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On dichotomy of Riesz products

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1. *Introduction*

(1.1) *Background.* Riesz products are very useful for the construction of singular measures on compact, Abelian groups. Under some circumstances, two Riesz products are either equivalent or singular in the measure–theoretic sense. Exact knowledge of these circumstances has been of major interest ever since the 1930s, when Riesz's famous example [8] was recognized as a fertile source of examples of singular continuous measures. Zygmund [11] showed that any Riesz product over a Hadamard dissociate subset of $\mathbb{N}$ is either a square integrable function or singular with respect to Lebesgue measure. Hewitt–Zuckerman [4] generalized these products to all compact, Abelian groups, introducing the notion of a dissociate subset. They extended Zygmund's result in certain cases. The next major step was taken by Brown–Moran [3] and Peyrière [6], [7], who showed that two Riesz products

$$\mu_a = \prod_{n\geq 0}(1 + a_n\chi_n + \overline{a}_n\overline{\chi}_n) \text{ and } \mu_b = \prod_{n\geq 0}(1 + b_n\chi_n + \overline{b}_n\overline{\chi}_n)$$

are mutually singular if

(α) $\sum_{n\geq 0} |a_n - b_n|^2 = \infty.$

The author [9] has improved another result of Brown–Moran [3] by showing that μ_a and μ_b are equivalent if

(β) $\sum_{n\geq 0} |a_n - b_n|^2/(1 - |a_n + b_n|) < \infty.$

He also extended these results to more general infinite products of functions. It follows from [3], proposition 2, that (α) and (β) are essentially the best possible conditions using the coefficients a_n and b_n alone. However, it is still not known whether two Riesz products on the circle over a general dissociate set are either equivalent or mutually singular.

We consider here dichotomy theorems for Riesz products and generalized Riesz products possessing a factorization property first considered by Peyrière [6]. As an application we give an affirmative answer to the question mentioned above in certain cases including the case of two Riesz products on the circle over a dissociate set of the form $\{r^k \mid k \geq 0\}$ ($r \geq 3$). We also give necessary and sufficient conditions for the occurrence of either case.

The results presented here are based on the author's extension [10] of Kakutani's dichotomy theorem [5]. This extension uses a 0-1 law as a major tool to establish dichotomy and to obtain necessary and sufficient criteria for absolute continuity and mutual orthogonality. Consequently, the main part of the present communication consists in establishing 0–1 laws for infinite products of functions (Section 3). The 0–1 laws used here are similar to the ergodic assumption appearing in [2] and [1], but do not require the countability condition occurring in their definition of ergodicity.

(1.2) *Notation and terminology.* G will be a compact, Abelian group, $\mathcal{B}$ the σ–algebra of its Borel sets, and λ normalized Haar measure on G. X denotes the (discrete) character group of G. $\mathcal{M}_1^+(G)$ will be the set of all probability measures on $(G, \mathcal{B})$, endowed with its weak* topology. $\mathfrak{C}_1^+(G)$ stands for the space of all continuous functions ≥ 0 on G with λ–integral 1. $\mathfrak{A}_1^+(G)$ denotes the subspace of all functions in $\mathfrak{C}_1^+(G)$ with absolutely summable Fourier series. For $f \in \mathfrak{A}_1^+(G)$, we will write

$$f' := |\widehat{f}|^\vee (= \sum_{\chi \in \mathsf{X}} |\widehat{f}(\chi) \mid \chi).$$

$\mathfrak{I}_1^+(G)$ means the set of all elements $f \in \mathfrak{A}_1^+(G)$ such that $\widehat{f} \geq 0$. Denoting by $\mathfrak{P}_1^+(G)$ the set of all functions in $\mathfrak{C}_1^+(G)$ such that $\widehat{f} \geq 0$, we obtain the inclusions

$$\mathfrak{P}_1^+(G) \subseteq \mathfrak{I}_1^+(G) \subseteq \mathfrak{A}_1^+(G) \subseteq \mathfrak{C}_1^+(G).$$

Let $(\mathsf{T}_n)_{n \geq 0}$ be a sequence of symmetric subsets of X containing the character 1. A sequence $(\chi_n)_{n \geq 0}$ such that $\chi_n \in \mathsf{T}_n$ for all n and $\chi_n = 1$ for almost all n will be called a (T_n)–*word* (or simply *word*, if no confusion seems possible). For $m \geq 0$, Ω_m will stand for the set of all words of length m, i.e. all words of the form $(\chi_0, \chi_1, \ldots, \chi_{m-1}, 1, 1, \ldots)$. Given $\chi \in \mathsf{X}$, $\Omega_m(\chi)$ will stand for the set of all words in Ω_m such that

$$\prod_{k=0}^{m-1} \chi_k = \chi$$

(the words in Ω_m that represent χ), and $\Omega(\chi)$ will mean the set of all words that represent χ. The sequence (T_n) will be called 1–*dissociate*, if the only word representing 1 is the trivial word $(1, 1, \ldots)$ and it will be called 2–*dissociate*, if each character in X has at most one representation as a word. A sequence $(\chi_n)_{n \geq 0} \subseteq \mathsf{X} \setminus \{1\}$ will be called 1–*dissociate* (*dissociate*), if the sequence $(\{1, \chi_n, \bar{\chi}_n\})_{n \geq 0}$ is 1–dissociate (2–dissociate).

