A Unified Hanani-Tutte Theorem for Level-Graphs

Bachelor Thesis of

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Abstract

A graph has a *planar* drawing if it can be drawn in the plane in such a way that its edges do not cross. Planarity can also be characterized by other properties of graphs, for example the circular ordering of edges at each vertex—also known as the *rotation system* of a graph—or the number of pairs of edges crossing an even number of times—the *even crossings* of the graph. The variants of the Hanani-Tutte Theorem concern themselves with the latter. The traditional Hanani-Tutte theorem—the *Strong Hanani-Tutte Theorem*—states that if a graph has a drawing such that every pair of independent edges—edges that have no common end-vertex—cross evenly, it also has a planar drawing. The *Weak Hanani-Tutte Theorem* requires a drawing where every pair of edges crosses evenly and expands the conclusion to include the existence of a planar drawing with a preserved rotation system. The generalized version—the *Unified Hanani-Tutte Theorem*—requires the lesser assumption of the Strong Hanani-Tutte Theorem and combines the conclusion of the strong and the weak version to conclude the existence of a planar drawing such that the rotation system of all vertices where all incident edges cross each other evenly is preserved.

We present two proofs of the Unified Hanani-Tutte Theorem in the plane, one by Pelsmajer et al. (2007) and the other by Fulek et al. (2017). We also consider *level-graphs*, graphs where each vertex is assigned a level. Levels assign each vertex a single number and are represented by horizontal lines. Then, a level-graph is *(level-)planar* if it has a planar drawing in the plane such that each vertex lies on the horizontal line corresponding to its level. We present a proof of the Strong Hanani-Tutte Theorem for level-graphs and by Fulek et al. (2013). The main contribution of this work is an extension of this proof to the Unified Hanani-Tutte Theorem for level-graphs.

Deutsche Zusammenfassung

Ein Graph hat eine *planare* Zeichnung, wenn er sich in der Ebene so darstellen lässt, dass sich keine seiner Kanten kreuzen. Abgesehen davon lässt sich die Planarität eines Graphen noch auf andere Art und Weisen charakterisieren, zum Beispiel durch die Anordnung der Kanten um ihre Endknoten—das *Rotationssystem* des Graphen—oder durch die Anzahl von Kantenpaaren, die sich gerade oft kreuzen—die *geraden Kreuzungen* des Graphen. Die Varianten des Hanani-Tutte Theorems beschäftigen sich mit letzterem. Die traditionelle Variante, das *Strong Hanani-Tutte Theorem*, besagt, dass ein Graph, der eine Zeichnung besitzt, in der sich jedes Paar unabhängiger Kanten gerade oft kreuzt, planar ist. Das *Weak Hanani-Tutte Theorem* setzt eine Zeichnung voraus, in der sich alle Kantenpaare gerade oft kreuzen und erweitert die Folgerung auf die Existenz einer planaren Zeichnung, in der das Rotationssystem gleich bleibt. Beide Theoreme sind verallgemeinert im *Unified Hanani-Tutte Theorem*, welches mit der schwächeren Voraussetzung des Strong Hanani-Tutte Theorems auskommt und dann die Existenz einer planaren Zeichnung ableitet, in der das Rotationssystem für alle Knoten, deren inzidente Kanten einander gerade oft kreuzen, unverändert bleibt.

Wir stellen zwei Beweise für das Unified Hanani-Tutte Theorem vor, von Pelsmajer et al. (2007) und von Fulek et al. (2017). Wir betrachten ebenfalls *Level-Graphen*—Graphen, deren Knoten je ein Level zugeordnet wird. Level ordnen jedem Knoten eindeutig eine Zahl zu und werden in Zeichnungen durch Horizontalen repräsentiert. Ein Level-Graph ist *(level-)planar* wenn er eine planare Zeichnung in der Ebene hat, sodass jeder Knoten des Graphen auf der Horizontalen liegt, die seinem Level entspricht. Wir stellen einen Beweis des Strong Hanani-Tutte Theorems für Level-Graphen von Fulek et al. (2013) vor. Der Hauptbeitrag der Arbeit ist dann die Erweiterung dieses Beweises auf einen Beweis des Unified Hanani-Tutte Theorem für Level-Graphen.

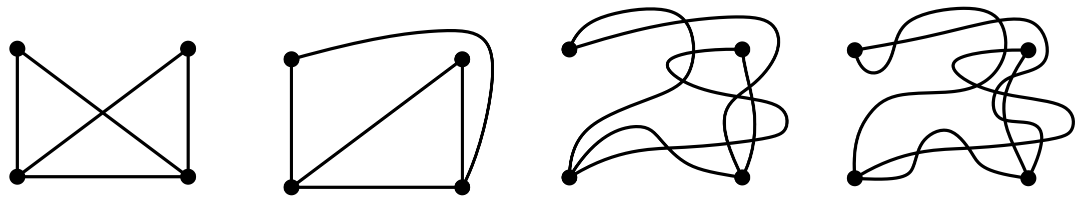
Contents

1. Introduction

When considering graphs and their representations in the plane, an essential problem is to draw them in such a way that edges only overlap when they share a vertex. Then, no edges in the drawing cross and we call it a *planar drawing* or an *embedding* (Figures 1.1a,b). We can formalize this with the concept of *crossing numbers*. The *crossing number* $cr(G)$ of a graph G is defined as the smallest number of crossings in any drawing of G. Graph G is then planar if $cr(G) = 0$. Planarity can also be characterized by other properties of graphs, for example the circular ordering of edges at each vertex—also known as the *rotation system* of a graph—or the number of pairs of edges crossing an even number of times—also known as the *even crossings* of the graph. The different versions of the Hanani-Tutte Theorem use the latter to characterize planar graphs. The traditional version is known as the Strong Hanani-Tutte Theorem [5, 21]. It states that if a graph G has a drawing such that every pair of independent edges cross an even number of time (Figure 1.1c), then it has a planar drawing (Figure 1.1b). We can formalize this with the *independent odd crossing number* $iocr(G)$, defined as the smallest number of oddly crossing pairs of independent edges—the independent odd crossings—in any drawing of G:

Theorem 1.1 (Strong Hanani-Tutte Theorem). *Let $G = (V, E)$ be a graph. If $iocr(G) = 0$ then $cr(G) = 0$.*

Another variant of the theorem, known as the Weak Hanani-Tutte Theorem, demands the stricter prerequisite of a drawing such that every pair of edges cross an even number of times (Figure 1.1d) and in return concludes the existence of not only a planar drawing but one such that the rotation system is preserved (Figure 1.1b) [4, 15, 16]. Again, it can be formalized by using crossing numbers. The *odd crossing number* $ocr(G)$ is defined as the smallest number of odd crossing pairs of any edges in any drawing of G.



(a) Non-planar

(b) Planar

(c) All independent edges cross evenly

(d) All edges cross evenly

Figure 1.1: A graph in the plane with different drawings.

Type	Weak H-T	Strong H-T	Unified H-T
Plane	✓[4, 15, 16]	✓[5, 21]	✓[12, 16]
Level	✓[9, 14]	✓[9]	✓
Radial (level)	✓[10]	✓[11]	?
Torus	✓[4, 17]	✓[13]	✗[8]
Orientable Surface Genus 2	✓[4, 17]	?	?
Orientable Surface Genus 3	✓[4, 17]	?	?
Orientable Surface Genus ≥ 4	✓[4, 17]	✗[8]	✗[8]
Projective Plane	✓[17]	✓[6, 18]	?
Non-orientable Surfaces	✓[17]	?	?

Table 1.1: Overview of the Hanani-Tutte variants and when they apply.

Theorem 1.2 (Weak Hanani-Tutte). *Let $G = (V, E)$ be a graph with a drawing realizing $ocr(G) = 0$. Then there is a drawing realizing $cr(G) = 0$ such that the rotation system is preserved.*

The description of the variants as *weak* and *strong* is misleading. The Strong Hanani-Tutte Theorem is not an actual generalization of the Weak Hanani-Tutte Theorem as it does not make statements about the rotation system. Instead, both variants are weak versions of their generalization, the *Unified Hanani-Tutte Theorem*, indirectly proven by Pelsmajer et al. [16] and later formulated and proven by Fulek et al. [12]. It requires the lesser assumption of the Strong Hanani-Tutte Theorem (Figure 1.1c) and concludes, in addition to the existence of the planar drawing (Figure 1.1b), the preservation of the rotation at *even vertices*—vertices where every incident edge crosses every other incident edge an even number of times.

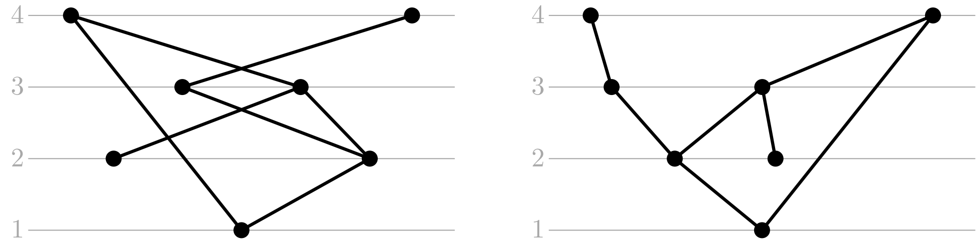
Theorem 1.3 (Unified Hanani-Tutte). *Let $G = (V, E)$ be a graph with a drawing realizing $iocr(G) = 0$. Then there is a drawing realizing $cr(G) = 0$ such that the rotation system at even vertices is preserved.*

Hanani-Tutte for other Types of Graphs

A natural follow-up question is to which other types of graphs the variants of the Hanani-Tutte Theorem apply. The version that has the most known results is the weak variant. It is not only proven for the plane, but also for x-monotone drawings and level-graphs [9, 14], radial (level-)planarity [10], as well as all orientable surfaces [4, 17] and all non-orientable surfaces [17], including the projective plane. The Strong Hanani-Tutte Theorem has, in addition to the plane, been proven for radial planarity [11], the torus [13] and the projective plane [6, 18]. Interestingly, it has been disproven for any orientable surface with a genus of 4 or more [8]. To our knowledge, it is yet to be solved whether the strong variant holds for the orientable surface of genus 2, the orientable surface of genus 3 as well as for non-orientable surfaces other than the projective plane. As a generalization of the other variants, the Unified Hanani-Tutte Theorem has the least results. Apart from the proof in the plane, there is only a counterexample for the torus [8]. Since the Strong Hanani-Tutte Theorem was disproven for orientable surfaces with a genus of 4 or more [8], it follows that the unified version does hold either. With this work, the Unified Hanani-Tutte Theorem is now also proven for level-graphs. Table 1.1 gives a compact overview. Further examples can be found in a survey by Schaefer (2013).

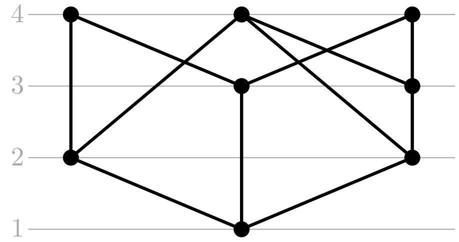
Level-Graphs and Level-Planarity

We take a closer look at *level-graphs*. A level-graph (G, l) is a graph $G = (V, E)$ where each vertex is assigned a level. Levels are represented by horizontal lines and can be

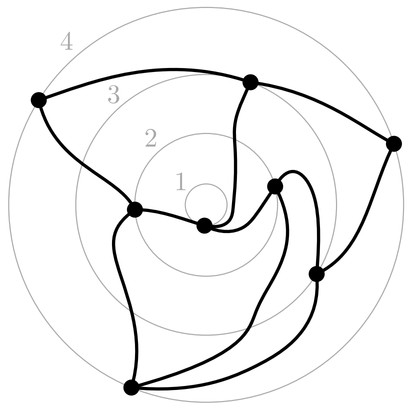


(a) Non-planar drawing (b) Level-planar drawing

Figure 1.2: A planar level-graph with different drawings.



(a) Non-planar drawing



(b) Radial level-planar drawing

Figure 1.3: A radial planar but not level-planar level-graph with different drawings.

ordered numerically. A level-graph is *level-planar* if it has a planar drawing that realizes the levelling of G (Figure 1.2). Pach and Tóth [14] as well as Fulek et al. [9] proved the characterisation of planarity by absence of odd crossing to hold for level-graphs by proving the Weak Hanani-Tutte Theorem. Fulek et al. [9] also proved the characterisation of planarity by absence of independent odd crossing by proving the Strong Hanani-Tutte Theorem. Randerath et al. [19] introduced another characterisation of level-planarity by considering the ordering of the vertices on each level and building a logical formula describing the following properties: The ordering of the vertices on each level is consistent, transitivity is upheld and lastly, for two independent edges starting and ending at the same level the lower-end vertices and the upper-end vertices have a corresponding order on the levels. By Randerath et al. [19] the transitivity of the ordering is not necessary characterize planarity and as a result 2-CNF can be built. Brückner et al. [3] proved that this result is equivalent to the Strong Hanani-Tutte Theorem for level-graphs [9].

