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Elliott-Morse measures and Kakutani's dichotomy theorem[1]

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Introduction

In 1963, the late Elliott and the late Morse published a paper [4] in which several related product measures were constructed for an arbitrary family of measures on an arbitrarily large set of measurable spaces. We will call measures of this type *Elliott–Morse measures* (Definition 1.5). This paper was written in the explicit but terse and uncompromising style, more easily accessible by a computer than by a human mind, that has become known as "morse code." Few people have apparently read it. This is a pity, as the paper contains a wealth of information.

By contrast, Kakutani's [11] famous dichotomy for product probabilities has a long history. It was specialized to measures on product groups (Hora [10]) and Gaussian measures on Hilbert space with identical correlation operators (Grenander [6], cf. also Skorohod [17]); it was extended to Gaussian measures with nonidentical correlation operators (Segal [16], Hajek [7], Feldman [5], Rozanov [15]), to product states on W^*–algebras (Bures [3]), to σ–finite "restricted" product measures (Hill [9], cf. also Yamasaki [18]), and to certain nonproduct measures (Ritter [13, 14]).

The question naturally arises whether a similar purity law as Kakutani's dichotomy can be obtained for a product probability with respect to Elliott–Morse measures. We show in this paper that the answer is affirmative in the case of the two Elliott–Morse measures considered here. There is, however, a major difference between the classical and the present case: In general, Elliott–Morse measures are not σ–finite and in the decomposition theorem of a probability with respect to a general measure, besides a singular part and a part with a density function, a third part appears on the scene which we will call (*relatively*) *diffuse* (Definition 2.2). Correspondingly, our main theorems say that a product probability may be either diffuse or singular, or it may possess a density function with respect to the two Elliott–Morse measures considered in this paper. (Theorems 3.3 and 3.8). As an application of this *trichotomy*, we are able completely to classify the Elliott–Morse measures according to their finiteness, σ–finiteness, and non–σ–finiteness (Theorem 3.10).

We conclude the paper with a number of applications; unfortunately, these exhibit the fact that in many interesting situations the diffuse and singular cases have the edge over the density case, thus limiting the practical applicability of Elliott–Morse measures to a high degree. We also discuss Elliott–Morse measures on Hilbert space in the light of Oxtoby's [12] theory of translation–invariant Borel measures on Polish groups.

1 Elliott–Morse measures

1.1. Elliott and Morse [4] developed several methods to define arbitrary Cartesian products of arbitrary (nonnegative) measures, thus extending the classical notion of a product probability measure to arbitrary measures. We will now sketch two of their constructions. Central to their method is the notion of a special product, the plus–product $\prod^+$ of a countable family of nonnegative real numbers.

[1]Dedicated to the memory of John Oxtoby

1.2. Definition. Let I be an at most countable index set and let $(a_n)_{n\in I}$ be a family of extended real numbers $a_n \in [0,\infty]$. Put $J := \{k \in I \mid a_k \le 1\}$. Thus the (finite or infinite) products $\prod_{k\in J} a_k$ and $\prod_{k\notin J} a_k$ are well defined. Define

$$\prod_{k\in I}^{+} a_k := (\prod_{k\in J} a_k)(\prod_{k\notin J} a_k).$$

Here we agree that $0\cdot\infty = \infty\cdot 0 = 0$ (and, of course, $a\cdot\infty = \infty\cdot a = \infty$ if $a > 0$) and that the empty product is 1. In contrast to the usual classical product, the plus–product attributes a value to *any* sequence $(a_k)_{k\in I}$ of extended real factors ≥ 0; manipulations with the plus–product are easy enough to make it a handy instrument in measure theory. We now compile some of its properties needed in the sequel.

1.3. Properties of the plus–product $\prod^{+}$.

(i) For a finite index set I, the plus–product $\prod^{+}$ coincides with the usual product.

(ii) If $a_k = 0$ for one k, then $\prod^{+} a_k = 0$.

(iii) If $b_k \ge a_k$ for all k, then $\prod^{+} b_k \ge \prod^{+} a_k$.

(iv) $\prod^{+} \sqrt{a_k} = \sqrt{\prod^{+} a_k}$.

(v) If $(I_\alpha)_{\alpha\in A}$ is a decomposition of I, where A is a *finite* index set, then

$$\prod_{k\in I}^{+} a_k = \prod_{\alpha\in A}(\prod_{k\in I_\alpha}^{+} a_k).$$

(This property breaks down for *infinite* A as the example $I = A = \mathbb{N}$, $a_k = 2^{(-1)^k}$ and $I_\alpha = \{2\alpha-1, 2\alpha\}$ shows. In particular, it is generally *not* true that

$$\prod^{+}(a_k b_k) = (\prod^{+} a_k)(\prod^{+} b_k) \text{ or } (\prod^{+} a_k)(\prod^{+} 1/a_k) = 1.)$$

We, however, have

(vi) If $0 < a_k < \infty$ and the product $\prod a_k$ exists in the *classical* sense (as a positive real number), then $\prod^{+}(a_k b_k) = (\prod a_k)(\prod^{+} b_k)$.

Now let $(X_k, \mathcal{A}_k, \eta_k)_{k\in I}$ be an at most countable family of measure spaces. (In this paper "measure" means an arbitrary σ–additive function on a σ–algebra with values in $[0,\infty]$.) Following Elliott and Morse, loc.cit., p. 253, we define

1.4. Definition. (a) A *box* for $(X_k, \mathcal{A}_k)_{k\in I}$ is a Cartesian product $\prod_{k\in I} B_k$ such that $B_k \in \mathcal{A}_k$.
(b) The *volume* (with respect to (η_k)) of a box $B = \prod_{k\in I} B_k$ is the extended real number

$$\mathrm{vlm}(B) := \prod^{+} \eta_k(B_k).$$

(c) A box B is *basic* (for the family (η_k)) if it has finite volume; it is *null* if its volume vanishes.

Note that an at most countable intersection of boxes is again a box, and it is basic or null if at least one of these boxes is basic or null, respectively. If a box B is basic and nonnull, then the volume of B is the *classical* product $\prod \eta_k(B_k)$. A box is null if and only if the subproduct $\prod_{\eta_k(B_k)<1}^{+} \eta_k(B_k)$ vanishes. Let $\mathcal{A}$ be the product of the σ–algebras $\mathcal{A}_k$ on the product space $X = \prod_{k\in I} X_k$.

1.5. Definition. We call *Elliott–Morse measure* (for the family (η_k)) any measure η on the measurable space $(X, \mathcal{A})$ such that

$$\eta(B) = \mathrm{vlm}(B)$$

for all (η_k)–basic boxes B.

