Bounding Crossings in Beyond-Planar Drawings Using the Density Formula

Bachelor Thesis of

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Statement of Authorship

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

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Abstract

Let $G = (V, E)$ be a simple connected undirected graph with a set of vertices V and a set of edges E. In a *drawing* Γ of a graph G, vertices are points in the plane and every edge is a continuous line, that does not cross itself and connects two such points. A graph is *planar* if it has a drawing where no edge is crossed. Planar graphs are visually pleasant, but rarely occur with real-world data sets. *Beyond-planar* graphs are not planar but instead have other restrictions to be more readable. In this thesis we take a look at *k-planar* graphs, where edges can be drawn with up to k crossings. More precisely, we discuss 2-layer 1-planar drawings and 2-layer 2-planar drawings of graphs. A *2-Layer drawing* consists of two parallel, usually horizontal lines that symbolize two layers. Vertices are positioned on either one of the two layers and edges can then only connect vertices that are on opposite layers. The dissertation "New Parameters for Beyond-Planar Graphs" [11] explores a number of bounds and similarities of 2-Layer drawings of beyond-planar graphs, but like most research has its focus mostly on edge densities and lower bounds on the number of crossings. In this thesis, the focus lies on finding upper bounds on the number of crossings by using the Density Formula [7], which relates the number of edges, vertices and crossings to the sizes of cells in a drawing. *Cells* are the separate contained areas of the plane that are left when removing all edges and vertices from a connected drawing. Proofs using the Density Formula rely mostly on counting the cells of smaller size, which is why a lot of this work is concerned with finding the possible smaller-sized celltypes in 2-layer 1-planar and 2-planar drawings of G. By deriving equations from the correlations between the cells and using linear programming, we provide new upper bounds on the number of crossings: For G with at least 3 vertices, we prove the upper bound $\frac{n-2}{2}$ for 2-layer 1-planar drawings and $\frac{10}{3}n - \frac{20}{3}$ for 2-layer 2-planar drawings of G. Moreover, we produce a bound $|E| \leq \frac{3}{2}n + \frac{4}{3}$ similar to the known edge density $\frac{3}{2}n - 2$ [4] of 2-layer 1-planar drawings of G.

Deutsche Zusammenfassung

Sei $G = (V, E)$ ein einfacher, zusammenhängender, ungerichteter Graph mit einer Menge von Knoten V und Kanten E. In einer *Zeichnung* Γ eines Graphen G sind Knoten Punkte in der Ebene und jede Kante ist eine durchgehende Linie, die sich nicht selbst schneidet und zwei solche Punkte verbindet. Ein Graph ist *planar*, wenn er eine Zeichnung hat, in der sich keine Kanten kreuzen. Planare Graphen sind optisch ansprechend, kommen jedoch bei realen Datensätzen selten vor. *Beyond-planar* Graphen sind nicht planar, sondern haben andere Eigenschaften, um besser lesbar zu sein. In dieser Arbeit betrachten wir *k-planare* Graphen, bei denen jede Kanten bis zu k Kreuzungen haben darf. Wir betrachten 2-Layer 1-planare Zeichnungen und 2-Layer 2-planare Zeichnungen von G. Eine *2-Layer Zeichnung* besteht aus zwei parallelen, meist horizontalen Linien, die zwei Layer bzw. Schichten symbolisieren. Knoten müssen auf diesen Schichten liegen und Kanten können nur Knoten verbinden, die sich auf unterschiedlichen Schichten befinden. Die Dissertation „New Parameters for Beyond-Planar Graphs" [11] liefert eine Reihe von Schranken und Zusammenhängen von 2-Layer-Zeichnungen von Beyond-Planar Graphen, konzentriert sich jedoch wie die meisten Forschungsarbeiten hauptsächlich auf die Kantendichte und einige untere Schranken der Anzahl an Kreuzungen. In dieser Arbeit liegt der Schwerpunkt auf der Ermittlung von oberen Schranken für die Anzahl an Kreuzungen unter Verwendung der "Density Formula" [7], die die Anzahl der Kanten, Knoten und Kreuzungen mit

Zellen in Verbindung bringt. *Zellen* sind die eingeschlossenen Bereiche der Ebene, die übrig bleiben, wenn alle Kanten und Knoten einer zusammenhängenden Zeichnung entfernt werden. Beweise, die die Density Formula verwenden, basieren hauptsächlich auf dem Zählen von Zellen kleiner Größen, weshalb sich der Großteil dieser Arbeit mit der Suche nach möglichen kleinen Zelltypen in 2-Layer 1-planaren und 2-planaren Zeichnungen von G befasst. Wir leiten Gleichungen aus den Zusammenhängen von Zellen ab und liefern durch Verwenden linearer Programmierung neue obere Schranken für die Anzahl an Kreuzungen: Wenn G mindestens 3 Knoten hat, gelten die oberen Schranken $\frac{n-2}{2}$ für Kreuzungen in 2-Layer 1-planare Zeichnungen und $\frac{10}{3}n - \frac{20}{3}$ für 2-Layer 2-planare Zeichnungen von G. Außerdem beweisen wir die Schranke $|E| \leq \frac{3}{2}n + \frac{4}{3}$, die nahe an der bekannten Kantendichte $\frac{3}{2}n - 2$ [4] of 2-Layer 1-planar Zeichnungen von G liegt.

Contents

1. Introduction

Graph theory formulates and tries to answer a number of interesting questions concerning graphs, their parameters and their properties. A graph $G = (V, E)$ is a set of vertices V and a set of edges E with every edge having two vertices as its endpoints. In a *drawing* Γ of a graph G, vertices are points in the plane and every edge is a continuous line, that does not cross itself and connects two such points. Edges can be directed or undirected, they can be weighted and may cross other edges. Graphs that share the same kinds of properties belong to the same *graph class*. There are, for example, *connected* graphs, where every vertex is accessible through a sequence of edges, there are *bipartite* graphs, whose vertex set consists of two independent sets and where edges only connect two vertices if they are in different sets, or *planar* graphs, where all edges can be drawn in such a way that there are no crossings. If a connected, bipartite or planar drawing Γ of G exists, then G is also called connected, bipartite or planar. The same applies in the other direction, if G has one of the mentioned properties, then there exists a drawing with the same property.

Clearly, how readable the drawings of a graph are for humans depends on its underlying data. Especially planar graphs are visually pleasant, because the absence of crossings makes it easy to identify connections fast [10]. However, with real-world data sets, it is seldom possible to use planar graphs, because the data is often too interconnected or because planar graphs are too limiting with only up to $3|V| - 6$ possible edges. In these cases, we need beyond-planar graphs to correctly and clearly model the data. *Beyond-planar* graphs are not planar but instead have other properties and restrictions to be more readable. Well-known examples include *RAC-graphs* where crossings need to be formed at a right angle, *quasiplanar* graphs where no three edges are allowed to pairwise cross each other or *k-planar* graphs, where edges can have up to k crossings.

Regardless of whether a graph is planar or beyond-planar, it is always useful to know about the maximum number of edges and crossings that are possible for a certain graph class. We call the upper bound on the number of edges over all possible drawings of a graph its *edge density*. When we talk about the maximum number of crossings, we also mean the maximum number that can be achieved over all drawings of G. As mentioned before, planar graphs are known to have edge density $3|V| - 6$, and since all edges have to be crossing-free in planar drawings, the maximum number of crossings for planar graphs is 0. A more extensive list of beyond-planar graph classes and their bounds can be found in the book "Beyond Planar Graphs" [12] and the dissertation "New Parameters for Beyond-Planar Graphs" [11]. In particular, chapter 7 of the dissertation, about drawing beyond-planar graphs on two layers, is relevant for this thesis.

A concise way to represent a bipartite graph is by placing its vertices on two separate layers. Drawing graphs on parallel lines was first introduced by Sugiyama [13], and using two layers has since been studied for numerous topics. To name a few, there is research on whether a bipartite graph has a biplanar subgraph [5], 2-Layer drawings of RAC-graphs [3] and k-planarity in 2-Layer drawings [11]. More formally, a *2-Layer drawing* consists of two parallel, usually horizontal lines that symbolize two layers. Much like with bipartite graphs, the vertices are implicitly split into two independent sets that are positioned on one of the two layers. Edges can then only connect vertices that are on different layers. The aforementioned dissertation [11] already explores a number of bounds and similarities of 2-Layer drawings of beyond-planar graphs, but has its focus put mostly on edge density and some lower bounds on the number of crossings. There is more research on lower bounds for the number of crossings than on upper bounds overall [1, 11, 9, 8].

The more crossings in a drawing, the less readable it becomes to a human viewer, hence the interest in lower bounds on crossing numbers. Still, it can be useful to also know of the "worst-case" numbers, which is why an upper bound on the number of crossings is interesting as well. In this thesis, we study 2-layer 1-planar drawings and 2-layer 2-planar drawings of G. To obtain the mentioned bounds, we use the result of a recently published paper that provides new means of proving upper bounds on the number of edges and crossings. "The Density Formula: One Lemma to Bound Them All" [7] introduces the *Density Formula*, which relates the number of edges, vertices and crossings to the sizes of cells in a drawing. *Cells* are the separate contained areas of the plane that are left when removing all edges and vertices from a connected drawing. The *size* of a cell c, written as $||c||$, is determined by counting the component-parts that lay in its boundary, namely edge-segments and vertex-incidences. The sets that consist of the same sized cells $||c|| = i$ are written as C_i. Proofs using the Density Formula rely mostly on counting the smaller-sized cells in drawings, which is a much more elemental task than the proceeding proofs which often deal with involved case distinctions. We now state the Density Formula [7]. Let $t \in \mathbb{R}$ and Γ be a connected drawing of $G = (V, E)$ with $|E| \geq 1$. Then the following holds:

$$|E| = t(|V| - 2) - \sum_{c \in C} \left(\frac{t-1}{4} ||c|| - t \right) - |X|$$

The formula is used after choosing a specific value for t, for example $t = 5$, which is also used later in this thesis:

$$\begin{aligned} |E| &= 5|V| - 10 - \sum_{c \in C} (||c|| - 5) - |X| \\ &= 5|V| - 10 + 2|C_3| + |C_4| - |X| - \sum_{c \in C_{\geq 5}} (||c|| - 5) \\ &\leq 5|V| - 10 + 2|C_3| + |C_4| - |X| \end{aligned}$$

Observe that it suffices to count the smaller-sized cells: Since $|C_3|$ and $|C_4|$ have positive factors, we obtain an upper bound on the number of edges by finding an upper bound on the number of small-sized cells. For $t = 5$, these are the cells of size 3 and 4. Moreover, after a simple transformation of the Density Formula, we can use it to bound the number of crossings as well:

$$|X| = t(|V| - 2) - \sum_{c \in C} \left(\frac{t-1}{4} ||c|| - t \right) - |E|$$

This thesis explores the usage of the Density Formula in 2-layer k-planar drawings in regard to finding upper bounds on the number of crossings.

In Chapter 2, we introduce important definitions and preliminaries to fully understand the tools that are used and to further facilitate working with cells on 2-layer drawings. We also give an explanation on linear programming.

In Chapter 3, we consider 2-Layer 1-Planar drawings. We determine the different types of smaller-sized cells that can occur, derive a number of inequalities and discuss multiple ways of using the Density Formula to obtain an upper bound on the number of crossings. We also reconfirm the already known edge density [4] by using a linear program.

Chapter 4 considers 2-layer 2-planar drawings. When exploring the possible cell types that, we put our focus on the configurations they create and again form rules and inequalities that are later used in a linear program. We discuss a possible upper bound on the number of crossings and prove a similar bound of $\frac{10}{3}n - \frac{20}{3}$.

Our main result is a new upper bound $\frac{|V|-2}{2}$ on the number of crossings, given a 2-layer 1-planar drawing of a simple, undirected, connected, maximal graph $G = (V, E)$ with $|V| \geq 3$. Furthermore, we outline a concept of a possible upper bound $|V| - 2$ for 2-layer 2-planar drawings of G and proof the bound $|X| \leq \frac{10}{3}n - \frac{20}{3}$. Additionally, we reproduce a similar bound $|E| \leq \frac{3}{2}n + \frac{4}{3}$ to the known edge density of G with $\frac{3}{2}|V| - 2$ for 2-layer 1-planar drawings [4].

2. Preliminaries

2.1. Graphs and Drawings

We consider undirected connected maximal graphs in the plane. Such a graph $G = (V, E)$ consists of a set of vertices V and a set of edges E, where $n = |V|$ denotes the number of vertices. A graph G is *connected* if every vertex of V can be reached through a sequence of edges from any other vertex. We call two vertices *adjacent*, if they are connected by an edge. We call two edges *adjacent* or *neighboring*, if they share a vertex as an endpoint. A *maximal* graph has as many edges as is possible to draw without violating any of its restrictions. We further only consider *simple* graphs, meaning their drawings contain neither loops nor duplicate edges connecting the same pair of vertices. Moreover, we assume that no three edges cross in the same point.

In a *drawing* Γ of a graph G, vertices are points in the plane and every edge is a continuous line that connects two points. We generally consider *simple drawings*, where any two edges have at most one point in common and no edge crosses itself. By definition, adjacent edges do not cross in simple drawings. A drawing where every edge is crossed at most k times is called *k-planar*. In this regard a crossing-free drawing with only uncrossed edges is called *planar*, whereas a drawing where each edge is crossed at most once is called *1-planar*. Therefore, planarity is a special case of k-planar drawings.

We split E into different sets, based on the amount of times an edge is crossed. So E_0 is the set of uncrossed edges, while edges in E_1 are crossed exactly once, edges in E_2 exactly twice and so on. For a k-planar drawing Γ of G it follows that $|E| = |E_0| + |E_1| + \cdots + |E_k|$. A crossed edge is divided into multiple *edge-segments* that are contained by crossing- and vertex-incidences. There are *outer* edge-segments contained by one vertex- and one crossing-incidence, and *inner* edge-segments with two crossing-incidences. Uncrossed edges are interpreted as *uncrossed* edge-segments with two vertex-incidences.

The set of crossings in Γ is denoted by X. When we talk about the maximum number of crossings, we mean the maximum over all possible drawings of G.

