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On Kakutani's Dichotomy Theorem for Infinite Products of not Necessarily Independent Functions

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1 Introduction

(1.1) Background. The background for the present communication is Kakutani's famous dichotomy theorem [10], viz.: if $(\Omega_n, \mathcal{F}^n)$ is a sequence of measurable spaces and P_n and Q_n are probability measures on $(\Omega_n, \mathcal{F}^n)$ such that Q_n is absolutely continuous with respect to P_n for all n, then either the product measure $\otimes Q_n$ is absolutely continuous with respect to the product measure $\otimes P_n$, or else these two measures are mutually singular. The first case occurs if and only if

$$\prod_n \int (dQ_n/dP_n)^{1/2} dP_n > 0.$$

It is interesting to ask, and for certain applications useful to know, whether the strong independence assumptions underlying Kakutani's theorem can be weakened. An indication that this is possible is provided by the Brown–Moran–Peyrière dichotomy theorem for Riesz products, [4], [12] and the author's generalization [14] of this theorem. In these papers, independence is replaced by group–theoretic dissociativity as introduced by Hewitt–Zuckerman [7]; see also [6]. Other results of this genre are the Jessen–Wintner purity law [9] (see also [19], p. 89 and [17], p. 98), its generalizations by Brown–Moran [3] and Kanter [11], and the results on extremal measures in Skorochod [16], §23. These theorems use ergodicity instead of independence.

The Brown–Moran–Peyrière theorem is proved by presenting a sufficient condition for equivalence and other condition for mutual singularity. If the set of Riesz products is suitably restricted, the criteria become complementary, thus yielding a dichotomy theorem for this set of measures. In contrast to this, the other theorems mentioned above are intrinsic dichotomy theorems, their proofs making essential use of 0–1 laws.

Brown [2] showed that certain Riesz products are ergodic. On the other hand it is well known that the dichotomy behavior of infinite products of measures can be derived from Kolmogorov's 0–1 law. As infinite products of measures as well as Riesz products are infinite products of functions in the sense of (2.4), it is natural to ask if one can find a dichotomy theorem for infinite products of functions, using 0–1 laws. We here give an affirmative answer to this question. We will deal with pure dichotomy statements as well as with necessary and sufficient conditions for the occurence of both cases. Some related topics are also taken up.

Kakutani's theorem and other related theorems, such as the Feldman–Hajek dichotomy theorem for Gaussian measures (see e.g. [16]) are proved by studying certain martingales arising from the projective system of the finite dimensional distributions of the measures. This method does not apply in our case of infinite products of functions since they are not projective limits in general. We have recourse to a different method using tail properties of infinite products of functions.

Most of the material presented here is couched in terms of infinite products of continuous functions on a compact space. It is possible to generalize almost all the results to the nontopological case. A discussion of this point is given in (7.8) and a sample theorem is stated in (7.9). This theorem contains Kakutani's theorem as a special case.

Applications, in particular to Riesz products and generalized Riesz products, will be given in a forthcoming paper [15].

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(1.2) Outline. Section 2 introduces the notion of an infinite product of uncorrelated functions. In Sects. 3 and 4, we deal with sufficient conditions for two such products to be absolutely continuous or mutually singular. In Sect. 5, we prove a fairly general dichotomy theorem for measures on measurable spaces. Section 6 is devoted to an application of this theorem to infinite products, while Sect. 7 deals with criteria for orthogonality and absolute continuity in the case of dichotomy of infinite products.

2 Infinite Products of Uncorrelated Functions

(2.1) Definitions. Let $(\Omega, \mathcal{F}, P)$ be a probability space, let A be an index set, let $\mathcal{E}$ be the net of all finite subsets of A, and let $(f_\alpha)_{\alpha\in A}$, $(g_\alpha)_{\alpha\in A}$, and $(f_{\alpha,l})_{\alpha\in A}$ $(1 \le l \le n)$ be families in $\mathcal{L}_1(P)$ such that the integrals in (a)–(d) make sense.

(a) We will say that (f_α) is (P-)*uncorrelated*, if, for all $\Phi \in \mathcal{E}$, we have

$$\int \prod_{\alpha\in\Phi} f_\alpha dP = \prod_{\alpha\in\Phi} \int f_\alpha dP.$$

(b) We will say that $(f_{\alpha,1}), \ldots, (f_{\alpha,n})$ are jointly (P-)*uncorrelated*, if, for all $\Phi \in \mathcal{E}$ and all mappings $\tau : \Phi \to [1, n]$, we have

$$\int \prod_{\alpha\in\Phi} f_{\alpha,\tau(\alpha)} dP = \prod_{\alpha\in\Phi} \int f_{\alpha,\tau(\alpha)} dP.$$

(c) We will say that (f_α) is *square* (P-)*uncorrelated*, if (f_α) and (f_α^2) are jointly uncorrelated.

(d) We will say that (f_α) and (g_α) are *jointly square* (P-)*uncorrelated*, if (f_α), (g_α), (f_α^2), (g_α^2), and $(f_\alpha g_\alpha)$ are jointly uncorrelated.

(2.2) Examples. (a) Examples of (square) uncorrelated families that are not independent occur in a natural way in Fourier analysis. If Δ is a dissociate subset in the dual group X of a compact Abelian group G, and if $(a_\chi)_{\chi\in\Delta}$ and $(b_\chi)_{\chi\in\Delta}$ are two families of complex numbers such that $|a_\chi| \le \frac{1}{2}$ and $|b_\chi| \le \frac{1}{2}$, then the families $(f_\chi)_{\chi\in\Delta}$ and $(g_\chi)_{\chi\in\Delta}$, where $f_\chi := (1 + a_\chi\chi + \bar{a}_\chi\chi^{-1})$ and $g_\chi := (1 + b_\chi\chi + \bar{b}_\chi\chi^{-1})$, are jointly square uncorrelated with respect to Haar measure on G. More generally, if $(T_\alpha)_{\alpha\in A}$ is a 1–dissociate system (see [14]) in X and if $(f_\alpha)_{\alpha\in A}$, $(g_\alpha)_{\alpha\in A}$ are two families of continuous functions on G such that the supports of the Fourier transform $\hat{f}_\alpha$ and $\hat{g}_\alpha$ are contained in T_α for all $\alpha \in A$, then the families (f_α) and (g_α) are jointly uncorrelated with respect to Haar measure on G.

(b) Affine transforms of jointly (square) uncorrelated families are jointly (square) uncorrelated. More explicitely, let $(f_{\alpha,1}), \ldots, (f_{\alpha,n})$ be n jointly uncorrelated families and let $l_{\alpha,k}$ have the form

$$l_{\alpha,k} = b_{\alpha,k,1}f_{\alpha,1} + \cdots + b_{\alpha,k,n}f_{\alpha,n} + c_{\alpha,k}$$

for $\alpha \in A$ and $k \in [1, t]$, where the $b_{\alpha,k,l}$'s and $c_{\alpha,k}$'s are complex numbers. Then the families $(l_{\alpha,1}), \ldots, (l_{\alpha,t})$ are jointly uncorrelated. Furthermore, let (f_α) and (g_α) be two jointly square uncorrelated families and let $m_{\alpha,k}$ have the form

$$m_{\alpha,k} = a_{\alpha,k}f_\alpha + b_{\alpha,k}g_\alpha + c_{\alpha,k}$$

for $\alpha \in A$ and $k \in \{1, 2\}$. Then the two families $(m_{\alpha,1})$ and $(m_{\alpha,2})$ are jointly square uncorrelated. The first of these propositions is proved by a straightforward computation, while the second one follows from it.

(c) There is an intimate connection between (strongly) multiplicative systems (see e.g. [17]) and (square) uncorrelated families. Every (strongly) multiplicative system is (square) uncorrelated. On the other hand, it follows from (b) that (f_α) is (square) uncorrelated if and only if $(f_\alpha - \int f_\alpha)$ is (strongly) multiplicative.