Bachmaier et al. [2] introduced a generalization of level-planarity called *radial (level-) planarity*. Radial planarity differs from level-planarity by representation of the levels. Instead of horizontal lines, the levels are represented as concentric circles. Consequentially, the edges, instead of being y-monotone, are drawn as monotone curves from inner to outer levels (Figure 1.3).

Overview

The main result of this work is a proof of the Unified Hanani-Tutte Theorem for level-graphs. To do such, we adapt the proof for the Strong Hanani-Tutte Theorem for x-monotone graphs by Fulek et al. [9]. First, in Chapter 2, we define the basic notations and terms we used throughout the thesis. In Chapter 3, we take a look at how the Unified Hanani-Tutte Theorem can be proven in the plane by introducing two such proofs. In Chapter 4, we introduce the proof of the Strong Hanani-Tutte Theorem our proof is based on—adapted to the terminology of level-graphs. Then, in Chapter 5 we show how to adapt the results of Section 4 to find a proof for the Unified Hanani-Tutte Theorem for level-graphs.

2. Preliminaries

2.1 General Terms and Techniques

Let $G = (V, E)$ be a graph with vertex-set V and edge-set E. Unless otherwise specified, G has no multi-edges.

We define a *drawing* D of G as a set of coordinates in the plane such that each vertex v is assigned a distinct point $(x(v), y(v))$ and each edge $uv \in E$ is represented as a continuous curve from $(x(u), y(u))$ to $(x(v), y(v))$. We only consider drawings such that a vertex and an edge only share a coordinate if they are incident and no edges has self-crossings. We can then define the planarity of G. A graph G is *planar* if it has a *planar drawing*—a drawing such that two edges have a common coordinate (x, y) if, and only if, they share an end-vertex v and $(x, y) = (x(v), y(v))$.

A vertex $v \in V(G)$ is *even* in a drawing D if all edges incident to v cross each other evenly, else v is *odd*. Note that in a planar drawing all vertices are even. An edge $e \in E(G)$ is *even* if it crosses every other edge evenly, else it is *odd*. A drawing of a graph is *even* if all edges in the drawing are even.

The *rotation* of a vertex v in a drawing describes the ordering of the edges incident to v, usually in a clockwise direction. The entirety of the rotation of all vertices in a drawing of a graph is called the *rotation system* of the graph. Two edges are *consecutive in the rotation at v* if no other edge lies in between them in either the clockwise or the counter-clockwise rotation at v.

We can modify the rotation at a vertex v such that two edges e_1 and e_2 cross each other and all other edges incident to v evenly. If e_1 and e_2 cross oddly, in the clockwise direction switch e_1 with its neighbour e' until $e' = e_2$. Then switch e_1 and e_2, introducing an additional crossing and making them cross evenly. Note that they are now consecutive in the rotation at w. Next, we modify the rotation further such that all other edges incident to w cross e_1 and e_2 evenly. Let e' be an edge incident to w with $e' \notin \{e_1, e_2\}$. Assume that e_2 lies in between e' and e_1 in clockwise direction. If e' crosses both e_1 and e_2 oddly switch it with its neighbouring edge until it was switched with both edges. If e' crosses e_1 evenly and e_2 oddly switch it with its neighbouring edge in clockwise direction until it was switched with e_2 but not yet with e_1. If e' crosses e_2 evenly and e_1 oddly switch it with its neighbouring edge in counter-clockwise direction until it was switched with e_1 but not yet with e_2. After repeating this step for each edge e that is incident to w with $e \notin \{e_1, e_2\}$, e_1 and e_2 cross each other and all other edges incident to v evenly.

Throughout this thesis, we often consider induced subgraphs of some other graph. Let H and H' be subgraphs of G. We define $H \oplus H'$ to be the subgraph of G induced by all vertices in $V(H) \cup V(H')$ and $H \ominus H'$ to be the subgraph of G induced by all vertices in $V(H) \setminus V(H')$. Further let W be a set of vertices in G. We define $H \oplus W$ to be the subgraph of G induced by the vertices in $V(H) \cup W$ and $H \ominus W$ to be the subgraph of G induced by the vertices in $V(H) \setminus W$. Those subgraphs then often need to be recombined together. We define *glueing* two graphs H and H' at vertex a_H in H and vertex a'_H in H' as identifying a_H with a'_H. The edges incident to a_H remain consecutive at the combined vertex as do the edges incident to a'_H. The exact place of the seam will be specified.

2.2 Level-Graphs

Next, we take a closer look at *level-graphs* and terms and techniques used in Chapters 4 and 5. A level-graph (G, l) is a graph $G = (V, E)$ with a function $l : V \to \mathbb{N}$ assigning a level to each vertex such that the end-vertices of an edge have different levels. In this Section as well as Chapters 4 and 5 we refer to level-Graphs simply as G. We define a drawing of a level-graph G analogous to the drawing of the graph in the plane with the additional constraint that $y(v) = l(v)$ for all $v \in V$ and each edge is represented as a y-monotone curve. Similarly, we define a drawing of G to be level-planar if the drawing is planar in the plane and $y(v) = l(v)$ for all $v \in V$. Then, G is *level-planar* if it has a level-planar drawing. In this Section as well as Chapters 4 and 5, we refer to level-planar drawings and graphs simply as planar.

We need to describe where an object—either a vertex or an edge—lies in respect to another object in a drawing. If e and f are two edges with a common lower end-vertex v, we say e is *left* of f in the rotation at v if e comes after f in the counter-clockwise rotation at v and *right* of f in the rotation at v if e comes after f in the clockwise rotation at v. Analogous, if e and f are two edges with a common upper end-vertex v, we say e is *left* of f in the rotation at v if e comes after f in the clockwise rotation at v and *right* of f in the rotation at v if e comes after f in the counter-clockwise rotation at v. A vertex v lies *left* of another vertex u with the same level if $x(v) < x(u)$ and *right* of u if $x(v) > x(u)$. A vertex v lies *left* of an edge e that has a coordinate $(x_e, l(v))$, if $x(v) < x_e$ on the horizontal line $y = l(v)$ and *right* of e if $x(v) > x_e$. An edge e with coordinate $(x(e), c)$ lies *left* of another edge e' with coordinate $(x(e'), c)$ on the horizontal line $y = c$ if $x(e) < x(e')$ and *right* of e' if $x(e) > x(e')$. Similarly, we say a vertex v lies *above* another vertex u if $y(v) > y(u)$, i.e. $l(v) > l(u)$. A vertex v lies *below* a vertex u if $y(v) < y(u)$, i.e. $l(v) < l(u)$. We say another vertex w lies *in between* v and u if $l(v) < l(w) < l(u)$ or $l(u) < l(w) < l(v)$. Whether w may have the same level as v or u will be specified. If an edge $e = uv$ only consists of coordinates in between u and v we call it *bounded* by u and v or $l(u)$ and $l(v)$.

When gluing the embedding of two subgraphs, we often insert one embedding into an inner face of another along an edge. Let G be a level-graph with an edge ab such that vertex a lies below vertex b and let H be a subgraph of G such that all vertices in G contains only vertices with levels higher than $l(a)$ and lower that $l(b)$. We can then insert a drawing D_H of H along ab into an embedding $D_{G \ominus H}$ of $G \ominus H$, without adding additional crossings, by resizing D_H. Without loss of generality, we insert D_H left of ab. To fit the drawing we count for each level i how many edges and vertices of H cross it and define that number as c_i. In $D_{G \ominus H}$, define d_i as the distance of ab to the next vertex or edge left of ab on level i. In D_H, for each level i, compress the distance between each edge or vertex on level i to $\frac{d_i}{c_i+1}$. Assume that the course of the edges in between levels moves in correspondence to the compression on the levels and define the obtained drawing as D'_H. Overall, on each level i, D'_H has width $d_i - 2\frac{d_i}{c_i+1}$ and can therefore, in $D_{G \ominus H}$, be fitted into the space between ab and the next vertex or edge left of ab on any level or horizontal line.

Fulek et al. [9, Thm 2.5, 2.6] proved that, if we insert an edge e into an inner face of an embedding of G, then there exists an embedding that includes e such that the rotation system is preserved. They also proved that any bounded edge e can be redrawn without changing the rest of the drawing such that e is monotone and the number of crossings in the drawing is unchanged.

For some of the arguments, we need to distinguish whether a vertex lies *inside* or *outside* of a cycle C. The curve representing C separates the plane into one or more regions, depending on how it crosses itself. All vertices lying in regions of limited area lie inside the cycle. Any edge that starts outside C can only end inside C if it crosses any edge in C oddly. It follows that, for every edge $h = uv$ not incident to C that crosses every edge in C evenly, that u is inside of C if, and only if, v is inside of C.

3. Unified Hanani-Tutte in the Plane

In the following, we introduce two proofs of the Unified Hanani-Tutte Theorem in the Plane. In this chapter, when we talk about embeddings or embeddings with preserved rotations, it always means an embedding with preserved rotations at all even vertices.

3.1 Proof by Removing Even Crossings

According to Fulek et al. [12] the Unified Hanani-Tutte Theorem directly follows from the proof of the Strong Hanani-Tutte Theorem by Pelsmajer et al. [16]. We give an overview of the proof and pay additional attention to the rotation system at even vertices.

Theorem 1.1 (Strong Hanani-Tutte Theorem)**.** *Let $G = (V, E)$ be a graph. If $iocr(G) = 0$ then $cr(G) = 0$.*

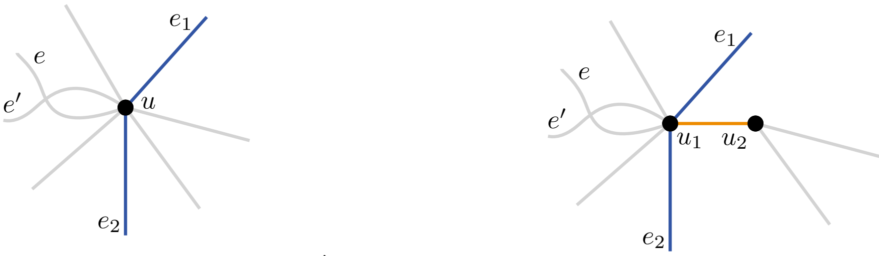
Since the proof given for this theorem includes the preservation of the rotation system at even vertices, what is actually proven is the Unified Hanani-Tutte Theorem.

Theorem 1.3 (Unified Hanani-Tutte)**.** *Let $G = (V, E)$ be a graph with a drawing realizing $iocr(G) = 0$. Then there is a drawing realizing $cr(G) = 0$ such that the rotation system at even vertices is preserved.*

Proof. Let D be a drawing of G such that there is no pair of independent edges crossing oddly. The basic idea is to find cycles in G and then make their edges even, marking them as processed after the procedure. Processed edges will always be crossing-free and contained in a cycle of crossing-free edges. Initially all edges in G are marked as unprocessed. We define the weight of Graph G as $w(G) = \sum_{v \in V} d(v)^3$ where $d(v)$ is the degree of vertex v in G. We prove the theorem by induction primarily over the weight $w(G)$ of G and secondarily over the number of unprocessed edges.

If all edges in D are even, the weak Hanani-Tutte theorem finishes the proof. Note that the rotation of all vertices is preserved.

Else there is at least one odd edge e with an end-vertex v. Define the edge crossing e oddly as e' and assume e' is incident to v, i.e. v is an odd vertex. Then, either e is a cut-edge—an edge that, if removed, separates the graph into two components—or e is contained in a cycle C. We distinguish between those two cases.



(a) Vertex u, oddly crossing edges e, e' and processed edges e_1, e_2.

(b) Vertex u is split into vertices u_1 and u_2.

Figure 3.1: Splitting vertex u.

Case 1: e is a cut-edge. Let u be the other end-vertex of e, i.e. $e = uv$. Then e separates G into two subgraphs G_v and G_u, containing v and u respectively, that, by inductive hypothesis, each have an embedding with preserved rotations, D_{G_v} and D_{G_u}. We can assume that v is on the outer face of D_{G_v} and that u is on the outer face of D_{G_u}. Else, for each subgraph, draw the embedding onto a sphere. Using a *stereographic projection* (e.g.[1]), we can project the embedding back onto the plane, excepting a single point of the sphere. By choosing this point to be in a face containing v or u respectively, the projected embedding in the plane has the same rotation system as the original embedding and v and u now lie on the outer face of D_{G_v} and D_{G_u}. If u is even, we need to take a closer look at the rotation at u. Choose the point to remove when re-projecting the embedding onto the plane to be inside the face whose boundary contains the two edges that were consecutive to e at u in the initial drawing. To obtain an embedding of G join D_{G_v} and D_{G_u} back together by re-inserting edge $e = uv$. Since in D_{G_u}, the outer face contains the two edges that were consecutive to e at u in the initial drawing, when we re-insert e it is now consecutive to the same edges as in the initial drawing and as such the rotation at u is preserved. As v is an odd vertex, we have found an embedding of G where the rotation at even vertices is preserved.

Case 2: e is contained in a cycle C. Then, there either is an odd vertex $u \in C$ such that u has an incident processed edge e_1 or for each odd vertex w in C, every edge incident to w is unprocessed. We distinguish between both cases.