In general, Elliott–Morse measures are not uniquely specified by their restrictions to all basic boxes. In their paper [4], Elliott and Morse constructed three outer measures, denoted by cpm, cnm, and prm, whose restrictions to $\mathcal{A}$ share the property of Definition 1.5. We will deal with their first two measures only and we now sketch their constructions.

1.6. First Elliott–Morse measure, $\eta^{(1)}$. The first outer measure, cpm, is the Hausdorff outer measure generated by the system of (η_k)–basic boxes and their volumes, i.e., for $Q \subseteq X$

$$\mathrm{cpm}(Q) := \inf \sum_{n=1}^{\infty} \mathrm{vlm}(B_n),$$

where the infimum is extended over all at most countable sequences $(B^{(n)})$ of (η_k)–basic boxes sucht that $Q \subseteq \cup B^{(n)}$ (cf. [4],6.1.8). (If there is no such covering of Q, the infimum is defined as ∞). By [4], Theorem 6.23, $\mathrm{cpm}(B) = \mathrm{vlm}(B)$ for all (η_k)–basic boxes $B \subseteq X$. Therefore, Part 1 of the proof of [4], Theorem 6.7 shows that all one–dimensional cylinders $A \times \prod_{k \neq i} X_k$, $A \in \mathcal{A}_i$, $i \in I$ are cpm–measurable. Since the system of cpm–measurable subsets of X is a σ–algebra, the same holds for all product measurable sets. Define $\eta^{(1)}$ as the restriction of cpm to the σ–algebra $\mathcal{A}$ of product measurable subsets of X. $\eta^{(1)}$ is an Elliott–Morse measure in the sense of Definition 1.5. We will call it the *first Elliott–Morse measure* generated by the family (η_k).

The measure $\eta^{(1)}$ may possess many measurable subsets of X which are trivial in the sense that they contain no subsets of positive, finite $\eta^{(1)}$–measure (cf. Example 1.8 in combination with Lemma 1.11 below). This defect suggests also looking at the *essential measure* associated with $\eta^{(1)}$. We will, however, not follow this idea, but will content ourselves with considering a related construction, the second Elliott–Morse measure.

Elliott and Morse [4] point out (cf., e.g., their introduction) that, in the case of an *uncountable* index set I (which we do not consider here), cpm suffers from another, more serious defect: Fubini's theorem does not hold for *integrable functions* (cf. Bledsoe and Morse [2]) and binary splitting of the product space in general. In the case of a *countable* index set, this defect is not present as the interested reader may verify. Befor stating Example 1.8, we fix some notations.

1.7. Notations and terminology. For $J \subseteq I$ we put $X_J = \prod_{k \in J} X_k$, we denote the product–$\sigma$–algebra on X_J by $\mathcal{A}_J$, and we denote the first Elliott–Morse measure for the subfamily $(\eta_k)_{k \in J}$ by $\eta_J^{(1)}$. A *nilcylinder* N for $(\eta_k)_{k \in I}$ is a subset of X of the form $N = N' \times X_{CJ}$, where $J \subseteq I$, $N' \in \mathcal{A}_J$, and $\eta_J^{(1)}(N') = 0$ (cf. [4], 6.1.12 and 5.1.5).

1.8. Example. There exists a nilcylinder N such that $\eta^{(1)}(N) = \infty$. In order to construct such a nilcylinder we use the index set $I = \mathbb{N}$. For $k \geq 3$, let $X_k = \{0, 1\}$ be endowed with the finest σ–algebra and let $\eta_k = \#$ be the counting measure. Furthermore, let $X_1 = X_2$ be the real interval $[0, 1]$ endowed with the Borel σ–algebra $\mathfrak{B}$, and let $\eta_1 = \eta_2$ be Lebesgue measure λ on $\mathfrak{B}$. It turns out that the nilcylinder $N = D \times \prod_{k \geq 3} X_k \subseteq X$, where D is the diagonal in $[0, 1] \times [0, 1]$, cannot be covered by an at most countable family of basic boxes and, thus, has infinite $\eta^{(1)}$–measure.

1.9. Second Elliott–Morse measure, $\eta^{(2)}$. For a countable product Elliott and Morse's second outer measure, cnm, is defined starting from cpm by punching out the nilcylinders;

$$\mathrm{cnm}(Q) := \inf_{C} \mathrm{cpm}(Q \setminus C) \quad (Q \subseteq X), \tag{1}$$

where C runs through all countable unions of (η_k)–nilcylinders (the interested reader may want to combine ([4] 6.15.11, 6.15.10, 6.4.5, 6.15.7, 3.1.15, 3.1.14, and 2.1.11)). (In [4] besides the nilcylinders, the so–called nilsets are also punched out; in the case of a countable index set I, any nilset is a countable union of boxes of volume zero and does not have to be taken into consideration). Define the *second Elliott–Morse measure* $\eta^{(2)}$, generated by the family (η_k) as the restriction of cnm to the σ–algebra $\mathcal{A}$ of all product measurable subsets of X.

1.10. Remarks. (a) For any (η_k)–basic box B, we have

$$\eta^{(2)}(B) = \mathrm{vlm}(B);$$

moreover, any set $A \in \mathcal{A}$ is cnm–measurable. In particular, $\eta^{(2)}$ is an Elliott–Morse measure in the sense of Definition 1.5.

(b) For the second Elliott–Morse measure $\eta^{(2)}$ the Fubini theorem holds for all $\eta^{(2)}$–integrable functions and any binary splitting of the product space. The interested reader will find proofs of the two foregoing remarks by combining ([4] 6.25, 6.15.11, the remark preceding Theorem 6.5, and Part 1 of the proof of Theorem 6.7).

(c) Fubini's theorem does not hold for infinite splitting in general: Consider $X_k = \{1, 2, 3, 4\}$, $\eta_k = \frac{1}{2}\#$, and $B := \{1\} \times X_2 \times \{1\} \times X_4 \times \ldots$. Then B is a basic box of volume zero, but $\mathrm{cnm}(\{1\} \times X_k) = 1$ for all k.

We will need the following sharpening of 1.10(a) in Sect. 3; it is also interesting by itself.