(d) Let $(\mathcal{T}^n)_{n\geq 0}$ be a decreasing sequence of sub–σ–algebras of $\mathcal{F}$ and let $(f_n)_{n\geq 0}$ and $(g_n)_{n\geq 0}$ be two sequences in $\mathcal{L}_1^+(P)$ adapted to $(\mathcal{T}^n)$. Suppose that $E_p(f_n \mid \mathcal{T}^{n+1}) = 1 = E_p(g_n \mid \mathcal{T}^{n+1})$ for all $n \geq 0$. (The symbol E_p indicates the conditional expectation with respect to the measure P.) Then the sequences (f_n) and (g_n) are jointly uncorrelated. There is a similar example with an increasing instead of a decreasing sequence of sub–σ–algebras.

(2.3) Notation. For the sake of simplicity we will restrict matters mainly to the case of a compact space K with its Borel σ–algebra $\mathcal{B}$ and a normed Radon measure P on K. By the standard extension procedure P will be considered a regular Borel measure on K. $\mathcal{C}(K)$ will mean the space of all continuous functions on K and we define

$$\mathcal{C}_1^+(K) := \{f \in \mathcal{C}(K) \mid f \geq 0, \int f dP = 1\}.$$

$\mathcal{M}_1^+(K)$ will denote the set of all normed Radon measures on K. Throughout, we will consider weak* topology on $\mathcal{M}_1^+(K)$.

If (f_α) is a family of complex–valued functions on K and if $\Phi \in \mathcal{E}$, we will write $f_\Phi := \prod_{\alpha\in\Phi} f_\alpha$.

Now suppose that (f_α) is an uncorrelated family in $\mathcal{C}_1^+(K)$. Then $f_\Phi P$ is obviously a normed Radon measure on K.

(2.4) Definition. Let (f_α) be an uncorrelated family in $\mathcal{C}_1^+(K)$. Each weak* cluster point of the net $(f_\Phi P)_{\Phi\in E} \subseteq \mathcal{M}_1^+(K)$ will be called an *infinite product generated by the family* (f_α).

(2.5) Counterexample. In general, the weak* limit of $(f_\Phi P)$ does not exist, not even in the independent case. The following example is due to R. Blumenthal (oral communication). I am grateful to him for permitting me to include it here.

Consider a P–independent sequence $(f_n) \subseteq \mathcal{C}_1^+(K)$ such that the sequence $(\prod_{l=0}^n f_l)$ converges to 0 P–a.e. (this is true if $\prod_{l=0}^\infty \int f_l^{\frac{1}{2}} dP = 0$). It follows that this sequence is not uniformly integrable and there exists a measurable set B such that the sequence

$$\left(\int_B \prod_{l=0}^n f_l dP\right) \geq 0 \tag{1}$$

diverges as n goes to infinity (see e.g. [5], 4.21.1). We now define $K' := K \times \{0, 1\}$ and two mappings $\theta : K \to K'$ and $\delta : K' \to K$ by $\theta(\chi) := (x, 1_B(x))$ and $\delta(\chi, y) := \chi$. We carry the structure on K over to K' by putting $P' := P_0$ and $f_n' = f_n \circ \delta$. As $\delta \circ \theta(\chi) = \chi(\chi \in K)$, the sequence (f_n') is P'–independent. On the other hand, $\phi : K' \to \mathbb{R}$ defined by $\phi(\chi, y) := y$ is a continuous function on K', we have $\phi\circ\theta = 1_B$, and therefore

$$\int \phi \prod_{l=0}^n f_l' dP' = \int_B \prod_{l=0}^n f_l dP$$

diverges in view of (1), i.e. $(\prod_{l=0}^n f_l')\, P'$ does not converge in the weak* sense.

We will make constant use of the following properties of the supports of infinite products.

(2.6) Lemma. (a) *Let (f_α) be uncorrelated and let Q_f be an infinite product generated by (f_α). Then, for any $\Phi \in \mathcal{E}$,*

(i) *$Q_f\{f_\Phi < \varepsilon\} \leq \varepsilon$ and*

(ii) *the set $\{f_\Phi > 0\}$ is a support of Q_f.*

(b) *Let* (f_α) *and* (g_α) *be jointly uncorrelated and let* Q_f *be an infinite product generated by* (f_α)*. Then we have for any* $\Phi \in \mathcal{E}$

(iii) $Q_f\{f_\Phi/g_\Phi < \varepsilon\} \le \varepsilon$.
(*Here* $\frac{f_\Phi}{g_\Phi}$ *is defined as 0 where* f_Φ *and* g_Φ *vanish.*)

Proof. Let $Q_f = \lim_\sigma f_{\Phi_\sigma} P$. Then (i) follows from the estimates

$$Q_f\{f_\Phi < \varepsilon\} \le \liminf_\sigma \int_{\{f_\Phi<\varepsilon\}} f_{\Phi_\sigma} dP \le \liminf_\sigma \int_{\{f_\Phi<\varepsilon\}} f_\Phi f_{\Phi_\sigma\setminus\Phi} dP \le \varepsilon, \tag{1}$$

while (ii) is an immediate consequence of (i). In order to prove (iii), one proceeds as in the proof of (1) to show that

$$Q_f\{0 < f_\Phi/g_\Phi < \varepsilon\} \le \varepsilon. \tag{2}$$

Estimate (iii) then follows from (2) and (ii).

(2.7) Definition. Let Q_g be an infinite product generated by an uncorrelated family $(g_\alpha)_{\alpha\in A}$ and let $\Phi \in \mathcal{E}$. Any cluster point R of $g_{\Psi_\tau\setminus\Phi} P)_\tau$, where (Ψ_τ) is any subset of E such that $Q_g = \lim_\tau g_{\Psi_\tau} P$, will be called a *tail measure* for Q_g and Φ.

(2.8) Remarks. (a) It is plain that Q_g is absolutely continuous with respect to R with Radon–Nikodym derivative g_Φ. In particular, any other tail measure R' for Q_g and Φ coincides with R on the set $\{g_\Phi > 0\}$.

(b) Now let $\Phi \in \mathcal{E}$ and suppose that we have $S\{g_\Phi = 0\} = 0$ for all infinite products S generated by $(g_\alpha)_{\alpha\in A\setminus\Phi}$. Any tail measure R for Q_g and Φ is then supported by the set $\{g_\Phi > 0\}$ and, by (a), there is only one such measure R. R is equivalent to Q_g. If $Q_g = \lim_\tau g_{\Psi_\tau} P$, then $R = \lim_\tau g_{\Psi_\tau\setminus\Phi} P$.

(2.9) Notation. The uniquely defined tail measure in (2.8.b) will be denoted by $R_{g,\Phi}$.

(2.10) Lemma. *Let* (f_α) *and* (g_α) *be jointly uncorrelated, let* $\Phi \in \mathcal{E}$*, and suppose that*

(α) $S\{g_\Phi = 0\} = 0$ *for all infinite products* S *generated by* $(g_\alpha)_{\alpha\in A\setminus\Phi}$*. Then we have*

(i) $(f_\Phi/g_\Phi)Q_g = f_\Phi R_{g,\Phi}$ *and*

(ii) $\int f_\Phi/g_\Phi dQ_g = 1$.

Proof. Using (2.8.a) we may write

$$f_\Phi Q_g = f_\Phi g_\Phi R_{g,\Phi}.$$

Equality (i) now follows from (2.6.ii) and (α), while (ii) follows from (i).

(2.11) Remarks. (a) We will repeatedly use the inequality $\int f_\Phi/g_\Phi dQ_g \le 1$, which holds without assuming (2.10.α).

(b) In the context of sequences instead of families of functions it is suitable to slightly restrict the notion of an infinite product. By an *infinite product* Q_g generated by an uncorrelated *sequence* $(g_n)_{n\ge 0}$ in $\mathcal{C}_1^+(K)$ we mean any weak* cluster point of the sequence $(g_{[0,n[}P)_{n\ge 0}$.

Let $m \ge 0$. Any cluster point R of $(g_{[m,n_\tau[}P)_\tau$, where (n_τ) is any subset of $\mathbb{N}$ such that $Q_g = \lim_\tau g_{[m,n_\tau[}P$, will be called a *tail measure* for Q_g and m. If we have $S\{g_{[0,m[} = 0\} = 0$ for all infinite products S generated by $(g_n)_{n\ge m}$, it is plain that any tail measure R for Q_g and m is then supported by the set $\{g_{[0,m[} > 0\}$ and there is only one such measure R (cf. (2.8)); it will be denoted by $R_{g,m}$. $R_{g,m}$ is equivalent to Q_g and if $Q_g = \lim_\tau g_{[0,n_\tau[}P$, then $R_{g,m} = \lim_\tau g_{[m,n_\tau[}P$.