Case 2.1: There is an odd vertex u in C with incident processed edge e_1. As e_1 is a processed edge, it is contained in a cycle C' of processed edges. Then, there is another processed edge e_2 incident to w. Note that all processed edges are crossing-free. Cycle C' divides the plane in two regions such that both e and e' are in the same region. If e were in a different region than e', in order to cross, at least one of them would have to cross one of the edges in C'. But, since all edges in C' are processed, they are crossing-free. As such, we can split u into two adjacent vertices u_1 and u_2 such that e and e' as well as every edge incident to u lying in the same region as e and e' is made incident to u_1 and the other edges incident to u are made incident to u_2 (Figure 3.1). This does not increase the number of crossings in the drawing. Now, since $d(u) = d(u_1) + d(u_2) - 2$, it follows that $d(u)^3 = (d(u_1) + d(u_2) - 2)^3 < d(u_1)^3 + d(u_2)^3$. Then, by induction the new graph has an embedding with preserved rotations. Contraction of u_1u_2 yields an embedding of G where the rotation at even vertices is preserved.

Case 2.2: For each odd vertex w in C, every edge incident to w is unprocessed. Then, for each such w repeat the following process. In a small neighbourhood of w, modify the rotation at w such that both edges incident to w contained in C cross evenly. After the modifications, each edge in C is even. Note that the number of pairs of independent edges crossing oddly was not changed. As proven by Pelsmajer et al. [16, Thm 2.1], in a graph in the plane, it is possible to remove all even crossings in a graph. Therefore, we can modify

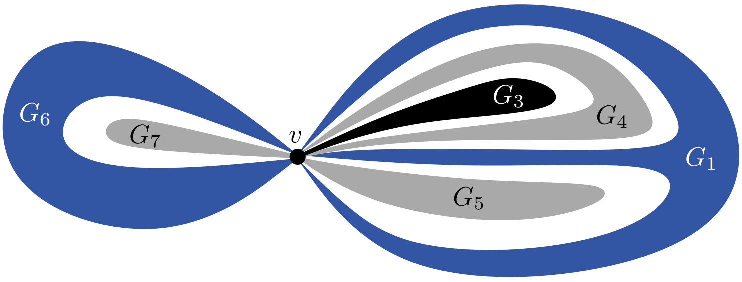


Figure 3.2: Lemma 3.1: G_3, G_5 and G_7 have consecutive connecting edges in the rotation at v (see Fig. 1 [12]).

our drawing such that each edge of C is crossing-free, there are no new odd crossings and crossing-free edges remain crossing-free. Then, we mark all edges in C as processed. The number of unprocessed edges has decreased and as such the inductive hypothesis can be applied to get an embedding of G such that the rotations at even vertices is preserved.

In all cases, we can find an embedding of G. □

3.2 Proof by Distinction According to Vertex-Connectivity

Fulek et al. [12] give a simpler proof for the Unified Hanani-Tutte Theorem.

Well-Formed Rotation Systems

During the main portion of the proof, Fulek et al. [9] make two statements regarding the rotation system. The first considers the rotation at a cut-vertex, if the graph has vertex-connectivity 1 and the second considers the rotations at a separating pair, if the graph has vertex-connectivity 2. Hence, let G be a graph with drawing D such that every pair of independent edges cross evenly.

First, consider the case that G has a cut-vertex v separating G into k components G_i.

Lemma 3.1 ([12, Claim A])**.** *If v is even in D, then there is an $i \in \{1 \ldots k\}$ such that the edges connecting G_i with v are consecutive in the rotation of v in D (Figure 3.1). Thus they form a well-defined linear order.*

Proof. Let $I = \{e_1, \ldots, e_n\}$ be a minimal interval of edges consecutive in the clockwise rotation of v such that I contains all edges of some component G_i incident to v and for each e_j let v_i be the other end-vertex. Suppose there is an edge $e_l \in I$ that connects v to G_j, $j \neq i$. By minimality, $l \neq 1$ and $l \neq n$ and as such, e_l lies after e_1 and before e_n in the clockwise rotation at v. Additionally, there is a connecting edge f from v to G_j that is not in I. Edge f lies before e_1 and after e_n in the clockwise rotation at v. Since $e_1 = vv_1$ and $e_n = vv_n$ are in the same connected component, there exists a path from v_1 to v_n that does not include v. This is true for e_l and f as well. Then there exists a cycle C including e_1, v and e_n as well as a cycle C' including e_l, v and f. Since in D every pair of independent edges crosses evenly and v is even, every edge of C crosses every edge of C' an even number of times. The curves representing C and C' cross at v. Since C and C' only share a single vertex, this implies that there is an edge in C that crosses an edge in C' oddly. Therefore, I can not contain any such e_l and as such there is a component G_i with consecutive edges in the rotation at v. □

Second, consider the case that G has a separating pair (u, v) separating G into k components G_i.

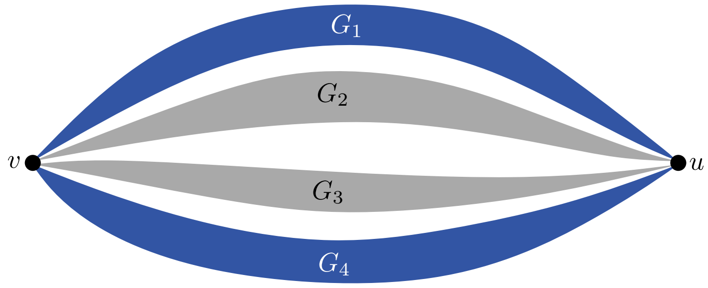


Figure 3.3: Lemma 3.2: v and u are even: The components G_i all have connecting edges consecutive in the rotation at v. The order of all G_i is inverse to v at u. (see Fig. 3 [12])

Lemma 3.2 ([12, Claim B])**.** *If v is even in D, then for each $i \in \{1 \ldots k\}$, the edges connecting G_i to v are consecutive in the rotation of v in D. This defines a well-defined linear order C_v of the graphs G_i around v. If both u and v are even, then the analogously defined order C_u is inverse to C_v (Figure 3.2).*

Proof. Let $a, c \in V(G_i)$, $b \in V(G_j)$ and $d \in V(G_{j'})$ with $i \notin \{j, j'\}$ such that the edges va, vb, vc and vd occur in the clockwise rotation at v in this order. Let C_i be a cycle in $G \ominus \{u\}$ extending path avc and let C_j be a cycle in $G \ominus G_i$ extending path bvd. Note that C_i and C_j share only vertex v. Since every pair of independent edges crosses evenly and v is even, every edge of C_i crosses every edge of C_j an even number of times. The curves representing C_i and C_j cross at v. Since C_i and C_j only share a single vertex, this implies that there is an edge in C_i that crosses an edge in C_j oddly. Therefore, such a C_j cannot exist and as such there is no such edge vb lying in between va and vc in the clockwise rotation at v. Therefore, the connecting edges of every G_i are consecutive in the rotation at v and as such the order C_v is well-defined

Next, assume both u and v are even. Let P_i, P_j and P_l be distinct paths from v to u through component G_m, $m \in \{i, j, l\}$ respectively. In each subgraph induced by paths P_i, P_j, P_l every pair of independent edges crosses evenly. By a local redrawing at internal vertices on the paths the drawing can be made even. The Weak Hanani-Tutte Theorem then implies that the cyclic order of the paths at v is inverse to the cyclic order of the paths at u. It follow that the cyclic order C_u at u is inverse to C_v at v. $\square$

Proof of the Unified Hanani-Tutte Theorem

Now, we can present the proof.

Theorem 1.3 (Unified Hanani-Tutte)**.** *Let $G = (V, E)$ be a graph with a drawing realizing $iocr(G) = 0$. Then there is a drawing realizing $cr(G) = 0$ such that the rotation system at even vertices is preserved.*

Proof. Let G be a graph with a drawing D such that every pair of independent edges cross evenly. We prove the theorem by induction over $n = |V|$.

The base case $n = 1$, a graph with only a single vertex, is trivial.

For the inductive step, we distinguish by vertex-connectivity.

<u>Case 1:</u> G is disconnected. Then, the statement follows for every component by inductive hypothesis, which can then be embedded next to each other.

Case 2: G has vertex connectivity 1. Then, there exists a separating vertex v. Define G_i to be the components of $G \ominus \{v\}$.

If v is odd in D, we do not need to pay attention to the rotation system at v. By inductive hypothesis, each $G_i \oplus v$ has an embedding D_{G_i}. To obtain an embedding of G, glue the different D_{G_i} at v in an arbitrary order.

Else, v is even in D. By Lemma 3.1 there is at least one component G_i whose connecting edges to v are consecutive in the rotation at v. Then, by inductive hypothesis, $G \ominus G_i$ has an embedding $D_{G \ominus G_i}$ and $G_i \oplus \{v\}$ has an embedding D_{G_i}. To obtain an embedding of G, glue D_{G_i} in an appropriate face of $D_{G \ominus G_i}$.

Case 3: G has vertex connectivity 2. Then, there exists a separating pair (u, v). Let G_i be the components of $G \ominus \{v, u\}$. By inductive Hypothesis, each $G_i \oplus \{u, v\}$ has an embedding $D_{G_i \oplus \{u,v\}}$.

If both v and u are odd, in order to obtain an embedding of G, glue all $D_{G_i \oplus \{u,v\}}$ at v arbitrarily and at u in a way that does not introduce any crossings.

Else, if v is even and u is odd, applying Lemma 3.1 to $G \ominus \{u\}$ and separating vertex v, there is a component G_j whose connecting edges to v are consecutive in the rotation at v. Then, by inductive hypothesis, $G \ominus G_j$ has an embedding $D_{G \ominus G_j}$ and $G_j \oplus \{u, v\}$ has an embedding $D_{G_j \oplus \{u,v\}}$. To obtain an embedding of G, glue $D_{G_j \oplus \{u,v\}}$ in the appropriate face of $D_{G \ominus G_j}$ at u and v. If v is odd and u is even, we can the argue analogously.

Else, both v and u are even. By inductive hypothesis, each $G_i \oplus \{u, v\}$ has an embedding $D_{G_i \oplus \{u,v\}}$. Define $G_0 = (\{u, v\}, \{uv\})$ if $uv \in G$. By Lemma 3.2 all $D_{G_i \oplus \{u,v\}}$ can be glued together in a way that preserves the rotations at both u and v without adding any crossings.

Case 4: G has vertex connectivity 3 or more. We show that it is possible to change the rotation of odd vertices locally to get an even drawing. The Weak Hanani-Tutte Theorem (Thm. 1.2) then finishes the proof.

Let v be an odd vertex in G and uv any arbitrary edge incident to v. It is possible to redraw every other edge incident to v in a small neighbourhood of v such that they cross uv evenly. For any edges f_1, f_2 consecutive in the rotation at v that cross oddly remove the odd crossing by swapping them in the rotation at v. Let $(u_0 = u, u_1, \ldots, u_{\deg(v)})$ be the end-vertices of the clockwise rotation of edges at v. After the adjustments, vu_0 crosses every vu_i evenly and every vu_i crosses vu_{i+1} evenly. Next, we prove that in this case v is an even vertex.

Suppose v is not even in G. Then, there is a pair of incident edges $vu_i = e_i, vu_j = e_j$ crossing oddly. Assume $i < j$. If there is more than one pair, choose the one with the least number of edges between them. Define edge $uv_k = e_k$ as the edge consecutive to e_i lying in between e_i and e_j and define $uv_0 = e_0$. Note that among e_0, e_i, e_k and e_j only e_i and e_j cross oddly. Using *Menger's Theorem* (e.g. [7]) and the fact that G has at least vertex-connectivity 3, there are vertex-disjoint paths P_1 from u_0 to u_k and P_2 from u_i to u_j. We can then define two cycles C and C'. Cycle C comprises of edges e_0 and e_k as well as path P_1. Cycle C' comprises of edges e_i and e_j as well as path P_2. Note that C and C' only share vertex v. Every edge of C crosses every edge of C' an even number of times. The curves representing C and C' cross at v. Since C and C' only share a single vertex, this implies that there is an edge in C that crosses an edge in C' oddly.

Overall, if G is disconnected or has vertex-connectivity 1 or 2, we can find an embedding for G by inductive hypothesis. Else, G has vertex-connectivity 3. Then, we can change the rotation of all odd vertices locally to get an even drawing and the theorem follows with the Weak Hanani-Tutte Theorem. □

4. Strong Hanani-Tutte for Level-Graphs

To later prove the Unified Hanani-Tutte Theorem for level-Graphs, we first present the proof for the Strong Hanani-Tutte Theorem for x-monotone Graphs [9], adapted for level-graphs.