1.11. Lemma. *Let $A \in \mathcal{A}$ have finite $\eta^{(1)}$–measure. Then $\eta^{(2)}(A) = \eta^{(1)}(A)$.*

Proof. We first show that the claim is true if $A = \bigcup_{k=1}^{n} B^{(n)}$ is a finite union of basic boxes. Let N be a countable union of nilcylinders. Putting for abbreviation $C^{(n)} := B^{(n)} \setminus N$, the sieve formula of Sylvester–Poincaré allows us to write

$$\eta^{(1)}\left(\bigcup_{k=1}^{n}(B^{(n)} \setminus N)\right) = -\sum_{j=1}^{n}(-1)^j \sum \eta^{(1)}(C_F), \tag{2}$$

where the inner sum is extended over all subsets $F \subseteq [1, n]$ with j elements and $C_F = \bigcap_{k\in F} C^{(k)}$. But $C_F = B_F \setminus N$, where $B_F = \bigcap_{k\in F} B^{(k)}$ is a basic box. Hence, by 1.10(a), we have

$$\eta^{(1)}(B_F) \geq \eta^{(1)}(C_F) \geq \eta^{(2)}(B_F) = \mathrm{vlm}(B_F) = \eta^{(1)}(B_F).$$

Therefore, we obtain from (2) by another application of the sieve formula

$$\eta^{(1)}\left(\bigcup_{k=1}^{n}(B^{(n)} \setminus N)\right) = -\sum_{j=1}^{n}(-1)^j \sum \eta^{(1)}(B_F) = \eta^{(1)}\left(\bigcup_{k=1}^{n} B^{(n)}\right). \tag{3}$$

The equality (3) implies $\eta^{(2)}(A) = \eta^{(1)}(A)$ if A is a finite union of basic boxes. By σ–continuity, the same is true if A is a countable union of basic boxes.

If A has finite first Elliott–Morse measure, then there is an at most countable family $(B^{(n)})$ of basic boxes such that $A \subseteq \cup B^{(n)}$ and $\eta^{(1)}(A) \geq \sum_n \mathrm{vlm}B^{(n)} - \varepsilon$. Therefore, we have

$$\begin{aligned}\eta^{(2)}(\cup B^{(n)} \setminus A) \leq& \eta^{(1)}(\cup B^{(n)} \setminus A) = \eta^{(1)}(\cup B^{(n)}) - \eta^{(1)}(A)\\ \leq& \sum_n \mathrm{vlm}B^{(n)} - \eta^{(1)}(A) \leq \varepsilon.\end{aligned}$$

Using this estimate and the first part of this proof, we finally obtain

$$\eta^{(2)}(A) = \eta^{(2)}(\cup B^{(n)}) - \eta^{(2)}(\cup B^{(n)} \setminus A) \geq \eta^{(1)}(\cup B^{(n)}) - \varepsilon \geq \eta^{(1)}(A) - \varepsilon.$$

The lemma now follows since ε was arbitrary.

2 A decomposition theorem

2.1. Explanation. Let μ be a probability measure and let η be an arbitrary (nonnegative) measure on a measurable space $(X, \mathcal{A})$. μ and η are called *mutually singular* if there is a measurable subset $A \subseteq X$ such that $\mu(A) = 1$ and $\eta(A) = 0$. An $\mathcal{A}$–measurable function $f : X \to \mathbb{R}_+$ is called a *density function* of μ with respect to η if

$$\mu(A) = \int_A f d\eta$$

for all measurable subsets $A \subseteq X$. As is usual, we write $\mu = f\eta$ in this case. If η is σ–finite, then by Radon–Nikodym's theorem, the existence of a density function is equivalent with *absolute continuity* of μ with respect to η, $\mu \ll \eta$, i.e., $\mu(A) = 0$ when $\eta(A) = 0$. If η is not σ–finite, then $\mu \ll \eta$ does not imply the existence of a density function; a standard example is provided by Lebesgue measure μ and the counting measure $\eta = \#$ on the real interval $[0, 1]$. In this case, the measure μ is not only absolutely continuous but even diffuse with respect to η in the sense of the following definition.

2.2. Definition. μ is *diffuse* with respect to $\eta(\mu \lll \eta)$ if $\mu(A) = 0$ whenever $\eta(A) < \infty$ (or, equivalently, when η is σ–finite on A).

This notion of diffuseness of μ w.r.t. η must not be mixed up with *diffuseness of a measure*, which means that no point carries μ–measure. The latter notion of diffuseness of a measure μ means diffuseness of μ w.r.t. counting measure in our sense.

There is the following sharpening of Lebesgue's classical decomposition theorem. It shows in particular that the absolutely continuous part in Lebesgue's theorem has a unique decomposition into a part with a density and a diffuse part. The proof is routine and we omit it.

2.3. Theorem. *The probability measure μ has a unique decomposition as the sum*

(i) $\mu = \mu_\perp + f\eta + \mu_d$,

where $\mu_\perp$ and η are mutually singular, f is a nonnegative and η–integrable function, and μ_d is diffuse with respect to η.

3 Trichotomy

3.1 Explanations. To avoid trivial discussions, we restrict matters to the case of a *countable* index set I from here on.

Let notation be as set up in Sect. 1 and suppose that for each $k \in I$ μ_k and ν_k are probability measures on the measurable space $(X_k, \mathcal{A}_k)$ such that $\mu_k = f_k \nu_k$ for some ν_k–integrable function $f_k \geq 0$ on X_k. Let $\mu = \otimes \mu_k$ and $\nu = \otimes \nu_k$ be the (uniquely defined) product probability measures of the sequences (μ_k) and (ν_k) on the product space $X = \prod X_k$. S. Kakutani [11] discovered the remarkable fact that, relative to Lebesgue's decomposition theorem, μ must be of pure type with respect to ν (cf. also Hewitt–Stromberg [8], p. 453, Theorem 22.36).

3.2. Dichotomy theorem (Kakutani [11]). (a) *Either*

(i) μ and ν *are mutually singular or*

(ii) $\mu = f\nu$ *for some ν–integrable function $f \geq 0$ on X.*

(b) (iii) *The first case occurs if and only if the infinite product*

(1) $$\prod_{k \in I} \int_{X_k} \sqrt{f_k} d\eta_k$$

vanishes.

(iv) *The second case occurs if and only if* (1) *is positive.*

(c) *In the second case, the density function f is obtained as an $\mathfrak{L}^1(\nu)$–limit of the net of finite partial products $(\prod_{k\in E} f_k)_{\#E<\infty}$.*

In the sequel, we will replace the basis measure ν by the first and second Elliott–Morse measures $\eta^{(1)}$ and $\eta^{(2)}$ and extend Kakutani's theorem to these measures. Thus let η_k be a nonnegative measure on the measurable space $(X_k, \mathcal{A}_k)$ such that $\mu_k = f_k\eta_k$ for some η_k–integrable function $f_k \geq 0$ on X_k. We first deal with the first Elliott–Morse measure $\eta^{(1)}$ generated by the sequence $(\eta_k)_k$ (cf. 1.6).