3 Absolute Continuity of Infinite Products

(3.1) Explanation. In this section we compile some results about absolute continuity of infinite products that can be proved by well known methods. (f_α) and (g_α) will denote two jointly uncorrelated families in $\mathcal{C}_1^+(K)$. Q_f and Q_g will denote two infinite products generated by (f_α) and (g_α), respectively.

(3.2) Lemma. *Suppose that*

(α) $\lim_{\Phi,\Psi} \int (f_{\Psi\setminus\Phi}/g_{\Psi\setminus\Phi})^{1/2} dQ_g = 1.$

Then the net $(f_\Phi/g_\Phi)^{1/2}$ converges in $\mathcal{L}_1(Q_g)$ to a function $h \in \mathcal{L}_2(Q_g)$.

Proof. We use the Cauchy–Schwarz inequality and (2.11.a) to estimate for $\Psi \supseteq \Phi$

$$\begin{aligned}[\int |(f_\Phi/g_\Phi)^{1/2} - (f_\Psi/g_\Psi)^{1/2}| dQ_g]^2 =& [\int (f_\Phi/g_\Phi)^{1/2} |1 - (f_{\Psi\setminus\Phi}/g_{\Psi\setminus\Phi})^{1/2}| dQ_g]^2 \\ \leq& 2(1 - \int (f_{\Psi\setminus\Phi}/g_{\Psi\setminus\Phi})^{1/2} dQ_g).\end{aligned}$$

Hence $(f_\Phi/g_\Phi)^{1/2}$ is a Cauchy net in $\mathcal{L}_1(Q_g)$ by (α). Again making use of (2.11.a) we see that this net is bounded in $\mathcal{L}_2(Q_g)$. The rest now follows from weak compactness of the unit ball in $\mathcal{L}_2(Q_g)$.

(3.3). We now deal with a sufficient condition for Q_f to be absolutely continuous with respect to Q_g. The method of its proof goes back to Kakutani [10] and has been applied to many other cases.

(3.4) Proposition. (a) *Suppose that (f_α), (g_α) and Q_g satisfy either the hypothesis*

(α) $\lim_{\Phi,\Psi} \int (f_\Phi/g_\Phi)(f_{\Psi\setminus\Phi}/g_{\Psi\setminus\Phi})^{1/2} dQ_g = 1$

or the hypotheses

(β) $\lim_{\Phi,\Psi} \int (f_{\Psi\setminus\Phi}/g_{\Psi\setminus\Phi})^{1/2} dQ_g = 1$

and (f_Φ/g_Φ) is uniformly Q_g–integrable.

Then the net (f_Φ/g_Φ) converges in $\mathcal{L}_1(Q_g)$.

(b) *If the net (f_Φ/g_Φ) converges in $\mathcal{L}_1(Q_g)$ then the following three statements are equivalent.*

(i) *Q_f is absolutely continuous with respect to Q_g and has Radon–Nikodym density* $\lim_\Phi f_\Phi/g_\Phi$*;*

(ii) $\lim_\Phi (f_\Phi/g_\Phi) Q_g = Q_f$;

(iii) *there exists a subnet (Φ_σ) of $\mathcal{E}$ such that* $\lim_\sigma (f_{\Phi_\sigma}/g_{\Phi_\sigma}) Q_g = Q_f$.

Proof. We use (2.11.a) to estimate for $\Psi \supseteq \Phi$

$$\int ((f_\Phi/g_\Phi)^{1/2} - (f_\Psi/g_\Psi)^{1/2})^2 dQ_g \leq 2(1 - \int (f_\Phi/g_\Phi)(f_{\Psi\setminus\Phi}/g_{\Psi\setminus\Phi})^{1/2} dQ_g.$$

Hence hypothesis (α) is equivalent to the convergence of the net $(f_\Phi/g_\Phi)^{1/2}$ in $\mathcal{L}_2(Q_g)$. But as all our functions are ≥ 0, this is equivalent to the convergence of (f_Φ/g_Φ) in $\mathcal{L}_1(Q_g)$.

If (β) is satisfied then, by Lemma (3.2), the net $(f_\Phi/g_\Phi)^{1/2}$ converges Q_g–stochastically. Hence (f_Φ/g_Φ) converges Q_g–stochastically and the assertion follows from uniform integrability.

Part (b) of the proposition is evident.

(3.5) Remarks. (a) Statement (3.4.iii) is of course satisfied when all the g_α's are 1.

(b) It may happen that the net (f_Φ/g_Φ) converges in $\mathcal{L}_1(Q_g)$ and its limit is not the Radon–Nikodym–density of Q_f with respect to Q_g. To obtain an example it is sufficient to consider a family $(f_\alpha) = (g_\alpha)$ that generates two infinite products Q_f and Q_g (cf. (2.5)).

(c) It may occasionally be tedious to verify Assumption (3.4.α) or (3.4.β). There is an "additive" condition for absolute continuity that is more convenient for applications. In some cases, this condition is equivalent to the "multiplicativite" condition (3.4.α), but it is weaker in general. This is in particular the case when the families of real numbers $(\inf_\chi(f_\alpha + g_\alpha)(\chi))_\alpha$ and $(\sup_\chi(f_\alpha + g_\alpha)(\chi))_\alpha$ are not bounded away from the marginal values 0 and ∞ respectively; see Brown–Moran [4] for a discussion of this point. Analogous criteria were given by Brown–Moran [4], Peyrière [13], and the author [14] for Riesz–products and generalized Riesz–products. The following proposition is proved in a similar way as [14], Satz 4.4, using tail measures. We will omit the details. $\| \|_u$ stands for the uniform norm.

(3.6) Proposition. *Suppose that*

(α) $\sum_{\alpha \in A} \|f_\alpha - g_\alpha\|_u^2 / \inf(f_\alpha + g_\alpha) < \infty$.

Then the net (f_Φ/g_Φ) converges in $\mathcal{L}_1(Q_g)$.

(3.7) Remark. If the families (f_α) and (g_α) are jointly square uncorrelated (2.1.d), one can replace (3.6.α) by the weaker condition

(α) $\sum_{\alpha \in A} \|f_\alpha - g_\alpha\|_2^2 / \inf(f_\alpha + g_\alpha) < \infty$,

where $\| \|_2$ is computed with respect to the measure P. The proof uses (2.2.b) and is the same as the proof of [4], Satz (4.4).

4 Mutual Orthogonality of Infinite Products

(4.1) Explanation. As in Sect. 3, (f_α) and (g_α) will mean two jointly uncorrelated families in $\mathcal{C}_1^+(K)$ and Q_f and Q_g will denote two infinite products generated by (f_α) and (g_α), respectively.

We will now deal with a general condition for mutual singularity of two infinite products.

(4.2) Proposition. *Suppose that for any $\varepsilon > 0$ there exists a set $\Phi \in \mathcal{E}$ such that*

(α) $\int (f_\Phi/g_\Phi)^{1/2} dQ_g \leq \varepsilon$.

Then Q_f and Q_g are mutually singular.

Proof. It follows from Hypothesis (α) that, for any $n \geq 1$, there exists a set $\Phi_n \in \mathcal{E}$ such that

$$\int (f_\Phi/g_\Phi)^{1/2} dQ_g \leq n^{-2}. \tag{1}$$

Using the Chebyshev–Markov inequality, we deduce from (1)

$$Q_g\{f_{\Phi_n}/g_{\Phi_n} < n^{-2}\} \geq 1 - n^{-1}. \tag{2}$$

On the other hand, by (2.6.iii), we have

$$Q_f\{f_{\Phi_n}/g_{\Phi_n} < n^{-2}\} \leq n^{-2}. \tag{3}$$

It finally follows from (2) and (3) that the set

$$U := \bigcap_{m \in N} \bigcup_{n \geq m} \{f_{\Phi_n}/g_{\Phi_n} < n^{-2}\}$$

is a support of Q_g, while it is a null set for Q_f.