Theorem 4.1 (Strong Hanani-Tutte for level-graphs)**.** *Let $G = (V, E)$ be a level-graph with level-assignment $l : V \to \mathbb{N}$. If G has a drawing such that every pair of independent edges crosses an even number of times, then G has a planar drawing.*

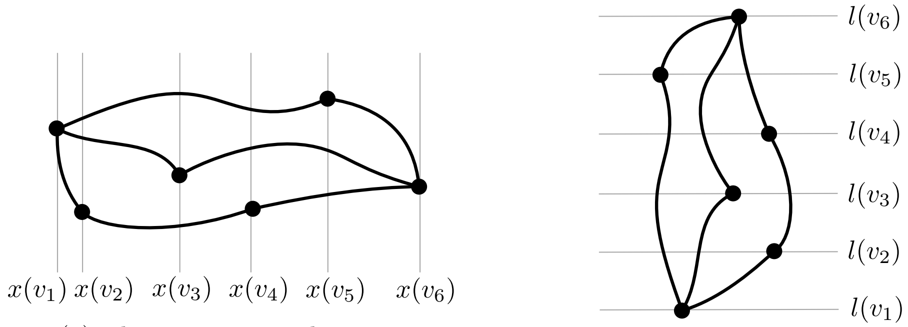
The theorem was proven by excluding the existence of a smallest counterexample in the sense that it has as few vertices as possible. Fulek et al. [9] give some properties for a possible smallest counterexample and also properties of level-graphs with a drawing such that all independent edges cross each other evenly. Using those, the existence of a smallest counterexample for the theorem can be disproven by contradiction. But before that, we take a look at the relationship between x-monotone drawings and level-graphs, particularly in the context of the Hanani-Tutte Theorems.

4.1 Monotone Drawings, Level-Graphs and Hanani-Tutte

Fulek et al. [9] consider x-monotone drawings of regular graphs—drawings such that every edge is monotone in x-direction and every vertical line contains at most one vertex. Both x-monotone drawings and level-graphs are based on a similar concept where the vertices are at least partially ordered.

We can represent an x-monotone drawing as a level-graph by rotating and stretching it partially (Figure 4.1). Let G be a graph with an x-monotone drawing D. First, we modify D by rotating it 90 degrees counter-clockwise, which results in a y-monotone drawing. Using D, we can define an order on the vertices where $v < v'$ if $x(v) < x(v')$. We then define a function $f : V \to \mathbb{N}$ such that $f(u) = 1$ for vertex $u = min\{v \in V\}$ and $f(v') = f(v) + 1$ for all $v, v' \in V$ if $v < v'$ and there is no vertex v'' such that $v < v'' < v'$. Then $f(v) < f(v') \Leftrightarrow v < v'$. We then modify D to a drawing D' by scaling its y-axis such that $y(v) = f(v)$ for each vertex. Then, D' is a drawing of level-graph (G, l) with $l = f$.

Representing a level-graph as an x-monotone drawing is not universally possible, but restricted by the number of vertices per level or x-coordinate respectively. A representation as an x-monotone drawing is only possible if each vertex has a unique level. Hence, let G be a level-graph with a level-assignment l and let D be a drawing of G. If each level has only a single vertex, we obtain an x-monotone drawing by simply rotating D 90 degrees



(a) The x-monotone drawing.

(b) The corresponding level-graph is obtained by rotating and partially stretching.

Figure 4.1: The modification of an x-monotone drawing to a level-graph.

clockwise. Else, if there is at least one level with more than one vertex, we can still use some properties of x-monotone drawings, the Hanani-Tutte Theorems included. We need to ensure that any modifications we make after converting the graph to an x-monotone drawing do not hinder the re-conversion to the original level-graph. After modifying the x-monotone drawing, it is possible that an edge acts as a barrier between a vertex and its original level (Figure 4.4). To prevent this, we insert temporary vertices and edges as placeholders (see [9, Sec. 4.2]) (Figure 4.2). Therefore, before converting the level-graph do the following. Define v_i^j to be the vertices on level i, as ordered in the original drawing from left to right. For each level k with more than one vertex and each vertex v_k^l such that there is no edge that has v_k^l as an upper end-vertex, insert a vertex v' just below level k and add edge $v'v_k^l$ without adding any crossing. Then, we modify D such that each vertex has a unique y-coordinate. On k, for each v_k^l with $l > 1$ we take a small surrounding and distort the inside along the y-axis such that $y(v_k^l) \neq l(v_k^l)$, $y(v_k^l) \neq y(v')$ for all other vertices v' with $l(v) = l(v')$ and $y(v_k^l) < y(v_k^{l+1})$. This drawing is y-monotone. We can obtain an x-monotone drawing by rotating the drawing 90 degrees clockwise.

We can use these connections between level-graphs and x-monotone drawings to, for example, explain how the Weak Hanani-Tutte Theorem for level-graphs follows from the Weak Hanani-Tutte Theorem for x-monotone drawings (Figure 4.3). Let G be a level-graph with a drawing such that every pair of edges crosses evenly. Using the described method, we can obtain a drawing of a level-graph with only a single vertex per level without adding or removing any crossings. This drawing is a y-monotone drawing, rotate it to obtain a x-monotone drawing. Then, with the Weak Hanani-Tutte Theorem for x-monotone drawings, we obtain an embedding with preserved rotations. To obtain an embedding of G revert the modifications. First, rotate the embedding of the x-monotone drawing counter-clockwise to make the embedding y-monotone. Then, by reverting the distortions we can move the vertices back to their original level. Since each distorted vertex v has an incident edge e crossing its original level and since the embedding is crossing-free this does not add or change any crossings and retains the monotony of any edge. Therefore, we have found an embedding of G with preserved rotations in respect to the initial drawing. Consequentially, the Weak Hanani-Tutte Theorem holds for level-graphs.

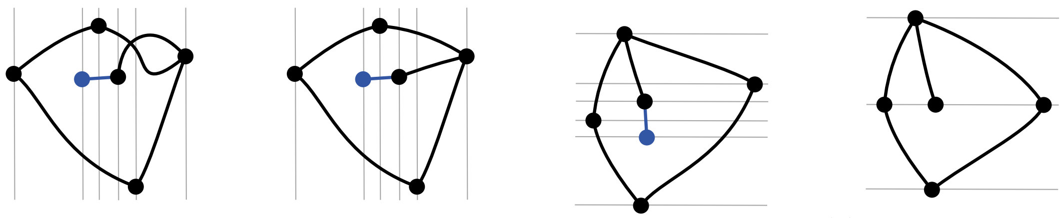
Theorem 4.2 (Weak Hanani-Tutte Theorem for level-graphs)**.** *Let $G = (V, E)$ be a level-graph with level-assignment $l : V \to \mathbb{N}$. If G has drawing such that every pair of edges cross an even number of time, then G has an embedding such that the rotation at all vertices is preserved.*

A similar approach could be used to conclude the Strong Hanani-Tutte Theorem for level-graphs from th Strong Hanani-Tutte Theorem for x-monotone drawings.

(a) The initial level-graph.

(b) Distortion of the vertices

Figure 4.2: The modification of a level-graph that has a level with more than one vertex. If needed, a placeholder is inserted.



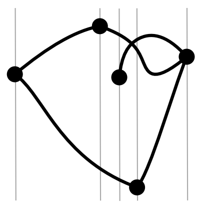
(a) The x-monotone drawing with placeholder.

(b) An embedding of the x-monotone drawing.

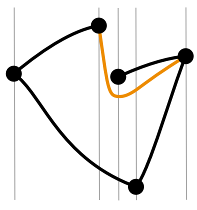
(c) The embedding as y-monotone drawing.

(d) Converted back into the original level-graph.

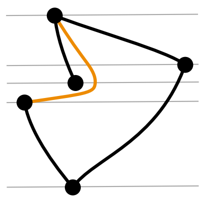
Figure 4.3: Continuation of Figure 4.2. An x-monotone drawing where the vertices on the same level were correctly modified.



(a) The x-monotone drawing without placeholder.



(b) An embedding of the x-monotone drawing.



(c) The embedding as y-monotone drawing. A reversion to the original graph is not possible.

Figure 4.4: Continuation of Figure 4.2. An x-monotone drawing where the vertices on the same level were not correctly modified.

4.2 Properties of a Smallest Counterexample

In the following we reiterate the properties for a smallest counterexample as given by Fulek et al. [9]. In preparation for the proof of the Unified Hanani-Tutte Theorem in Section 5 we can consider that by allowing multi-edges in G the Lemmata in this section hold for multi-graphs as well.

Lemma 4.3 (Lemma 3.3 i [9])**.** *Let G be a smallest counterexample to Theorem 4.1. Then G is connected.*

Proof. Let G be a smallest counterexample to Theorem 4.1 with a drawing such that every pair of independent edges crosses an even number of times. If G is not connected, by minimality, its components themselves have embeddings with preserved rotations. Define the different components as G_i. We can pull the components apart such that each G_i completely lies to the left of G_{i+1} and embed them separately. The different components do not intersect each other and as such we found an embedding of G. □

Hence, for the rest of this section assume that all smallest counterexamples to Theorem 4.1 are connected.

Lemma 4.4 (Lemma 3.3 ii [9])**.** *Let G be a smallest counterexample to Theorem 4.1. Then G has no connected subgraph H such that*

1. *H has only the neighbours a and b.*
2. *H lies completely between the vertices a and b.*
3. *$G \ominus H$ has other vertices apart from a and b.*

Proof. Let G be a smallest counterexample to Theorem 4.1 with a drawing such that every pair of independent edges crosses an even number of times and consider such a H. Then, let $E_a \subsetneq E(G)$ be the set of edges connecting a to H and let $E_b \subsetneq E(G)$ be the set of edges connecting b to H. If $ab \notin E(G)$ insert ab by taking any path P from a to b in H and drawing ab next to P. Then, ab is bounded by $l(a)$ and $l(b)$ and we can redraw ab to be monotone while leaving the number of crossings unchanged. Since each edge in $G \ominus H$ crosses each edge in H evenly, ab has no odd crossings with any edge of $G \ominus H \ominus \{a, b\}$. Then, by minimality of G, $G \ominus H$ has an embedding $D_{G \ominus H}$. Similarly we can obtain an embedding D_H for $H \oplus \{a, b\}$. To obtain an embedding of G glue D_H to $D_{G \ominus H}$ at a and b along ab. If $ab \notin E(G)$ originally, remove it. □

Lemma 4.5 (Lemma 3.3 iii [9])**.** *Let G be a smallest counterexample to Theorem 4.1. If G has a cut-vertex a and $G \ominus \{a\}$ has a component H that lies completely above a, then*

a) *H contains only a single vertex b.*

b) *G has no edge ac such that c lies above b.*

c) *G has no connected subgraph H' such that*

   1. *H' fully lies between the vertices a and b.*
   2. *H' has neighbouring vertex a.*
   3. *all other neighbouring vertices of H' lie above b.*

Proof. Let G be a smallest counterexample to Theorem 4.1 with a drawing such that every pair of independent edges crosses an even number of times and consider such a H.

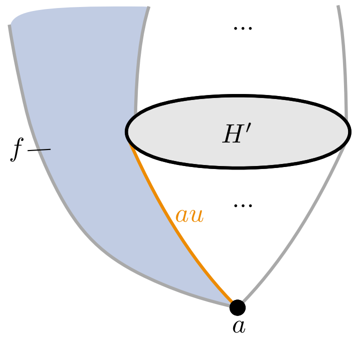
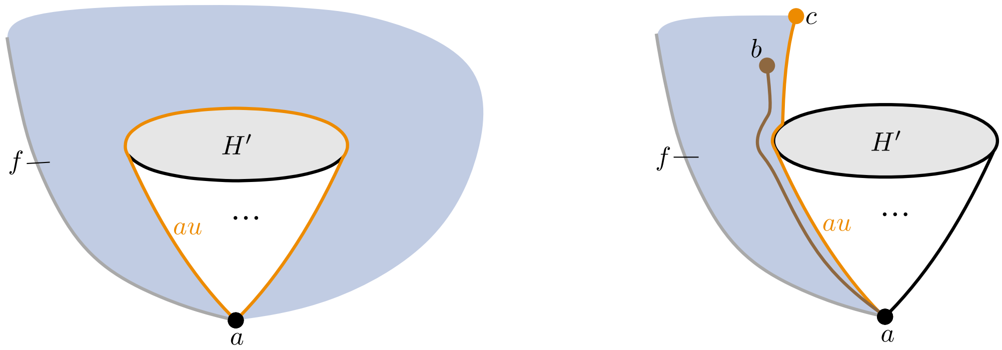


Figure 4.5: Vertex a, subgraph H', leftmost connecting edge au and the face f let to au.



(a) The boundary walk ends where it started, contradicting the assumptions regarding H'.

(b) The boundary walk exits H' to vertex c above b. Edge ab can be embedded along the boundary walk.