Now let (T_n) be 1–dissociate and let (f_n) be a sequence in $\mathfrak{C}_1^+(G)$ such that

$$\operatorname{supp} \widehat{f}_n := \{\chi \in \mathsf{X} \mid \widehat{f}_n(\chi) \neq 0\} \subseteq \mathsf{T}_n$$

for all n. We will write

$$f_{m,n} := \prod_{k=m}^{n-1} f_k.$$

1–dissociativity of (T_n) implies that $\int f_{m,n} d\lambda = 1$ for all m, n such that $m \leq n$; any weak* cluster point of the sequence $(f_{0,n}\lambda)$ will be called a *generalized Riesz product* generated by the sequence (f_n). If (f_n) is a sequence in $\mathfrak{I}_1^+(G)$, then the weak* limit exists (see [9]). Let μ be a generalized Riesz product generated by (f_n) and let $\mathbf{m} \geq 0$. Then any cluster point ρ of the net $(f_{m,n_\tau}\lambda)_\tau$, where $(n_\tau)_\tau$ is any subset of $\mathbb{N}$ such that

$$\mu = \lim_\tau f_{0,n_\tau} \lambda,$$

will be called a *tail measure* for μ and m. We have $\mu = f_{0,m}\rho$.

If f_n has the form $f_n = 1 + a_n \chi_n + \bar{a}_n \bar{\chi}_n$ for all n, where (χ_n) is a 1–dissociate sequence and the a_n's are complex numbers of modulus $\leq \frac{1}{2}$, then μ is called a *Riesz product*. Since, in this case, we have $f_n \in \mathfrak{I}_1^+(G)$, the sequence (f_n) generates exactly one Riesz product $\mu = \lim_n f_{0,n}\lambda$ and tail measures are uniquely defined.

Given a probability space $(\Omega, \mathcal{F}, P)$ a sub–σ–algebra $\mathcal{T}$ of $\mathcal{F}$, and a function $f \in \mathcal{L}_1(P)$, the symbol $E_P(f \mid \mathcal{T})$ denotes the conditional expectation of f with respect to $\mathcal{T}$ and P. P will be called *trivial* on a sub–σ–algebra $\mathcal{T} \subseteq \mathcal{F}$, if $P(T) = 0$ or 1 for all $T \in \mathcal{T}$.

2. *A previous result*

In [10], theorem 7.7, we extended Kakutani's dichotomy theorem [5] on product measures to a non–independent case. The following theorem is a specialization of this result and is the basis for our main dichotomy theorems (4.2 and 4.4) for Riesz products. Let $(\mathcal{T}^n)$ be a decreasing sequence of sub–σ–algebras of $\mathcal{B}$ generated by continuous functions on G and let $\mathcal{T}^\infty := \bigcap_{n \geq 0} \mathcal{T}^n$.

(2.1) THEOREM. *Let (f_n) and (g_n) be two sequences in $\mathfrak{C}_1^+(G)$ adapted to $(\mathcal{T}^n)$ and let μ_f and μ_g be generalized Riesz products generated by (f_n) and (g_n), respectively. Suppose that*

(α) *μ_g is trivial on $\mathcal{T}^\infty$,*

(β) *$E_\lambda(f_n \mid \mathcal{T}^{n+1}) = 1 = E_\lambda(g_n \mid \mathcal{T}^{n+1})$ for all $n \geq 0$,*

(γ) *$\sigma\{g_{0,n} = 0\} = 0$ for all $\mathbf{n} \geq 0$ and all generalized Riesz products generated by $(g_k)_{k\geq n}$, and that*

(δ) *there exist two subnets (m_σ) and (n_τ) of $\mathbb{N}$ such that $\lim_\tau g_{0,n_\tau}\lambda = \mu_g$ and*

$$\lim_{\sigma,\tau} f_{0,m_\sigma} g_{m_\sigma,n_\tau}\lambda = \mu_f.$$

(a) The following statements are equivalent.