(4.3) Remarks. (α) Hypothesis (4.2.α) implies that the net $(f_\Phi/g_\Phi)^{1/2}$ converges to 0 in $\mathcal{L}_1(Q_g)$. To see this, let $\varepsilon > 0$ and choose $\Phi \in \mathcal{E}$ such that

$$\int (f_\Phi/g_\Phi)^{1/2} dQ_g \leq \varepsilon^2. \tag{1}$$

Given $\Psi \in \mathcal{E}$ such that $\Psi \supseteq \Phi$, we deduce from Hölder's inequality, (2.11.a), and (1) the estimates

$$\begin{aligned}\int (f_\Psi/g_\Psi)^{1/2} dQ_g \leq & [\int (f_\Psi/g_\Psi)^{1/3} dQ_g]^{3/4} [\int (f_\Psi/g_\Psi) dQ_g]^{1/4} \\ \leq & [\int (f_\Psi/g_\Psi)^{1/3} dQ_g]^{3/4} \\ = & [\int (f_\Psi/g_\Psi)^{1/3} (f_{\Psi\setminus\Phi}/g_{\Psi\setminus\Phi})^{1/3} dQ_g]^{3/4} \\ \leq & [\int (f_\Psi/g_\Psi)^{1/2} dQ_g]^{1/2} [\int f_{\Psi\setminus\Phi}/g_{\Psi\setminus\Phi} dQ_g]^{1/4} \\ \leq & \varepsilon.\end{aligned}$$

(b) As in the case of Proposition (3.4) it may be tedious to verify the orthogonality condition (4.2.α). We will therefore mention three other conditions. The first one will be formulated in terms of sequences rather than families.

(4.4) Notation. In the following proposition, $(\mathcal{T}^n)_{n\geq 0}$ denotes a decreasing sequence of σ–algebras generated by continuous functions such that the sequences (f_n) and (g_n) are adapted to $(\mathcal{T}_n)$ (e.g. $\mathcal{T}^n = \mathcal{A}\{f_n, f_{n+1}, \ldots; g_n, g_{n+1}, \ldots\}$). Throughout the sequel, the symbol $E_Q(h/\mathcal{T})$ will denote the conditional expectation of the function h with respect to the σ–algebra $\mathcal{T}$ and the measure Q.

(4.5) Proposition. (a) *If for some $k \geq 1$ we have*

(i) $\prod_{n\geq 0} \operatorname{ess\,sup} E_{Q_g}((f_n/g_n)^{1/2}/\mathcal{T}^{n+k}) = 0$,

then the measures Q_f and Q_g are mutually singular.

(b) *Suppose that the sequences (f_n) and (g_n) satisfy the conditions*

(α) $E_P(f_n/\mathcal{T}^{n+1}) = 1 = E_P(g_n/\mathcal{T}^{n+1})$ $(n \geq 0)$ *and*

(β) *there are two constants $c > 0$ and C such that $g_n \geq c$ and $f_n + g_n \leq C$ for all $n \geq 0$.*

Then for any $k \geq 1$ (i) is equivalent to each of the following two equalities.

(ii) $\prod_{n\geq 0} \operatorname{ess\,inf} E_{Q_g}((f_n/g_n)^2/\mathcal{T}^{n+k}) = \infty$;

(iii) $\sum_{n\geq 0} \operatorname{ess\,inf} E_{Q_g}((f_n - g_n)^2/\mathcal{T}^{n+k}) = \infty$.

Proof. By (i) there is an integer $0 \leq l < k$ such that

$$\prod_{n\geq 0} \operatorname{ess\,sup} E_{Q_g}((f_{nk+l}/g_{nk+l})^{1/2}/\mathcal{T}^{(n+1)k+l}) = 0. \tag{1}$$

Condition (4.2.α) now follows from (1) and the estimate

$$\begin{aligned}&\int \prod_{n=0}^{r} (f_{nk+l}/g_{nk+l})^{1/2} dQ_g \\ \leq & \operatorname{ess\,sup} E_{Q_g}((f_l/g_l)^{1/2}/\mathcal{T}^{k+l}) \int \prod_{n=1}^{r} (f_{nk+l}/g_{nk+l})^{1/2} dQ_g \\ \leq & \ldots \\ \leq & \prod_{n=0}^{r} \operatorname{ess\,sup} E_{Q_g}((f_{nk+l}/g_{nk+l})^{1/2}/\mathcal{T}^{(n+1)k+l}),\end{aligned}$$

completing the proof of (a).

In order to prove (b) we first show that

$$E_{Q_g}((f_n/g_n)/\mathcal{T}^{n+1}) = 1. \tag{2}$$

Note that (α) implies

$$E_P(g_0 \dots g_n/\mathcal{T}^{n+1}) = 1 = E_P(g_0 \dots g_{n-1} f_n/\mathcal{T}^{n+1}). \tag{3}$$

Now let $(g_{0,n_\tau}P)_\tau$ be a subnet of the sequence $(g_{0,n}P)_n$ with weak* limit Q_g. On account of (3), we may write for any continuous, $\mathcal{T}^{n+1}$–measurable function ϕ (note that $g_n > 0$)

$$\begin{aligned}\int \phi f_n/g_n dQ_g =& \lim_\tau \int \phi g_0 \dots g_{n-1} f_n g_{n+1,n_\tau} dP \\ =& \lim_\tau \int \phi g_{n+1,n_\tau} dP \\ =& \lim_\tau \int \phi g_{0,n_\tau} dP \\ =& \int \phi dQ_g,\end{aligned}$$

i.e., (2) holds (cf.(4.4)).

We now show the equivalence of (i)–(iii). By (2) and by Hölder's inequality for conditional expectations we have

$$\begin{aligned}1 =& E_{Q_g}((f_n/g_n)^{2/3}(f_n/g_n)^{1/3}/\mathcal{T}^{n+k}) \\ \le& [E_{Q_g}((f_n/g_n)^2/\mathcal{T}^{n+k})]^{1/3}[E_{Q_g}((f_n/g_n)^{1/2}/\mathcal{T}^{n+k})]^{2/3}.\end{aligned}$$

Thus (ii) holds, if (i) is satisfied.

Now suppose that (ii) is satisfied. Then (iii) immediately follows from the estimate

$$\begin{aligned}1 + c^{-2}E_{Q_g}((f_n - g_n)^2/\mathcal{T}^{n+k}) \ge& 1 + E_{Q_g}((f_n - g_n)^2/g_n^2/\mathcal{T}^{n+k}) \\ =& E_{Q_g}((f_n/g_n)^2/\mathcal{T}^{n+k}).\end{aligned}$$

To prove that (iii) implies (i) note that there exists a constant $K > 0$ such that the inequality

$$(f_n/g_n)^{1/2} \le 1 + \frac{1}{2}(f_n/g_n - 1) - K(f_n/g_n - 1)^2$$

obtains for all $n \ge 0$, as (f_n/g_n) is uniformly bounded by (β). Thus, again using (2) and (β), we have

$$E_{Q_g}((f_n/g_n)^{1/2}/\mathcal{T}^{n+k}) \le 1 - C^{-2}KE_{Q_g}((f_n - g_n)^2/\mathcal{T}^{n+k}),$$

and (i) follows if (iii) is satisfied.

(4.6). There is a second orthogonality condition, like Proposition (4.5), but using ascending instead of descending sequences of σ–algebras. We leave its obvious formulation to the reader. If the families (f_α) and (g_α) are jointly square uncorrelated, one obtains a somewhat better result. It is proved in essentially the same way as the one given in [14] for 2–dissociate systems and uses a method that goes back to [4] and [13]. We state it here.

(4.7) Theorem. *Suppose that the families (f_α) and (g_α) are jointly square uncorrelated and that*

(α) $\sum_{\alpha\in A} \|f_\alpha - g_\alpha\|_2^2/\sup(f_\alpha + g_\alpha) = \infty$.

Then Q_f and Q_g are mutually singular.

(The norm $\| \|_2$ is computed with respect to the measure P.)

5 A General Dichotomy Theorem

(5.1) Terminology. As a major tool to establish dichotomy theorems for measures we will assume the validity of a 0–1 law; we will say that a probability measure Q on a σ–algebra $\mathcal{F}$ is *trivial* on a sub–σ–algebra $\mathcal{T}$ of $\mathcal{F}$, if $Q(T) = 0$ or 1 for all $T \in \mathcal{T}$. Our first dichotomy theorem can be applied to general probability measures on an abstract measurable space $(\Omega, \mathcal{F})$ and reads as follows.