The *edge density* of a graph G, is the maximum number of edges over all possible drawings of G. For example, 1-planar graphs have an edge density of $4n - 8$ [9]. Since we assume our graphs to be maximal, G always has as many edges as possible, in other words, the number of edges is equal to its edge density.

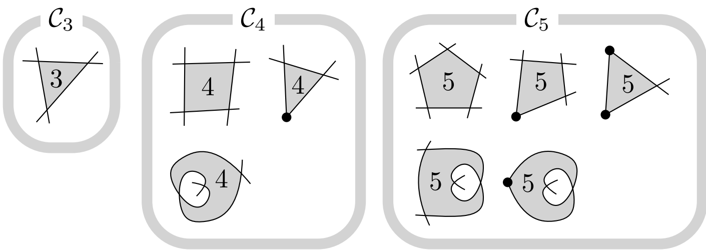


Figure 2.1.: Taken from [7], all celltypes with size less than 6 in a simple connected drawing of a graph G with $n \geq 3$. The bottom row are degenerate cells that do not appear in simple drawings.

2.1.1. Cells

In this thesis we want to derive upper bounds on the number of crossings and edges using the density lemma and thus need to look at the occurring cells in Γ. *Cells* are the contained areas of the plane that are left when removing all edges and vertices from a connected drawing. The size of a cell c is denoted by $||c||$. Sets of cells with the same size i are written as C_i. Figure 2.1 shows an illustration of all cells with a size smaller than 5, the top row showing all cells that can occur in a simple drawing. To determine the size of a cell, the edge-segment-incidences and vertex-incidences that form its boundary are counted. Note that we do not count crossing-incidences here. Looking at the rightmost cell of C_5 in Figure 2.1, its cyclic boundary consists of a vertex-incidence, an uncrossed edge-segment-incidence followed by another vertex-incidence, an outer edge-segment-incidence, then a crossing-incidence, another outer edge-segment and then we already reached the vertex we started on. Counting only the relevant incidences of this boundary, we can confirm the cell has a size of 5.

We use small pictograms to refer to the different types of cells. The pictogram  for example, stands for cells with the aforementioned boundary. In the actual drawing, the cell can be transformed or rotated, but as long as its boundary consists of the same incidences in the specified order, it is addressed by the -pictogram. Using  in an equation represents the number of cells of that celltype, while using it in a sentence is referring to the type of cell.

The bottom row of Figure 2.1 shows examples of *degenerate cells*, where the same edge- or vertex-incidence appears multiple times in the boundary of a cell. Observe how the edges need to either cross themselves or have more than one point in common with another edge. Since this would be violating the restrictions of simple drawings and graphs, these types of edges are irrelevant for the rest of this work.

Theorem 2.1. Density Formula [7]
Let $t \in \mathbb{R}$ and Γ be a connected drawing of a graph $G = (V, E)$ with $|E| \geq 1$. Then

$$|E| = t(n-2) - \sum_{c \in C} \left(\frac{t-1}{4} ||c|| - t \right) - |X|.$$

In this thesis we use the formula with $t = 5$:

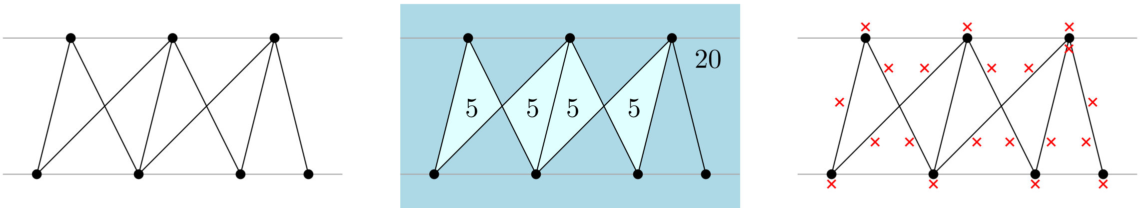
$$|E| = 5n - 10 - \sum_{c \in C} (||c|| - 5) - |X|. \tag{2.1}$$

2.1.2. 2-Layer Drawings

A *2-Layer drawing* consists of two parallel, usually horizontal lines that symbolize two layers. Vertices have to be positioned on one of the layers and edges can only connect

Class	Edge Density	Ref.	Lower Bound Crossings	Ref.	Upper Bound Crossings	Ref.
planar	$n-1$	[6]	0	def.	0	def.
1-planar	$\frac{3}{2}n-2$	[4]	$0.295\frac{m^3}{n^2}$	[11]	$\frac{\|E\|}{2}$	def.
2-planar	$\frac{5}{3}n-\frac{7}{3}$	[11]	$0.295\frac{m^3}{n^2}$	[11]	$\frac{2\|E\|}{2}$	def.
k-planar	$\frac{125}{96}\sqrt{kn}$	[11]	$0.295\frac{m^3}{n^2}$	[11]	$\frac{k\|E\|}{2}$	def.

Table 2.1.: Bounds on the number of edges and crossing in 2-Layer drawings. The edge density for k-planar graphs is valid for $k \geq 6$. The lower bound on the number of crossings is valid for $n \geq 9$ vertices and $m \geq \frac{125}{48}n$ edges.



(a) Drawing Γ of G.

(b) Present cells and their sizes. Inner cells light blue, outer cell blue.

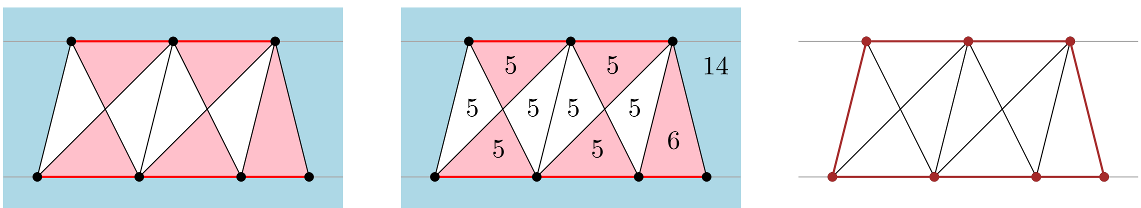
(c) Counted incidences for the size of the outer cell marked.

Figure 2.2.: Simple 2-layer 1-planar drawing Γ of a connected maximal graph G.

vertices from opposite layers. There are other common restrictions on this type of drawing, for example that edges are drawn as straight lines [3, 13], which we mostly adhere to in this thesis. However, to help make the case distinction in Section 4.1 as visibly clear as possible, we also use some curved lines. This does not alter the correctness of our proofs.

Refer to Table 2.1 for an overview of the relevant currently known edge densities and bounds on the number of crossings for 2-layer drawings. The displayed upper bounds on the number of crossings are derived by the fact that every edge can be crossed at most k times and since a crossing consists of two edges, $\frac{1}{2}|E|k$ is the maximum amount of crossings in a k-planar drawing. After inserting the corresponding numbers for k, we get the upper bounds claimed in the table.

After removing all edges and vertices of a connected drawing, there are not only the *inner cells* left that were fully enclosed by edges, but also one outer area spanning over the rest

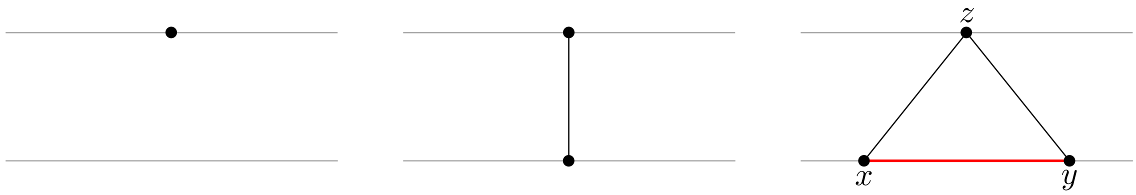


(a) Augmented maximal drawing Γ^+.

(b) Cells in Γ_1^+ and their sizes

(c) Boundary of the new outer cell accentuated.

Figure 2.3.: Augmented maximal 2-layer 1-planar drawing Γ^+ of a connected simple maximal graph G. Edges of E_{hor} marked red. Boundary cells highlighted red, new outer cell blue.



(a) $n = 1$ (b) $n = 2$ (c) $n = 3$

Figure 2.4.: Augmented 2-layer k-planar drawing Γ^+ of a connected simple maximal graph G for the cases of $n \leq 3$ vertices. Edges of E_{hor} marked red.

of the plane. We call this outside area the *outer cell*. See Figure 2.2 for a distinction between inner and outer cells. Especially Figure 2.2c shows the difficulty that the outer cell imposes: Because of the constraint on 2-layer drawings that adjacent vertices do not share a layer, we have a lot of "open" spaces. These spaces contribute to the size of the outer cell, making it cumbersome to work with. Observe how there are edge-segments and vertices counted more then once, since they appears multiple times in the boundary. We primarily want to work with smaller cells, as to have a better grasp on how to bound them, which is why we use the following workaround.

To make it easier to work with the outer cell of Γ, we instead look at the *augmented drawing* Γ^+, which has additional horizontal edges E_{hor} between consecutive vertices of a layer. By adding these boundary edges E_{hor}, the previous outer cell is divided into boundary cells and an outer cell of smaller size, see Figures 2.2a, 2.3a and 2.3b. This means that the number of edges in G, and therefore in Γ, is different to the number of edges in Γ^+.

Observation 1. *For a k-planar drawing Γ of a graph $G = (V, E)$ it follows that*

$$|E| = |E_0| + |E_1| + \cdots + |E_k|. \tag{2.2}$$

For the augmented drawing Γ^+ it follows that

$$|E| = |E_{\text{hor}}| + |E_0| + |E_1| + \cdots + |E_k|. \tag{2.3}$$

We will from now on assume that $n \geq 3$. For the sake of completeness, we now look at the augmented maximal drawings of a k-planar graph with less then 3 vertices: If there is only one vertex, because an edge needs two endpoints and G is simple, there are no edges and therefore no crossings, see Figure 2.4a for an example. If G has two vertices, we can assume that the vertices lay on opposite layers and are connected by an edge, as depicted in Figure 2.4b. This assumption is reasonable, because G is maximal. The described edge is the only possible edge in Γ^+ that fulfills our restriction on G. Because there is only one edge and G is simple, there can be no crossings. Note that for both cases, there are no edges of E_{hor}, which is why Γ and Γ^+ are equal here.

Observation 2. *In a 2-layer augmented drawing Γ^+ of a k-planar graph G with edges E, crossings X and n vertices, it holds that for $n = 1$: $|E| = 0$ and $|X| = 0$, while for $n = 2$: $|E| = 1$ and $|X| = 0$.*

Let us also look at the smallest possible case for our assumption of $n \geq 3$. If $n = 3$ and G is maximal, we can rule out the case that all three vertices are on the same layer with no edges joining them. We are thus left with two vertices on one layer and one vertex on the other. Let x and y be the two vertices on the same layer and the other vertex be called z. We can now draw two edges, connecting x to z and y to z. These edges can always be drawn independent of k, because they are crossing-free. Because the two present edges are neighbors, they cannot form a crossing, leaving us with zero crossings once again. There are also no further edges that can be drawn without violating restrictions on G or Γ. In this case, the only difference between Γ and Γ^+ is a single additional edge of E_{hor} that connects x and y on their shared layer. For an example of the described situation, see Figure 2.4c.

Observation 3. *In the augmented 2-layer drawing Γ^+ of a k-planar graph G with $n = 3$ vertices, there are $|E| = 3$ edges and $|X| = 0$ crossings.*

The two leftmost vertices on opposite layers can always be connected by an uncrossed edge, and because G is maximal, in every drawing Γ of G this edge is present. The same applies for the two rightmost vertices. Because of our assumption that $n \geq 3$, we can safely say that these are two different edges. As the example in Figure 2.3c shows, the boundary of the outer cell therefore consists of the two outmost vertical edges that connect the opposite layers, n vertices from G and n connecting boundary edges of E_{hor}.

Observation 4. *In every 2-layer drawing Γ of a k-planar graph G, there is an edge connecting the two leftmost vertices of both layers and a different edge connecting the two rightmost vertices of both layers.*

Observation 5. *The equations $|E_{\text{hor}}| = n-2$ and $||\text{outerCell}|| = 2n$ hold for the augmented 2-layer drawing Γ^+ of a k-planar graph G*

From Observations 3 and 5 and the fact that we assume G to be maximal with $n \geq 3$, it follows that the size of the outer cell is strictly larger than 4.

Observation 6. *In a 2-layer augmented drawing Γ^+ of a k-planar graph G with $n \geq 3$ vertices, it holds that $||\text{outerCell}|| > 4$.*

2.2. Linear Programming

Linear programming is a reliable method to obtain an optimal solution for a set of linear functions. In this thesis we will form a number of linear inequality constraints based on the relations between cells, edges and crossings. The so-called decision variables must be non-negative and follow these constraints. In our case, we use the components of 2-layer drawings as our decision variables. We then want to find a non-negative vector that fulfills all functions and is maximal for either $|X|$ or $|E|$. We now go through an example process of using a linear program (LP), like it is done later in Sections 3.1.2 and 4.2.