(i) μ_f is absolutely continuous with respect to μ_g;

(ii) μ_f and μ_g are not mutually singular;

(iii) there is a $\varepsilon > 0$ such that, for all $n \geq 0$, we have $\int (f_{0,n}/g_{0,n})^{\frac{1}{2}} d\mu_g \geq \varepsilon$;

(iv) $\lim_{m,n\to\infty} \int (f_{m,n}/g_{m,n})^{\frac{1}{2}} d\mu_g = 1$.

(b) If the equivalent statements (i)–(iv) hold, then the sequence $(f_{0,n}/g_{0,n})$ converges in $\mathcal{L}_1(\mu_g)$ to the Radon–Nikodym density $d\mu_f/d\mu_g$.

3. *Zero–one laws*

In order to apply Theorem (2.1) to Riesz products and generalized Riesz products we need a 0–1 law for these measures. We first give a simple but general 0–1 law that extends Kolmogorov's 0–1 law in the product case.

(3.1) Lemma. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\mathcal{T}$ be a sub–σ–algebra of $\mathcal{F}$, and let $\mathcal{C}$ be a total subset of $\mathcal{L}_2(P)$. The following two statements are equivalent.*

(a) P is trivial on $\mathcal{T}$;

(b) $\int \psi E_P(\bar{\psi} \mid \mathcal{T}) dP \leq |\int \psi dP|^2$ for all $\psi \in C$.

Proof. We only have to show that (b) implies (a). By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |\int 1 E_P(\psi \mid \mathcal{T}) dP|^2 &\leq \int E_P(\psi \mid \mathcal{T}) E_P(\bar{\psi} \mid \mathcal{T}) dP \\ &= \int \psi E_P(\bar{\psi} \mid \mathcal{T}) dP. \end{aligned} \tag{1}$$

It follows from (b) that equality holds in (1). Thus $E_P(\psi \mid \mathcal{T})$ is constant P–a.e. for all $\psi \in \mathcal{C}$. Since $\mathcal{C}$ is total in $\mathcal{L}_2(P)$, we see that $E_P(f \mid \mathcal{T})$ is constant P–a.e. for all $f \in \mathcal{L}_2(P)$, i.e., P is trivial on $\mathcal{T}$.

We will next apply Lemma (3.1) to Riesz products and generalized Riesz products. To do this, we need a definition that goes back to Peyrière [6] in the case of dissociate sets.

(3.2) *Definition.* Let $(\mathsf{T}_n)_{n\geq 0}$ be a sequence of symmetric subsets of X such that $1 \in \mathsf{T}_n$ for all $n \geq 0$. We will say that the sequence (T_n) is *factorizing*, if

(i) $\mathsf{T}_n \cap \langle \bigcup_{m>n} \mathsf{T}_m \rangle = \{1\}$ for all $n \geq 0$.

(b) We will say that a sequence $(\chi_n)_{n\geq 0}$ in X is *factorizing*, if the sequence of subsets $(\{1, \chi_n, \bar{\chi}_n\})_{n\geq 0}$ is factorizing. (The symbol $\langle \mathsf{Y} \rangle$ indicates the subgroup generated by a subset $\mathsf{Y} \subseteq \mathsf{X}$.)

(3.3) *Examples.* (a) Let $(G_n)_{n\geq 0}$ be a sequence of compact Abelian groups and let

$$G = \prod_{n\geq 0} G_n$$

be the product group. Let X_n be the dual group of G_n. Then $\mathsf{T}_n := \mathsf{X}_n$ can be considered as a subgroup of the dual group $\mathsf{X} = \bigoplus_{n\geq 0} \mathsf{X}_n$ of G. The sequence (T_n) is factorizing.

(b) The archetype of a factorizing sequence (r_n) in $\mathbb{Z}$ is given by

$$r_n = \prod_{k=1}^{n} l_k,$$

where the l_k's are integers ≥ 2.

We now introduce some notations.

(3.4) *Notation and terminology.* In what follows, (T_n) will denote a *factorizing* sequence of symmetric subsets of X such that $1 \in \mathsf{T}_n$ for all n. We will write T^n for the subgroup $\langle \bigcup_{\mathbf{m}\geq n} \mathsf{T}_m \rangle$ of X and H_n for the annihilator $A(G, \mathsf{T}^n)$ of T^n in G $(n \geq 0)$. (H_n) is an increasing sequence of compact subgroups of G. λ_n will denote Haar measure on H_n. In the obvious way, λ_n will also be considered as a Borel measure on G. $\overline{\mathcal{T}}^n$ stands for the sub–σ–algebra of $\mathcal{B}$ consisting of all H_n–invariant Borel sets in G, $\mathcal{T}^n \subseteq \overline{\mathcal{T}}^n$ stands for the sub–σ–algebra of $\mathcal{B}$ generated by T^n, and $\overline{\mathcal{T}}^\infty$ and $\mathcal{T}^\infty$ denote the σ–algebras $\bigcap_{n\geq 0} \overline{\mathcal{T}}^n$ and $\bigcap_{n\geq 0} \mathcal{T}^n$, respectively.