(5.2) Abstract dichotomy theorem. *Let $(\mathcal{T}^n)_{n\geq 0}$ be a decreasing sequence of sub–σ–algebras of $\mathcal{F}$ and let $\mathcal{T}^\infty := \bigcap_{n\geq 0} \mathcal{T}^n$. Let Q_1 be a probability measure and Q_2 be a measure ≥ 0 on $\mathcal{F}$. Suppose that*

- (α) *Q_1 is trivial on $\mathcal{T}^\infty$ and*
- (β) *there exists a support L of Q_1 such that for any Q_2–null set N and any $n \geq 0$ there is a representative v_n of the conditional probability $Q_1(N/\mathcal{T}^n)$ such that $\int_L v_n dQ_2 = 0$.*

Then the following statements are equivalent.

- *(i) Q_1 is absolutely continuous with respect to Q_2;*
- *(ii) Q_1 and Q_2 are not mutually singular.*

Proof. We only have to show that (ii) implies (i). Suppose that there exists a set $N \in \mathcal{F}$ such that

$$Q_2(N) = 0 \tag{1}$$

and

$$Q_1(N) > 0. \tag{2}$$

Let N_n be the $\mathcal{T}^n$–measurable set

$$N_n := \{v_n > 0\}, \tag{3}$$

where v_n is a function as in Hypothesis (β). In particular, we have

$$Q_2(N_n \cap L) = 0. \tag{4}$$

By definition of v_n and N_n we have

$$Q_1(N \setminus N_n) = \int_{\complement N_n} 1_N dQ_1 = \int_{\complement N_n} v_n dQ_1 = 0. \tag{5}$$

We finally define $T := \liminf_{n\to\infty} N_n$. As $T \in \mathcal{T}^\infty$, we deduce from (2), (5), and Hypothesis (α) that $T \cap L$ is a support of Q_1, while (4) implies that $T \cap L$ is a Q_2–null set, i.e., $Q_1 \perp Q_2$.

(5.3) Remarks. (a) Suppose that Hypothesis (5.2.β) is satisfied with $L = \Omega$. Then the set T in the proof of (5.2) separates Q_1 and Q_2. As $T \in \mathcal{T}^\infty$, we have shown that Statements (5.2.i) and (5.2.ii) are equivalent to the following two statements.

- (i) $Q_1 \mid \mathcal{T}^\infty$ is absolutely continuous with respect to $Q_2 \mid \mathcal{T}^\infty$;
- (ii) $Q_1 \mid \mathcal{T}^\infty$ and $Q_2 \mid \mathcal{T}^\infty$ are not mutually singular.
  (To show that (5.2.ii) implies (i) use again (5.2.α).)

(b) Let Δ be a countable group of automorphisms of the measurable space $(\Omega, \mathcal{F})$. A probability measure Q_1 on $\mathcal{F}$ is said to be Δ–*ergodic*, if for any Δ–invariant set $F \in \mathcal{F}$ we have $Q_1(F) = 0$ or 1. A measure $Q_2 \geq 0$ on $\mathcal{F}$ is said to be Δ–*quasi–invariant*, if Q_2 is equivalent to all its images $\delta(Q_2)(\delta \in \Delta)$. It is well known that in this situation either Q_1 is absolutely continuous with respect to Q_2, or these two measures are mutually singular. To see that this result follows from (5.2) let $\mathcal{T}^n = \mathcal{T}$ be the σ–algebra of all Δ–invariant measurable subsets of $\Omega (n \geq 0)$. Then triviality of Q_1 on $\mathcal{T}$ is just ergodicity. On the other hand let N be a Q_2–null set and define the $\mathcal{T}$–measurable set $M := \bigcup_{\delta\in\Delta} \delta(N)$. If $Q_1(N/\mathcal{T})$ is any version of the conditional probability, then $v_n := 1_M Q_1(N/\mathcal{T})$ has the properties required in (5.2.β).

(c) In certain cases of measurable groups and vector spaces dichotomy results of ergodic type are discussed in Brown–Moran [3], Lemma 2, Skorokhod [16], p. 129, Corollary 1, and Kanter [11], Lemma 2.1. These are special cases of (b). All these results contain part of the Jessen–Wintner purity law [9], see also [17]. Of course the dichotomy statement of Kakutani's theorem is also contained in (5.2). A similar result appearing in Jacobs [8] does not seem to be related to tail σ–fields. I thank Prof. K. Jacobs for pointing out the paper [8] to me.

6 Dichotomy for Infinite Products

(6.1) Explanation. The purpose of this section is to specialize Theorem (5.2) to the case of infinite products. We will return to the setting of Sects. 3 and 4, however, we will consider sequences rather than families. So let $(f_n)_{n\geq 0}$ and $(g_n)_{n\geq 0}$ be two jointly uncorrelated sequences in $\mathcal{C}_1^+(K)$ adapted to a decreasing sequence $(\mathcal{T}^n)_{n\geq 0}$ of σ–algebras generated by continuous functions on K (e.g. $\mathcal{T}^n = \mathcal{A}\{f_n, f_{n+1}, \dots; g_n, g_{n+1}, \dots\}$), and let $\mathcal{T}^\infty := \bigcap_{n\geq 0} \mathcal{T}^n$ be the terminal σ–algebra. For abbreviation we will write $f_{m,n} := \prod_{l=m}^{n-1} f_l$ and $g_{m,n} := \prod_{l=m}^{n-1} g_l$. Infinite products and tail measures are as defined in (2.11.b). Q_f and Q_g will denote two infinite products generated by (f_n) and (g_n), respectively. We will write P_n for the measure $f_{0,n}P$.

(6.2) Counterexample. In general it is, of course, not true that two infinite products Q_f and Q_g as above are either absolutely continuous or mutually singular. A counterexample is obtained by considering $K = [0,1]$, $P = \lambda$ (Lebesgue measure), and the two sequences

$$f_n := \begin{cases} 1 + r_n & \text{on} \quad [0, \frac{1}{2}[\\ \\ 1 & \text{on} \quad [\frac{1}{2}, 1] \end{cases}, \quad g_n = 1 \quad (n \geq 1),$$

where r_n is the Rademacher function defined by $r_n(x) = \operatorname{sgn} \sin(2^n 2\pi x)$. It is clear that $(\prod_{l=1}^n f_l)\,\lambda$ converges to the measure $Q_f := \frac{1}{2}\varepsilon_0 + 1_{[\frac{1}{2},1]}\lambda$ in the weak* sense.

(6.3) Lemma. *Let $n \geq 0$, let R be a tail measure for Q_g and n define $M_n := f_{0,n}R$. Suppose that for any $c \in \mathcal{C}(K)$ there is a continuous version of the conditional expectation $E_{P_n}(c/\mathcal{T}^n)$. Then for any bounded, Borel–measurable function h on K there is a common version of the conditional expectations $E_{M_n}(h/\mathcal{T}^n)$, $E_{Q_f}(h/\mathcal{T}^n)$, and $E_{P_n}(h/\mathcal{T}^n)$.*

Proof. Let c' be a continuous version of the conditional expectation $E_{P_n}(c/\mathcal{T}^n)$ for $c \in \mathcal{C}(K)$ and let $R = \lim_\tau g_{n,n_\tau}P$, $Q_g = \lim_\tau g_{0,n_\tau}P$. For any continuous, $\mathcal{T}^n$–measurable function ϕ on K, we then have

$$\begin{aligned} \int c'\phi dM_n &= \lim_\tau \int c'\phi f_{0,n} g_{n,n_\tau} dP \\ &= \lim_\tau \int c\phi f_{0,n} g_{n,n_\tau} dP \\ &= \int c\phi dM_n. \end{aligned} \tag{1}$$

As these functions ϕ generate $\mathcal{T}^n$ (6.1), (1) signifies that c' is a version of $E_{M_n}(c/\mathcal{T}^n)$; analogously, c' is also a version of $E_{Q_f}(c/\mathcal{T}^n)$.