Generally, all constraints are gathered in a matrix $M \in \mathbb{R}^{r,c}$ and a solution vector $b \in \mathbb{R}^{r,1}$. The decision-variables are defined in component vector $k \in \mathbb{R}^{1,c}$, where $k = (k_1, \ldots, k_c)$ and $k_1 = (a_{11}, \ldots, a_{r1})$, …, $k_c = (a_{1c}, \ldots, a_{rc})$. The goal is to find a non-negative vector $x \in \mathbb{R}^c$ that fulfills all constraints and is maximal for either $kx = k_1x_1 + \cdots + k_cx_c$ or a chosen variable of x.

$$\begin{array}{cc} & \begin{array}{cccc} x_1 & \ldots & x_c & b \end{array} \qquad\qquad \textit{constraints} \\ M = & \left(\begin{array}{ccc|c} a_{11} & \ldots & a_{1c} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{r1} & \ldots & a_{rc} & 0 \end{array}\right) \begin{array}{c} a_{11}x_1 + \cdots + a_{1c}x_c \leq 0 \\ \vdots \\ a_{r1}x_1 + \cdots + a_{rc}x_c \leq 0 \end{array} \end{array}$$

To make the example more precise to our use-cases of finding upper bounds on the number of crossings and edges, let us use a few exemplary constraints. We set up a number of rules through inequalities in relation to zero. We want the LP to be scalable with the number of vertices n, so we need the include the equality $n = 1$. We do this by splitting the equality into two linear inequalities $n \leq 1$ and $-n \leq 1$, which together are equivalent to $n = 1$. Let vector $x = (x_1, \ldots, x_n)$ have x_1 as "variable" n, x_2 as $|E|$ and x_3 as $|X|$. After transforming all inequalities in relation to zero, we get the solution vector $b = (1, 1, 0, 0)$ and have a finished matrix M_0.

$$M_0 = \begin{array}{c} \begin{array}{cccc} n & |E| & |X| & \end{array} \\ \left(\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ -2 & 1 & 0 & 0 \end{array}\right) \end{array} \begin{array}{r} \textit{constraints} \\ n \leq 1 \\ -n \leq 1 \\ |X| - |E| \leq 0 \\ |E| - 2n \leq 0 \end{array}$$

Given the case we want to obtain an upper bound on the number of crossings, the LP computes a non-negative vector v that fulfills all constraints while maximizing the "variable" $|X|$. In our example, we get the optimal vector $v = (1, 2, 2)$. Because we wrote the LP to scale with n and work with "$\leq$", the number 2 of the vector that is meant for "variable" $|X|$ should actually be interpreted as the bound $|X| \leq 2n$. To better comprehend the result of the LP, and to confirm its correctness, we also compute the dual LP. For our example, we get the vector $d = (2, 0, 1, 1)$, which gives us an insight into which constraints were multiplied by what factor in order to get the result of $|X| \leq 2n$. With d, we can retrace the calculation by skipping the equations used for $n = 1$ to get the correct scalable result. We then multiply each constraint by the factor given in the dual d and then sum up the inequalities like in the following:

$$\begin{aligned} & 1(|X| - |E|) + 1(|E| - 2n) \leq 0 \\ \Leftrightarrow & X + (-1 + 1)E - 2n \leq 0 \\ \Leftrightarrow & X - 2n \leq 0 \\ \Leftrightarrow & X \leq 2n \end{aligned}$$

We confirmed the result given by the LP and have thus proven an exemplary upper bound on $|X|$ that scales with n.

3. 2-Layer 1-Planar Drawings

To differentiate whether a cell is a boundary cell, we further categorize cells of the same type based on their uncrossed edge. In this regard, we assign colors so that red denotes elements of E_{hor} and green stands for edges of E_0. A -cell for example, can either be a boundary cell or not , as depicted in Figure 3.1. The cells occurring in the mentioned figure are actually the only possible celltypes to exist in an augmented 2-layer 1-planar drawing, as we will prove in the following lemma.

Lemma 3.1. *In an augmented 2-layer 1-planar drawing Γ^+ of a graph G, all inner cells have type or , more specifically there exist only inner cells of types , and .*

Proof. Consider a cell c that is incident to a crossing. Let e and e' denote the edges that form the crossing and let u and v be the endpoints of e and e' incident to c, respectively. Because of 1-planarity, the edge uv can be drawn crossing-free in the interior of c, illustrated in Figure 3.2 and Figure 3.3a. If u and v are on opposite layers, by the maximality of the drawing it follows that uv is contained in Γ^+. Furthermore, c is bounded by uv and the edge-segments of e and e' incident to u and v, respectively. For an illustration of this scenario, see Figure 3.2. It follows that c must be a cell of type . Otherwise u and v are on the same layer. If there is no other vertex on the layer between u and v, the vertices are connected by a boundary edge of E_{hor}. This would mean c is a -cell, as depicted

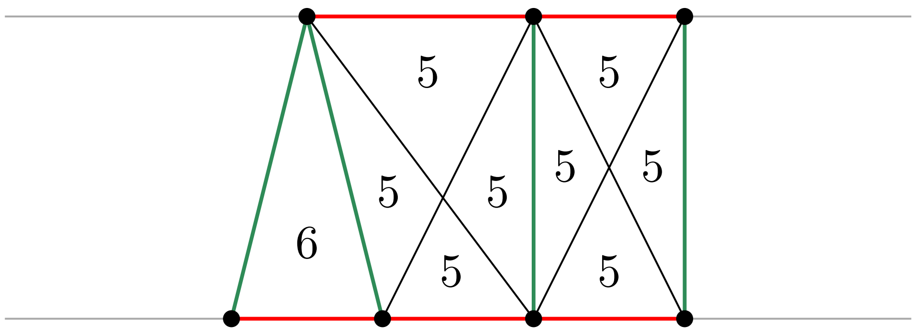
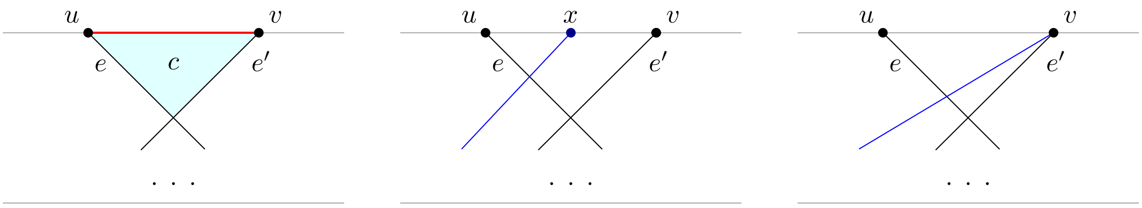


Figure 3.1.: Γ^+ with edges of E_{hor} colored red and edges of E_0 colored green. Example of same cell type having different colored uncrossed edges, resulting in a difference of cell types and .

Figure 3.2.: Cell c incident to a crossing, with its incident vertices u and v on opposite layers.



(a) (b) (c)

Figure 3.3.: Left: Cell c incident to a crossing, with its incident vertices u and v on the same layer. Middle and Right: Impossible cases due to contradiction of 1-Planarity.

in Figure 3.3a. Assume for the sake of contradiction, that there is a vertex x between u and v. Since G is connected, x is incident to a vertical edge which must cross either e or e', contradicting the 1-planarity of the drawing. Figure 3.3b illustrates this conflict. For the same reason, u and v cannot have any more incident edges running between e and e', see Figure 3.3c. Consequently, cells that are incident to a crossing are of type , moreover, they always form the subgraph .

Consider an interior cell c and assume that it is not a -cell. We show that c must be a cell of type . By the argumentation above, all edges incident to c are crossing-free. Let uv and vw be two edges on the boundary of c, consecutive around the shared vertex v. The only two possibilities are that either uv is a boundary edge and vw an uncrossed edge of E_0 or that both uv and vw are uncrossed edges in E_0. No other options are possible, since c is not the outer cell and G is a connected graph, so uv and vw cannot both be boundary edges of E_{hor}. In the first possible case, uv has both its endpoints on the same layer and vw is an uncrossed edge of E_0. Because of 1-planarity, we can draw a third uncrossed edge connecting the vertices of endpoints u and w, see Figure 3.4. By the maximality of the drawing, c is a cell of type . The only other possibility would be for both uv and vw to be uncrossed edges in E_0, v being on one layer and u and w on the other, cf. Figure 3.5a. If there is no vertex on the same layer as u and w lying between them, then a third crossing-free edge uw of E_{hor} can be drawn, illustrated in Figure 3.5b. Assuming there is such a vertex called x, since G is connected, it would have to be connected to the other layer via an edge g. With uv and vw being crossing-free and g not in E_{hor}, edge g needs to have the endpoints x and v as depicted in Figure 3.5c. This is a contradiction to our earlier assumption that uv and vw are consecutive edges around v. We have thus shown that any cell that is not the outer cell or a -cell is a -cell. □

With this, we have completed an important step to successfully use the density lemma, in order to bound the number of crossings and edges for 2-layer 1-planar drawings of graphs. We now establish some equations regarding our limited types of cells and edges in Γ^+, to further aid us in achieving those bounds.

Figure 3.4.: $uv \in E_{\text{hor}}$ and $vw \in E_0$ consecutive edges on the boundary of c.

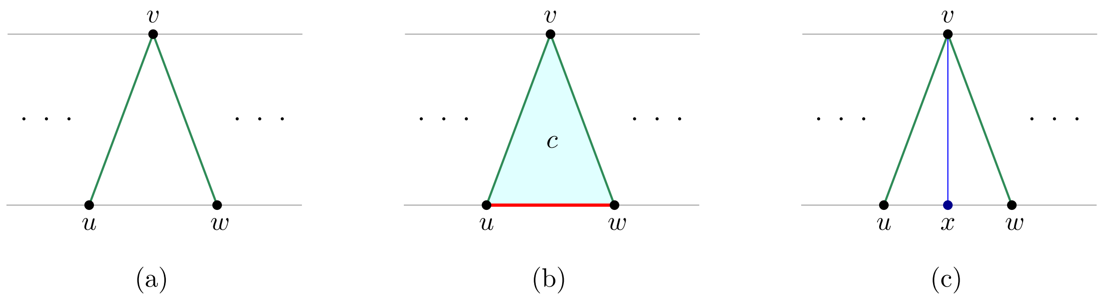


Figure 3.5.: Left, Middle: $uv, vw \in E_0$ consecutive edges on the boundary of c. Right: Contradiction to consecutiveness.

Lemma 3.2. *In the augmented 2-layer 1-planar drawing Γ^+ of a graph G, the following equations hold true:*

$$|E| = |E_{\text{hor}}| + |E_0| + |E_1| \tag{3.1}$$

$$||\text{outerCell}|| = 2n \tag{3.2}$$

$$|E_{\text{hor}}| = n - 2 \tag{3.3}$$

$$|E_{\text{hor}}| = {\color{red}\triangle_5} + {\color{red}\triangle_6} \tag{3.4}$$

$$|E_0| = \frac{1}{2}{\color{green}\triangle_5} + {\color{red}\triangle_6} + 1 \tag{3.5}$$

Proof. Equation (3.1) follows from Observation 1, while (3.2) and (3.3) can be found in Observation 5.

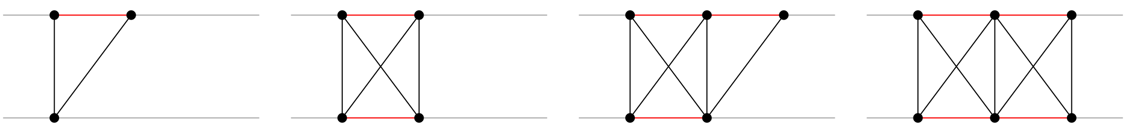
Since there can only be cells of type $\triangle_5$ or $\triangle_6$, see Lemma 3.1, boundary edges of E_{hor} can only be incident to either a ${\color{red}\triangle_5}$-cell or a ${\color{red}\triangle_6}$-cell. This concludes equation (3.4).

For equation (3.5), we count the incidences of uncrossed edges with inner cells. Every uncrossed edge that is not incident to the outer cell has two such incidences, whereas the two uncrossed edges incident to the outer cell have one such incidence. This makes for a total of $2|E_0| - 2$ incidences. On the other hand every ${\color{green}\triangle_5}$ is incident to one edge of E_0 while ${\color{red}\triangle_5}$ is incident to none and ${\color{red}\triangle_6}$ is incident to two. Therefore the number of incidences is ${\color{green}\triangle_5} + 2{\color{red}\triangle_6}$, resulting in the following equation:

$$2|E_0| - 2 = {\color{green}\triangle_5} + 2{\color{red}\triangle_6}$$
$$\Leftrightarrow |E_0| = \frac{1}{2}{\color{green}\triangle_5} + {\color{red}\triangle_6} + 1 \qquad \square$$

Lemma 3.3. *In the augmented 2-layer 1-planar drawing Γ^+ of a graph G, the following equations hold true:*

$$\triangle_5 = {\color{green}\triangle_5} + {\color{red}\triangle_5} \tag{3.6}$$



(a) $n = 3 \quad |X| = 0$ (b) $n = 4 \quad |X| = 1$ (c) $n = 5 \quad |X| = 1$ (d) $n = 6 \quad |X| = 2$

Figure 3.6.: Augmented 2-layer 1-planar drawing Γ^+ of G with n vertices and $|X|$ crossings.

$$ = \tag{3.7}$$

$$2|X| = \tag{3.8}$$

$$|E_1| = 2|X| \tag{3.9}$$

Proof. Equation (3.6) follows from the definition of the mentioned celltype.

As discussed in Lemma 3.1, a cell of type only appears at a subgraph , which consists of an equal amount of -cells and -cells. This proves equation (3.7).

To show (3.8), we count the number of crossings compared to -cells. A crossing consists of 4 , so together with (3.7) we get:

$$\begin{gathered} 4|X| = \quad = \quad + \quad = 2 \\ \Leftrightarrow 2|X| = \end{gathered}$$

Lastly, counting the incidences of crossings and cut edges gives us (3.9):

$$|E_1| = 2|X|$$ □

3.1. Bounding Crossings

To get a better feeling of what bound to aim for, we take a look at the possible celltypes. In Lemma 3.1, we found that there can only exist inner cells of type and . We further observed that -cells only occur in a certain type of subgraph , explained in more detail in the proof of said Lemma. As one can easily see by looking at the celltypes, -cells are the only ones forming crossings, or more precisely crossings only occur at -subgraphs. The maximal number of crossings should therefore equal the number of -subgraphs that fit into Γ^+. The -subgraph contains four vertices and when drawing , the first one does indeed need at least four vertices in G. Every additional -subgraph only needs two more vertices, since it can make use of the already existing ones. To get a clearer picture of the explained situation, see Figure 3.6. This means that there can fit up to $\frac{n-2}{2}$ -subgraphs in Γ^+, making $|X| \leq \frac{n-2}{2}$ a plausible bound. Because the edges of E_{hor} are uncrossed, their existence does not make a difference to the number of crossings, which is why the bound should also hold for "normal" simple 2-layer 1-planar drawings of G. In the following we prove that bound to be correct. First, we prove it by using the Density Formula in a by-hand calculation, and then show an alternative method using a linear program.