(3.5) Lemma. *(a) Let (χ_n) be a word such that $\prod_{k\geq 0}^{m-1} \chi_k \in \mathsf{T}^m$ for an $m \geq 0$. Then $\chi_k = 1$ for $k < m$.*

(b) (T_n) is 1–dissociate.

Proof. (a) Suppose that there exists an index $k < m$ such that $\chi_k \neq 1$. Let l be the least among these indices. Then we have

$$\prod_{k\geq l} \chi_k \in \mathsf{T}^m \text{ and } \chi_l \in \prod_{k>l} \bar{\chi}_k \mathsf{T}^m \subseteq \mathsf{T}^{l+1}.$$

This contradicts the factorization property. Part (b) is an immediate consequence of (a).

(3.6) *Explanation.* Let (g_n) be a sequence in $\mathfrak{C}_1^+(G)$ such that $\operatorname{supp} \widehat{g}_n \subseteq \mathsf{T}_n$ for all $n \geq 0$. Let μ be a generalized Riesz product generated by (g_n) (see (1.2) and (3.5(b))). The following lemma says that tail measures for μ are uniquely defined and gives a representation of tail measures.

(3.7) Lemma. *Let m be an integer ≥ 0.*

(a) There is only one tail measure ρ_m for μ and m.

(b) $\rho_m = \mu * \lambda_m$.

Proof. Note that $\widehat{g}_{0,m}(\chi) = 0$, if χ is not represented by a word in Ω_m. We thus obtain by (3.5(a)).

$$\widehat{g}_{0,m}(\chi)\widehat{\lambda}_m(\chi) = \widehat{g}_{0,m}(\chi)1_{\mathsf{T}^m}(\chi) = \begin{cases} 1 & \chi = 1, \\ 0 & \chi \neq 1, \end{cases} \tag{1}$$

i.e.

$$g_{0,m} * \lambda_m = 1. \tag{2}$$

Since the functions g_l such that $l \geq m$ are H_m–invariant, (2) implies for $n \geq m$ and $x \in G$

$$\begin{aligned} g_{0,n} * \lambda_n(x) = &\int g_{0,m}(x-y) g_{m,n}(x-y) \lambda_m(dy), \\ =& g_{m,n}(x)(g_{0,m} * \lambda_m)(x), \\ =& g_{m,n}(x). \end{aligned}$$

If $\mu = \lim_\tau g_{0,n_\tau}\lambda$, then (3) shows that

$$\lim_\tau g_{m,n_\tau}\lambda = \lim_\tau g_{0,n_\tau} * \lambda_m = \mu * \lambda_m.$$

This proves the lemma.

We now obtain a representation of the conditional expectation occurring in (3.1(b)).

(3.8) Lemma. *For any bounded, Borel–measurable function h on G and any $m \geq 0$ we have* *(i)* $E_\mu(h \mid \overline{\mathcal{T}}^m) = (hg_{0,m}) * \lambda_m$.

Proof. We use Lemma (3.8) to compute for $E \in \overline{\mathcal{T}}^m$

$$\begin{aligned}\int 1_E h d_\mu &= \int\int 1_E(x+y)h(x+y)g_{0,m}(x+y)\mu(dx)\lambda_m(dy)\\ &= \int 1_E(x)[\int h(x+y)g_{0,m}(x+y)\lambda_m(dy)]\mu(dx)\\ &= \int 1_E[(hg_{0,m}) * \lambda_m]d\mu.\end{aligned}$$

Since $(hg_{0,m}) * \lambda_m$ is $\overline{\mathcal{T}}^m$–measurable, the lemma follows.

We finally obtain a 0–1 law for generalized Riesz products in the case of factorization.

(3.9) Proposition. *Let (g_n) be a sequence in $\mathfrak{C}_1^+(G)$ such that* $\operatorname{supp}\widehat{g}_n \subseteq \mathsf{T}_n$ *for all n and let*

$$\mu = \lim_{\tau\in D} g_{0,n_\tau}\lambda$$

be a generalized Riesz product generated by (g_n). The following two statements are equivalent.