3.3. Trichotomy theorem (for the first Elliott–Morse measure). (a) *Exactly one of the following cases occurs.*

(i) *μ is diffuse with respect to $\eta^{(1)}$,*

(ii) *μ and $\eta^{(1)}$ are mutually singular,*

(iii) *$\mu = f\eta^{(1)}$ for some $\eta^{(1)}$–integrable function $f \geq 0$ on X.*

(b) (iv) *The first case occurs if and only if $\mu(B) = 0$ for all (η_k)–basic boxes B.*

(v) *The second case occurs if and only if there is a basic box $B = \prod_{k\in I} B_k$ for (η_k) such that $\mu(B) > 0$ and the infinite product*

$$\prod_{k\in I}^{+} \int_{B_k} \sqrt{f_k}\, d\eta_k \qquad (1)$$

vanishes.

(vi) *The third case occurs if and only if there is a basic box $B = \prod_{k\in I} B_k$ for (η_k) such that $\mu(B) > 0$ and the infinite product* (1) *is positive.*

Proof. For abbreviation we will write $\eta := \eta^{(1)}$; we also identify the index set with the set $\mathbb{N}$ of natural numbers in this proof. We proceed along the following selfexplanatory logic tree.

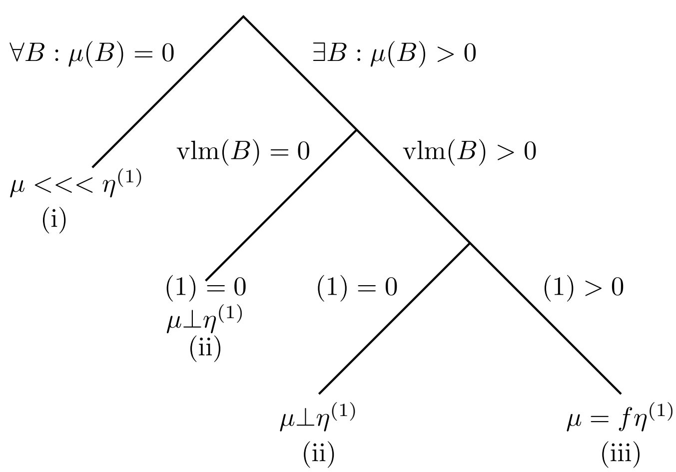

Figure 1:

Let us deal first with case (i) and suppose that $\mu(B) = 0$ for all (η_k)–basic boxes B. Let A be in $\mathcal{A}$ and finite for η. The set A can be covered by a countable family $(B^{(n)})_{n\in\mathbb{N}}$ of (η_k)–basic boxes. By our hypothesis, we have

$$\mu(A) \leq \mu(\bigcup_{n\in\mathbb{N}} B^{(n)}) \leq \sum_{n\in\mathbb{N}} \mu(B^{(n)}) = 0.$$

That is, μ is diffuse with respect to η.

Both cases (ii) and (iii) materialize when there is an (η_k)–basic box $B = \prod_{k\in\mathbb{N}} B_k$ for which $\eta(B)$ is positive. We suppose first that $\prod^{+} \eta_k(B_k) = \mathrm{vlm}(B) = 0$. Since $\prod_{k\in\mathbb{N}} \mu_k(B_k) = \mu(B) > 0$, we must have $\mu_k(B_k) > 0$ and (by absolute continuity) $\eta_k(B_k) > 0$ for all $k \in \mathbb{N}$, and so $\prod^+ \eta_k(B_k)$ converges to 0, but all finite partial products are positive; in particular $\prod^+_{k>n} \eta_k(B_k) = 0$ for all n. By an elementary property of infinite products, we have

$$\lim_{n\to\infty} \prod_{k=n}^{\infty} \mu_k(B_k) = 1. \tag{2}$$

Next define $B^{(n)}$ as $\prod_{k\leq n} X_k \times \prod_{k>n} B_k$. Plainly we have $B^{(n)} \subseteq B^{(n+1)}$ and, by (2), $\lim_{n\to\infty} \mu(B^{(n)}) = 1$. Note finally that

$$\mathrm{vlm}(B^{(n)}) = \prod_{k\leq n} \eta_k(X_k) \cdot \prod^+_{k>n} \eta_k(B_k).$$

Since $\prod^+_{k>n} \eta_k(B_k) = 0$ we have $\mathrm{vlm}(B^{(n)}) = 0$ for all n, and so $\mu(\bigcup_{n\in\mathbb{N}} B^{(n)}) = 1$ and $\eta(\bigcup_{n\in\mathbb{N}} B^{(n)}) = 0$. That is, μ and η are mutually singular. We also have by the Cauchy–Schwarz inequality and by 1.3

$$\prod_k^+ \int_{B_k} \sqrt{f_k} d\eta_k \leq \prod_k^+ \sqrt{\eta_k(B_k) \int_{B_k} f_k d\eta_k} \leq \prod_k^+ \sqrt{\eta_k(B_k)} = \sqrt{\mathrm{vlm}(B)} = 0.$$

Suppose next that $\mathrm{vlm}(B) > 0$: As B is a basic box, we have $\mathrm{vlm}(B) < \infty$ by definition. Consequently, we must have $0 < \eta_k(B_k) < \infty$ for all k. We define two probability measures on X. For $k \in \mathbb{N}$, let ν_k be the probability measure on X_k such that

$$\nu_k(A) = \frac{\mu_k(A \cap B_k)}{\mu_k(B_k)} \text{ for } A \in \mathcal{A}_k.$$

Let θ_k be defined similarly by

$$\theta_k(A) = \frac{\eta_k(A \cap B_k)}{\eta_k(B_k)}.$$

Let $\nu = \bigotimes_{k\in\mathbb{N}} \nu_k$ and $\theta = \bigotimes_{k\in\mathbb{N}} \theta_k$. Write $g_k = \frac{\eta_k(B_k)}{\mu_k(B_k)} f_k$, so that $\nu_k = g_k \theta_k$.