Theorem 3.4. *Let Γ be a simple 2-layer 1-planar drawing of a connected maximal graph G with n vertices. For $n \geq 3$, Γ has at most $\frac{n-2}{2}$ crossings.*

3.1.1. By-Hand Calculation

Proof. To make the density lemma applicable, G needs to have at least one edge, meaning that $n \geq 2$ should hold true. From our restriction of $n \geq 3$ and Lemma 3.1, it follows that $|C_3|$ and $|C_4|$ are 0 and the only cells with their sizes greater or equal 5 are the celltypes $\triangle_5$, $\triangle_6$ and the outer cell. With this we can start rewriting the Density Formula with $t = 5$ from Equation (2.1) as follows:

$$
\begin{aligned}
|E| &= 5n - 10 - \sum_{c \in C} (||c|| - 5) - |X| \\
&= 5n - 10 + 2|C_3| + |C_4| - |X| - \sum_{c \in C \geq 5} (||c|| - 5) \\
&= 5n - 10 - |X| - \triangle_5(5-5) - \triangle_6(6-5) - (||\text{outerCell}|| - 5) \\
&= 5n - 10 - |X| - \triangle_6 - (2n - 5) \\
&= 3n - 5 - |X| - \triangle_6
\end{aligned}
$$

Since we want to prove a bound for the number of crossings, from here on out we solve for $|X|$ and use fitting equations from Lemma 3.2 and 3.3 to arrive at the desired bound. The specific equation that is used in a step is mentioned on the right-hand side.

$$
\begin{aligned}
|X| &= 3n - 5 - |E| - \triangle_6 \\
&= 3n - 5 - |E_{\text{hor}}| - |E_0| - |E_1| - \triangle_6 && (3.1) \\
&= 3n - 5 - (n-2) - (\frac{1}{2}{\color{green}\triangle_5} + \triangle_6 + 1) - 2|X| - \triangle_6 && (3.3), (3.5), (3.9) \\
&= 2n - 3 - (\frac{1}{2}{\color{green}\triangle_5} + \triangle_6 + 1) - 2|X| - \triangle_6 \\
&= 2n - 4 - \frac{1}{2}{\color{green}\triangle_5} - \triangle_6 - (|X| + |X|) - \triangle_6 \\
2|X| &= 2n - 4 - \frac{1}{2}{\color{green}\triangle_5} - \triangle_6 - |X| - \triangle_6 \\
&= 2n - 4 - \frac{1}{2}{\color{green}\triangle_5} - \triangle_6 - ({\color{red}\triangle_5} + \triangle_6) && (3.8) \\
&= 2n - 4 - \frac{1}{2}{\color{green}\triangle_5} - \triangle_6 - |E_{\text{hor}}| && (3.4) \\
&= 2n - 4 - \frac{1}{2}{\color{green}\triangle_5} - \triangle_6 - (n-2) && (3.3) \\
&= n - 2 - \frac{1}{2}{\color{green}\triangle_5} - \triangle_6 \\
&\leq n - 2 \\
\Leftrightarrow |X| &\leq \frac{(n-2)}{2} && \square
\end{aligned}
$$

3.1.2. Linear Program

Alternatively, we can also prove a similar bound to Theorem 3.4 by using a linear program (LP). This method is especially useful for more complex graphs, where the amount of inequalities would otherwise be overwhelming to use in a by-hand calculation. For an explanation of how we use the LP, see the example in Section 2.2. We write the LP using the equations of Lemmas 3.2 and 3.3 and the Density Formula with $t = 5$ from Equation (2.1). Each equation needs to be rewritten into a linear inequality in relation to zero. While doing so, we also avoid fractions and ignore constants. For example, we rewrite Equation (3.5) as follows:

$$|E_0| = \frac{1}{2}\triangle^{\times}_{5g} + \triangle_{6} + 1 \rightarrow \begin{cases} 2|E_0| - \triangle^{\times}_{5g} - 2\triangle_{6} & \leq 0 \\ -2|E_0| + \triangle^{\times}_{5g} + 2\triangle_{6} & \leq 0 \end{cases}$$

We further consider $n = 1$, meaning that we are interested in 100 percent of vertices. The result will therefore scale with the number of vertices. The adjusted inequalities and Density Formula finally give us matrix M_1.

$$M_1 = \begin{array}{c} \begin{array}{ccccccccc} N & |X| & |E| & |E_{\text{hor}}| & |E_0| & |E_1| & \triangle^{\times}_{5r} & \triangle^{\times}_{5g} & \triangle_{6} \end{array} \\ \left(\begin{array}{ccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \begin{array}{l} \\ n=1 \\ (2.1) \\ (3.1) \\ (3.1) \\ (3.3) \\ (3.4) \\ (3.4) \\ (3.5) \\ (3.5) \\ (3.7) \\ (3.7) \\ (3.8) \\ (3.9) \\ (3.9) \end{array}$$

To prove an upper bound on $|X|$, we need to maximize $|X|$ in the LP. We get the result of $|X| \leq \frac{n}{2}$ which is basically our proven bound of $|X| \leq \frac{n-2}{2}$ only without the constant of -1. After computing the dual of the LP, we get the vector

$$d = (\frac{1}{2} \quad \frac{1}{6} \quad 0 \quad \frac{1}{6} \quad 0 \quad 0 \quad \frac{1}{6} \quad 0 \quad \frac{1}{12} \quad \frac{1}{6} \quad 0 \quad \frac{1}{4} \quad 0 \quad \frac{1}{6}).$$

Using d in a by-hand calculation to confirm the bound given by the LP, we also add the ignored constants back in to get an even more precise bound:

$$\begin{aligned} & \frac{1}{6}(|E| - 3n + |X| + \triangle_{6}) + \frac{1}{6}(-|E| + |E_{\text{hor}}| + |E_0| + |E_1|) \\ & + \frac{1}{6}(-|E_{\text{hor}}| + \triangle^{\times}_{5r} + \triangle_{6}) + \frac{1}{12}(-2|E_0| + \triangle^{\times}_{5g} + 2\triangle_{6} + 2)) \\ & + \frac{1}{6}(\triangle^{\times}_{5g} - \triangle^{\times}_{5r}) + \frac{1}{4}(2|X| - \triangle^{\times}_{5g}) + \frac{1}{6}(-|E_1| + 2|X|) && \leq 0 \\ \Leftrightarrow \quad & (\frac{1}{6} + \frac{2}{4} + \frac{2}{6})|X| - \frac{3}{6}n + \frac{2}{12} \\ & + (\frac{1}{6} - \frac{1}{6})\triangle^{\times}_{5r} + (\frac{1}{12} + \frac{1}{6} - \frac{1}{4})\triangle^{\times}_{5g} + (\frac{1}{6} + \frac{1}{6} + \frac{2}{12})\triangle_{6} \\ & + (\frac{1}{6} - \frac{1}{6})|E| + (\frac{1}{6} - \frac{1}{6})|E_{\text{hor}}| + (\frac{1}{6} - \frac{2}{12})|E_0| + (\frac{1}{6} - \frac{1}{6})|E_1| && \leq 0 \\ \Leftrightarrow \quad & |X| - \frac{1}{2}n + \frac{1}{6} + \frac{1}{2}\triangle_{6} && \leq 0 \\ \Leftrightarrow \quad & |X| \leq \frac{1}{2}n - \frac{1}{6} - \frac{1}{2}\triangle_{6} \\ \Leftrightarrow \quad & |X| \leq \frac{n}{2} - \frac{1}{6} \end{aligned}$$

We reconfirmed the bound $|X| \le \frac{n}{2}$ calculated by the LP, and even got it closer to Theorem 3.4 by adding back the constants of the used inequalities. This bound, while not being equal to the theorem, is an upper bound on the proven bound $|X| \le \frac{n-2}{2} = \frac{n}{2} - 1 \le \frac{n}{2} - \frac{1}{6}$ and thus correct.

3.2. Edge Density

We can use the same LP as in Section 3.1.2 to also bound the number of edges for augmented 2-layer 1-planar drawings on G. We do this by maximizing $|E|$ instead of $|X|$. The known edge density for 1-planar graphs that can be drawn in two layers is $\frac{3}{2}n - 2$ [4]. Adding $|E_{\text{hor}}| = n-2$ to that bound (Observation 5), we receive the upper bound for the augmented drawing $|E| \le \frac{5}{2}n - 4$. Disregarding the constants like before, the LP gets us the correct bound of $\frac{5}{2}n$ with the following vector for the dual of the LP:

$$d = (\frac{5}{2} \quad \frac{2}{3} \quad \frac{1}{3} \quad 0 \quad \frac{1}{2} \quad 0 \quad \frac{1}{6} \quad \frac{1}{6} \quad 0 \quad \frac{1}{6} \quad 0 \quad 0 \quad \frac{1}{3} \quad 0)$$

Reconfirming the obtained bound by using d and M_1 in a by-hand calculation, while adding back the ignored constants of the inequalities the constraints are based on:

$$\begin{aligned}
& \frac{2}{3}(|E| - 3n + |X| + \triangle_6) + \frac{1}{3}(|E| - |E_{\text{hor}}| - |E_0| - |E_1|) \\
& + \frac{1}{2}(|E_{\text{hor}}| - n + 2) + \frac{1}{6}(-|E_{\text{hor}}| + \cancel{\triangle}_5 + \triangle_6) \\
& + \frac{1}{6}(2|E_0| - \cancel{\triangle}_5 - 2\triangle_6 - 2) + \frac{1}{6}(\cancel{\triangle}_5 - \cancel{\triangle}_5) + \frac{1}{3}(|E_1| - 2|X|) && \le 0 \\
\Leftrightarrow\ & (\frac{2}{3} + \frac{1}{3})|E| + (-\frac{6}{3} - \frac{1}{2})n + (1 - \frac{2}{6}) + (\frac{2}{3} - \frac{2}{3})|X| \\
& + (\frac{2}{3} + \frac{1}{6} - \frac{2}{6})\triangle_6 + (\frac{1}{6} - \frac{1}{6})\cancel{\triangle}_5 + (-\frac{1}{6} + \frac{1}{6})\cancel{\triangle}_5 \\
& + (-\frac{1}{3} + \frac{1}{2} - \frac{1}{6})|E_{\text{hor}}| + (-\frac{1}{3} + \frac{2}{6})|E_0| + (-\frac{1}{3} + \frac{1}{3})|E_1| && \le 0 \\
\Leftrightarrow\ & |E| - \frac{5}{2}n + \frac{2}{3} + (\frac{4}{6} - \frac{1}{6})\triangle_6 && \le 0 \\
\Leftrightarrow\ & |E| \le \frac{5}{2}n - \frac{2}{3} - \frac{1}{2}\triangle_6 \\
\Leftrightarrow\ & |E| \le \frac{5}{2}n - \frac{2}{3}
\end{aligned}$$

We proved $|E| \le \frac{5}{2}n - \frac{2}{3}$ which is an upper bound on $\frac{5}{2}n - 4$. Subtracting the number of edges $|E_{\text{hor}}| = n - 2$ that are custom to the augmented drawing Γ^+, we obtain the bound $\frac{3}{2}n + \frac{4}{3}$ for 2-layer 1-planar drawings of G. This is correct given $|E| \le \frac{3}{2}n - 2 \le \frac{3}{2}n + \frac{4}{3}$.

Theorem 3.5. *A simple connected maximal 1-planar graph G with $n \ge 3$ vertices that can be drawn on two layers has at most $\frac{3}{2}n + \frac{4}{3}$ edges.*

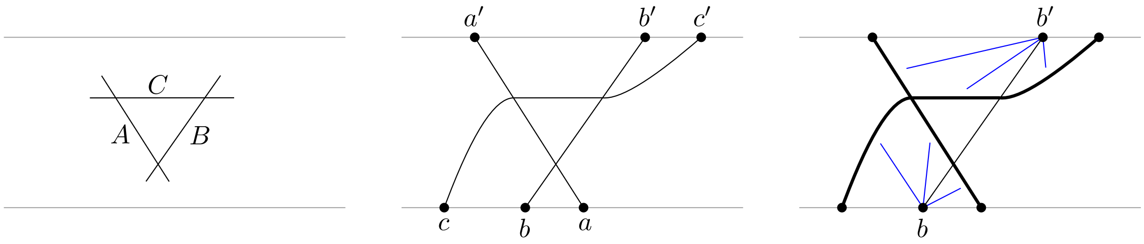
4. 2-Layer 2-Planar Drawings

In this chapter, we consider the augmented 2-layer drawing Γ^+ of G to be 2-planar. To get an overview on what kind of celltypes are possible in Γ^+, we go through the celltypes with sizes 5 or less and check whether they can be drawn without violating any constraints on G and Γ. The first row of Figure 2.1 shows the celltypes we want to check. We will derive a number of inequalities on the relations between celltypes, edges and crossings, and use them together with the Density Formula in a linear program to bound the number of crossings.