(a) *μ is trivial on $\overline{\mathcal{T}}^\infty$;*

(b) $\lim_{\sigma\in D}\lim_{\tau\in D} \sum_{\substack{\chi\in\psi\mathsf{T}^{n_\sigma}\\ \chi\neq\psi}} \widehat{g}_{0,n_\sigma}(\chi)\widehat{g}_{0,n_\tau}(\bar{\chi}) = 0$ *for all $\psi \in \mathsf{X}$.*

Proof. We will show that (b) is equivalent to (3.1(b)) for $(\Omega, \mathcal{F}, P) = (G, \mathcal{B}, \mu)$, $\mathcal{T} = \overline{\mathcal{T}}^\infty$, and $\mathcal{C} = \mathsf{X}$. By (3.8), we have

$$\begin{aligned}\int \psi E_\mu(\bar{\psi} \mid \overline{\mathcal{T}}^\infty)d\mu &= \lim_m \int \psi E_\mu(\bar{\psi} \mid \overline{\mathcal{T}}^m)d\mu\\ &= \lim_m \int \psi[\bar{\psi}g_{0,m} * \lambda_m]d\mu\\ &= \lim_m\lim_\tau \int \psi[\bar{\psi}g_{0,m} * \lambda_m]g_{0,n_\tau}d\lambda. \qquad (1)\end{aligned}$$

Using the identity

$$[\bar{\psi}g_{0,m} * \lambda_m]^\wedge(\chi) = 1_{\mathsf{T}^m}(\chi)\widehat{g}_{0,m}(\psi\chi), \qquad (2)$$

we obtain

$$\begin{aligned}\int \psi[\bar{\psi}g_{0,m} * \lambda_m]g_{0,n_\tau}d\lambda &= \sum_{\chi\in\mathsf{T}^m} \widehat{g}_{0,m}(\psi\chi)\int \psi\chi g_{0,n_\tau}d\lambda\\ &= \sum_{\chi\in\mathsf{T}^m} \widehat{g}_{0,m}(\psi\chi)\widehat{g}_{0,n_\tau}(\overline{\psi\chi}). \qquad (3)\end{aligned}$$

It follows from (1) and (3) that

$$\begin{aligned}\int \psi E_\mu(\bar{\psi} \mid \overline{\mathcal{T}}^\infty)d\mu &= \lim_{\sigma\in D}\lim_{\tau\in D} \sum_{\chi\in\psi\mathsf{T}^{n_\sigma}} \widehat{g}_{0,n_\sigma}(\chi)\widehat{g}_{0,n_\tau}(\bar{\chi})\\ &= \widehat{\mu}(\psi)\widehat{\mu}(\bar{\psi}) + \lim_{\sigma\in D}\lim_{\tau\in D} \sum_{\substack{\chi\in\psi\mathsf{T}^{n_\sigma}\\ \chi\neq\psi}} \widehat{g}_{0,n_\sigma}(\chi)\widehat{g}_{0,n_\tau}(\bar{\chi}). \qquad (4)\end{aligned}$$

Since (3.1(b)) in the present case is equivalent to

$$\int \psi E_\mu(\bar{\psi} \mid \mathcal{T}^\infty) = \widehat{\mu}(\psi)\widehat{\mu}(\bar{\psi}) \qquad (5)$$

by the Cauchy–Schwarz inequality, (4) shows that (3.1(b)) is equivalent to (b).

(3.10) *Remark.* If (T_n) is 2–dissociate (see (1.2)) as well as factorizing (3.9(b)) can be rewritten in a simpler form. In this case, we have for $m \le n$ and any $\chi \in \mathsf{X}$

$$\widehat{g}_{0,m}(\chi)\widehat{g}_{0,n}(\bar{\chi}) = |\widehat{g}_{0,m}(\chi)|^2$$

(consider the two cases $\Omega_m(\chi) = \emptyset$ and $\Omega_m(\chi) \ne \emptyset$), and Statement (3.9(b)) is thus equivalent to the statement

(α) $\lim_m \sum_{\substack{\chi \in \psi \mathsf{T}^m \\ \chi \ne \psi}} |\widehat{g}_{0,m}(\chi)|^2 = 0$ for all $\psi \in \mathsf{X}$.

We mention two corollaries of Proposition (3.9). For the first corollary note that a sequence $(g_n) \subseteq \mathfrak{I}_1^+(G)$ (see (1.2)) such that $\operatorname{supp} \widehat{g}_n \subseteq \mathfrak{T}_n$ $(n \ge 0)$ generates exactly one generalized Riesz product $\mu = \lim_n g_{0,n}\lambda$ (see [9]).

(3.11) Corollary. *Let (g_n) be a sequence in $\mathfrak{I}_1^+(G)$ such that $\operatorname{supp}\widehat{g}_n \subseteq \mathsf{T}_n$ for all n and let μ be the generalized Riesz product generated by (g_n). Suppose that*

(α) $\bigcap_{m \ge 0} \mathsf{T}^m = \{1\}$ *and that*

(β) $\widehat{\mu}$ *vanishes at infinity.*

Then μ is trivial on $\overline{\mathfrak{T}}^{\infty}$.