Figure 4.6: The boundary walk of f starting at a and continuing through au.

a) Let H be a component of $G \ominus \{a\}$ such that $l(a) < l(v)$ for all vertices v in H and let b be the vertex with maximal level in H. If $|V(H)| > 1$ we get a contradiction using Lemma 4.4 and choosing H in Lemma 4.4 as $H \ominus \{b\}$. It follows that $|V(H)| = 1$ and as such $H = \{b\}$.

b) Consider an edge ac in G with $l(b) < l(c)$. By minimality of G, the subgraph $G \ominus \{b\}$ has an embedding $D_{G\ominus\{b\}}$. We can then obtain an embedding of G by inserting vertex b and edge ab into $D_{G\ominus\{b\}}$ along ac.

c) Consider such a H'. We can assume that H' is an induced subgraph. Due to minimality of G, subgraph $G \ominus \{b\}$ has an embedding $D_{G\ominus\{b\}}$. Next we show ho to embed ab in $D_{G\ominus\{b\}}$. Define au to be the edge leftmost at a connecting a to H' and define f to be the face left of au containing au in its boundary (Figure 4.5). Assume that ab lies to the left of au. Then, starting at a and continuing through au, follow the boundary of f until we come across a vertex c that is not contained in H'. If $c = a$ (Figure 4.6a) then H' has no neighbours v above b, which contradicts our assumptions. Therefore, c must lie above b (Figure 4.6b). Add edge ab and vertex b to $D_{G\ominus\{b\}}$ by drawing it inside the face f along the walk defined by the boundary of f from a to c through H' and stopping at level $l(b)$. We can redraw ab to be monotone without changing the crossings and therefore we can embed a crossing-free edge ab. Then, we have found an embedding of G with preserved rotations.

□

To make it more clear that when Lemma 4.5 is used, all assumptions are met and it can in fact be applied, we extend the Lemma to exclude the existence of a component H' of $G \ominus \{a\}$.

Lemma 4.6 (Extension of Lemma 4.5)**.** *If G has a cut-vertex a and $G \ominus \{a\}$ has a component H that lies completely above a, then*

a) H contains only a single vertex b.

b) G has no edge ac such that c lies above b.

c) G has no connected subgraph $\overline{H}$ such that

1. $\overline{H}$ is a component of $G \ominus \{a\}$.

2. $\overline{H}$ fully lies above vertex a.

Proof. Let G be a smallest counterexample to Theorem 4.1 with a drawing such that every pair of independent edges crosses an even number of times. Items a) and b) apply by Lemma 4.5.

To prove item c), suppose there is such a $\overline{H}$. Define $\hat{H}$ as the subgraph induced by all vertices of $\overline{H}$ with a level lower than $l(b)$. Then, either $\hat{H}$ is connected or $\hat{H}$ has more than one component. If $\hat{H}$ is connected, $\hat{H}$ lies above a and below b, a is neighbouring vertex of $\overline{H}$ and $\overline{H}$ has no other neighbouring vertices below b. This contradicts Lemma 4.5 with $H' = \hat{H}$. Else, $\hat{H}$ has more than one component. Define $\hat{H}_i$ as the components of $\hat{H}$. Then there exists a component $\hat{H}_j$ of $\hat{H}$ that has neighbour a and has no other neighbours below b. By definition, all vertices of $\hat{H}_j$ lie between a and b. This contradicts Lemma 4.5 with $H' = \hat{H}_j$. □

4.3 Properties of Level-Graphs with Independent Even Crossings

In this section we consider a specific situation: Let G be a level-graph such that all independent edges cross an even number of times. Assume that G has an odd vertex v_0 with three incident edges e_1, e_2 and e_3 such that e_3 lies between e_1 and e_2. The edges e_1 and e_2 cross oddly and e_3 crosses both e_1 and e_2 evenly (Figure 4.9). This situation arises in the main part of the proof (Section 4.4). There, we have a drawing of G with a minimal number of odd crossings, which results in the defined constellation of edges for any remaining odd crossings.

Lemma 4.7 (Lemma 3.4 [9])**.** *For arbitrary $l_R > l(v_0)$, define G' as the graph induced by all vertices of G lying between levels $l(v_0)$ (excluded) and l_R (included). Let G'_i be the component of G' that contains v_i. If $l(v_i) > l_R$, then $G'_i = \emptyset$.*

Suppose that G'_1, G'_2, G'_3 are pairwise disjoint and that for every $i = 1, 2, 3$ there is a path P_i (in G) from v_0 through e_i to some vertex v'_i that lies above level l_R such that all vertices of P_i lie above v_0 (Figure 4.7). If $G'_i = \emptyset$, then define $E(P_i) = \{e_i\}$.

Then each G'_i has no neighbours (in G) below v_0.

Sketch Proof. The lemma is proven by contradiction. If there were a neighbour v' of G'_i below v_0, there would be a path P'_i from v_i to v' that, apart from v', fully lies in G'_i. Using the fact that each time two edges e and f cross, they switch order on a horizontal line and the fact that all pairs of independent edges cross an even number of times, each assignment of $\{i, j, k\}$ to $\{1, 2, 3\}$ can be lead to a contradiction such that the last edge of P'_i must pass both left and right of v_0. □

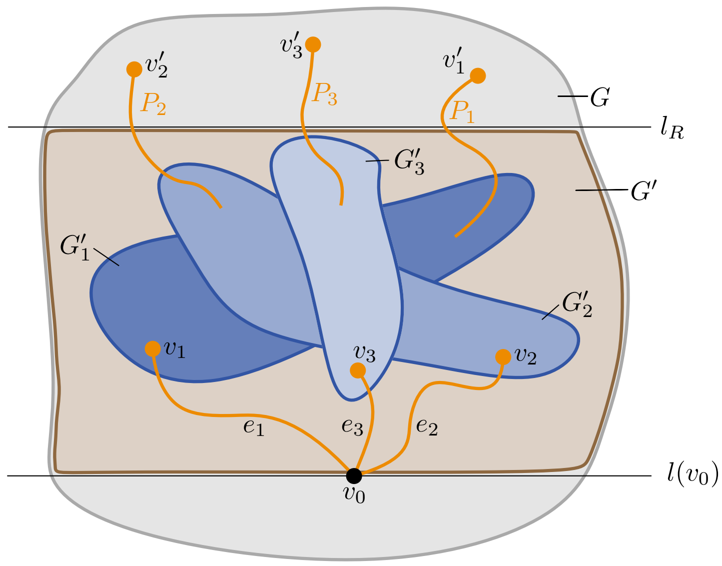


Figure 4.7: Graph G, induced subgraph G' between $l(v_0)$ and l_R, vertices v_1, v_2, v_3 contained in components G_1', G_2', G_3' as well as paths P_1', P_2', P_3' ending over l_R. (see Fig. 6 [9])

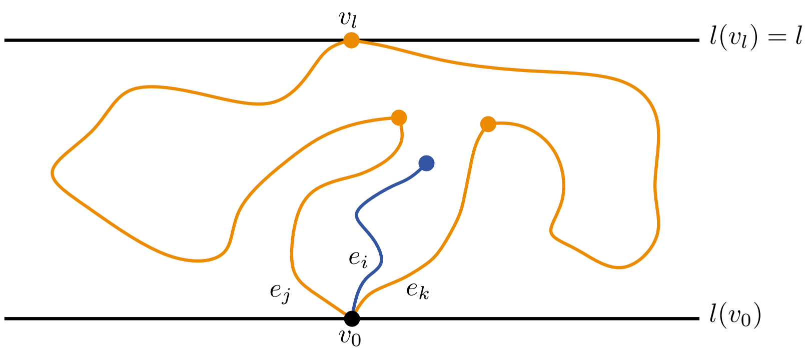


Figure 4.8: Edges e_i, e_j, e_k, cycle C containing e_j, e_k and its uppermost vertex v_l.

Lemma 4.8 (Lemma 3.5 [9]). *Suppose that for some distinct $j, k \in \{1, 2, 3\}$, there is a cycle C that contains e_j and e_k such that every vertex of C lies above v_0. Let v_l be the vertex on C with the highest level (Figure 4.8). Let i be the index such that $\{i, j, k\} = \{1, 2, 3\}$ and suppose that v_i is not in C. Let G'_i be the component of $G \ominus C$ that contains v_i .*

Then every vertex v of G'_i lies between v_0 and v_l (both excluded).

Sketch Proof. The lemma is proven by contradiction. Suppose that v_i lies above v_l. This can be disproven by leading each assignment of $\{i, j, k\}$ to $\{1, 2, 3\}$ to a contradiction such that v_l lies both left and right of e_i. As such v_i lies between v_0 and v_l (both excluded). Vertex v_i lies between the paths P_j and P_k. Then, supposing there is a path P'_i in G'_i to a vertex either below v_0 or above v_l, it can be deduced that all vertices in P'_i are all either to the left or to the right of both P_j and P_k, contradicting the placement of v_i. □

After careful consideration of the full proofs, it can be deduced that both Lemma 4.7 and Lemma 4.8 hold for multi-graphs as well.

4.4 Proof of the Strong Hanani-Tutte Theorem

Using the properties established in this chapter, Fulek et al. [9] prove the Strong Hanani-Tutte Theorem for level-graphs by disproving the existence of a smallest counterexample G.

Theorem 4.1 (Strong Hanani-Tutte for level-graphs). *Let $G = (V, E)$ be a level-graph with level-assignment $l : V \to \mathbb{N}$. If G has a drawing such that every pair of independent edges crosses an even number of times, then G has a planar drawing.*

Proof. Let G be a smallest counterexample to Theorem 4.1 with a drawing D such that every pair of independent edges crosses an even number of times and the number of pairs of edges crossing oddly is minimal.

If D has no odd crossing pair of edges then the Weak Hanani-Tutte Theorem for level-graphs (Thm. 4.2) finishes the proof.

Else, D has a pair of edges crossing oddly. By Lemma 4.3 G is connected. Since all independent edges cross evenly, this pair has a common end-vertex v_0. We can assume v_0 is the lower end-vertex. Note that v_0 is an odd vertex. Define e_1, e_2 as the pair of odd-crossing edges incident to v_0 such that there are as few edges between both in clockwise rotation at v_0 as possible and such that e_1 lies to the left of e_2. By minimality of the number of odd crossings, e_1 and e_2 are not consecutive in the rotation at v_0. Otherwise this odd crossing could be resolved by switching e_1 and e_2 in the rotation at v_0. By choice of e_1 and e_2, there are edges incident to v_0 lying between e_1 and e_2 that cross each other as well as e_1 and e_2 evenly. Let e_3 be such an edge (Figure 4.9). Define v_1, v_2, v_3 as the upper end-vertices of e_1, e_2, e_3 and let G_0 be the subgraph of G induced by all vertices lying above v_0. We distinguish by the placement of v_1, v_2 and v_3 in the components of $G_0 \ominus v_0$.

<u>Case 1:</u> All vertices of $\{v_1, v_2, v_3\}$ are in different components of $G_0 \ominus v_0$.

For $i \in \{1, 2, 3\}$ let G_i be the component of $G_0 \ominus v_0$ that contains v_i and let v'_i be the vertex with the highest level in G_i. Assign $\{1, 2, 3\}$ to $\{i, j, k\}$ such that v'_i is the lowest of $\{v'_1, v'_2, v'_3\}$. Then define G'_i, G'_j and G'_k as the components of the subgraph induced by all vertices lying between v_0 and v'_i that contain v_i, v_j and v_k respectively. If v_j lies above v'_i, then define $G'_j = \emptyset$. Define G'_k analogous. Thus, we can apply Lemma 4.7 with $l_R = l(v'_i)$, which then states that G'_i, G'_j, G'_k have no neighbours below v_0. Note that by definition of

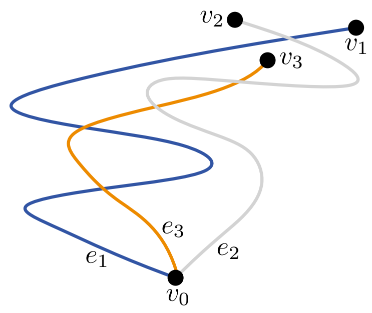


Figure 4.9: Edges e_1, e_2, e_3 and end-vertices v_1, v_2, v_3. The pairs e_1, e_3 and e_2, e_3 cross evenly and the pair e_1, e_2 crosses oddly.

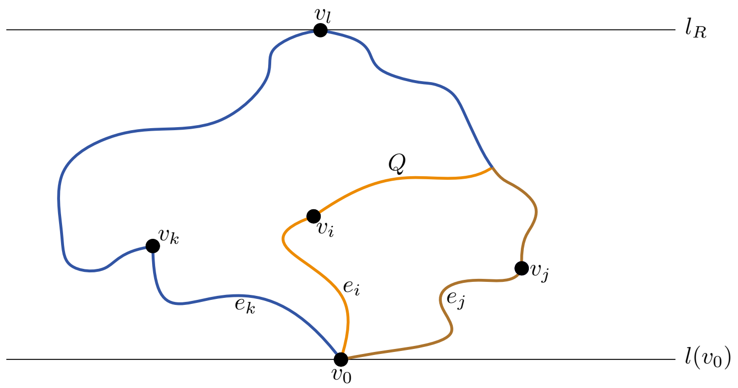


Figure 4.10: Lower bound $l(v_0)$, upper bound $l(v_l)$ edges e_i, e_j, e_k and path Q.

l_R and v_i, G_i has no neighbours in G below v_0 and is therefore a component of $G \ominus \{v_0\}$. Also note that by definition of l_R and v_i, G_i lies on and below l_R. Overall $G_i' = G_i$ and as such G_i' is a component of $G \ominus \{v_0\}$. We can then apply Lemma 4.6 with $a = v_0$ and $H = G_i'$. It follows that G_i' only contains a single vertex and therefore $v_i = v_i'$ (4.6 a)). It also follows that v_j and v_k lie below v_i and below v_i' (4.6 b)). Therefore $G_j' \neq \emptyset$ and $G_k' \neq \emptyset$. Overall, both G_j' and G_k' are connected components of $G \ominus \{v_0\}$ that fully lie above v_0, contradicting Lemma 4.6.