4.1. Celltypes, Configurations and Trails

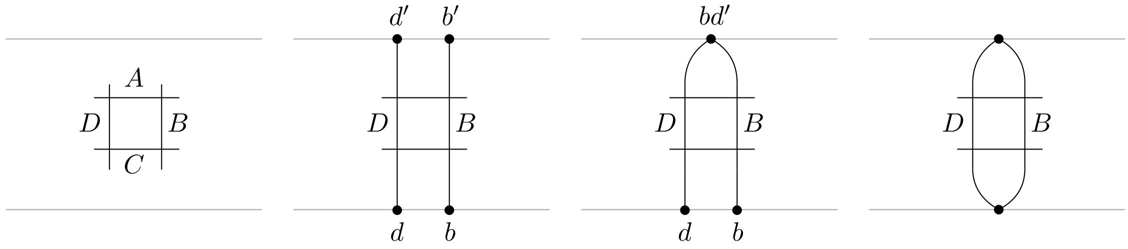
A *configuration* is a kind of subgraph that has a fixed set of cells that always occur in a specific arrangement. More formally, we take the planarization Λ of Γ^+, where all crossings are changed to vertices in order to get a new planar drawing of Γ^+. A connected labeled embedded subgraph of Λ, with labels specifying whether a vertex is representing a crossing of Γ^+ or is a vertex of G, is then called a *configuration*. We speak of the same *type* of configuration if two of these embedded labeled graphs are isomorphic. They are also of the same type if the cell-arrangement is only mirrored but otherwise has the same neighboring cells. For visual clarity of presented figures, illustrations of configurations are not labeled. If a celltype can indeed occur in Γ^+, we try to see if there is a correlating configuration that we can use to obtain inequalities on partaking celltypes. In this regard, we are especially interested to find all possible *trails* of a celltype. *Trails* are special types of configurations. They are a sequence of cells, each pair connected by a shared inner edge-segment in their boundary. A trail starts at a cell with at least one inner edge-segment incidence that is not a $\boxed{4}$-cell, and then traverses to the next cell that shares an inner edge-segment. From here, we have either encountered a $\boxed{4}$-cell and keep traversing to the next cell sharing the opposite inner edge-segment, or we have reached a cell that is not a $\boxed{4}$-cell. In the latter case, the trail is complete, otherwise the traversal continues until a different celltype is reached. A trail that start at a $\boxed{6}$-cell and ends in a cell of type $\triangle_4$ is written as $\triangle_6 \leftrightarrow \triangle_4$. The number of such trails in a drawing is written as $(\triangle_6 \leftrightarrow \triangle_4)$ and is equivalent to $(\triangle_4 \leftrightarrow \boxed{6})$. Note that this notation includes trails with $\boxed{4}$-cells as well as those without.

To facilitate talking about 2-layer drawings, we call one layer the "top" layer and the other the "bottom" layer. This naming does not have any effect on the correctness of the argumentation. For this chapter, we further introduce the convention that edges are labeled



(a) (b) (c)

Figure 4.1.: Trying to fit a -cell into Γ^+. The thickened edges are already crossed twice and would be crossed by the blue edges, which would violate 2-planarity.



(a) (b) (c) (d) Violating restrictions on Γ

Figure 4.2.: Trying to fit a -cell into Γ^+

by upper case letters, while their vertices are labeled with the corresponding lower case letter. Depending on whether a vertex connected to edge A is on the "top" or "bottom" layer, it would be called a' or a, respectively. A vertex on the "top" layer shared by edges A and B would be called ab'. Of course, there may be exceptions to this rule.

4.1.1. Celltypes of C_3

For the set of cells with size 3, we only need to check if -cells are possible to draw in Γ^+. Let A, B and C be the edges of the three inner edge-segment-incidences of a -cell, as illustrated in Figure 4.1a. Since Γ^+ is a 2-layer drawing, edges need to have an endpoint on both layers, and because the three edges A, B and C all cross each other, they cannot share any vertices. The vertex a' on the "top" layer connects edge A to its vertex a on the "bottom" layer. The vertices connected by B and C are named by the same convention, see Figure 4.1b. The edges are each already part of two crossings in , so because of the 2-planarity of Γ^+, they cannot be crossed anymore. Taking all this into consideration, Figure 4.1b already shows the only possibility to correctly fit a -cell in a 2-layer drawing, with the exception of mirroring the components horizontally or vertically. However, since A and C are both already crossed twice, edge B and its vertices are now isolated from the rest of the drawing, see Figure 4.1c. This is a contradiction to G being a connected graph, meaning -cells cannot exist in Γ and thus neither in Γ^+.

4.1.2. Celltypes of C_4

The set C_4 contains the two celltypes and we need to check.

Starting with the celltype with four inner edge-segments in their boundary, -cells. Let the edges belonging to the four edge-segment-incidences be called A, B, C and D, named in alphabetical order, clockwise in the boundary of , cf. Figure 4.2a. The edges B and D can either have separate endpoints from another on both layers, or they can share

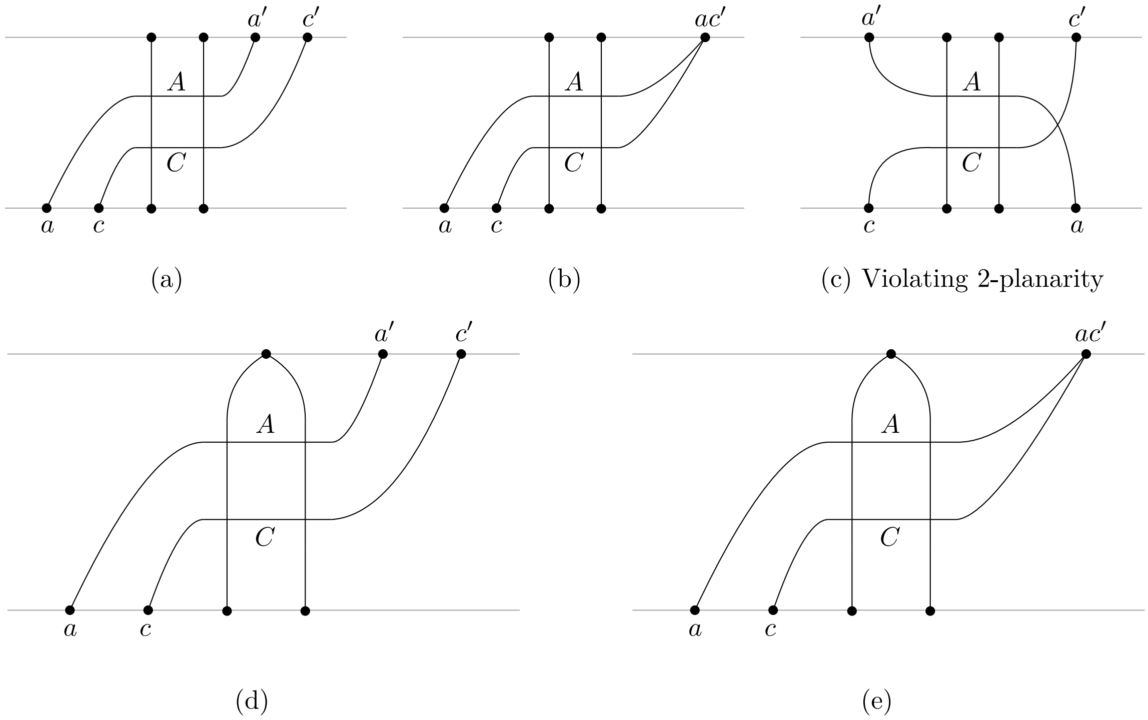


Figure 4.3.

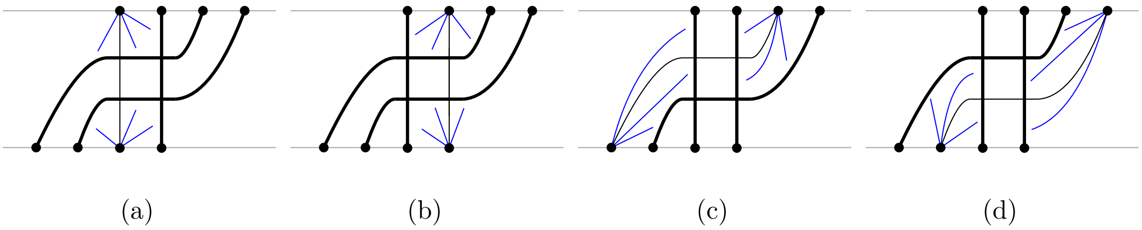
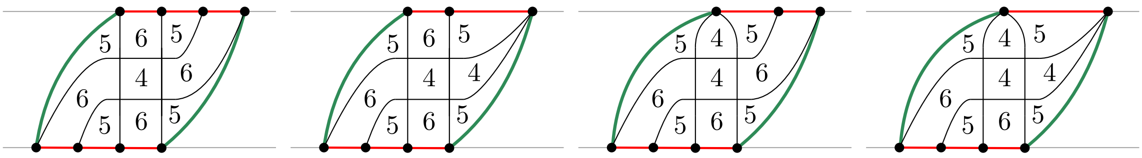


Figure 4.4.: The thickened edges are already crossed twice and would be crossed by the blue edges, which would violate 2-planarity.

one vertex. Because Γ^+ is simple, B and D cannot share both endpoints, as they would in Figure 4.2d. If the edges are not neighbors, let the vertices on the "top" layer connected by the edges B and D be called b' and d', respectively. The vertices on the "bottom" layer are called b and d, adhering to the explained naming convention, see Figure 4.2b. In case the edges share a vertex, let that vertex be named after the earlier explained convention, depending on which layer it is shared on, refer to Figure 4.2c for an example.

Regardless of whether B and D share a vertex or not, they cannot be part of any more crossings in order to comply with the 2-planarity of Γ^+. The same goes for edges A and C, which are also both already crossed twice. It follows that they too can have either separate endpoints or share one vertex on a layer, as the examples illustrated in Figure 4.3 show.

It is always possible to draw in the uncrossed edges H_1 and H_2 connecting the two left-most and two right-most vertices, respectively. In the area between H_1 and H_2, there cannot be any additional vertices or edges to the ones already mentioned, as one can clearly see when looking at Figure 4.4. Because G is connected, there cannot be any sole vertices, but edges connecting any additional vertex would need to reach the other layer. However, any additional edge would need to cross one of the already present edges of E_2, violating the 2-planarity of Γ^+. The configuration shown on Figure 4.5a ist not possible in Γ^+, because we assume G to be connected and without the edges of E_{hor}, not all vertices can be reached.



(a) impossible (b) (c) (d)

Figure 4.5.: (a) impossible configuration if G is connected, (b)-(d) The three types of configurations possible for a $\boxplus_4$-cell in Γ^+.

It follows that a $\boxplus_4$-cell in Γ^+ is always part of one of the three configuration depicted in Figures 4.5b to 4.5d, that have either only one or no $\square_6 \leftrightarrow \square_6$-trail. We can derive a number of trails and inequalities from these configurations by counting the incidences of celltypes:

Lemma 4.1. *In the augmented 2-layer 2-planar drawing Γ^+ of a graph G, the following inequalities hold:*

$$4\boxplus_4 \leq \triangle_5 \tag{4.1}$$

$$2\boxplus_4 \leq \square_6 \tag{4.2}$$

$$2\boxplus_4 \leq (\square_6 \leftrightarrow \triangle_4) + (\square_6 \leftrightarrow \square_6) \tag{4.3}$$

Proof. Because we have shown that a $\boxplus_4$-cell occuring in Γ^+ is always part of one of the three configurations illustrated in Figures 4.5b to 4.5d, the equations (4.1) and (4.2) follow from counting the celltype-incidences in these configurations and taking the minimum over all cases. In all four configurations, there are exactly four $\triangle_5$-cells, and depending on whether the edges incident to $\boxplus_4$ share vertices or not, there are always at least two $\square_6$-cells present.

For (4.3), we know from the proven possible configurations that $\boxplus_4$-cells can only occur in trails $\square_6 \leftrightarrow \triangle_4$ and $\square_6 \leftrightarrow \square_6$. Because each $\boxplus_4$-cell is incident to two trail, we add a factor of 2. Moreover, these types of trails could possibly also exist without including $\boxplus_4$-cells, which is why the inequality follows instead of an equality. □

To check the possible occurrence of a $\triangle_4$-cell, let ab be the vertex in its boundary positioned on the "bottom" layer, and let A and B be the neighboring edges at ab responsible for the outer edge-segment-incidences in $\triangle_4$. Further, let C be the edge forming the inner edge-segment-incidence in $\triangle_4$ that crosses both A and B, see Figure 4.6a. Both edges A and B need to have an endpoint on the other layer. Let these vertices be called a' and b' like in Figure 4.6b, conforming with our naming convention. The vertices need to be separate, because there would otherwise be a contradiction to our assumption that Γ^+ is a simple drawing. The violating scenario is depicted in Figure 4.6c. It is clear that C needs two have an endpoint on each layer. Because Γ^+ is simple and C already crosses both A and B, edge C cannot share vertices a' or b'. For the same reason, C cannot cross these edges again, which finally lands us on the only possibilities depicted in Figure 4.6d and Figure 4.6e, with the exception of mirroring them horizontally. As shown in these figures, C has its vertices c' and c outside of the area enclosed by A and B. We have now successfully shown that it is possible for a $\triangle_4$-cell to occur in Γ^+.

