 0 on K,
- has the same distribution of signs as F on the connected components of K,
- has $\deg(F_1) \le \deg(F)$,
- is relatively prime with F, and
- satisfies $|F_1(x)| < |F(x)|$ for all $x \in K$.

(If $\beta_1,\ldots,\beta_t$ are the real roots of F with multiplicities $\nu_1,\ldots,\nu_t$ then $F = \gamma \cdot G \cdot \prod_{j=1}^{t}\left(T-\beta_j\right)^{\nu_j}$ with $\gamma \in \mathbb{R}^\times$ and G a product of monic irreducible quadratic factors. There is a small positive real number ε such that $\left(\beta_j-\varepsilon,\beta_j+\varepsilon\right)\cap K = \varnothing$ for every j. Pick some $\eta_j \in \left(\beta_j-\varepsilon,\beta_j+\varepsilon\right)\setminus\{\beta_j\}$. There is a small positive real number δ such that $F_1 = \delta\cdot\gamma\cdot\prod_{j=1}^{t}\left(T-\eta_j\right)^{\nu_j}$ meets all the requirements.)

The properties of F_1 imply that $0 < F_1 \cdot \prod_{i=1}^{r}\left(\varepsilon_i\left(T-\alpha_i\right)\right)^{\mu_i} < s$. The first convexity property yields a polynomial H such that $F_1 \cdot \prod_{i=1}^{r}\left(\varepsilon_i\left(T-\alpha_i\right)\right)^{\mu_i} = H \cdot s$. Canceling all linear factors of the form $T-\alpha_i$ one gets $F_1 = H \cdot F$. Since F and F_1 are relatively prime it follows that F must be constant.

So far it has been shown that the polynomial s is hyperbolic and all its roots are in K. Hence, up to a positive constant factor, it can be written in the form $s = \prod_{i=1}^{r}\left(\varepsilon_i\left(T-\alpha_i\right)\right)^{\mu_i}$.

Suppose that $\alpha_j \in K_1$ and that $\mu_j \ge 2$. There is some $\varepsilon > 0$ such that $\left(\alpha_j-\varepsilon,\alpha_j+\varepsilon\right)\cap K = \{\alpha_j\}$. Pick any element $\beta_j \in \left(\alpha_j-\varepsilon,\alpha_j+\varepsilon\right)\setminus\{\alpha_j\}$ and define $G = \left(T-\alpha_j\right)\cdot\left(T-\beta_j\right)$. There is a positive constant γ such that $0 \le \gamma\cdot G \le \left(T-\alpha_j\right)^2$, hence

$$0 \le \gamma \cdot G \cdot \prod_{i\ne j}\left(\varepsilon_i\cdot\left(T-\alpha_i\right)\right)^{\mu_i}\cdot\left(\varepsilon_j\cdot\left(T-\alpha_j\right)\right)^{\mu_j-2} \le s.$$

Now $\gamma \cdot G \cdot \prod_{i\ne j}\left(\varepsilon_i\cdot\left(T-\alpha_i\right)\right)^{\mu_i}\cdot\left(\varepsilon_j\cdot\left(T-\alpha_j\right)\right)^{\mu_j-2}$ is a multiple of s, but the multiplicity of the root α_j is μ_j for s and μ_j-1 for $\gamma \cdot G \cdot \prod_{i\ne j}\left(\varepsilon_i\cdot\left(T-\alpha_i\right)\right)^{\mu_i}\cdot\left(\varepsilon_j\cdot\left(T-\alpha_j\right)\right)^{\mu_j-2}$. This is impossible, and therefore isolated points of K cannot be multiple roots of s.

Next, assume that $\alpha_j \in K_3$ and that μ_j is odd. Then s changes sign at α_j, and this contradicts the hypothesis that s is non-negative on K. Thus, s has even multiplicity at inner points of K.

Now let $s = \prod_{i=1}^{r} \left(\varepsilon_i \left(T - \alpha_i \right) \right)^{\mu_i}$ be a polynomial that is non-negative on *K*, has roots only in *K*, with multiplicity 1 for isolated points and with even multiplicity for inner points. It is claimed that *s* has the first convexity property: Suppose that $0 \le F \le s$. It suffices to prove that each α_j is a root of *F* with multiplicity at least μ_j.

If $\alpha_j \in K_1$ then $F(\alpha_j) = 0$, and the multiplicity is at least $\mu_j = 1$.

If $\alpha_j \in K_2 \cup K_3$, then $F(\alpha_j) = 0$, and there is a half open interval *I* (either $I = \left[\alpha_j, \alpha_j + \varepsilon \right)$ or $I = \left(\alpha_j - \varepsilon, \alpha_j \right]$) with $0 \le F|_I \le s|_I$. This implies that the multiplicity of α_j as a root of *F* is at least μ_j.

Altogether, the non-negative polynomials in $\mathbb{R}[T]$ with the first convexity property have been determined completely. Now the Proposition shows: The positive polynomials *s* for which $\mathbb{R}[T]$ contains all bounded elements of $\mathbb{R}[T]_s$ are those that have only real roots, all of them in *K*, but not isolated, and with even multiplicity if a root is an inner point of *K*.