Now let (c_k) be a sequence in $\mathcal{C}(K)$ converging to h in $\mathcal{L}_1(M_n)$, $\mathcal{L}_1(Q_f)$, and $\mathcal{L}_1(P_n)$. Passing to a subsequence we may suppose that c'_k converges to $E_{M_n}(h/\mathcal{T}^n)$, $E_{Q_f}(h/\mathcal{T}^n)$, and $E_{P_n}(h/\mathcal{T}^n)$ M_n–, Q_f–, and P_n–a.e., respectively. The function h' defined by

$$h'(x) := \begin{cases} \lim_{k\to\infty} c'_k(x) & \text{where the limit exists,} \\ 0 & \text{else} \end{cases}$$

has the required properties.

Combining (5.2) and (6.3) we obtain the following theorem.

(6.4) Dichotomy Theorem for Infinite Products. *Suppose that*

(α) *Q_f is trivial on $\mathcal{T}^\infty$,*

(β) *for any $c \in \mathcal{C}(K)$ and any $n \geq 0$ there is a continuous version of the conditional expectation $E_{P_n}(c/\mathcal{T}^n)$, and*

(γ) *$S\{g_{0,n} = 0\} = 0$ for all $n \geq 0$ and all infinite products S generated by $(g_k)_{k \geq n}$.*

Then either Q_f is absolutely continuous with respect to Q_g or else these two measures are mutually singular.

Proof. We have to show that Hypotheses (β) and (γ) imply (5.2.β) for the measures $Q_1 := Q_f$ and $Q_2 := Q_g$. The set $L := \bigcap_{n \geq 0}\{f_n > 0\}$ is a support of Q_f (see (2.6.ii)).

Let N be a Q_g–null set and let v_n be a common version of the conditional probabilities $M_n(N/\mathcal{T}^n)$ and $Q_f(N/\mathcal{T}^n)$ according to Lemma (6.3). We may then write

$$\begin{aligned}\int_L v_n dQ_g &\leq \int v_n g_{0,n}/f_{0,n} dM_n \\ &= \int v_n E_{M,n}((g_{0,n})/f_{0,n}/\mathcal{T}^n) dM_n \\ &= \int_N E_{M_n}((g_{0,n}/f_{0,n})/\mathcal{T}^n) dM_n \\ &= \int_N E_{M_n}((g_{0,n}/f_{0,n})/\mathcal{T}^n) f_{0,n} dR_{g,n}. \qquad (1)\end{aligned}$$

As $R_{g,n}$ is absolutely continuous with respect to Q_g by hypothesis (γ) (cf. (2.11.b)), we obtain $\int_L v_n dQ_g = 0$ as desired.

(6.5) Explanation. T. Tjur's theory [18] on conditional distributions can be applied to our situation. We obtain a sufficient condition for (6.4.β). We briefly outline one of his results. Suppose $\mathcal{T}^n$ is generated by the set $\{\phi_\iota \mid \iota \in I_n\}$ of continuous functions on K. Let L_n be the product space $X_{\iota \in I_n} \phi_\iota(K)$ and let $t_n : K \to L_n$ be the canonical mapping defined by $t_n(x) : (\phi_\iota(x))_{\iota \in I_n}$. For $y \in t_n(\operatorname{supp} P_n)$, let N_y be the net

$$N_y := \{(U, V) \mid U \text{ open neighborhood of } y, V \text{ open subset of } U \text{ such that } V \cap t_n(\operatorname{supp} P_n) \neq \emptyset\},$$

ordered by the relation

$$(U_1, V_1) \leq (U_2, V_2) \text{ if and only if } U_2 \subseteq U_1.$$

Finally, let D_n be the set of all $y \in t_n(\operatorname{supp} P_n)$ such that the weak* limit

$$P_n^y := \lim_{(U,V) \in N_y} [P_n(t_n^{-1}(V))]^{-1} 1_{t_n^{-1}(V)} P_n$$

exists in $\mathcal{M}_1^+(K)$. Then the following result holds.

(6.6) Proposition. *Suppose that $D_n = t_n(\operatorname{supp} P_n)$ for some $n \geq 0$. Then for any $c \in \mathcal{C}(K)$ there exists a continuous version of the conditional expectation $E_{P_n}(c/\mathcal{T}^n)$.*

Proof. By hypothesis and an account of [18] (21.1), (20.5), and (20.3) the mapping

$$y \to \int c(x) Q_n^y(dx) \qquad (1)$$

is continuous on $t_n(\operatorname{supp} P_n)$ and a version of the conditional expectation $E_{P_n}(c/t_n)$ of c under the hypothesis t_n there. Let d_n be an extension of the mapping (1) to a continuous mapping on L_n. Then $d_n \circ t_n$ is a version of the conditional expectation $E_{P_n}(c/\mathcal{T}^n)$.

7 Criteria for Absolute Continuity and Orthogonality in the Case of Dichotomy

In this section general hypotheses are as described in (6.1). We will now deal with criteria for orthogonality and absolute continuity in the case of dichotomy. We first need some lemmas.

(7.1) Lemma. *Suppose that*

(α) $\lim_{m,n} \int (f_{m,n}/g_{m,n})^{1/2} dQ_g = 1$ *and that*

(β) *there exists a real number* $\gamma < 1$ *such that for all* $m \geq 0$ *we have* $Q_g\{f_{0,m} = 0\} \leq \gamma$.

Then the function h *of Lemma (3.2) (in the case of sequences) is not a* Q_g*–null function.*

Proof. We proceed by contradiction. Suppose that we have

$$\int h dQ_g = 0. \tag{1}$$

By (α) there exists an m such that for all $n \geq m$ we have

$$\int (f_{m,n}/g_{m,n})^{1/2} dQ_g > \frac{1}{2}(1 + \gamma^{1/2}), \tag{2}$$

and by (β) there exists a strictly positive ε such that

$$[Q_g\{f_{0,m}/g_{0,m} < \varepsilon\}]^{1/2} \leq \gamma^{1/2} + \frac{1}{4}(1 - \gamma^{1/2}). \tag{3}$$

Using (1) we may choose an index $n \geq m$ such that

$$\int (f_{0,n}/g_{0,n})^{1/2} dQ_g \leq \frac{1}{4}\varepsilon^{1/2}(1 - \gamma^{1/2}). \tag{4}$$

Using the Cauchy–Schwarz inequality, (4), (3), and (2.11.a), we now estimate as follows.

$$\begin{aligned}
\int (f_{m,n}/g_{m,n})^{1/2} dQ_g =& \int_{\{f_{0,m}/g_{0,m} \geq \varepsilon\}} + \int_{\{f_{0,m}/g_{0,m} < \varepsilon\}} \\
\leq& \varepsilon^{-1/2} \int (f_{0,n}/g_{0,n})^{1/2} dQ_g + [Q_g\{f_{0,m}/g_{0,m} < \varepsilon\}]^{1/2} [\int f_{m,n}/g_{m,n} dQ_g]^{1/2} \\
\leq& \frac{1}{4}(1 - \gamma^{1/2}) + \gamma^{1/2} + \frac{1}{4}(1 - \gamma^{1/2}) \\
=& \frac{1}{2}(1 + \gamma^{1/2}).
\end{aligned}$$

This contradicts (2) and proves the lemma.

The following lemma replaces Fatou's lemma in the case of a stochastically convergent net of integrable functions. We formulate it in the setting of an abstract probability space $(\Omega, \mathcal{F}, P)$.

(7.2) Lemma. *Suppose that the net* (h_σ) *in* $\mathcal{L}_1^+(P)$ *converges* P*-stochastically to a limit* $h \in \mathcal{L}_1^+(P)$. *Then we have*

$$\int h dP \leq \liminf_\sigma \int h_\sigma dP.$$

Proof. We have for all $\delta > 0$

$$\int (h - h_\sigma) dP = \int_{\{h - h_\sigma \leq \delta\}} + \int_{\{h - h_\sigma > \delta\}} \leq \delta + \int_{\{h - h_\sigma > \delta\}} h dP. \tag{1}$$

As (h_σ) converges stochastically to h, (1) implies

$$\limsup_\sigma \int (h - h_\sigma) dP \leq 0.$$

The following lemma is a weakening of Proposition (3.4).