Kakutani's dichotomy theorem shows that either ν and θ are mutually singular, or $\nu = g\theta$, where g is the limit in $\mathfrak{L}^1(\theta)$ of the finite products $\prod_{k=1}^n g_k$. Singularity occurs if and only if the product

$$\prod^+_{k\in\mathbb{N}} \int_{B_k} \sqrt{g_k} d\theta_k = \prod_{k\in\mathbb{N}} \int_{B_k} \sqrt{g_k} d\theta_k \tag{3}$$

vanishes and absolute continuity if and only if it is positive. (Observe that all factors in (3) are less than or equal to (1).) The product (3) is equal to

$$\prod^+_{k\in\mathbb{N}} \frac{1}{\sqrt{\mu_k(B_k)\eta_k(B_k)}} \int_{B_k} \sqrt{f_k} d\eta_k. \tag{4}$$

Since $\prod_{k\in\mathbb{N}} \mu_k(B_k)$ and $\prod^+_{k\in\mathbb{N}} \eta_k(B_k)$ are positive and finite, (4) is equal to

$$\begin{aligned} &\frac{1}{\prod_{k\in\mathbb{N}} \sqrt{\mu_k(B_k)}} \cdot \frac{1}{\prod^+_{k\in\mathbb{N}} \sqrt{\eta_k(B_k)}} \prod^+_{k\in\mathbb{N}} \int_{B_k} \sqrt{f_k} d\eta_k \\ =&\mu(B)^{-1/2} \mathrm{vlm}(B)^{-1/2} \prod^+_{k\in\mathbb{N}} \int_{B_k} \sqrt{f_k} d\eta_k. \end{aligned}$$

Thus (3) is equal to a positive constant times (1).

We must now extend the dichotomy $\nu \perp \theta$ or $\nu = g\theta$ to the original measures μ and η. To this end, we construct an increasing sequence $B^{(n)}$ of basic boxes containing B for which

$$\lim_{n\to\infty} \mu(B^{(n)}) = 1. \tag{5}$$

First choose a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers such that

$$\prod_{k\geq k_n} \mu_k(B_k) \geq 1-\frac{1}{n}. \tag{6}$$

For $k \geq k_n$, let $B_k^{(n)} := B_k$. For $k < k_n$, use $\mu_k = f_k\eta_k$ to choose $\mathcal{A}_k$–measurable sets $B_k^{(n)}$ such that

$$B_k^{(n+1)} \supseteq B_k^{(n)} \supseteq B_k, \tag{7}$$

$$\eta_k(B_k^{(n)}) < \infty, \tag{8}$$

and

$$\prod_{k<k_n} \mu_k(B_k^{(n)}) \geq 1-\frac{1}{n}. \tag{9}$$

Define $B^{(n)}$ as $\prod_{k=1}^{\infty} B_k^{(n)}$. By (7) we have $B^{(n+1)} \supseteq B^{(n)}$; by (8) each $B^{(n)}$ is an (η_k)–basic box; by (9) and (6), (5) holds. Now define $\nu_k^{(n)}$, $\theta_k^{(n)}$, $\nu^{(n)}$, $\theta^{(n)}$, and $g_k^{(n)}$ for the sequence $(B_k^{(n)})_{k\in\mathbb{N}}$ as ν_k, θ_k, ν, θ, and g_k were defined for the original sequence $(B_k)_{k\in\mathbb{N}}$. A well–known uniqueness theorem for finite measures (see, e.g., Bauer [1], p.34, Satz 5.5) shows that $\nu^{(n)}$ is equal to $\frac{1}{\mu(B^{(n)})}\mu \mid B^{(n)}$. An analogous statement holds for the relationship between the measures $\theta^{(n)}$ and $\eta \mid B^{(n)}$.

Now observe that, by $\mu_k(B_k) > 0$, all integrals $\int_{B_k}\sqrt{f_k}d\eta_k$ are positive. Hence, if the product (1) vanishes, then $\prod_{k=1}^{\infty}{}^{+} \int_{B_k^{(n)}}\sqrt{f_k}d\eta_k = 0$ for all n. Thus if the product (1) vanishes, Kakutani's theorem shows that $\mu \mid B^{(n)}$ and $\eta \mid B^{(n)}$ are mutually singular for all n. The inequalities (6) and (9) imply that $\lim_{n\to\infty}\mu(B^{(n)}) = 1$. Since η and μ are singular on $B^{(n)}$, they are singular on $\bigcup_{n=1}^{\infty} B^{(n)}$, which is to say that μ and η are singular.

Finally, we deal with the case in which the product (1) is positive. By the aforementioned relationship between (3) and (4), all of the products $\prod_{k=1}^{\infty} \int_{B_k^{(n)}}\sqrt{g_k^{(n)}}d\theta_k^{(n)}$ are positive. Once again Kakutani's theorem shows that $\nu^{(n)} \ll \theta^{(n)}$ and so $\mu \mid B^{(n)} \ll \eta \mid B^{(n)}$ for all n. It follows that $\mu \ll \eta \mid \bigcup_{n=1}^{\infty} B_k^{(n)}$. As $\bigcup_{n=1}^{\infty} B^{(n)}$ is σ–finite with respect to η, there is a density function f such that $\mu = f\eta$.

3.4. Remarks. (a) The proof of Theorem 3.3 shows that the criterion (v) for case (ii) can also be cast as follows (cf. the logic tree at the beginning of the proof of Theorem 3.3):
Singularity occurs if and only if *one* of the following holds:

- (v') there is an (η_k)–basic box B such that $\mu(B) > 0$ and $\mathrm{vlm}(B) = 0$;
- (v") there is an (η_k)–basic box $B = \prod_{k\in I} B_k$ such that $\mu(B) > 0$, $\mathrm{vlm}(B) > 0$ and the infinite product 3.3(1) vanishes.

(b) If μ and $\eta^{(1)}$ are mutually singular, then the infinite product

$$\prod_{k\in I}{}^{+} \int_{C_k}\sqrt{f_k}d\eta_k \tag{1}$$

vanishes for all (η_k)–basic boxes $C = \prod_{k\in I} C_k$ such that $\mu(C) > 0$. (If we had (1)> 0 for one such box, then we would be in case (iii) of Theorem 3.3.) A similar argument shows
(c) If $\mu = f\eta^{(1)}$, then the infinite product (1) is positive for *all* (η_k)–basic boxes $C = \prod_{k\in I} C_k$ such that $\mu(C) > 0$.

3.5 We turn now to the second Elliott–Morse measure $\eta^{(2)}$ generated by the family $(\eta_k)_k$ of measures on X_k (cf. 1.9). For a subset $J \subseteq I$, we again write $\eta_J^{(i)}$ for the ith Elliott–Morse measure on X_J defined by the subfamily $(\eta_k)_{k\in J}$ of $(\eta_k)_{k\in I}$. In a similar way, we define the subproduct μ_J of the product probability μ.