Proof. Let μ' be the generalized Riesz product generated by the sequence (g'_n) (see (1.2)). We put

$$g'_{0,n} := \prod_{k=0}^{m-1} g'_k.$$

By Lemma (3.8) we have for all $m \ge 0$ and all $\psi \in \mathsf{X}$ (see also (3.9(2)))

$$\begin{aligned} \sum_{\chi \in \psi \mathsf{T}^m} |\widehat{g}_{0,m}(\chi)| \le & \sum_{\chi \in \psi \mathsf{T}^m} \widehat{g}'_{0,m}(\chi) \\ =& \big[(\bar{\psi} g'_{0,m}) * \lambda_m\big](0) \\ =& E_{\mu'}(\bar{\psi} \mid \overline{\mathfrak{T}}^m) \\ \le & 1. \end{aligned} \tag{1}$$

Since $\widehat{g}_{0,m} \in \mathfrak{l}_1(\mathsf{X})$ and since the sequence $(\widehat{g}_{0,n})_n$ converges boundedly to $\widehat{\mu}$, we have for any $m \ge 0$

$$\lim_n \sum_{\substack{\chi \in \psi \mathsf{T}^m \\ \chi \ne \psi}} \widehat{g}_{0,m}(\chi)\widehat{g}_{0,n}(\bar{\chi}) = \sum_{\substack{\chi \in \psi \mathsf{T}^m \\ \chi \ne \psi}} \widehat{g}_{0,m}(\chi)\widehat{\mu}(\bar{\chi}). \tag{2}$$

Condition (3.9(b)) now follows from (2), (1), (α), and (β).

The proof of the following corollary uses a refinement of an argument appearing in Brown [1], p. 235. $|E|$ stands for the cardinal number of a (finite) set E.

(3.12) Corollary. *Let (g_n) be a sequence in $\mathfrak{C}_1^+(G)$ such that $\operatorname{supp}\widehat{g}_n \subseteq \mathsf{T}_n$ for all n and let μ be a generalized Riesz product generated by (g_n). Suppose that*

(α) $\bigcap_{m \ge 0} \mathsf{T}^m = \{1\}$,

(β) $\sup |\{(\chi_k) \in \Omega_m \mid \psi \ne \prod_{k=0}^{m-1} \chi_k \in \psi \mathsf{T}^m\}| < \infty$

for all $\psi \in \mathsf{X}$, and that

(γ) *there is a constant $\gamma < 1$ such that $\|\widehat{g_n - 1}\|_\infty \le \gamma$ for almost all n.*

Then μ is trivial on $\overline{\mathcal{T}}^{\infty}$.

Proof. Given a character ψ, we define

$$\Lambda_m := \left\{ (\chi_k) \in \Omega_m \mid \psi \neq \prod_{k=0}^{m-1} \chi_k \in \psi \mathsf{T}^m \right\}$$

and

$$l_m := \min_{(\chi_k)\in\Lambda_m} |\{k \mid \chi_k \neq 1\}| \quad (m \geq 1).$$

We will show that

$$\lim_{m\to\infty} l_m = \infty. \tag{1}$$

Let m and n be two integers such that $1 \leq m \leq n$ and let $(\chi_0, \dots, \chi_{n-1}, 1, 1, \dots) \in \Lambda_n$. We then have

$$\prod_{k=0}^{m-1} \chi_k = \prod_{k\geq 0} \chi_k \prod_{k\geq m} \bar{\chi}_k \in \psi \mathsf{T}^n \mathsf{T}^m = \psi \mathsf{T}^m. \tag{2}$$

If $\prod_{k=0}^{m-1} \chi_k$ were equal to ψ, we would have

$$\psi \prod_{k\geq m} \chi_k \in \psi \mathsf{T}^n$$

and Lemma (3.5(a)) would imply

$$\prod_{k\geq m} \chi_k = 1,$$

contrary to

$$\prod_{k\geq 0} \chi_k \neq \psi.$$

Thus we have

$$\prod_{k=0}^{m-1} \chi_k \neq \psi$$

and (2) implies that $(\chi_0, \dots, \chi_{m-1}, 1, 1, \dots) \in \Lambda_n$.

Given an integer $m \geq 1$, Assumptions (α) and (β) together imply that there exists an integer $n > m$ such that $\Lambda_m \cap \Lambda_n = \emptyset$. The reasoning above now shows that $l_n > l_m$, i.e. (1) holds.