Next, the question will be answered for which positive polynomials $s \in \mathbb{R}[T]$ (with partial order P_K as above) the extension $\mathbb{R}[T] \subseteq \mathbb{R}[T]_s$ is convex. Of course, if this is the case then $\mathbb{R}[T]$ contains the bounded elements of $\mathbb{R}[T]_s$, and *s* must be one of the polynomials described above. It is conceivable that convexity of the extension $\mathbb{R}[T] \subseteq \mathbb{R}[T]_s$ entails additional restrictions for *s*. However, this is not the case:

Let *s* be a polynomial that is non-negative on *K*, has only real roots, all of them in *K*, but not isolated, and with even multiplicity if a root is an inner point of *K*. Now assume that there are $0 < a, b \in \mathbb{R}[T]$ and $1 \le n \in \mathbb{N}$ such that $0 \le a \le b \cdot s^n$. Then, as before, every root α of *s* (with multiplicity μ) is a root of *a* with multiplicity at least $n \cdot \mu$. Thus, *a* is a multiple of s^n.

The discussion shows that $\mathbb{R}[T] \subseteq \mathbb{R}[T]_s$ is a convex extension if and only if $\mathbb{R}[T]$ contains the bounded elements of $\mathbb{R}[T]_s$. **Ω**

Example 31

Consider the polynomial ring $\mathbb{R}[T]$ with the sums of squares as partial order (i.e., the positive cone is formed by the positive semi-definite polynomials). Then the extension $\mathbb{R}[T] \subseteq \mathbb{R}[T]_{T^2}$ is convex. Now consider the intermediate ring $\mathbb{R}[T]\left[\frac{(T-1)^2}{T^2} \right]$. The bounded elements of $\mathbb{R}[T]_{T^2}$ are contained in $\mathbb{R}[T]$, hence also in $\mathbb{R}[T]\left[\frac{(T-1)^2}{T^2} \right]$. It will be shown that $\frac{1}{T^2}$ belongs to the convex hull of $\mathbb{R}[T]\left[\frac{(T-1)^2}{T^2} \right]$, but not to the ring itself.

First assume that $\frac{1}{T^2} \in \mathbb{R}[T]\left[\frac{(T-1)^2}{T^2} \right]$, say $\frac{1}{T^2} = F_0 + F_1 \cdot \frac{(T-1)^2}{T^2} + \ldots + F_k \cdot \left(\frac{(T-1)^2}{T^2} \right)^k$ with $F_0, \ldots, F_k \in \mathbb{R}[T]$. Pick a representation of this type with *k* as small as possible. It is clearly impossible that $k=0$. If $k=1$, one obtains the equation $1=F_0\cdot T^2+F_1\cdot(T-1)^2$ of polynomials, and this also leads to a contradiction immediately. So, assume that $k\geq 2$. Then there is an equation $T^{2\cdot k-2}=F_0\cdot T^{2\cdot k}+\ldots+F_{k-1}\cdot(T-1)^{2\cdot k-2}\cdot T^2+F_k\cdot(T-1)^{2\cdot k}$. All summands, except for the last one, are divisible by T^2, which implies that $F_k=T^2\cdot G$. It follows that $T^{2\cdot k-2}=F_0\cdot T^{2\cdot k}+\ldots+F_{k-1}\cdot(T-1)^{2\cdot k-2}\cdot T^2+G\cdot(T-1)^{2\cdot k}\cdot T^2$, and T^2 can be cancelled from the equation. This yields

$$\frac{1}{T^2}=F_0+F_1\cdot\frac{(T-1)^2}{T^2}+\ldots+\left(F_{k-1}+G\cdot(T-1)^2\right)\cdot\left(\frac{(T-1)^2}{T^2}\right)^{k-1},$$

and this contradicts the minimality of *k*. Thus, $\frac{1}{T^2}\notin\mathbb{R}[T]\left[\frac{(T-1)^2}{T^2}\right]$.

Finally, the inequality $0\leq\frac{1}{T^2}\leq 2+2\cdot\frac{(T-1)^2}{T^2}$ shows that $\frac{1}{T^2}$ is in the convex hull of $\mathbb{R}[T]\left[\frac{(T-1)^2}{T^2}\right]$, i.e., the convex hull of $\mathbb{R}[T]\left[\frac{(T-1)^2}{T^2}\right]$ is $\mathbb{R}[T]_{T^2}$. **Ω**

The list of examples can be continued. In a polynomial ring with several variables, again with the partial order consisting of the polynomials that are positive semi-definite on a closed semi-algebraic, there is a similar relationship between elements with first convexity property and the geometry of the semi-algebraic set as the one observed in the first example above.

Many more results concerning these questions are available, the investigations are still in progress. The approach to convexity questions described above, i.e., asking for the convex extensions of a given poring, will also improve the understanding of convex subrings and real holomorphy rings.