3.6. Example. We will see that trichotomy also holds for the second Elliott–Morse measure $\eta^{(2)}$ (cf. 3.8). It is, however, *not* true that $\mu \lll \eta^{(2)}$, $\mu\perp\eta^{(2)}$ or $\mu = f\eta^{(2)}$ if the same relation holds with respect to $\eta^{(1)}$. Let us show that it is possible to have $\mu \lll \eta^{(1)}$ and $\mu\perp\eta^{(2)}$ at the same time. Consider $X_k = \{0,1\}$ for all $k \in \mathbb{Z}$, the coin-tossing probability μ on $\{0,1\}^{\mathbb{Z}}$, and

$$\eta_k := \begin{cases} \delta_0 + \delta_1, & (k \le 0) \\ (1-\alpha)\delta_0 + \alpha\delta_1 & (k > 0) \end{cases}$$

where $0 < \alpha < 1$ and $\alpha \neq \frac{1}{2}$. A subset $B = \prod_{k\in\mathbb{Z}} B_k$ is a basic box for $(\eta_k)_k$ if $\#B_k \le 1$ for all but finitely many $k \le 0$ or if $\#B_k \le 1$ for infinitely many $k > 0$. In both cases, $\#\{k \in \mathbb{Z}/\#B_k \le 1\} = \infty$ and hence $\mu(B) = 0$. By Theorem 3.3, $\mu \lll \eta^{(1)}$. On the other hand, by Kakutani's dichotomy, the subproducts $\mu_{\mathbb{N}}$ and $\eta_{\mathbb{N}}$ are mutually singular probability measures. Let A' be a measurable subset of $\prod_{k>0} X_k$ such that $\mu_{\mathbb{N}}(A') = 1$ and $\eta_{\mathbb{N}}(A') = 0$. Then the subset $A := (\prod_{k\le 0} X_k) \times A'$ is an (η_k)–nilcylinder and $\eta^{(2)}(A) = 0$. But $\mu(A) = 1$ and hence μ and $\eta^{(2)}$ are mutually singular. This case, however, is the only exception as we will now see. We precede the next theorem with a short lemma.

3.7. Lemma. *Suppose that no subset $J \subseteq I$ exists for which $\mu_J\perp\eta_J^{(1)}$. Then $\mu(N) = 0$ for all (η_k)–nilcylinders N.*

Proof. Assume on the contrary that $\mu(A' \times X) > 0$ for some nilcylinder $A' \times X_{CJ}$, $A' \subseteq X_J$. Since $\mu = \mu_J \otimes \mu_{CJ}$, we have $\mu_J(A') > 0$ and of course $\eta_J^{(1)}(A') = 0$. This denies the relation $\mu_J \ll \eta_J^{(1)}$. Applying the Trichotomy Theorem 3.3 to the measures μ_J and $\eta_J^{(1)}$, we obtain $\mu_J\perp\eta_J^{(1)}$ – a contradiction to our hypothesis.

Using this lemma the interested reader can verify the following

3.8. Trichotomy theorem (for the second Elliott–Morse measure) (a) *Exactly one of the following cases occurs.*

(i) *μ is diffuse with respect to $\eta^{(2)}$,*

(ii) *μ and $\eta^{(2)}$ are mutually singular,*

(iii) *$\mu = f\eta^{(2)}$ for some $\eta^{(2)}$–integrable function $f \ge 0$ on X.*

(b)(iv) *The first case occurs if and only if $\mu \lll \eta^{(1)}$ and there exists no (infinite) subset $J \subseteq I$ such that $\mu_J\perp\eta_J^{(1)}$.*

(v) *The second case occurs if and only if there exists an (infinite) subset $J \subseteq I$ such that μ_J and $\eta_J^{(1)}$ are mutually singular.*

(vi) *The third case occurs if and only if $\mu = f\eta^{(1)}$ for some $f \in \mathfrak{L}^1_+(\eta)$ and, in this case, f is also a version of the density function of μ with respect to $\eta^{(2)}$.*

The trichotomy theorems make it easy to classify the Elliott–Morse measures according to their finiteness, σ–finiteness, or non–σ–finiteness. We first state a lemma without proof.

3.9. Lemma. *Let $C_k \in A_k$, $\eta_k(C_k) < \infty$ for all $k \in I$, $C = \prod_{k\in I} C_k$. Then*

(a) $\eta^{(1)}(C) = \eta^{(2)}(C) = \mathrm{vol}(C)$;

(b) *If* $\mathrm{vol}(C) = \infty$ *then C is not σ–finite for $\eta^{(1)}$ and $\eta^{(2)}$.*

A measure η on a measurable space $(X,\mathcal{A})$ is called *semi–regular* if, for all $A\in\mathcal{A}$, we have $\eta(A)=\sup\eta(C)$, where the supremum runs over all $C\in\mathcal{A}$, $C\subseteq A$ with finite η–measure; equivalently, η is semi–regular if any measurable set of infinite measure possesses a measurable subset of finite, positive measure. If a measure η is *not* semi–regular, then X contains a measurable subset A of infinite η–measure such that any measurable subset C of A with finite η–measure is η–null. Using Lemma 3.10 and distinguishing the cases where all η_k's are semi–regular and the contrary one can prove the following

3.10. Theorem. *Let $I_0:=\{k\in I\mid \eta_k \text{ is finite}\}$, $I_1:=\{k\in I\mid \eta_k \text{ is } \sigma\text{–finite and nonfinite}\}$, and $I_2:=\{k\in I\mid \eta_k \text{ is non–}\sigma\text{–finite}\}$.*

(a)(i) *$\eta^{(1)}=0$ if and only if $\prod^+_{k\in I_0}\eta_k(X_k)=0$;*

(ii) *$\eta^{(1)}$ is finite and nonvanishing if and only if $0<\prod^+_{k\in I_0}\eta_k(X_k)<\infty$ and $I_1=I_2=\emptyset$;*

(iii) *$\eta^{(1)}$ is σ–finite and nonfinite if and only if $0<\prod^+_{k\in I_0}\eta_k(X_k)<\infty$, I_1 is finite and nonvoid, and $I_2=\emptyset$.*

(b) *The same statements hold with $\eta^{(1)}$ replaced by $\eta^{(2)}$.*

4 Examples

The previous theorems are illuminated by three examples. The first two serve to illustrate trichotomy; the second demonstrates the bad behavior that an Elliott–Morse measure may exhibit; the last example is concerned with invariant measures on Hilbert space.