Let us now find common configurations for the celltype $\triangle_4$. We do this by going through the already present vertices one by one, and checking if there can be any more vertices or

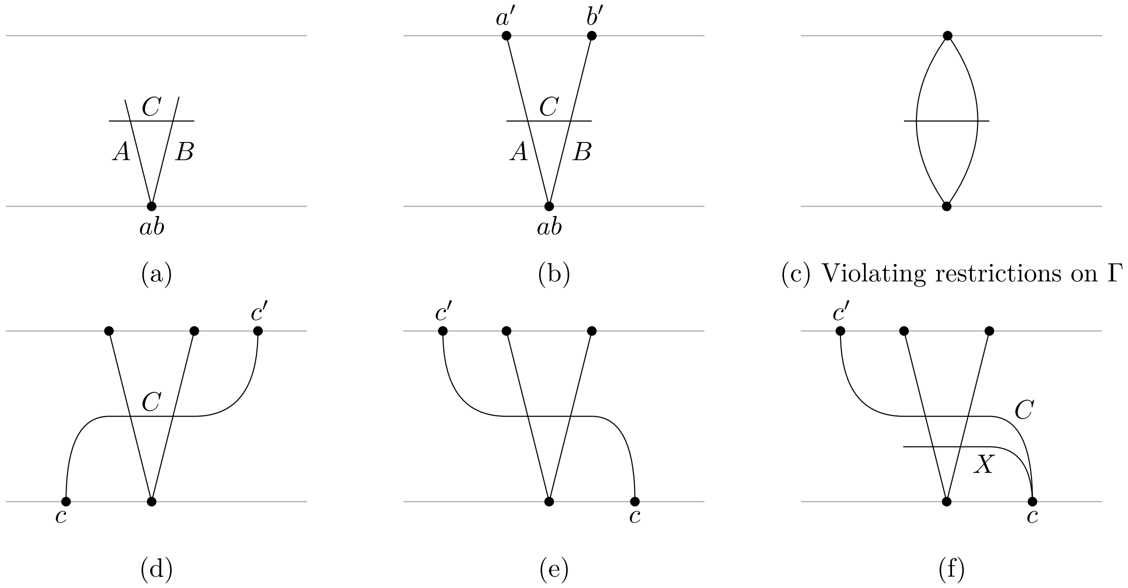
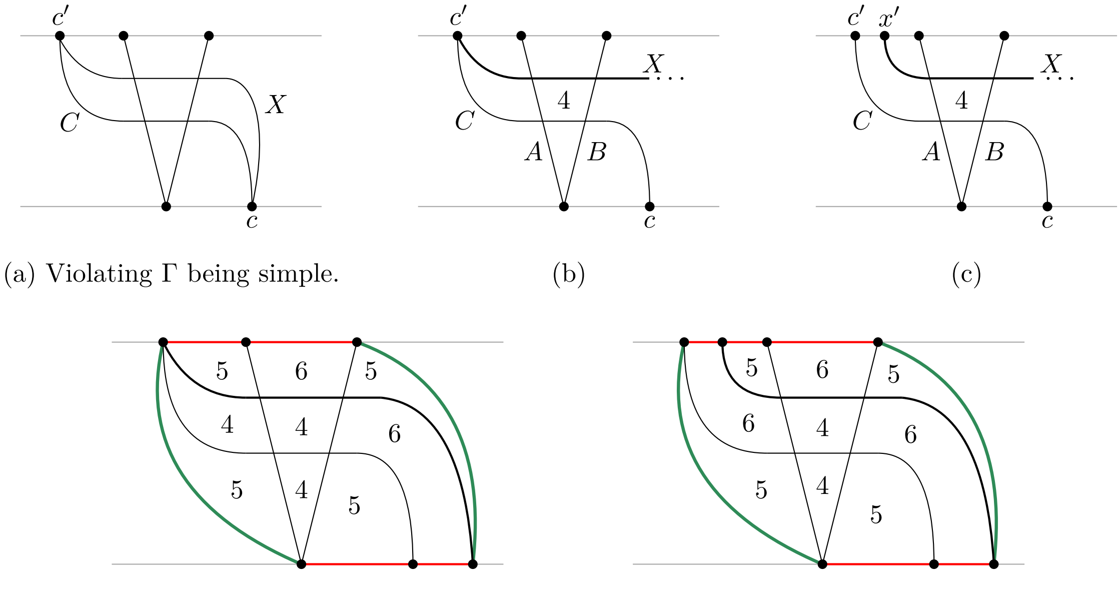


Figure 4.6.: Trying to fit a $\triangle$-cell into Γ^+

edges added without violating constraints. Firstly, we assume that there is no edge X left of C that is connected to c as depicted in Figure 4.6f. In other words, we assume Figure 4.6e and a fixed $\triangle$-cell as our starting scenario. Secondly there can be no additional edges at ab interfering with the $\triangle$-cell by crossing $C \in E_2$, because of Γ^+ being 2-planar. That means we only need to look at the vertices on the layer with vertices a', b' and c'. For the following argumentation, let that layer be the "top" layer. The argumentation still holds for the mirrored cases.

If c' has an additional edge X, because of the drawing being 2-planar, X cannot cross C, but instead has to cross both A and B to reach the other layer. This is illustrated in Figure 4.7b. The same applies if c' has no further edges, but there is instead an additional vertex x' connected to X, positioned between c' and a', like in Figure 4.7c. No matter whether X is connected to c' or x', there can be no further vertices or edges interfering with the formed configuration. This is because both A and B are now crossed twice, and thus cannot let any more edges reach the other layer without violating the 2-planarity of Γ^+. Together with the fact that we have formed a $\boxplus$-cell in both scenarios, it follows that the $\triangle$-cell must be part of a $\boxplus$-configuration. For X being connected to c', because Γ^+ is simple, C and X cannot share another point in common, see Figure 4.7a. Thus, X must have its own vertex on the opposite layer. However, if X is connected to a new vertex x' instead of c', it can be connected to c. These two scenarios form the same type of configuration depicted in Figure 4.7d. A third and last case is for X to be connected to x' and also have its own vertex on the other layer, different from the currently present ones. This results in the configuration shown in Figure 4.7e. Note that these are the same types of configuration as in Figures 4.5a and 4.5d.

If c' has no additional edges crossing A and B, let a' have an additional edge X. For X to reach the other layer without violating any constraints, it needs to cross B, see Figure 4.8a. The only other possibility, crossing $C \in E_2$, would be a contradiction to the drawing being 2-planar. Further, Figure 4.8b shows how there can be no more vertices or edges present in the area between c' and b' contributing to the configuration, because they would need to cross either C or B to lawfully connect to the other layer, contradicting either Γ^+ being 2-planar or G being connected. It is clear that no matter what happens to X after it crosses B, X is able to reach the other layer without violating constraints, resulting in the configuration depicted in Figure 4.8c. Assuming the case that neither c'



(a) Violating Γ being simple.

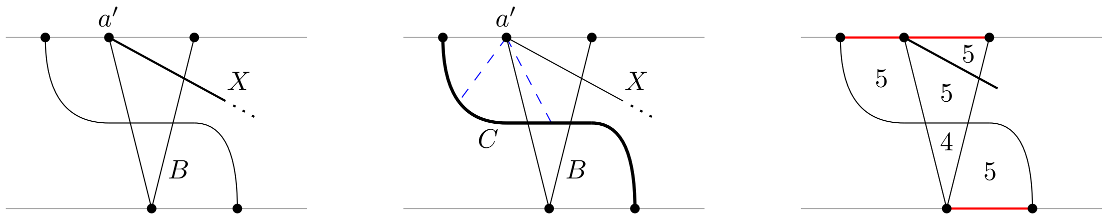
(b)

(c)

(d)

(e)

Figure 4.7.

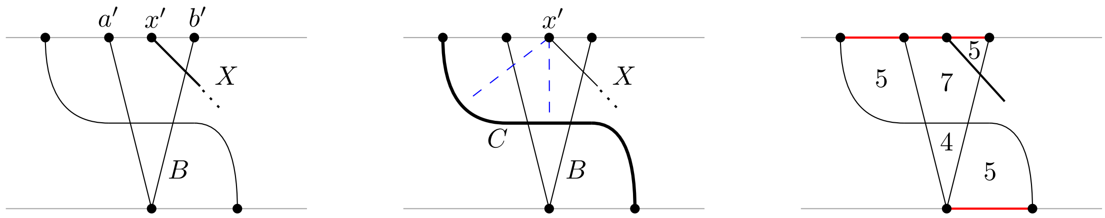


(a)

(b)

(c)

Figure 4.8.

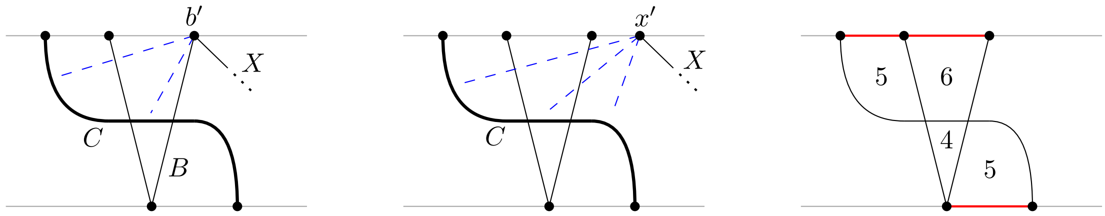


(a)

(b)

(c)

Figure 4.9.



(a) (b) (c)

Figure 4.10.

or a' have any additional edges violating our starting scenario, but there is instead an additional vertex x' between a' and b', the new vertex must be connected to the rest of the drawings by an edge X, because G is connected. Since Γ^+ is a 2-layer drawing, the second vertex of X must be on the opposite layer. Because C is already crossed twice and Γ^+ is 2-planar, it follows that X needs to cross B, as illustrated in Figure 4.8a. Since both C and B are now in E_2, there can be no further vertices and edges in the area between c' and b', as to not violate 2-planarity or G being connected. Figure 4.9c shows the resulting configuration.

The last possible case is for neither c' nor a' to have additional edges contributing to the starting scenario and for there to be no additional vertices present in the area between c' and b'. The vertex b' or a new vertex x' would then be the last available option to have an additional edge X make a change to our configuration. However, as shown in Figures 4.10a and 4.10b, since C is already crossed twice and Γ^+ is 2-planar, any edges outgoing from b' or x' need to go away from the subgraph, not cutting B or A, and therefore not making any difference to the present cells. This results in the last possible type of configuration illustrated in Figure 4.10c.

Lemma 4.2. *In the augmented 2-layer 2-planar drawing Γ^+ of a graph G, the following inequalities hold:*

$$\triangle_4 = (\triangle_4 \leftrightarrow \Diamond_5) + (\triangle_4 \leftrightarrow \square_6) + (\triangle_4 \leftrightarrow \hat{\square}_7) \tag{4.4}$$

$$\triangle_4 - \underline{\triangle_5} \leq 0 \tag{4.5}$$

Proof. We have shown that a $\triangle_4$-cell drawn in Γ^+ is always part of one of five possible types of configurations. The trails $\triangle_4 \leftrightarrow \Diamond_5$, $\triangle_4 \leftrightarrow \square_6$ and $\triangle_4 \leftrightarrow \hat{\square}_7$ can be derived from these configurations. Since there are no $\triangle_4 \leftrightarrow \triangle_4$-trails possible in a simple drawing, we can simply count the incidences of $\triangle_4$-cells and the mentioned trails to obtain Equation (4.4).

Equation (4.5) stems from the fact that in every type of $\triangle_4$-configuration, for each $\triangle_4$-cell there is at least one $\underline{\triangle_5}$-cell present. □

4.1.3. Celltypes of C_5

The set C_5 contains three different celltypes that we should check. We have already seen in our previous observations that $\underline{\triangle_5}$- and $\Diamond_5$-cells can be present in Γ^+, for example in the $\triangle_4$-configuration depicted in Figure 4.8c. Let us look at the only celltype with size five left in C_5, which is cells with five inner edge-segment-incidences and no vertex-incidence. To show that this type of cell cannot exist in a 2-planar Γ^+ drawing, it suffices to look at only three of the five edges. None of the five edge-incidences can connect to the same edge, because they are each already crossed twice and would otherwise violate the 2-planarity of the drawing. Let A and B be two edges that share a crossing and let C be the edge

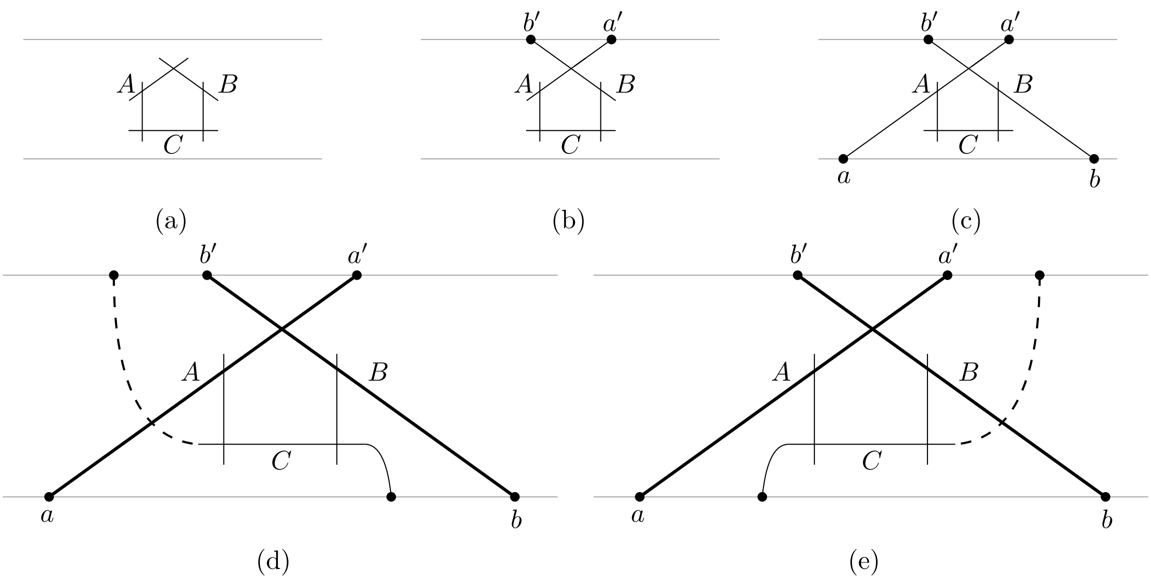


Figure 4.11.: Trying to draw a cell with five inner edge-segment-incidences in a 2-layer 2-planar drawing Γ^+ of G.

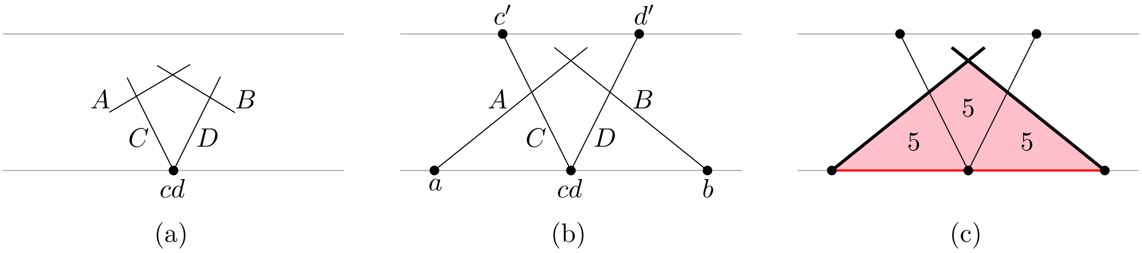


Figure 4.12.: Drawing a cell of size 5 with three crossing-incidences in a 2-layer 2-planar drawing Γ^+ of G. Thickened edges cannot be crossed due to 2-planarity of Γ^+. Cells highlighted red are fixed components of any configuration with a -cell.

opposite to that crossing, not sharing a crossing-incidence with A nor B in the boundary on the cell. See Figure 4.11a for an example drawing of the mentioned and labeled edges. Since A and B are sharing a crossing-incidence, they already have a point in common and can thus not share any vertices for the drawing to remain simple. Furthermore they need to have an endpoint on each layer, resulting in a subgraph like the one illustrated in Figure 4.11c. It is important to keep in mind that both A and B are already crossed twice and cannot have any more crossings in order for the drawing to comply with 2-planarity. However, C also needs to have endpoints on both layers and must cross either A or B to reach the opposite layer. Figure 4.11 illustrates how C must violate 2-planarity in order to be drawn. It follows that a cell of size 5 with five inner edge-segment-incidences cannot occur in a 2-planar drawing Γ^+ of G.