Case 2. At least two vertices of $\{v_1, v_2, v_3\}$ are in the same component of $G_0 \ominus v_0$.

Let l be the smallest level such that the subgraph induced by all vertices of G lying between the levels $l(v_0)$ (excluded) and l (included) has a component that contains at least two vertices of $\{v_1, v_2, v_3\}$. Then there is a cycle C that contains e_j and e_k such that $\{e_j, e_k\} \subsetneq \{e_1, e_2, e_3\}$ as well as a vertex v_l on level l such that all vertices of the cycle lie between v_0 and v_l, both included (Figure 4.10). If $v_l = e_n$ for a $n \in \{1, 2, 3\}$ then we can assume that $e_n = vv_n$ is a part of cycle C. Let $i \in \{1, 2, 3\}$ such that $i \neq k$ and $i \neq j$, then it follows that $v_i \neq v_l$. Suppose there is path Q from v_i to C such that the vertices of Q lie completely between v_0 and v_l, excluding both. Then there exists a cycle C' that includes v_0, e_i, Q and part of $C \ominus \{v_l\}$ such that it either contains e_j or e_k. But then all vertices in C' lie between v_0 (included) and v_l (excluded). This contradicts the choice of l as the smallest level such that a cycle with any two of $\{e_1, e_2, e_3\}$ completely lies on and below l. Supposing $V(Q) = \{v_i\}$ implies that v_i is not contained in C.

Next, let G_i' be the component of $G \ominus C$ that contains v_i. By Lemma 4.8 G_i' lies between v_0 and v_l (both excluded). The previous paragraph also implies that G_i' has no neighbours in $C \ominus \{v_0, v_l\}$. Let v_i' be the vertex with the highest level in G_i' and define G_j', G_k' according to Lemma 4.7 with $l_R = l(v_i')$. By choice of l and as v_i' lies below l, it follows that $G_j' \neq G_k'$.

We further distinguish by adjacency of v_l to G_i':

<u>Case 2.1:</u> G'_i is not adjacent to v_l

Since v_i is a vertex of G'_i, v_i is in a different component of $G_0 \ominus v_0$ than both v_j and v_k. Note that G'_i neither has a neighbour below $l(v_0)$ nor a neighbour above or on l nor a neighbour in $C - \{v_0, v_l\}$. As such, the only neighbour of G'_i is v_0. Further note that G'_i lies completely between the levels $l(v_0)$ and $l(v_l)$ (both excluded). Therefore G'_i is a component of $G \ominus \{v_0\}$. We can therefore apply Lemma 4.6 with $a = v_0$ and $H = G'_i$. It follows that G'_i only contains a single vertex and therefore $v_i = v'_i$ (4.6 a)). It also follows that v_j and v_k lie below v_i and below v'_i (4.6 b)). Therefore neither G'_j or G'_k are empty. Overall, both G'_j and G'_k are connected components of $G \ominus \{v_0\}$ that fully lie above v_0, contradicting Lemma 4.6.

<u>Case 2.2:</u> G contains an edge from G'_i to v_l

Then G'_i has only neighbours v_0 and v_l, lies completely between its neighbours and $G \ominus G'_i$ contains at least all vertices in $C \ominus \{v_0, v_l\}$ and is therefore not empty, contradicting Lemma 4.4. $\square$

Overall, after minimizing the number of odd crossings in a smallest counterexample G, either there are none left and according to the Weak Hanani-Tutte Theorem (Thm 4.2) G has an embedding, or there is at least one odd crossing left. Then, for each complete case distinction we make, each case can be lead to a contradiction. Therefore, after minimizing the number of odd crossings for G, none will be left and as such G has an embedding. It follows that G is not a counterexample to Theorem 4.1 and since G was defined to be the smallest counterexample, there exist none at all. Therefore, the Unified Hanani-Tutte Theorem for level-graphs (Thm 4.1) was proven.

5. Unified Hanani-Tutte for Level-Graphs

In order to prove the Unified Hanani-Tutte Theorem for level-graphs, we expand and adapt the proof for the Strong Hanani-Tutte Theorem for level-graphs (Thm 4.1).

Theorem 5.1 (Unified Hanani-Tutte for level-graphs). *Let $G = (V, E)$ be a multi-level-graph with level-assignment $l : V \to \mathbb{N}$. If G has drawing such that every pair of independent edges crosses an even number of times, then G has an embedding such that the rotation at all even vertices is preserved.*

In this chapter, when we talk about an embedding or an embedding with preserved rotation, it always means an embedding such that the rotation at all even vertices is preserved.

5.1 Properties of a Smallest Counterexample

The properties for a smallest counterexample to the Strong Hanani-Tutte Theorem (Section 4.2)apply to a smallest counterexample to the Unified Hanani-Tutte Theorem as well. The proofs remove and then re-insert one or more edges into the rotation at a vertex v. Therefore, we need to distinguish for those v whether it is an even or an odd vertex in the initial drawing. If all such vertices v are odd, we do not need to pay attention to the rotation at those v and as such the proof given for the corresponding lemma in Section 4.2 is valid for this case. Therefore, we only take a closer look at the cases where at least one such v is an even vertex.

Lemma 5.2 (Analogous to Lemma 4.3). *Suppose that G is a smallest counterexample to Theorem 5.1. Then G is connected.*

Proof. The Lemma can be proven analogous to Lemma 4.3. □

Hence, assume that all smallest counterexamples to Theorem 5.1 are connected.

Lemma 5.3 (Analogous to Lemma 4.4). *Suppose that G is a smallest counterexample to Theorem 5.1. Then G has no connected subgraph H such that*

1. *H has only the neighbours a and b.*
2. *H lies completely in between the vertices a and b.*
3. *$G \ominus H$ has other vertices apart from a and b.*

Proof. Let G be a smallest counterexample to Theorem 4.1 with a drawing such that every pair of independent edges crosses an even number of times. We expand on the proof of Lemma 4.4. Suppose such a graph H exists. The basic idea is to use the minimality of G to find embeddings for subgraphs $G \ominus H$ and $H \oplus \{a, b\}$ and glue them back together to find an embedding of G and therefore contradict the assumption that G is a counterexample. We need to ensure that the rotation at the vertices where we glue the embedding of the subgraphs back together is preserved for even vertices. Therefore, we differentiate between four separate cases for the vertices a and b and whether they are even (Figure 5.2) or odd (Figure 5.1).

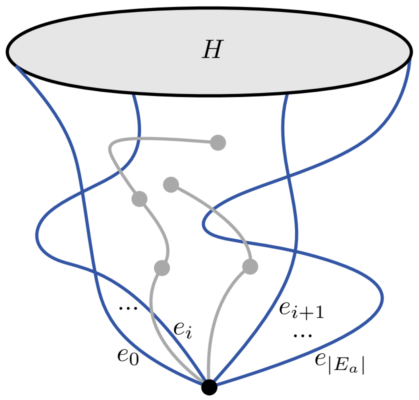
Let $E_a \subsetneq E$ be the set of edges connecting H to a and let $E_b \subsetneq E$ be the set of edges connecting H to b.

<u>Case 1:</u> If neither a nor b are even, we do not need to pay further attention to the rotation of edges at a or b in order to fulfil Theorem 5.1. Therefore, the proof for the corresponding lemma in Section 4.2 (Lemma 4.4) is valid.

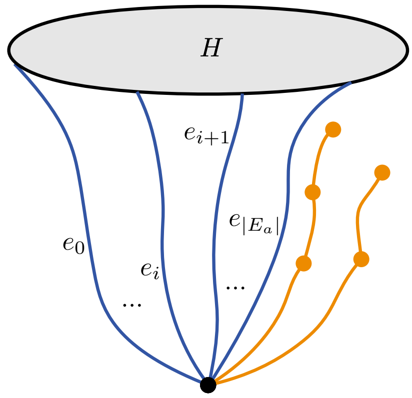
<u>Case 2:</u> If a is even and b is odd, define $\{e_1, \ldots, e_{|E_a|}\}$ as the edges in E_a as appearing in clockwise order at a and for each $i \in \{1, \ldots, |E_a|\}$ define v_i as the other endpoint of e_i.

If the edges in E_a are not consecutive in the rotation at a, there is at least one edge in between e_i and e_{i+1} for least one index i. In order to deal with these, we show that all edges in between e_i and e_{i+1} connect to a subgraph G_i of G that only contains and is even induced by all vertices that do not lie in H and are contained inside a cycle C_i defined by the edges e_i, e_{i+1} and any arbitrary path P_i that connects v_i and v_{i+1} and fully lies in H. Hence, assume that there are l_i edges in between e_i and e_{i+1} in the clockwise rotation at a. Define the edges that lie in between e_i and e_{i+1} in the rotation at a as $e_i^j,\ j \in \{1, \ldots, l_i\}$. Further, for all e_i^j define v_i^j to be its other end-vertex. Since a is even, e_i^j crosses both e_i and e_{i+1} evenly and therefore every v_i^j lies in between e_i and e_{i+1}. Additionally, since by definition of e_i^j, no v_i^j is adjacent to any vertex in H, v_i^j also lies in between $l(a)$ and P_i. Therefore v_i^j is contained in the region defined by C_i with limited area and as such v_i^j lies inside of C_i. Let v be a vertex that is not adjacent to H or C_i. Then, any edge incident to v crosses any edge in C_i evenly. It follows that, if v is inside C_i, every end-vertex of any edge incident to v lies inside of C_i as well. Analogously, if v lies outside of C_i, every end-vertex of any edge incident to v lies outside of C_i as well.

By definition of H and e_i^j, there is no path from a to H that includes e_i^j. Note that there is then also no such path that includes any v_j and therefore no vertex on such a path, excepting a, is incident any e_j. It follows that no vertex on such a path, excepting a and v_i^j, is adjacent to either C_i or H. For any fixed $i \in \{1, \ldots, |E_a|\}$, we can then deduce that every path that starts at a and continues through a e_i^j only contains vertices on the inside of C_i. Analogously, every path that starts at a and contains neither any vertex of C_i nor any v_i^j, contains, apart from a, only vertices lying outside of C_i. For $i \neq k$, it follows that any vertex not in H that is contained inside of C_i lies outside of C_k. It also follows that any vertex not in H that lies on the inside of C_k lies outside of C_i. Note that the areas inside of C_i and C_k need not be disjunct. Now we can define G_i as the subgraph of G induced by all vertices contained inside C_i that are not part of H. There is no vertex $z \in G$ that does not lie in H or any G_i such that z lies inside a cycle C_k since every possible path from a to z that does not contain e_i, e_{i+1} or any e_i^j starts outside of C_i. As argued before, all vertices on such a path, including z, lie outside of C_i as well. Overall, all edges e_i^j lying in between e_i and e_{i+1} connect a to a subgraph G_i of G containing all connected components of $G \ominus \{a\}$ that are connected to a by the edges lying in between e_i and e_{i+1} in the rotation at a. Define the union of all G_i as $\mathcal{G}$.

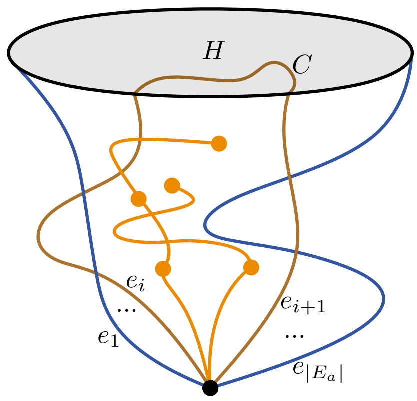


(a) Subgraph H is connected to a by edges e_k. There is at least one non-consecutive pair of edges e_i, e_{i+1}.

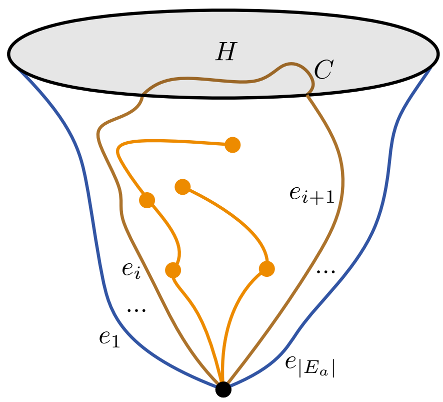


(b) The resulting embedding.

Figure 5.1: Uneven vertex a, subgraph H and non-consecutive connecting edges. The rotation can be changed arbitrarily.



(a) Subgraph H is connected to a by edges e_k. There are non-consecutive e_i and a_{i+1} that, together with a path through h Form cycle C_i.



(b) The resulting embedding. The rotation at a is preserved.

Figure 5.2: Even vertex a, subgraph H and non-consecutive connecting edges.