(7.3) Lemma. *Let (f_n), (g_n), Q_f, and Q_g satisfy the following three conditions.*

(α) $\lim_{m,n} \int (f_{m,n}/g_{m,n})^{1/2} dQ_g = 1$;

(β) *there exists a real number $\gamma < 1$ such that for all $m \geq 0$ we have $Q_g\{f_{0,m} = 0\} \leq \gamma$;*

(γ) *there exists a subnet (m_σ) of $\mathbb{N}$ such that*

$$\lim_\sigma (f_{0,m_\sigma}/g_{0,m_\sigma}) Q_g = Q_f.$$

Then Q_f and Q_g are not mutually singular.

Proof. By Lemma (3.2) the sequence $(f_{0,m}/g_{0,m})^{1/2}$ converges in $\mathcal{L}_1(Q_g)$ to a function $h \in \mathcal{L}_2(Q_g)$. It follows that the sequence $(f_{0,m}/g_{0,m})$ converges to h^2 Q_G–stochastically. Hence, using Lemma (7.2) and (γ), we may estimate for any positive, continuous function ϕ on K

$$\int \phi h^2 dQ_g \leq \liminf_\sigma \int \phi (f_{m_\sigma}/g_{m_\sigma}) dQ_g = \int \phi dQ_f,$$

i.e., $h^2 Q_g \leq Q_f$. As $h^2 Q_g$ is different from zero (7.1), Q_f and Q_g are not mutually singular.

We now come to the key lemma of this section.

(7.4) Lemma. *Suppose that*

(α) *Q_g is trivial on $\mathcal{T}^\infty$,*

(β) *there exists a real number $p > 1$ such that for all $\Phi \in \mathcal{E}$ there exists an integer $n \geq 0$ with the property $E_{Q_g}((g_\Phi/f_\Phi)/\mathcal{T}^n) \in \mathcal{L}_p(Q_g)$, and*

(γ) *there exists a real number $\gamma < 1$ such that for all $m \geq 0$ there is a finite subset $\Phi \subseteq [m, \infty[$ such that*

(i) $\int (g_\Phi/f_\Phi)^{1/2} dQ_g < \gamma$.

Then for all $\varepsilon > 0$ and all $m \geq 0$ there is a finite subset $\Psi \subseteq [m, \infty[$ such that

(ii) $\int (f_\Psi/g_\Psi)^{1/2} dQ_g < \varepsilon$.

Proof. It is sufficient to prove (ii) for $\varepsilon = \gamma^{(2p-1)/p}$. Choose $\Phi \subseteq [m, \infty[$ such that (i) holds. As the sequence

$$E_{Q_g}((f_\Phi/g_\Phi)^{1/2}/\mathcal{T}^l)_{l \geq n} \tag{1}$$

is a backward martingale with respect to $(\mathcal{T}^l)$, it converges to

$$E_{Q_g}((f_\Phi/g_\Phi)^{1/2}/\mathcal{T}^\infty) \tag{2}$$

Q_g–a.e. As Q_g is trivial on $\mathcal{T}^\infty$, (2) is equal to $\int (f_\Phi/g_\Phi)^{1/2} dQ_g$ Q_g–a.e. Hypothesis (β) implies by Jensen's inequality that $E_{Q_g}((f_\Phi/g_\Phi)^{1/2}/\mathcal{T}^n) \in \mathcal{L}_{2p}(Q_g)$. Hence (1) is uniformly integrable in $\mathcal{L}_{2p}(Q_g)$, i.e., (1) converges to $\int (f_\Phi/g_\Phi)^{1/2} dQ_g$ in $\mathcal{L}_{2p}(Q_g)$ as $l \to \infty$. Thus, by (i), there exists an index $l > \max \Phi$ such that

$$\| E_{Q_g}((f_\Phi/g_\Phi)^{1/2}/\mathcal{T}^l) \|_{2p, Q_g} < \gamma. \tag{3}$$

Again applying (γ) we may now choose a finite subset $\Lambda \subseteq [l, \infty[$ such that

$$\int (f_\Lambda/g_\Lambda)^{1/2} dQ_g < \gamma. \tag{4}$$

Observing (2.11.a) and (4) and applying Hölder's inequality we estimate

$$\begin{aligned} &\int (f_\Lambda/g_\Lambda)^{p/(2p-1)} dQ_g \\ =&\int (f_\Lambda/g_\Lambda)^{1/(2p-1)} (f_\Lambda/g_\Lambda)^{(p-1)/(2p-1)} dQ_g \\ \leq&[\int f_\Lambda/g_\Lambda dQ_g]^{1/(2p-1)} [\int (f_\Lambda/g_\Lambda)^{1/2} dQ_g]^{(2p-2)/(2p-1)} \\ \leq&\gamma^{(2p-2)/(2p-1)}. \end{aligned} \tag{5}$$

We finally put $\Psi := \Phi \cup \Lambda$ and apply Hölder's inequality, (3), and (5) to obtain

$$\begin{aligned} \int (f_\Psi/g_\Psi)^{1/2} dQ_g &= \int (f_\Phi/g_\Phi)^{1/2} (f_\Lambda/g_\Lambda)^{1/2} dQ_g \\ &= \int E_{Q_g}((f_\Phi/g_\Phi)^{1/2}/\mathcal{T}^l)(f_\Lambda/g_\Lambda)^{1/2} dQ_g \\ &\leq \gamma^{(2p-1)/p}. \end{aligned}$$

This proves Lemma (7.4).

We combine (4.2), (6.4), (7.3), and (7.4) to prove the following result. As in Sect. 6, P_n stands for the measure $f_{0,n}P$.

(7.5) First Dichotomy Criterion for Infinite Products. Suppose that

- (α) both measures Q_f and A_g are trivial on $\mathcal{T}^\infty$,
- (β) for any $c \in \mathcal{C}(K)$ and any $n \geq 0$ there is a continuous version of the conditional expectation $E_{P_n}(c/\mathcal{T}^n)$,
- (γ) $S\{g_{0,n} = 0\} = 0$ for all $n \geq 0$ and all infinite products generated by $(g_k)_{k \geq n}$,
- (δ) there exist two subnets (m_σ) and (n_τ) of $\mathbb{N}$ such that $\lim_\tau g_{0,n_\tau} P = Q_g$ and $\lim_{\sigma,\tau} f_{0,m_\sigma} g_{m_\sigma,n_\tau} P = Q_f$, and
- (ε) there exists a real number $p > 1$ such that for all $\Phi \in \mathcal{E}$ there exists an integer $n \geq 0$ with the property $E_{Q_g}((f_\Phi/g_\Phi)/\mathcal{T}^n) \in \mathcal{L}_p(Q_g)$.

Then the following four statements are equivalent.

- (i) Q_f is absolutely continuous with respect to Q_g;
- (ii) Q_f and Q_g are not mutually singular;
- (iii) there is an $\varepsilon > 0$ suth that for all $n \geq 0$ we have

$$\int (f_{0,n}/g_{0,n})^{1/2} dQ_g \geq \varepsilon;$$

- (iv) $\lim_{m,n\to\infty} \int (f_{m,n}/g_{m,n})^{1/2} dQ_g = 1$ and there exists a real number $\gamma < 1$ such that $Q_g\{f_{0,m} = 0\} \leq \gamma$ for all $m \geq 0$.

Proof. Statements (i) and (ii) are equivalent by (6.4). If (iii) does not hold, then Q_f and Q_g are mutually singular by (4.2). Now suppose that (iii) holds. Then we deduce from Hölder's inequality and (2.11.a) for any integer $n \geq 0$ and any subset $\Psi \subset [0, n[$

$$\begin{aligned}\varepsilon^2 \leq & [\int (f_{0,n}/g_{0,n})^{1/4}(f_{0,n}/g_{0,n})^{1/4} dQ_g]^2 \\ \leq & [\int (f_{0,n}/g_{0,n})^{1/3} dQ_g]^{3/2} \\ = & [\int (f_\Psi/g_\Psi)^{1/3}(f_{[0,n[\setminus\Psi}/g_{[0,n[\setminus\Psi})^{1/3} dQ_g]^{3/2} \\ \leq & \int (f_\Psi/g_\Psi)^{1/2} dQ_g.\end{aligned}$$

Hence (7.4.ii) does not hold, i.e. (7.4.γ) fails; thus the first half of (iv) is proved. On the other hand, we have for all $m \geq 0$ by the Cauchy–Schwarz inequality and by (2.11.a)

$$\varepsilon^2 \leq [\int 1_{\{f_{0,m}>0\}}(f_{0,m}/g_{0,m})^{1/2} dQ_g]^2 \leq Q_g\{f_{0,m} > 0\}.$$

This completes the proof of (iv). The implication from (iv) to (ii) is just Lemma (7.3). Note that (7.3.γ) follows from (γ) and (δ) by (2.11.b).