4.1. Finite factors. Let X_k be the two point set $\{0,1\}$, $\mu_k=\frac{1}{2}\#$, and let $\eta_k=\alpha\delta_0+\beta\delta_1$ where δ_0 and δ_1 are the Dirac measures on 0 and 1, respectively, and where α and β are positive real numbers such that $\alpha,\beta\le 1$ and $\alpha+\beta\ge 1$. By Theorem 3.3, we have in this case

(i) $\mu\lll\eta^{(1)}$ if $\alpha+\beta>1$,

(ii) $\mu\perp\eta^{(1)}$ if $\alpha+\beta=1$, $\alpha\ne\beta$, and

(iii) $\mu=\eta^{(1)}$ if $\alpha+\beta=1$, $\alpha=\beta$.

The same is true if $\eta^{(1)}$ is replaced by the second Elliott–Morse measure, $\eta^{(2)}$.

4.2. Infinite products of Gauß and Lebesgue measures. Let $X_k=\mathbb{R}$, $\mu_k=\gamma_{0,v_k}$ be the one–dimensional, centered Gauß measure with variance v_k, let λ be Lebesgue measure on $\mathbb{R}$, and let $\eta_k:=\alpha_k\lambda$ be a positive, real multiple of λ. Here we have the following fact:

(a) Either μ is diffuse with respect to $\eta^{(1)}$, or μ and $\eta^{(1)}$ are mutually singular.

(b) The first case occurs if and only if $\mu(B)=0$ for all (η_k)–basic boxes B of the form $\prod_k[-b_k,b_k]$.

In order to prove this fact, by applying homothetic transformations on the factors $\mathbb{R}$, it is sufficient to deal with the case $\alpha_k=1$. Let us first suppose that the condition in (b) is satisfied, and let $C=\prod_k C_k$ be any basic box. By Theorem 3.3, we have to show that $\mu(C)=0$ in order to show diffuseness of μ w.r.t. $\eta^{(1)}$. Defining $b_k:=\lambda(C_k)/2$ we have

$$\mathrm{vlm}\Big(\prod_k[-b_k,b_k]\Big)=\prod{}^+ 2b_k=\prod_k{}^+\lambda(C_k)=\mathrm{vlm}(C)<\infty,$$

that is, $\prod_k[-b_k,b_k]$ is a basic box; we thus have by hypothesis

$$\mu(C)=\prod_k\mu_k(C_k)\le\prod_k\mu_k[-b_k,b_k]=0$$

and μ is diffuse w.r.t. $\eta^{(1)}$. We suppose now that there exists a basic box $B = \prod_k [-b_k, b_k]$ such that $\mu(B) > 0$. If

$$\text{vlm}(B) = \prod^{+} 2b_k \tag{1}$$

vanishes, then we have singularity by Remark 3.4(a)(v'). On the other hand, if (1) is positive, we use 3.4(v") to prove that $\mu \perp \eta^{(1)}$. In order to apply 3.4(v"), it remains to show that the infinite product 3.3(1) vanishes. Since (1) is finite and positive, we have

$$\lim_k b_k = \frac{1}{2} \tag{2}$$

and therefore, by $\mu(B) > 0$,

$$\lim_k v_k = 0. \tag{3}$$

Writing for abbreviation $I_k(a, b) := \int_a^b \sqrt{f_k(x)}dx$, where

$$f_k(x) = \frac{1}{\sqrt{2\pi v_k}} e^{-x^2/2v_k}$$

is the density function of γ_{0,v_k} with respect to λ, we show that

$$\rho_k := \frac{I_k(-b_k, b_k)}{I_k(-b_k/2, b_k/2)} = \frac{I_k(0, b_k)}{I_k(0, b_k/2)} \tag{4}$$

tends to 1 as $k \to \infty$. This is most easily done by proving that

$$\frac{1}{\rho_k - 1} = \frac{I_k(0, b_k/2)}{I_k(b_k/2, b_k)} \geq \frac{I_k(0, \infty) - I_k(b_k/2, \infty)}{I_k(b_k/2, \infty)} = \frac{I_k(0, \infty)}{I_k(b_k/2, \infty)} - 1 \tag{5}$$

tends to ∞. But $I_k(0, \infty) = \sqrt[4]{\pi v_k/2}$ and $I_k(a, \infty) \leq \frac{2v_k}{\sqrt[4]{2\pi v_k} a} e^{-a^2/4v_k}$; hence (5) exceeds the number $\frac{\sqrt{\pi} b_k}{\sqrt[4]{v_k}} e^{b_k^2/16v_k} - 1$ and (2), (3) show that (5) converges to ∞ and thus (4) converges to 1. Therefore, to finish the proof, we may choose a subsequence $(k_n) \subseteq \mathbb{N}$ such that

$$\prod_n \rho_{k_n} < \infty \tag{6}$$

and estimate the infinite product 3.3(1) using 1.3, (4), and the Cauchy–Schwarz inequality.

$$\begin{aligned}
\prod_k^{+} \int_{-b_k}^{b_k} \sqrt{f_k(x)}dx &= \prod_n^{+} I_{k_n}(-b_{k_n}, b_{k_n}) \prod_{k \notin \{k_n\}}^{+} I_k(-b_k, b_k) \\
&= \prod_n^{+} \left(\rho_{k_n} I_{k_n} \left(-\frac{b_{k_n}}{2}, \frac{b_{k_n}}{2} \right) \right) \prod_{k \notin \{k_n\}}^{+} I_k(-b_k, b_k) \\
&= \prod_n \rho_{k_n} \prod_n^{+} I_{k_n} \left(-\frac{b_{k_n}}{2}, \frac{b_{k_n}}{2} \right) \prod_{k \notin \{k_n\}}^{+} I_k(-b_k, b_k) \\
&\leq \prod_n \rho_{k_n} \prod_n^{+} \sqrt{b_{k_n}} \prod_{k \notin \{k_n\}}^{+} \sqrt{2b_{k_n}} \\
&= \prod_n \rho_{k_n} \sqrt{\prod_n^{+} b_{k_n}} \sqrt{\prod_{k \notin \{k_n\}}^{+} 2b_{k_n}}.
\end{aligned}$$

But the middle factor in the last line, and therefore also the infinite product 3.3(1), vanishes by (2).

4.3. Translation invariant measures on Hilbert space. The infinite product of Lebesgue measures already used in Part 4.2 may be employed to construct translation invariant Borel measures on Hilbert space.

Before giving details, we will interrupt the study of Elliott–Morse measures to include some known facts about translation invariant Borel measures on Polish groups. We thank S. Graf for a discussion of this subsection.

4.3.1. Translation invariant Borel measures on Polish groups. To our knowledge, Oxtoby [12] was the first author to study Borel measures invariant under all translations on a Polish group G with its Borel σ–algebra $\mathfrak{B}$. (A topological group is called *Polish*, if it is separable and if there is a complete metric that generates its topology.) His results, *inter alia*, are as follows.