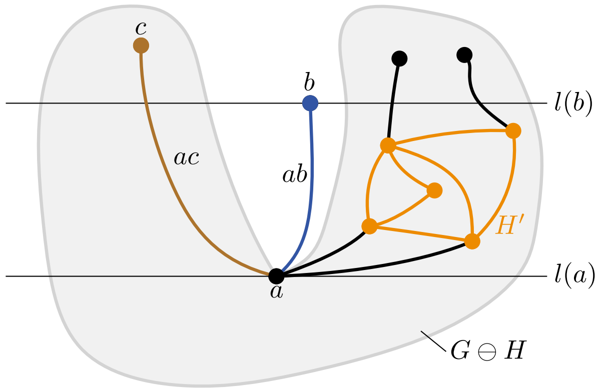


Figure 5.3: Graph G has a cut-vertex a. Subgraph $G \ominus \{a\}$ has component $H = \{b\}$. Edge ac and subgraph H' can not occur in a smallest counterexample.

To finish this case, we need to ensure that the edge ab is contained in G at a convenient position. If $ab \notin G$ or if $ab \in G$ such that ab is not consecutive to any edge in E_a, we insert an edge ab by taking any path P from a through H to b and drawing the edge ab in the region just to the left of P. Edge ab is then bounded by $l(a)$ and $l(b)$ and has, analogous to P, no odd crossings with any edge of $G \ominus H \ominus \mathcal{G}$ that is not incident to a or b. Then, $G \ominus H \ominus \mathcal{G}$, including ab, has no pair of independent edges that cross oddly. We can redraw ab as a monotone edge without changing the crossings of ab with any edge in $G \ominus H \ominus \mathcal{G}$. Therefore, we get a monotone edge ab that crosses any edge in $G \ominus H \ominus \mathcal{G}$ evenly.

By minimality the induced subgraph of G obtained by removing H and all G_i has an embedding $D_{G \ominus H \ominus \mathcal{G}}$ with preserved rotations. Similarly, we can obtain an embedding D_H for $H \oplus \{a, b\}$ as well as an embedding D_{G_i} for each existing $G_i \oplus \{a\}$. To obtain an embedding of G we first insert H by glueing D_H to $D_{G \ominus H \ominus \mathcal{G}}$ at a and b just to the right or left of ab, depending on its previous position in the initial drawing and obtain an embedding $D_{G \ominus \mathcal{G}}$ of $G \ominus \mathcal{G}$. For each defined G_i, to embed D_{G_i} into $D_{G \ominus \mathcal{G}}$, define f_i as the face of $D_{G \ominus \mathcal{G}}$ that contains e_i and e_{i+1} in its boundary. We can then embed an edge t from a to the highest vertex in f_i and embed D_{G_i} along t. Remove t again. If an edge ab was inserted previously remove it as well. Then, we have found an embedding of G with preserved rotations.

<u>Case 3:</u> If b is even and a is odd we can argue analogously to to the case 'if a is even and b is odd'.

<u>Case 4:</u> If both a and b are even, we can argue similarly to the procedure used for a in the previous case. If the edges in E_a are not consecutive in the rotation at a, define subgraphs G_i^a analogous to G_i in the previous case where necessary. If the edges in E_b are not consecutive in the rotation at b, define G_i^b for b analogous to G_i^a for a. As done previously, add ab, if necessary, get the embeddings of the subgraphs $G \ominus H \ominus \mathcal{G}$, $H \oplus \{a, b\}$ as well as embeddings for all $G_i^a \oplus \{a\}$ and all $G_i^b \oplus \{b\}$. As before, to get an embedding for G glue the embeddings at a and b taking into account the original rotation at those vertices. If necessary remove ab again. □

Lemma 5.4 (Analogous to Lemma 4.5)**.** *Suppose that G is smallest counterexample to Theorem 5.1. If G has a cut-vertex a and $G \ominus \{a\}$ has a component H that lies completely above a, then*

a) H contains only a single vertex b.

b) G has no edge ac such that c lies above b.

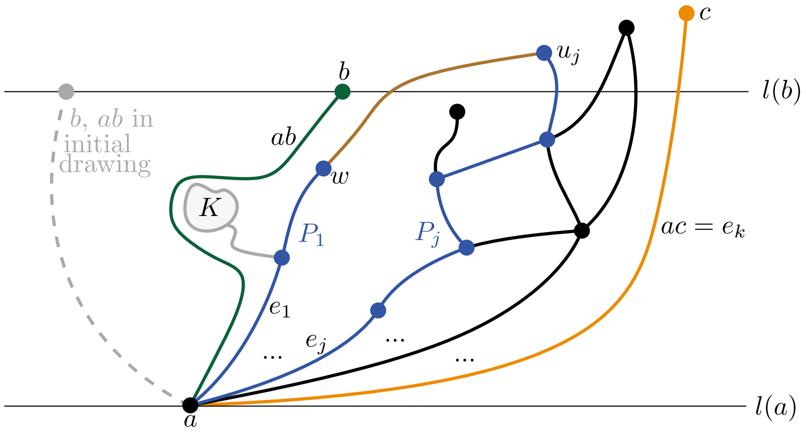


Figure 5.4: Part of the embedding of G with vertex a, Paths P_1 and P_j as well as edges ac and wu_j.

c) G has no connected subgraph H' such that

1. H' fully lies in between the vertices a and b.

2. H' has neighbouring vertex a.

3. all other neighbouring vertices of H' lie above b.

Proof. Let G be a smallest counterexample to Theorem 4.1 with a drawing such that every pair of independent edges crosses an even number of times and suppose such a graph H exists. We expand on the proof of Lemma 4.5.

a) Since Lemma 4.4 holds for the Unified Hanani-Tutte Theorem as well (Lem. 5.3), we can argue analogously to the proof of the corresponding lemma in Section 4.2 (Lem. 4.5 a)).

b) The basic idea is to obtain an embedding of $G \ominus \{b\}$ by minimality of G and then find a way to embed b and ab correctly. Consider a possible edge ac in G with $l(b) < l(c)$. If there are more such edges choose the one with the least amount of edges incident to a in between ac and ab in the rotation at a. We assume that ac lies to the right of ab. Then, we distinguish for a whether it is even or odd.

Case 1: If a is odd, we do not need to pay further attention to the rotation at a and therefore we can argue analogously to the proof for the corresponding lemma in Section 4.2 (Lem. 4.5 b)).

Case 2: If a is even, we first embed $G \ominus \{b\}$. By minimality, $G \ominus \{b\}$ has an embedding $D_{G\ominus\{b\}}$ with preserved rotation. If ac is consecutive to ab in the rotation at a, to obtain an embedding of G, we can embed b and ab in $D_{G\ominus\{b\}}$ along but left of ac. Else, ac is not consecutive to ab at a. Then, we construct a path in $D_{G\ominus\{b\}}$ along which we can embed ab as follows (Figure 5.4). In the initial drawing, let $e_1, ..., e_k$ be the edges between ab and ac in the clockwise rotation at a such that e_1 is consecutive to ab and $e_k = ac$ and let $v_1, \dots, v_k$ be their upper end-vertices.

In $D_{G\ominus\{b\}}$, if there are one or more paths from a through e_1 to a vertex v that lies above b, define the leftmost of those as P_1. Then we can embed ab and b just to the left of P_1 and add end-vertex b at level $l(b)$. If ab is crossing-free we are done. Else, by construction of P_1, for every edge $e_u = ux$ crossing ab, u lies in P_1 and x lies to the left of P_1. Since we chose the leftmost of all suitable paths, the highest vertex reachable through a path from u through e_u still lies below b. Next, we take a look at only those vertices that lie between $l(a)$ and $l(b)$ (both excluded). In the original drawing, ab separates those vertices into two vertex-sets V_l and V_r—depending on whether they lie left or right of ab in the initial drawing. No vertex u_l in V_l is adjacent to any vertex u_r in V_r since every edge $u_l u_r$ connecting u_l and u_r would need to cross ab oddly. Now, since a is an even vertex and the rotation at a is preserved in $D_{G \ominus \{b\}}$, all e_i lie right of ab in the original drawing as well as in $D_{G \ominus \{b\}}$. As such all v_i with $l(v_i) \leq l(b)$ lie in V_r. Consequently any vertex along a path in between $l(a)$ and $l(b)$ (both excluded) that starts at v_i only contain vertices in V_r, including u and any paths containing e_u. We return to considering $D_{G \ominus \{b\}}$. By definition of P_1 as leftmost possible path, it overall follows that e_u connects a subgraph K—defined as the component of $G \ominus \{u\}$ containing x—to P_1. To remove the crossing of ab and e_u, we can lay ab around K without introducing additional crossings. Then, ab is bounded by $l(a)$ and $l(b)$ and we can redraw ab as a monotone edge without changing the crossings in the drawing. We have then obtain an embedding of G such that the rotation is preserved.

Else, there is no path from a through e_1 to a vertex v above b. Then define P_1, similarly to the previous paragraph, as the leftmost of all possible paths that start at a, continue through e_1 and end at the highest vertex reachable through such a path. We then want to find a way to continue P_1 to reach a vertex above b without introducing additional crossings. Hence, let j be the lowest index such that there is a path from a through e_j to a vertex above b. Define the lowest such vertex as u_j and define P_j to be the leftmost path from a through e_j to u_j. Since $e_k = ac$ and if $j = k$ then $u_j = c$ and $P_j = \{ac\}$, there is always at least one such such index j. Next, let w be the uppermost vertex on P_1. Note that no edges cross each other and w is connected to u_j through a. Therefore, there is a path from w to u_j that is bounded by $l(a)$ and $l(u_j)$ along which we can embed an edge wu_j. We can then draw a monotone edge from w to u_j. As such, we can continue P_1 by adding an edge wu_j in a way that does not add any crossing. Define the resulting path as P_1'. As argued before, we can draw ab crossing-free along P_1' and add end-vertex b at level $l(b)$. As such, we have found an embedding of G with preserved rotation.

Overall, if G contains an edge ac and $l(b) < l(c)$ we can always find a way to embed G such that the rotation is preserved.

c) Consider such a H'. By minimality of G, we can obtain an embedding $D_{G \ominus \{b\}}$ of $G \ominus \{b\}$. Analogous to the proof of the corresponding lemma in Section 4.2 (Lem. 4.5 c)), in $D_{G \ominus \{b\}}$, do the following. Define au to be the edge from a to H' leftmost at a and define f to be the face to the left of au containing au in its boundary (Figure 4.5). We know that there is a path P that goes through H' from a to a vertex c lying above b that is defined by walking along the boundary of f.

If a is odd we can embed ab along P, analogous to the proof for Lemma 4.5 c).

If a is even and ab consecutive to au at a in the initial drawing, we can embed ab into $D_{G \ominus \{b\}}$ in the same was as if a was odd.

If a is even and ab is not consecutive to au in the initial drawing, let e_1 be the consecutive edge to the left of ab in the rotation at a. Now, using the same method as in the proof for Lemma 5.4 b), either there is a path P_1 from a through e_1 to a vertex above b or we can construct such a path P_1'. Instead of considering an edge ac we can consider a path from a to c. Then, we can embed ab into $D_{G \ominus \{b\}}$ along either P_1 or P_1'.

Altogether, we have again found an embedding for G.

□

Lemma 5.5 (Extension of Lemma 5.4, analogous to Lemma 4.6)**.** *If G has a cut-vertex a and $G \ominus \{a\}$ has a component H that lies completely above a, then*

a) H contains only a single vertex b.

b) G has no edge ac such that c lies above b.

c) G has no connected subgraph $\overline{H}$ such that

1. $\overline{H}$ is a component of $G \ominus \{a\}$.

2. $\overline{H}$ fully lies above vertex a.

Proof. Items a) and b) apply by Lemma 5.4. Item c) can be proven analogous to corresponding Lemma 4.6. □

5.2 Proof of the Unified Hanani-Tutte Theorem

Theorem 5.1 (Unified Hanani-Tutte for level-graphs)**.** *Let $G = (V, E)$ be a multi-level-graph with level-assignment $l : V \to \mathbb{N}$. If G has drawing such that every pair of independent edges crosses an even number of times, then G has an embedding such that the rotation at all even vertices is preserved.*

The main portion of the proof of the Unified Hanani-Tutte Theorem is matches the proof of the Strong Hanani-Tutte Theorem (Section 4.4). Though, since we consider multi-graphs, we need to ensure Section 4.4 applies for multi-graphs as well. The proof starts with a smallest counterexample G for Theorem 5.1 and a drawing of G with a minimal number of odd crossings such that all independent edges cross evenly. If there are no odd crossings left, the Weak Hanani-Tutte Theorem for level-graphs finishes the proof. Else, this results in the situation described in Section 4.3: There are three edges e_1, e_2, e_3 that have a common lower end-vertex v_0 such that e_3 lies in between e_1 and e_2. The edges e_1 and e_2 cross oddly and e_3 crosses both e_1 and e_2 evenly. In order to be able to apply Lemma 4.7 and Lemma 4.8, the upper end-vertices of e_1, e_2 and e_3 must be distinct. Then, the proof is analogous to Section 4.4. Hence:

Lemma 5.6. *Let G be a smallest counterexample to Theorem 5.1 with a drawing such that the number of odd crossings is minimal. If there is an odd vertex v_0 left, then there are three incident edges e_1, e_2 and e_3 such that e_3 lies in between e_1 and e_2 in the rotation at v_0, e_1 and e_2 cross oddly and e_3 crosses both e_1 and e_2 evenly. Then the other end-vertices of e_1, e_2 and e_3—v_1, v_2 and v_3—are distinct.*

Proof. Let G be a smallest counterexample to Theorem 4.1 with a drawing such that every pair of independent edges crosses an even number of times and the number of pairs of edges crossing oddly is minimal. Further let v_0 be an odd vertex. Then there is an incident pair of odd-crossing edges e_1 and e_2. By minimality of the number of odd crossings, e_1 and e_2 are not consecutive in the rotation at v_0. Therefore, there is at least one incident edge e_3 that lies in between e_1 and e_2 and crosses both e_1 and e_2 evenly (Figure 4.9). We can assume that v_0 is the lower-end vertex of e_1, e_2 and e_3. We show that $v_1 = v_2, v_1 \neq v_3$ as well as $v_1 = v_3, v_1 \neq v_2$ as well as $v_2 = v_3, v_2 \neq v_1$ and finally $v_1 = v_2 = v_3$ always lead to a contradiction.