Since we already know that -cells can occur in Γ^+, we now try to also gain insight on what kind of configurations they form. For this, let C and D be the edges responsible for the two outer edge-segment-incidences and cd be the vertex-incidence they share in the boundary of a -cell. We consider the layer cd is positioned on to be the "bottom" layer. Let A and B be the two inner edge-segment-incidences in the boundary, A sharing a crossing-incidence with C and B sharing a crossing-incidence with D. See Figure 4.12a for an example drawing of the named components. Edges C and D have a vertex on the "bottom" layer and thus need endpoints on the "top" layer. Because the edges already have the point cd in common, they need to have their own vertices on the other layer,

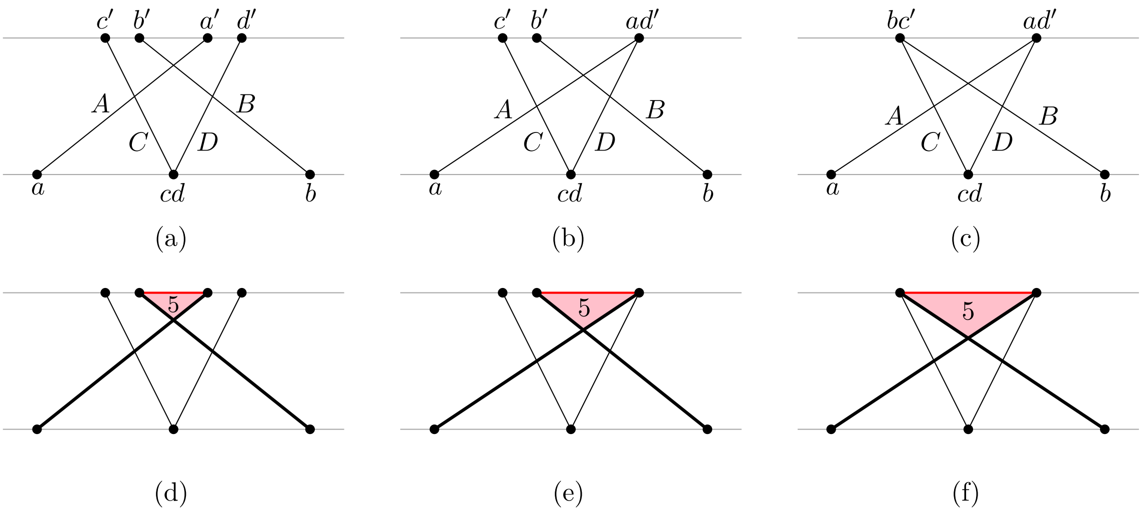


Figure 4.13.: The possibilities of sharing vertices on the top layer in Γ^+. Thickened edges cannot be crossed due to 2-planarity of the drawing, -cell marked in red is fixed component of configurations with a -cell.

called c' and d' by our naming convention. Edges A and B are both already crossed twice and must not be part of another crossing to comply with the 2-planarity of the drawing. Furthermore, because A and B each already cross one of the edges C and D, they cannot share the vertex cd. This results in A and B needing their own vertices on the layer which cd lies on, here the "bottom" layer. Let these vertices be called a for A and b for B, as shown in Figure 4.12b. The part of the drawing marked in Figure 4.12c is already a fixed configuration, since there can be no more crossings on A and B, meaning there can be neither additional vertices on the layer between a and b nor can there be outgoing edges from the area in between a, b and the crossing formed by A and B. This means that we only have to look at the layer on which c' and d' lie on, the "top" layer, to find the possible configurations for .

Edge B is not crossing C, and A is not crossing D, so these pairs could share vertices on the "top" layer. Thus, we need to check the following three scenarios: All four edges have their own vertex, either A D or B C share a vertex, or both A D and B C share vertices. Figures 4.13a to 4.13c show these possibilities. Note, that because the argumentation still holds for the mirrored cases, it does not matter whether A D or B C share a vertex, the scenario is the same. Due to the 2-planarity of Γ^+, there is always a -cell between the vertices of A and B on the top layer, as shown in Figures 4.13d to 4.13f. There can be no vertices in this space, since they would need to have an edge connect them to the other layer, which is impossible without crossing either A or B and violating the 2-planarity of Γ^+. For the same reason, the already existing vertices do not have any additional edges crossing through A or B. Based on this, we only need to check if there are any additional edges or vertices possible outside of this fixed space.

Assuming all four edges have separate vertices on the top layer, let the vertices be called a', b', c' and d' according to our naming convention, cf. Figure 4.13a. As we already established, there are no more edges crossing A and B, neither from any of the mentioned vertices, nor from any additional vertices. The only edges that are still possible to be crossed without violating any restriction are C and D. There could be crossings occurring through additional edges at b' or a' like in Figure 4.14a or through new vertices x and y lying in between c', b' or a', d', respectively, cf. Figure 4.14b. There can, however, be only one more edge crossing C and D each, before the 2-planarity would be violated again. This leaves us with five possible configurations: No additional edges (Figure 4.14c),

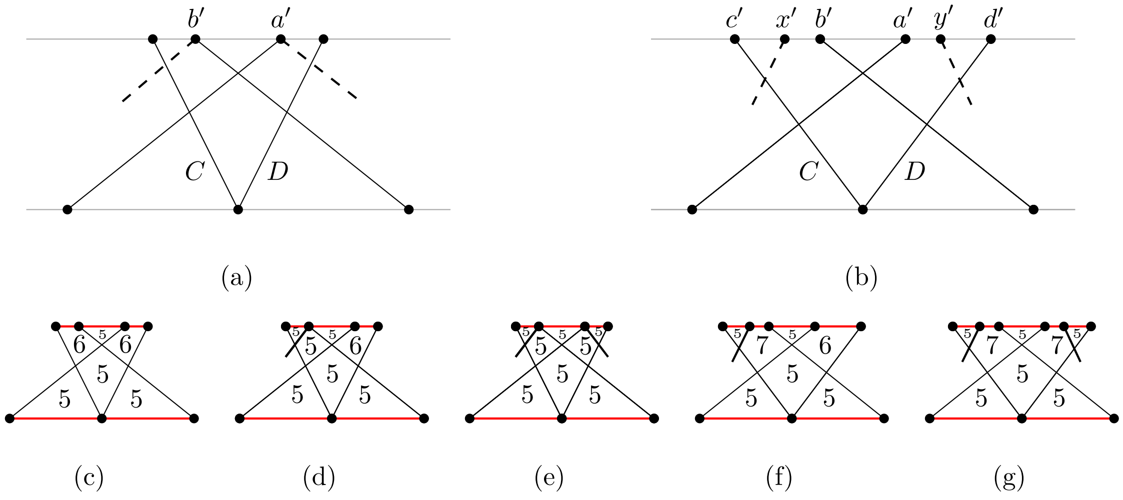


Figure 4.14.

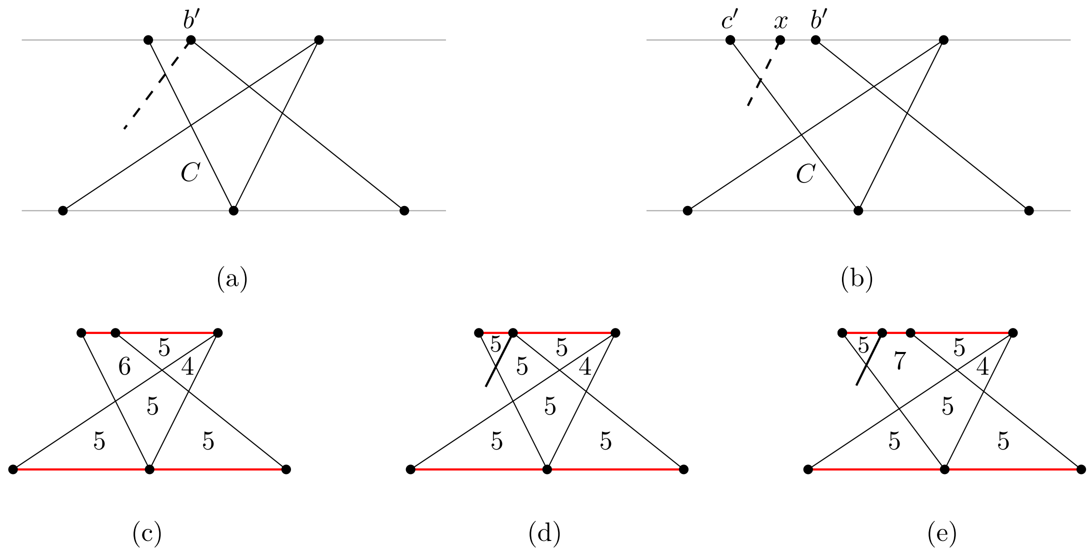
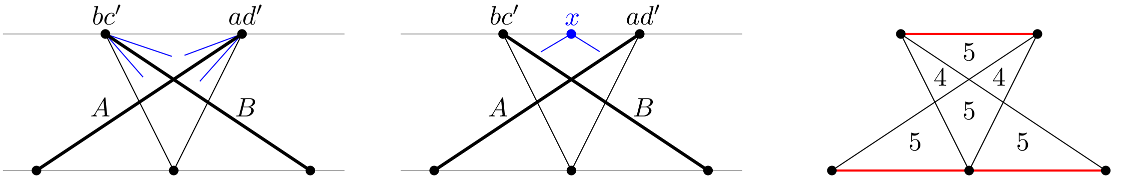


Figure 4.15.

additional edges at only one of the vertices a', b' or at both of them (Figure 4.14d), additional edges at both vertices a' and b' (Figure 4.14e) or lastly one to two additional vertices with edges crossing through C and/or D (Figures 4.14f and 4.14g).

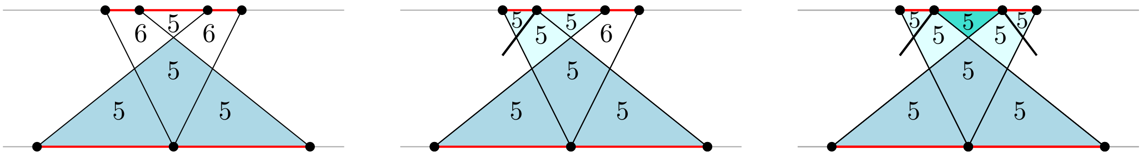
Assume that two edges of A, B, C and D have separate vertices on the "top" layer and the other two share a vertex. The following argumentation holds regardless of whether A and D or B and C share a vertex. We assume that A and D share a vertex ad'. Let B and C have separate vertices b' and c', respectively, as depicted in Figure 4.13b. There can be no additional edges, one additional edge at b' or an additional vertex x with an edge crossing through C. These are the three possible configurations for this scenario, illustrated in Figures 4.15c to 4.15e.

The last case is where A, D as well as B and C share vertices on the top layer. Let these vertices be called ad' for edges A and D, and bd' for edges B and D, cf. Figure 4.13c. Like we already argued for Figure 4.13f, there can neither be an additional vertex x' nor additional edges disturbing the 5-cell on the top layer between ad' and bd', because Γ^+ is 2-planar and A and B are both already crossed twice. Figures 4.16a and 4.16b illustrates the explained problem. It follows that in case both A and B share their vertices with C



(a) (b) (c)

Figure 4.16.: Blue edges are not possible without violating 2-planarity of Γ^+.



(a) (b) (c)

Figure 4.17.

and D on the top layer, the configuration including the -cell looks like the drawing in Figure 4.16c. There could still be edges at the outmost vertices that lead away from the configurations, but since they do not interfere with the present cells, the argumentation holds.

Now that we found all possible configurations for -cells in Γ^+, we are interested in deriving equations from them.

Lemma 4.3. *In the augmented 2-layer 2-planar drawing Γ^+ of a graph G, the following inequality holds:*

$$\leq \tag{4.6}$$

$$(\ \leftrightarrow\ ) + (\ \leftrightarrow\ ) + (\ \leftrightarrow\ ) + (\ \leftrightarrow\ ) = 2 \tag{4.7}$$

Proof. Looking at the "safe spaces" marked in Figure 4.12c and the bottom row of Figure 4.13, one could assume that by counting the incidences of those fixed celltypes, the inequality $4 \leq$ would follow. However, when checking the types of configuration for the -cell, we encounter the problem highlighted in Figure 4.17: Firstly, we cannot cross-charge a -cell with the -cell sharing its crossing-incidence, because that -cell might already be accounted for by another -cell sharing its outer edge-segment-incidence. Secondly we cannot even cross-charge a -cell with the two neighboring -cells, each sharing one of the outer edge-segments in its boundary, because we run the risk of incorrectly double counting the -cells if there is another -cell connected to it. We can, however, cross-charge a -cell with one of the two -cells connected by an outer edge-segment, as this leaves the second one free to be cross-charged to another . This leaves us with Equation (4.6).

Through the possible configurations in Γ^+ that a -cell must occur in, we know that there are exactly four different types of trails including -cells. Because a -cell has two inner edge-segment-incidences, each -cell is part of two trails. Thus, Equation (4.7) follows from counting the incidences of trails in correlation to the incidences of trails including -cells. □

4.1.4. Other Celltypes and Equations

We know from the previous observations of celltypes that and -cells can occur in Γ^+ without having to explicitly check. Because the density lemma [7] is about bounding the number of smaller-sized cells, we do not need to further discuss these cells of size 6 and 7. Moreover, because  and -cells can occur in 1-planar drawings of G (Lemma 3.1) and a 1-planar drawing is a special case of a 2-planar drawing, it follows that these celltypes can also occur in a 2-layer 2-planar drawing.