4.3.2. Theorem. [12], p. 220, Theorem 3. *In any nondiscrete Polish group G, there exists a left–invariant Borel measure $\nu \neq 0$ such that $\nu(\{x\}) = 0$ for all $x \in G$.*

4.3.3. Theorem. [12], p. 217, Theorem 2. *If G is not locally compact and $\nu \neq 0$ is a left–invariant Borel measure in G, then any nonempty open set contains a compact set which is the union of uncountably many disjoint mutually congruent compact sets of (equal) finite positive ν–measure.*

In the light of Theorem 4.3.3, it is clear that if G is not locally compact and ν is a left–invariant Borel measure in G, then no nonempty open set can be σ–finite; moreover there can be no left–invarinat *Radon* measure on $\mathfrak{B}$ different from the measure 0. (A measure ν on $\mathfrak{B}$ is called a *Radon measure*, if

(i) $\nu(K) < \infty$ for all compact subsets $K \subseteq G$,

(ii) $\nu(D) = \sup\{\nu(K) \mid K \text{ compact} \subseteq D\}$ for all $D \in \mathfrak{B}$.)

If the Polish group G is the Hilbert space ℓ^2, invariant measures on the Borel σ–algebra may be constructed by using Elliott–Morse measures: Let $(\alpha_k)_{\alpha\in\mathbb{N}}$ be any sequence of positive numbers and let η_k be the multiple $\eta_k(dx) := \alpha_k dx$ of Lebesgue measure in the real line. Denote by ν the restriction of the first Elliott–Morse measure $\eta^{(1)}$ to the subspace $\ell^2 \subseteq \mathbb{R}^{\mathbb{N}}$. Since the restriction of the product σ–algebra on $\mathbb{R}^{\mathbb{N}}$ (where each factor is endowed with its Borel σ–algebra) to ℓ^2 is the Borel σ–algebra on ℓ^2, ν is a Borel measure on ℓ^2. Since $\eta^{(1)}$ is plainly translation invariant, so is ν. In the following proposition, we will call two Borel measures in ℓ^2 *equivalent* if one is a positive, real multiple of the other.

4.3.4. Proposition. (a) If the sequence $(\alpha_n^{-1})_n$ is square summable, then ν does not vanish.

(b) Two measures, ν and $\bar{\nu}$, generated by two such sequences (α_n) and $(\bar{\alpha}_n)$ in the way described above are equivalent if and only if

$$\prod{}^{+}(\alpha_n/\bar{\alpha}_n) \in]0, \infty[. \tag{1}$$

Proof. (a) If $\sum_n \frac{1}{\alpha_n^2} < \infty$ then the (η_k)–basic box $B := \prod_n \left[0, \frac{1}{\alpha_n}\right]$ is contained in ℓ^2. Since $\eta^{(1)}(B) = \operatorname{vlm}(B) = 1$, we have $\nu \neq 0$. This proves Part (a).

To prove Part (b), let us suppose first that (1) is satisfied and hence the plus–product in (1) is classical. If $\bar{\nu}$ is generated by the sequence $(\bar{\eta}_k)$ and if $B = \prod_k B_k$ is an (η_k)–basic box, we may compute, using 1.3(vi),

$$\begin{aligned}\eta^{(1)}(B) &= \prod_k{}^{+} \eta_k(B_k) = \prod_k{}^{+} \alpha_k\lambda(B_k) = \prod_k(\alpha_k/\bar{\alpha}_k)\prod_k{}^{+} \bar{\alpha}k\lambda(B_k)\\ &= \prod_k(\alpha_k/\bar{\alpha}_k)\bar{\eta}^{(1)}(B).\end{aligned}$$

Since this equality is true for all (η_k)–basic boxes, we see that the system of (η_k)–basic boxes coincides with the system of $(\bar{\eta}_k)$–basic boxes. As both measures ν and $\bar{\nu}$ are restrictions of first Eliott–Morse measures, we have in addition $\nu = \prod{}^{+}_k(\alpha_k/\bar{\alpha}_k)\bar{\nu}$.

Suppose now that $\nu = \alpha\bar{\nu}$ for some α, $0 < \alpha < \infty$. Then for the (η_k)–basic box $B = \prod_k[0, \alpha_k^{-1}]$ we obtain

$$1 = \nu(B) = \alpha\bar{\nu}(B) = \alpha\prod_k{}^{+}(\bar{\alpha}_k/\alpha_k).$$

Therefore $\prod^{+}(\alpha_k/\bar{\alpha}_k)$ is the classical product $\prod(\alpha_k/\bar{\alpha}_k)$ and equals α; the proof of Part (b) is finished.

4.3.5. Remarks. (a) Since the first and second Elliott–Morse measures coincide on basic boxes, Proposition 4.3.4 remains true if ν and $\bar{\nu}$ are interpreted as the restrictions of second Elliott–Morse measures to ℓ^2.

(b) It is interesting to note that non–σ–finiteness of any nonempty ball with respect to ν (which follows from Oxtoby's Theorem 4.3.3) can be derived from trichotomy. Indeed, let $\mu_k = \gamma_{0,v_k}$ be the one–dimensional Gaussian distribution with mean 0 and variance $v_k = \alpha_k^{-2}$, where α_k^{-1} is square summable and let $\eta_k(dx) = \alpha_k dx$. Then we have $(v_k) \in \ell^1$, and it is well known that μ is concentrated on ℓ^2 (cf. Skorohod [17], Chap.1, §5). It is therefore sufficient to show that μ is diffuse with respect to $\eta^{(1)}$. Let B be any (η_k)–basic box of the form $\prod_k[-b_k, b_k]$. In view of 4.2(b), we show that $\mu(B) = 0$. If $b_k = 0$ for one k, then plainly $\mu(B) = 0$. In the opposite case, we have $\alpha_k b_k \leq 1$ for infinitely many indices k since $\mathrm{vlm}(B) < \infty$. Denoting by J the subset of these indices k, we therefore have

$$\sum_{k\in J}\mu_k(]b_k,\infty[) = \sum_{k\in J}\frac{1}{\sqrt{2\pi v_k}}\int_{b_k}^{\infty} e^{-t^2/2v_k}dt = \sum_{k\in J}\frac{1}{\sqrt{2\pi}}\int_{\alpha_k b_k}^{\infty} e^{-s^2/2}ds = \infty.$$

Hence $\mu(B) \leq \prod_{k\in J}\mu_k([-b_k, b_k]) = 0$, i.e., we have verified the criterion for diffuseness given in 4.2(b).

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