<u>Case 1:</u> $v_1 = v_2$, $v_1 \neq v_3$ (Figure 5.5)

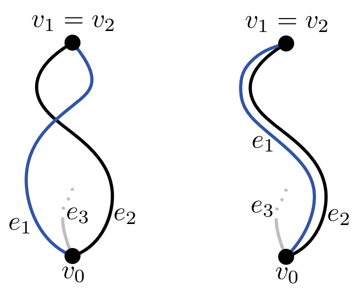


Figure 5.5: The upper end-vertices of e_1 and e_2—v_1 and v_2—are the same. We can embed e_1 along e_2 and reduce the overall number of odd crossings.

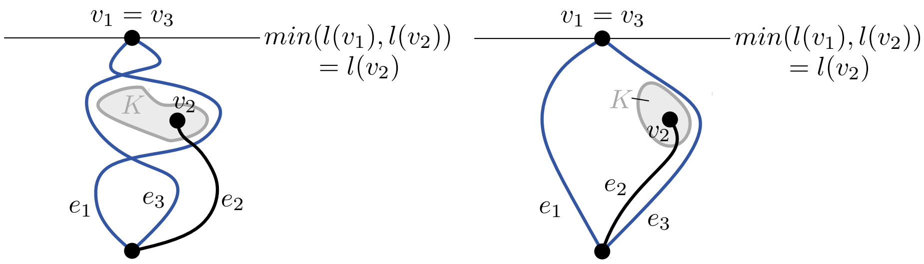


Figure 5.6: The upper end-vertices of e_1 and e_3—v_1 and v_3—are the same. The end-vertex v_2 of e_2 lies below $v_1 = v_3$. Every vertex inside of cycle $C = \{e_1, e_2\}$ is adjacent only to vertices inside of C, to v_0 or to v_1. Subgraphs $G \ominus K$ and $K \oplus \{v_0\}$ can be embedded separately and glued back together to get an embedding of G.

Then, $v_1 = v_2$ is an odd vertex and the rotation at v_1 and v_0 can be changed arbitrarily. Therefore, we can resolve the odd crossing by removing one of e_1, e_2 and embedding it along the other edge. By taking the edge e_i with the higher number of odd crossings and embedding it along $e_j, j \neq i$ we can obtain a drawing of G where the overall number of odd crossings is decreased, contradicting the minimality of the number of odd crossings in the initial drawing.

<u>Case 2:</u> $v_1 = v_3$, $v_1 \neq v_2$

We know that e_3 starts to the right of e_1 and to the left of e_2 at v_0. Since e_3 crosses e_2 evenly, e_3 lies to the left of e_2 at level $\min(l(v_2), l(v_3))$. By definition, e_2 starts to the right of e_1. Since e_2 crosses e_1 oddly, e_2 lies to the right of e_1 at $\min(l(v_1), l(v_2)) = \min(l(v_2), l(v_3))$.

If $\min(l(v_2), l(v_3)) = l(v_3)$, since $v_1 = v_3$, e_2 lies to the right of e_1 and to the left of e_3 at level $l(v_3)$, contradicting the fact that on level $l(v_3)$ e_1 and e_3 have the same coordinate $(x(v_3), y(v_3))$.

Else $\min(l(v_2), l(v_3)) = l(v_2)$. Then, at level $l(v_2)$, since e_2 crosses e_1 oddly and e_3 evenly, v_2 lies in between e_1 and e_3. Since e_1 and e_3 start with e_3 lying to the right of e_1, but e_2 must cross e_1 oddly and e_3 evenly, e_1 and e_3 cross oddly in between $l(v_0)$ and $l(v_2)$. Since overall e_1 and e_3 cross evenly, they need to cross oddly again between $l(v_2)$ and $l(v_1)$. Therefore, v_2 lies in an area limited by cycle $C = \{e_1, e_3\}$ and as such inside C. Note that for every edge c of C and every edge $h = uv$ that is not incident to C and crosses c evenly it follows that u lies inside of C if, and only if, v lies inside of C. As such, any path containing v_2 either contains only vertices inside of C or it contains a vertex in C, meaning vertex v_0 or v_1. Therefore, each components of $G \ominus \{v_0, v_1\}$ either consists exclusively of vertices on the inside of C or it consists exclusively of vertices on the outside of C. We define the union of the components lying inside C as subgraph K. By previous argument, K consists exclusively of vertices that lie inside of C. Note that K can only be adjacent to v_1 if v_1 is odd. By minimality of G, subgraphs $G \ominus K$ and $K \oplus \{v_0, v_1\}$ have an embeddings with preserved rotations, $D_{G\ominus K}$ and D_K respectively. We can then obtain an embedding of G by flattening D_K as needed and inserting it into $D_{G\ominus K}$ along either e_1 or e_3 (Figure 5.6).

<u>Case 3:</u> $v_2 = v_3$, $v_1 \neq v_3$

We arrive at a contradiction analogous to the previous case.

<u>Case 4:</u> $v_1 = v_2 = v_3$

In this case e_1 lies to the left of e_2 and e_3 lies left of e_2 and right of e_1 in the rotation at v_0. Since e_1 and e_2 cross oddly and e_3 crosses both e_1 and e_2 evenly, regarding the rotation at $v_1 = v_2 = v_3$, e_3 must still lie to the left of e_2 and to the right of e_1, but e_2 must lie to the left of e_1, which in turn demands that e_2 lies to the left of e_1, a contradiction. $\square$

6. Conclusion

In this thesis, we considered the different variants of the Hanani-Tutte Theorem with a special emphasis on the Unified Hanani-Tutte Theorem. First, we took a look at how the unified Hanani-Tutte Theorem can be proven in the plane. Then, we considered level-planarity and presented a proof of the Strong Hanani-Tutte Theorem for level-graphs. Afterwards, we showed how to adapt this proof to the Unified Hanani-Tutte Theorem for level-graphs. Overall,the main result of this work.

Open Questions

The following questions immediately follow from our work: Is it possible…

- … to use a different approach to find an easier and/or more direct proof for the Unified Hanani-Tutte Theorem for level-graphs?
- … to modify the proof of Theorem 5.1 (Chapter 5) to include radial level-planarity?

As mentioned in the Introduction, there are still many open Hanani-Tutte problems (table in Figure 1.1), especially on non-orientable surfaces. There are other interesting questions regarding the variants of the Hanani-Tutte Theorem: Is there any type of drawing where…

- … the weak Hanani-Tutte Theorem does not hold?
- … either $ocr = 0$ is equivalent to $cr = 0$ but the rotation system cannot generally be preserved?
- … —in addition to orientable surfaces of genus at least 4—the Strong Hanani-Tutte Theorem does not hold?

Bibliography

[1] T. A. Apostol. *Mathematical analysis.* Reading, Mass. Addison-Wesley, 2nd edition, 1974. ISBN 0201002884.

[2] C. Bachmaier, F.-J. Brandenburg, and M. Forster. Radial level planarity testing and embedding in linear time. *Journal of Graph Algorithms and Applications*, 9(1):53–97, 2005. doi: 10.7155/jgaa.00100.

[3] G. Brückner, I. Rutter, and P. Stumpf. Level planarity: Transitivity vs. even crossings. *Graph Drawing and Network Visualization*, 11282:39–52, 2018. doi: 10.1007/978-3-030-04414-5_3.

[4] G. Cairns and Y. Nikolayevsky. Bounds for generalized thrackles. *Discrete & Computational Geometry*, 23(2):191–206, 2000. doi: 10.1007/PL00009495.

[5] C. Chojnacki. Über wesentlich unplättbare kurven im dreidimensionalen raume. *Fundamenta Mathematicae*, 23(1):135–142, 1934. doi: 10.4064/fm-23-1-135-142.

[6] É. C. de Verdière, V. Kaluza, P. Paták, Z. Patáková, and M. Tancer. A direct proof of the strong hanani-tutte theorem on the projective plane. *Graph Drawing and Network Visualization*, 9801:454–467, 2016. doi: 10.1007/978-3-319-50106-2_35.

[7] R. Diestel. *Graph Theory.* Springer Berlin / Heidelberg, Berlin, Heidelberg, 5th edition, 2017. ISBN 3662536218. doi: 10.1007/978-3-662-53622-3.

[8] R. Fulek and J. Kynčl. Counterexample to an extension of the hanani-tutte theorem on the surface of genus 4. *Combinatorica*, 39:1267–1279, 2019. doi: 10.1007/s00493-019-3905-7.

[9] R. Fulek, M. J. Pelsmajer, M. Schaefer, and D. Štefankovič. Hanani-tutte, monotone drawings, and level-planarity. In J. Pach, editor, *Thirty Essays on Geometric Graph Theory*, pages 263–287. Springer New York, New York, 2013. ISBN 978-1-4614-0110-0. doi: 10.1007/978-1-4614-0110-0_14.

[10] R. Fulek, M. J. Pelsmajer, and M. Schaefer. Hanani-tutte for radial planarity. *Graph Drawing and Network Visualization*, 9411:99–110, 2015. doi: 10.1007/978-3-319-27261-0_9.

[11] R. Fulek, M. J. Pelsmajer, and M. Schaefer. Hanani-tutte for radial planarity ii. *Graph Drawing and Network Visualization*, 9801:468–481, 2016. doi: 10.1007/978-3-319-50106-2_36.

[12] R. Fulek, J. Kynčl, and D. Pálvölgyi. Unified hanani–tutte theorem. *The Electronic Journal of Combinatorics*, 24(3), July 2017. doi: 10.37236/6663.

[13] R. Fulek, M. J. Pelsmajer, and M. Schaefer. Strong Hanani-Tutte for the Torus. In K. Buchin and E. Colin de Verdière, editors, *37th International Symposium on Computational Geometry (SoCG 2021)*, volume 189 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 38:1–38:15, Dagstuhl, Germany, 2021. Schloss Dagstuhl

– Leibniz-Zentrum für Informatik. ISBN 978-3-95977-184-9. doi: 10.4230/LIPIcs.SoCG.2021.38.

[14] J. Pach and G. Tóth. Monotone drawings of planar graphs. *Journal of graph theory*, 46(1):39–47, 2004–05. doi: 10.1002/jgt.10168.

[15] J. Pach and G. Tóth. Which crossing number is it anyway? *Journal of Combinatorial Theory, Series B*, 80(2):225–246, 2000. ISSN 0095-8956. doi: https://doi.org/10.1006/jctb.2000.1978.

[16] M. J. Pelsmajer, M. Schaefer, and D. Štefankovič. Removing even crossings. *Journal of combinatorial theory. Series B*, 97(4):489–500, 2007. doi: 10.46298/dmtcs.3430.

[17] M. J. Pelsmajer, M. Schaefer, and D. Štefankovič. Removing even crossings on surfaces. *Electronic notes in discrete mathematics*, 29:85–90, 2007. doi: 10.1007/978-3-319-50106-2_36.

[18] M. J. Pelsmajer, M. Schaefer, and D. Stasi. Strong hanani-tutte on the projective plane. *SIAM journal on discrete mathematics*, 23(3):1317–1323, 2009. doi: 10.1137/08072485X.

[19] B. Randerath, E. Speckenmeyer, E. Boros, P. L. Hammer, A. Kogan, K. Makino, B. Simeone, and O. Cepek. A satisfiability formulation of problems on level graphs. *Electronic notes in discrete mathematics*, 9:269–277, 2001. doi: 10.1016/S1571-0653(04)00327-0.

[20] M. Schaefer. Toward a theory of planarity: Hanani-tutte and planarity variants. *Journal of graph algorithms and applications*, 17(4):367–440, 2013. doi: 10.7155/jgaa.00298.

[21] W. T. Tutte. Toward a theory of crossing numbers. *Journal of Combinatorial Theory, Series A*, 8(1):45–53, 1970. doi: 10.1016/S0021-9800(70)80007-2.