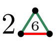
Lemma 4.4. *In the augmented 2-layer 2-planar drawing* Γ^+ *of a graph* G*, the following inequality holds:*

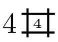
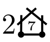
( $\leftrightarrow$ ) + ( $\leftrightarrow$ ) + ( $\leftrightarrow$ ) $\leq$  (4.8)

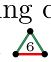
Proof. Equation (4.8) can be derived by adding up all trails taken from the configurations up until now, where -cells are included. Because we have not explicitly checked if there are any other trails for this celltype, the equation is only an inequality rather than an equality. ☐

We have derived a number of equations on how celltypes are associated with one another. In the following two lemmas, we further correlate celltypes to edges and crossings.

Lemma 4.5. *In the augmented 2-layer 2-planar drawing* Γ^+ *of a graph* G*, the following inequalities hold true:*

2  +  $\leq 2|E_0| - 2$ (4.9)

 + 4  + 2  +  + 2  $\leq 2|E_2|$ (4.10)

Proof. In the context of what is being considered, uncrossed edge-segments that are not in E_{hor} occur in the celltypes  and . Every edge-segment is incident to two cells, with the exception of the two uncrossed edges taking part in the boundary of the outer cell, see Observation 4. So counting the incidences, this would give us an equality between $2|E_0| - 2$ and 2  + , but since there could be cells with uncrossed edges and size larger than 5 that we do not check for in this thesis, we instead obtain the inequality (4.9).

Inner edge-segments have two crossing-incidences. The edge belonging to that edge-segment cannot have any more crossings in a 2-planar drawing. It follows that the number of cells with an inner edge-segment-incidence correlates to the number of edges in E_2. Since an inner edge-segment is always incident to two cells, the inequality (4.10) holds when counting all incidences of celltypes with inner edge-segments in their boundary. ☐

Lemma 4.6. *In the augmented 2-layer 2-planar drawing* Γ^+ *of a graph* G*, the following inequalities hold true:*

$$|E| = |E_{\text{hor}}| + |E_0| + |E_1| + |E_2| \tag{4.11}$$

$$|E_1| + |E_2| \leq \frac{2}{3}|E| \tag{4.12}$$

$$2|X| \leq 2(|E_1| + |E_2|) \tag{4.13}$$

2  +  $\leq |X|$ (4.14)

(a) $n = 4 \quad |X| = 1$ (b) $n = 5 \quad |X| = 3$ (c) $n = 6 \quad |X| = 4$ (d) $n = 7 \quad |X| = 5$

Figure 4.18.: Observations on how many crossings can fit into a 2-layer 2-planar drawing Γ of G without violating constraints. Heavier black edges "changed" vertex or are new to the drawing of the subfigure before.

Proof. Equation (4.11) follows from Observation 1 and a proof for (4.12) can be found in Theorem 3 of [2]. Equation (4.13) follows from the fact that every edge in a simple 2-planar drawing is crossed at most twice and every crossing is formed by two different edges. Lemma 8.2 in [7] shows how $3 + 2 + \leq |X|$ holds for 2-planar drawings of G. Since there can be no -cells drawn in Γ^+, Equation (4.14) follows.

□

Last but not least, we specify and rewrite the Density Formula with $t = 5$ from Equation (2.1) in fitting augmented 2-layer 2-planar drawings of G. The inequality holds as no cells of C_3 can be drawn in Γ^+.

$$
\begin{aligned}
|E| &= 5n - 10 - \sum_{c \in C} (||c|| - 5) - |X| \\
&= 5n - 10 + 2|C_3| + |C_4| - |X| - \sum_{c \in C \geq 5} (||c|| - 5) \\
&\leq 5n - 10 - |X| + \quad +
\end{aligned}
$$

$$
\Leftrightarrow |E| - 5n - 10 + |X| - \quad - \quad \leq 0 \tag{4.15}
$$

4.2. Bounding Crossings

To get an idea of what a realistic upper bound on the number of crossings looks like, it helps to try and draw up some "worst-case" scenarios complying with restrictions on the augmented 2-layer 2-planar drawing Γ^+ of G. For $n = 3$ vertices in G there are no crossings possible (Observation 3), so we start the discussion with $n = 4$, see Section 4.2. Describing the drawing and its celltypes starting from the "left" side, the outmost edge is always uncrossed, as explained in Observation 4. In our mission to maximize the number of crossings in the drawing, the leftmost cell incident to the uncrossed edge-segment is always a -cell. After this cell, it depends on the number of vertices how much the "middle" of the drawing can be filled with trails, or like in $n = 4$, it gets closed of by another -cell with its uncrossed edge-segment-incidence being the outmost edge of the drawing. Otherwise, if the number of vertices allow it, the type of trail that immediately follows the -cells on the border of the drawing is a $\leftrightarrow$-trail. The trail connects to by an outer edge-segment of . The smallest number of vertices for this trail to appear is $n = 5$, where there is one $\leftrightarrow$-trail on either -cell-border, and they share their -cell-incidence, see Section 4.2. After the border configuration is established, with an increasing number of vertices follows an increasing number of $\leftrightarrow$-trails filling the area in between the two established $\leftrightarrow$-trails. It does not matter whether n is an even or odd number, this arrangement of cells and trails is always possible, see Section 4.2. Thus, Figure 4.18 shows how there can be $n - 2$ crossings drawn in Γ^+ for $n > 4$ without violating any constraints. The remainder of this thesis discusses the verification of the bound using the Density Formula.

The amount of inequalities we have found in this chapter would be quite overwhelming to use in a by-hand calculation, which is why we use a linear program (LP) to do the calculation for us. For an explanation of how the LP is used in this thesis, see the example in Section 2.2 or the use-case in Section 3.1.2. We use the equations of Lemmas 4.1 to 4.6 and eq. (4.15) and the reworked Density Formula with $t = 5$ from Equation (4.15). Further, because 1-planar drawings are a special type of 2-planar drawings, we can also use Equations (3.3) and (3.6) from Chapter 3. The mentioned equations are rewritten into linear inequalities in relation to zero, while avoiding fractions and ignoring constants. We consider $n = 1$, so our result scales with the number of vertices.

We use the "variables" for $n, X, |E|, |E_{\mathrm{hor}}|, |E_0|, |E_1|, |E_2|$ and more variables for each celltype and type of trail that occur in the inequalities. We finally get matrix M_2, see Section A. To prove an upper bound on $|X|$, we need to maximize $|X|$ in the LP. We get the result of $|X| \leq \frac{10}{3}n$ and after computing the dual of the LP, we get vector

$$d = (\frac{10}{3} \quad \frac{2}{3} \quad 0 \quad 0 \quad 0 \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{3} \quad 0 \quad \cdots \quad 0).$$

Because d shows us exactly which inequalities were used to obtain the bound, doing a by-hand calculation to confirm the correctness of the LP is a feasible task now. We also include the previously ignored constants to further improve the bound:

$$\begin{aligned}
& \frac{2}{3}(|E| - 5n - 10 + |X| - \triangle - \boxplus) + \frac{1}{2}(2|X| - 2|E_1| - 2|E_2|) \\
& + \frac{2}{3}(\triangle + \boxplus - |X|) + \frac{1}{3}(3|E1| + 3|E2| - 2|E|)) && \leq 0 \\
\Leftrightarrow \quad & (\frac{2}{3} + 1 - \frac{2}{3})|X| - \frac{10}{3}n - \frac{20}{3} + (-\frac{4}{3} \\
& + (-\frac{2}{3} + \frac{2}{3})\triangle + (-\frac{2}{3} + \frac{2}{3})\boxplus + (\frac{2}{3} - \frac{2}{3})|E| + (-1+1)|E_1| + (-1+1)|E_2| && \leq 0 \\
\Leftrightarrow \quad & |X| - \frac{10}{3}n + \frac{20}{3} \leq 0 \\
\Leftrightarrow \quad & |X| \leq \frac{10}{3}n - \frac{20}{3} \\
\Leftrightarrow \quad & |X| \leq \frac{10}{3}n - \frac{20}{3}
\end{aligned}$$

With this we have confirmed that the bound $|X| \leq \frac{10}{3}n$ given by the LP. The bound obtained by the by-hand calculation is even slightly better by a constant of $-\frac{20}{3}$.

Theorem 4.7. *Let Γ be a simple 2-layer 2-planar drawing of a connected maximal graph G with n vertices. For $n \geq 3$, Γ has at most $\frac{10}{3}n - \frac{20}{3}$ crossings.*

5. Conclusion

This thesis discussed using the Density Formula [7] to find upper bounds on the number of crossings in simple 2-layer k-planar drawings of a connected maximal graph G. More precisely, we examined 2-layer 1-planar and 2-layer 2-planar drawings of G. To make use of the density lemma, we analyzed what types of cells can occur in these drawings and derived a number of equations concerning the relations between cells, edges and crossings. We first showed an intuitive explanation of possible bounds for the number of crossings and then used our equations to obtain upper bounds through linear programming. In doing so we proved new upper bounds for the number of crossings: $\frac{n-2}{2}$ for 2-layer 1-planar drawings and $\frac{10}{3}n - \frac{20}{3}$ for 2-layer 2-planar drawings. We further reproduced a similar bound $|E| \leq \frac{3}{2}n + \frac{4}{3}$ to the known edge density $\frac{3}{2}n - 2$ [4] of 2-layer 1-planar drawings of G.

Although we were not able to prove the exact bounds we developed informally, we still showed how to make use of the density formula not only in regard to bounding the number of edges, but also the number of crossings. Although our linear program did not fully make use of it, in Section 4.1 we laid the groundwork in finding correlations between trail types and cells, which could be helpful in getting an even better bound.

Class	Edge Density	Ref.	Lower Bound Crossings	Ref.	Upper Bound Crossings	Ref.
planar	$n-1$	[6]	0	def.	0	def.
1-planar	$\frac{3}{2}n-2$	[4]	$0.295\frac{m^3}{n^2}$	[11]	$\frac{n-2}{2}$	Theorem 3.4
2-planar	$\frac{5}{3}n-\frac{7}{3}$	[11]	$0.295\frac{m^3}{n^2}$	[11]	$\frac{10}{3}n-\frac{20}{3}$	Theorem 4.7
k-planar	$\frac{125}{96}\sqrt{k}n$	[11]	$0.295\frac{m^3}{n^2}$	[11]	$\frac{k\|E\|}{2}$	def.

Table 5.1.: Bounds on the number of edges and crossing in 2-Layer drawings. The edge density for k-planar graphs is valid for $k \geq 6$. The lower bound on the number of crossings is valid for $n \geq 9$ vertices and $m \geq \frac{125}{48}n$ edges. The upper bounds on the number of crossings for 1-planar and 2-planar drawings are valid for $n \geq 3$.

Bibliography

[1] J. Balogh, B. Lidický, S. Norin, F. Pfender, G. Salazar, and S. Spiro. Crossing numbers of complete bipartite graphs. *Procedia Computer Science*, 223:78–87, 2023.

[2] M. A. Bekos, P. Bose, A. Büngener, V. Dujmović, M. Hoffmann, M. Kaufmann, P. Morin, S. Odak, and A. Weinberger. On k-planar graphs without short cycles, 2024.

[3] E. Di Giacomo, W. Didimo, P. Eades, and G. Liotta. 2-layer right angle crossing drawings. *Algorithmica*, 68(4):954–997, November 2012.

[4] W. Didimo. Density of straight-line 1-planar graph drawings. *Information Processing Letters*, 113(7):236–240, April 2013.

[5] P. Eades and S. Whitesides. Drawing graphs in two layers. *Theoretical Computer Science*, 131(2):361–374, 1994.

[6] F. HARARY. A new crossing number for bipartite graphs. *Utilitas Math*, 1:203–209, 1972.

[7] M. Kaufmann, B. Klemz, K. Knorr, M. M. Reddy, F. Schröder, and T. Ueckerdt. The density formula: One lemma to bound them all, 2023.

[8] R. Marti and M. Laguna. Heuristics and meta-heuristics for 2-layer straight line crossing minimization. *Discrete Applied Mathematics*, 127(3):665–678, May 2003.

[9] J. Pach and G. Tóth. Graphs drawn with few crossings per edge. *Combinatorica*, 17(3):427–439, September 1997.

[10] H. C. Purchase, R. F. Cohen, and M. James. *Validating graph drawing aesthetics*, pages 435–446. Springer Berlin Heidelberg, 1996.

[11] T. Schneck. New parameters for beyond-planar graphs, 2020.

[12] T. T. Seok-Hee Hong, editor. *Beyond Planar Graphs: Communications of NII Shonan Meetings*. Springer Singapore, 2020.

[13] K. Sugiyama, S. Tagawa, and M. Toda. Methods for Visual Understanding of Hierarchical System Structures. *IEEE Transactions on Systems, Man and Cybernetics*, SMC-11(2):109–125, 1981.

Appendix

A. Appendix Section 1

Matrix M_2 without the variables and constraints concerning trails, as they are not used by the LP. The variables for the number of cells of the same type are named based on their size and number of crossing-incidences. For example, the number of -cells is represented by c5cr1,  by c5cr3 and so on. The variable c5cr1green denotes the number of -cells and c5cr1red denotes the number of -cells.

$$
M_2 = \begin{array}{c}
\begin{array}{cccccccccccccccc}
N & |X| & |E| & |E_{\mathrm{hor}}| & |E_0| & |E_1| & |E_2| & \mathrm{c4cr2} & \mathrm{c4cr4} & \mathrm{c5cr1} & \mathrm{c5cr1green} & \mathrm{c5cr1red} & \mathrm{c5cr3} & \mathrm{c6cr0} & \mathrm{c6cr2} & \mathrm{c7cr3}
\end{array} \\
\left(\begin{array}{cccccccccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 4 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
\begin{array}{l}
\\
n = 1 \\ (4.11) \\ (4.11) \\ (3.3) \\ (4.13) \\ (4.14) \\ (4.12) \\ (4.15) \\ (4.10) \\ (4.9) \\ (3.6) \\ (3.6) \\ (4.1) \\ (4.2) \\ (4.6) \\ (4.5)
\end{array}
$$