Let c_ψ be the supremum in (β). Using (β) and (γ), we may estimate as follows.

$$\begin{aligned}
|\sum_{\substack{\chi\in\psi\mathsf{T}^m \\ \chi\neq\psi}} \widehat{g}_{0,m}(\chi)\widehat{g}_{0,n}(\bar{\chi})| &\leq \sum_{\substack{\chi\in\psi\mathsf{T}^m \\ \chi\neq\psi}} |\widehat{g}_{0,m}(\chi)| \\
&= \sum_{\substack{\chi\in\psi\mathsf{T}^m \\ \chi\neq\psi}} \left| \sum_{(\chi_k)\in\Omega_m(\chi)} \prod_{k=0}^{m-1} \widehat{g}_k(\chi_k) \right| \\
&\leq \sum_{(\chi_k)\in\Lambda_m} \left| \prod_{k=0}^{m-1} \widehat{g}_k(\chi_k) \right| \\
&\leq \sum_{(\chi_k)\in\Lambda_m} \gamma^{l_m} \\
&\leq c_\psi \gamma^{l_m}.
\end{aligned} \tag{3}$$

Condition (3.9(b)) now follows from (3) and (1).

Theorem (2.1) together with Proposition (3.9) and its corollaries give rise to several similar dichotomy theorems for Riesz products and generalized Riesz products. We will here state two sample results. We first give a sufficient condition for Assmption (2.1(δ)).

(4.1) LEMMA. *Let (T_n) be a 1–dissociate sequence of symmetric subsets of X and let (f_n) and (g_n) be two sequences in $\mathfrak{I}_1^+(G)$ such that* $\operatorname{supp}\widehat{f}_n \subseteq \mathsf{T}_n$ *and* $\operatorname{supp}\widehat{g}_n \subseteq \mathsf{T}_n$ *for all n. Let μ_f and μ_g be the generalized Riesz products generated by (f_n) and (g_n), respectively. Then* $\lim_{m,n} f_{0,m} g_{m,n} \lambda = \mu_f$.

Proof. Let $\chi \in \mathsf{X}$. By [9], (3.3), the family

$$\Big(\prod_{k \geq 0} \widehat{f}_k(\chi_k) \Big)_{(\chi_k) \in \Omega(\chi)}$$

is absolutely summable and

$$\widehat{\mu}_f(\chi) = \sum_{(\chi_k) \in \Omega(\chi)} \prod_{k \geq 0} \widehat{f}_k(\chi_k).$$

The lemma now follows from the equality

$$(f_{0,m} g_{m,n})^\wedge(\chi) = \sum_{(\chi_k) \in \Omega_n(\chi)} \prod_{k=0}^{m-1} \widehat{f}_k(\chi_k) \prod_{k=m}^{n-1} \widehat{g}_k(\chi_k).$$

We next give a first dichotomy theorem for generalized Riesz products. Notation is as in (3.4).

(4.2) THEOREM. *Let (T_n) be a factorizing sequence of symmetric subsets of X containing the character 1 and let (f_n) and (g_n) be two sequences in $\mathfrak{I}_1^+(G)$ such that* $\operatorname{supp}\widehat{f}_n \subseteq \mathsf{T}_n$ *and* $\operatorname{supp}\widehat{g}_n \subseteq \mathsf{T}_n$ *for all n. Let μ_f and μ_g be the generalized Riesz products generated by (f_n) and (g_n), respectively. Suppose that*

(α) $\bigcap_{m \geq 0} \mathsf{T}^m = \{1\}$,

(β) *$\widehat{\mu}_g$ vanishes at infinity, and that*

(γ) *the functions g_n are nowhere zero.*

(a) The following statements are equivalent.

(i) μ_f is absolutely continuous with respect to μ_g;

(ii) μ_f and μ_g are not mutually singular;

(iii) there is an $\varepsilon > 0$ such that, for all $n \geq 0$, we have $\int (f_{0,n}/g_{0,n})^{\frac{1}{2}} d\mu_g \geq \varepsilon$;

(iv) $\lim_{m,n \to \infty} \int (f_{m,n}/g_{m,n})^{\frac{1}{2}} d\mu_g = 1$.

(b) If the equivalent statements (i)–(iv) hold, then the sequence $(f_{0,n}/g_{0,n})$ converges in $\mathcal{L}_1(\mu_g)$ to the Radon–Nikodym density $d\mu_f/d\mu_g$.

Proof. Since $\mathfrak{T}^\infty \subseteq \overline{\mathfrak{T}}^\infty$, (2.1($\alpha$)) follows from Corollary (3.11), α, and β. By Lemma (3.8) and by (3.7(2)), we have $E_\lambda(f_n \mid \overline{\mathfrak{T}}^{n+1}) = f_n * \lambda_{n+1} = 1$ and similarly

$$E_\lambda(g_n \mid \overline{\mathfrak{T}}^{n+1}) = 1.$$

Thus (2.1(β)) is satisfied. Since (2.1(δ)) is satisfied by Lemma (4.1), the theorem follows from Theorem (2.1).