(7.6) Remarks. (a) It is easy to see that the conditional expectation appearing in Hypothesis (7.5.ε) for $n = 1 + \max \Phi$ is bounded (and (7.5.ε) is satisfied) when $\|E_{Q_g}(g_k^{-1}/T^{k+1})\|_{\infty,Q_g} < \infty$ for all $k \geq 0$. This is in particular the case when $E_P(g_k/\mathcal{T}^{k+1}) = 1 (k \geq 0)$; cf. (4.5.2).

(b) Hypothesis (7.5.γ) and (7.5.ε) are trivially satisfied when all the g_n's are strictly positive. Furthermore, Hypothesis (7.5.δ) is satisfied when $g_n = 1$ for all $n \geq 0$.

(c) One obtains a slightly better result, if the sequences (f_n) and (g_n) are not only jointly uncorrelated, but satisfy the stronger assumption $E_P(f_n/\mathcal{T}^{n+1}) = 1 = E_p(g_n/\mathcal{T}^{n+1}) (n \geq 0)$. The proof of our next theorem makes no use of Theorem (5.2) or Theorem (6.4).

(7.7) Second Dichotomy Criterion for Infinite Products. Suppose that

(α) Q_g is trivial on $\mathcal{T}^\infty$,

(β) $E_P(f_n/\mathcal{T}^{n+1}) = 1 = E_p(g_n/\mathcal{T}^{n+1})$ for all $n \geq 0$,

(γ) $S\{g_{0,n} = 0\} = 0$ for all $n \geq 0$ and all infinite products generated by $(g_k)_{k \geq n}$,

and

(δ) there exist two subnets (m_σ) and (n_τ) of $\mathbb{N}$ such that $\lim_\tau g_{0,n_\tau} P = Q_g$ and $\lim_{\sigma,\tau} f_{0,m_\sigma} g_{m_\sigma,n_\tau} P = Q_f$.

(a) The following four statements are equivalent.

(i) Q_f is absolutely continuous with respect to Q_g;

(ii) Q_f and Q_g are not mutually singular;

(iii) there is an $\varepsilon > 0$ such that for all $n \geq 0$, we have $\int (f_{0,n}/g_{0,n})^{1/2} dQ_g \geq \varepsilon$;

(iv) $\lim_{m,n\to\infty} \int (f_{m,n}/g_{m,n})^{1/2} dQ_g = 1$.

(b) If the equivalent statements (i)–(iv) hold, then the sequence $(f_{0,n}/g_{0,n})$ converges in $\mathcal{L}_1(Q_g)$ to the Radon–Nikodym density dQ_f/dQ_g.

Proof. Statement (iii) follows from Statement (ii) by (4.2). As in the case of Equality (4.5.2), Assumptions (β) and (γ) imply

$$E_{Q_g}((f_\Phi/g_\Phi)/\mathcal{T}^n) = 1 \tag{1}$$

for all $\Phi \in \mathcal{E}$ and $n \geq 1 + \max \Phi$. Thus, using Lemma (7.4), (α), and (1), we deduce Statement (iv) from Statement (iii) in the same way as (7.5.iv) from (7.5.iii). Equation (1) and (iv) together imply 3.4.α). We now use Proposition (3.4) to show the implication from (iv) to (i) and Part (b) of the theorem. Note that (3.4.iii) follows from (γ) and (δ) by (2.8).

(7.8) Remark. The results of this communication admit the following generalization. Suppose that $\mathcal{S}$ is a (real) algebra and a vector lattice of bounded functions on a set Ω such that $1 \in \mathcal{S}$. Let $\mathcal{F}$ be the σ–algebra generated by $\mathcal{S}$ and let $\mathcal{M}(\mathcal{F})$ be the set of all finite signed measures on $\mathcal{F}$. It follows from the monotone class theorem that $(\mathcal{S}, \mathcal{M}(\mathcal{F}))$ is a separated dual system with respect to integration. Let P be a probability measure in $\mathcal{M}(\mathcal{F})$ and let

$$\mathcal{S}_1^+ := \{\phi \in S \mid \phi \geq 0 \text{ and } \int \phi dP = 1\}.$$

Let $(f_n)_{n\geq 0}$ and $(g_n)_{n\geq 0}$ be jointly P–uncorrelated families in $\mathcal{S}_1^+$, adapted to a decreasing sequence $(\mathcal{T}^n)_{n\geq 0}$ of σ–algebras generated by functions in $\mathcal{S}$. Suppose for simplicity that $g_n^{-1} \in \mathcal{S}$ for all $n \geq 0$. By an infinite product generated by (f_n) we mean a $\sigma(\mathcal{M}(\mathcal{F}), \mathcal{S})$–cluster point of the sequence $(f_{0,n}P)_{n\geq 0}$ in $\mathcal{M}_1^+(\mathcal{F})$. Then, *mutatis mutandis*, all the foregoing results remain true if $(K, \mathcal{B}, P)$ is replaced by $(\Omega, \mathcal{F}, P)$, $\mathcal{C}(K)$ is replaced by $\mathcal{S}$, and weak* topology on $\mathcal{M}(K)$ is replaced by $\sigma(\mathcal{M}(\mathcal{F}), \mathcal{S})$. In most cases it is also possible to apply H. Bauer's representation [1] of abstract integrals by means of Radon measures to deduce the results in the abstract case from those in the topological case. We state here a sample result; it corresponds to Theorem (7.5).

(7.9) Theorem. *Let notation be as in (7.8) and let Q_f and Q_g be infinite products generated by the sequences (f_n) and (g_n), respectively. Suppose that*

- (α) *both measures Q_f and Q_g are trivial on $\mathcal{T}^\infty$,*
- (β) *for any $s \in S$ and any $n \geq 0$ there is a version of $\mathcal{S}$ of the conditional expectation $E_{P_n}(s/\mathcal{T}^n)$, and*
- (γ) *there exist two subnets (m_σ) and (n_τ) of $\mathbb{N}$ such that $\lim_\tau g_{0,n_\tau}P = Q_g$ and $\lim_{\sigma,\tau} f_{0,m_\sigma} g_{m_\sigma,n_\tau}P = Q_f$.*

Then Statements (7.5.i)–(7.5.iv) are equivalent in this situation.

(7.10) Remark (Kakutani's dichotomy theorem [10]). Let $(\Omega_n, \mathcal{F}_n, P_n)_{n\geq 0}$ be a sequence of probability spaces and suppose that $h_n \in \mathcal{L}_1^+(P_n)$ satisfies $\int h_n dP_n = 1$ for all $n \geq 0$. Define $\Omega := X\Omega_n$, $\mathcal{F} := \otimes \mathcal{F}_n$, $P := \otimes P_n$, $f_n(x_0, x_1, \dots) := h_n(x_n)$ for $(x_0, x_1, \dots) \in \Omega$, and $g_n = 1 (n \geq 0)$. Let S be the set of all bounded, $\mathcal{F}$–measurable functions on Ω that depend only on a finite number of coordinates. Then the sequence $(f_n)_{n\geq 0}$ generates exactly one infinite product, namely the product measure $Q_f = \otimes h_n P_n$. Let $\mathcal{T}^n (n \geq 0)$ be the tail σ–algebras of $\mathcal{F}$. Kolmogorov's 0–1 law no asserts that P is trivial on $\mathcal{T}^\infty$, and Kakutani's dichotomy theorem follows from the abstract version of Theorem (7.7). Kakutani's theorem is also contained in the abstract version (7.9) of Theorem (7.5), as the function

$$w_n(x_0, x_1, \dots) := \int \dots \int t(y_0, \dots, y_{n-1}, x_n, x_{n-1}, \dots) \prod_{k=0}^{n-1} h_k(y_k) P_0(dy_0) \dots P_{n-1}(dy_{n-1})$$

is a version of $E_{P_n}(t/\mathcal{T}^n)$, where t is an arbitrary bounded, $\mathcal{F}$–measurable function on Ω.

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