(4.3) *Explanation.* Theorem (4.2) has only little interest in the case of Riesz products over a dissociate set. In this case, Assumptions (4.2(β)) and (4.2(γ)) imply $\inf_{n,x} g_n(x) > 0$, and the Brown–Moran–Peyrière dichotomy theorem (see (1.1)) applies without assuming factorization. Using Corollary (3.12) we, however, obtain a theorem that gives new information even in the classical case of Riesz products over $(4^k)_{k \geq 0}$.

(4.4) THEOREM. *Let (T_n) be a factorizing sequence of symmetric subsets of X containing the character 1 and let (f_n) and (g_n) be two sequences in $\mathfrak{I}_1^+(G)$ such that* $\operatorname{supp}\widehat{f}_n \subseteq \mathsf{T}_n$ *and* $\operatorname{supp}\widehat{g}_n \subseteq \mathsf{T}_n$ *for all n. Let μ_f and μ_g be the generalized Riesz products generated by (f_n) and (g_n), respectively. Suppose that*

(α) $\bigcap_{m\geq 0} \mathsf{T}^m = \{1\}$,

(β) $\sup_{m\geq 1} \left|\left\{(\chi_k) \in \Omega_m \mid \psi \neq \prod_{k=0}^{m-1} \chi_k \in \psi\mathsf{T}^m\right\}\right| < \infty$ *for all* $\psi \in \mathsf{X}$,

(γ) *there is a constant* $\gamma < 1$ *such that* $\|\widehat{g_n - 1}\|_\infty \leq \gamma$ *for almost all* n, *and that*

(δ) *the functions* g_n *are nowhere zero.*

Then Statements (4.2(a)) and (4.2(b)) hold.

Proof. As in the proof of (4.2), (2.1(β)) follows from (3.8) and (3.7(2)). The theorem now follows from (2.1), (3.12), and Lemma (4.1).

(4.5) *Illustrations.* (a) Let (r_n) be as in (3.3(b)) and let

$$\mu_a := \prod_{n\geq 0}(1 + a_n \exp(2\pi i r_n t) + \bar{a}_n \exp(-2\pi i r_n t)),$$
$$\mu_b := \prod_{n\geq 0}(1 + b_n \exp(2\pi i r_n t) + \bar{b}_n \exp(-2\pi i r_n t)) \qquad (0 \leq t < 1)$$

be two Riesz products over (r_n), where (a_n) and (b_n) are sequences of complex numbers of modulus $< \frac{1}{2}$. Then Theorem (4.4) says that μ_a is equivalent to μ_b or these two measures are mutually singular and give necessary and sufficient conditions for the occurrence of both cases. To show (4.4(β)) note that

$$\sum_{k=0}^{m-1} r_k \leq \sum_{k=0}^{m-1}(r_{k+1} - r_k) < r_m,$$

so that the supremum is at most 2. This result is also a sharpening of Peyrière's result [7] on Riesz products over the 1–dissociate sequence $(2^k)_{k\geq 0}$.

(b) Let λ be Lebesgue measure on $[0,1]$ and let $r_1 := \operatorname{sgn}\sin 2\pi x$ $(0 \leq x \leq 1)$ be the first Rademacher function. It is plain that $\mu_f := \lambda$ and $\mu_g := (1 + r_1)\lambda$ are Riesz products on $[0,1]$ (identified with the Cantor group $\{0,1\}^{\mathbb{N}}$ in the usual way). Since μ_f is neither absolutely continuous nor singular with respect to μ_g, this example shows that Assumptions (4.2(γ)) and (4.4(δ)) may not be omitted. In some cases, however, these assumptions may be relaxed. Suppose that the sequence (T_n) is 2–dissociate (see(1.2)) as well as factorizing and replace (4.4(γ)) by the stronger condition

(α) there exist two constants $p \geq 1$ and $K < 2$ such that $\|\widehat{g}_n\|_p^p \leq K$ almost all n. Satz (5.3) in [9] then asserts that μ_g and all its tail measures are continuous. Hence Condition (2.1(γ)) is satisfied if the sets $\{g_n = 0\}$ are countable, i.e. Theorem (4.4) remains true in this case. In particular, we have

(4.6) Proposition. *Any two Riesz products on* $\mathbb{T}$ *over a dissociate set of the form* $\{r^n \mid n \geq 0\}$ $(r \geq 3)$ *are either equivalent or mutually singular and the criteria for absolute continuity given in (4.2) are valid. In this case the ratios* $(f_{0,n}/g_{0,n})$ *converge in* $\mathfrak{L}_1(\mu_b)$ *to the Radon–Nikodym derivative* $(d\mu_a/d\mu_b)$.

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