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On Integral Representation and the Choquet Boundary for Convolution Algebras of Measures

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Abstract. We show that the Choquet boundary of a convolution algebra of measures is contained in the set of generalized characters of idempotent modulus. We then give a number of sufficient conditions for Choquet boundary points and determine the Choquet boundary for some examples, including the examples of Hewitt–Kakutani and Simon. Finally we state and prove a theorem of Bochner's type for *L*-algebras generated by a single measure.

1. Introduction

It is well known that Bochner's theorem on representation of normalized, positive definite functions on an abelian, locally compact group G may be deduced from Choquet's integral representation theorem (see e. g. [12]). However, as the group algebra $\mathbb{L}^1(G)$ is *symmetric*, its Choquet and Šilov boundaries both coincide with its Gelfand space (i. e., the dual group $\hat{G}$) and there cannot be an integral representation of *all* elements of the closed convex hull of the Gelfand space (i. e., the normalized, positive definite functions on G) over a proper subset of the Gelfand space. For the same reason, Bochner's theorem may be deduced from Krein–Milman's theorem or also from uniform density in $\mathfrak{C}_0(\hat{G})$ of the space of all Fourier transforms of integrable functions on G (theorem of Stone–Weierstraß). The situation is different if we are dealing with representations of functionals on *asymmetric* L-subalgebras $\mathscr{M}$ of the convolution algebra $\mathbf{M}(G)$ of all bounded measures on an abelian, locally compact group G. Here the Choquet boundary of the Gelfand space Δ of $\mathscr{M}$ is not even closed in general and we need the full strength of Choquet's theorem to obtain such representations. It follows from a theorem of H. Bauer that the Šilov boundary of Δ is the closure of its Choquet boundary ([2], see also [3]).

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In view of Choquet's and Bauer's theorems exact knowledge of the Choquet boundary of Δ is of major importance. The Šilov and the strong boundaries of Δ have been studied in the past by many authors. Yet it seems that there is not much known about its Choquet boundary.

Since TAYLOR [18] has shown that the strong boundary of an L-algebra of measures is contained in the set H of generalized characters of idempotent modulus and that H itself is a boundary in the sense that every Gelfand transform assumes its maximum modulus at an element of H, in view of Bauer's Maximum Principle, it is natural to ask if H contains the Choquet boundary. We answer this question in the affirmative (3.1).

Given a convex embedding (see 2.3), Choquet boundary points are closely related to extreme points. The closed convex hull of Δ in $\mathcal{M}'$ is the projective limit of the closed convex hulls of the Gelfand spaces of the elementary L-algebras contained in $\mathcal{M}$. It is obvious that each generalized character which is extreme coordinatewise is also extreme. The converse is not true in general. Roughly speaking it is only true that an extreme point can be *approximated* coordinatewise by extreme points (CHOQUET [7], see also D. A. EDWARDS [8]). We deal with this question in (3.2)—(3.4).

We then reduce the problem of deciding if a given $h \in H$ is in the Choquet boundary of Δ to the same problem on a certain compact subset of Δ, a subset which is itself the Gelfand space of a factor algebra of $\mathcal{M}$ (3.11). Our attention has been drawn in this direction by BROWN and MORAN's paper [6]. The reduction (3.11) gives rise to a number of sufficient conditions for $h \in H$ to be in the Choquet boundary, one of them (3.15) purely topological. The others depend on the construction of certain Gelfand transforms. In Section 3 we use mainly methods from Banach algebra and convexity theory.

In Section 4 we determine the Choquet boundary for discrete $\mathcal{M}$ and for Hewitt–Kakutani's and Simon's examples. It turns out that these Choquet boundaries are minimal, i. e., they consist of the set of generalized characters of unit modulus.

Section 5 deals with an application of Choquet's integral representation theorem in the situation of an L-algebra generated by a single measure. In particular we obtain essentially an extension of Bochner's theorem also in the asymmetric case (5.6).

We thank G. BROWN for a very useful discussion on the subject matter of this paper during his stay in Erlangen.

2. Notations and Terminology

(2.1) *L-subalgebras of* $\mathbf{M}(G)$. The standard references here are [9] and [19]. In what follows, G will denote a locally compact, abelian group with Haar measure λ and $\mathbf{M}(G)$ the algebra of finite, complex-valued, regular Borel measures on G with convolution as multiplication. We denote the point mass at $x \in G$ by δ_x. An *L-subspace* B of $\mathbf{M}(G)$ is a closed subspace that is hereditary with respect to absolute continuity $\ll$, i. e., if $\mu \in B$ and $\nu \in \mathbf{M}(G)$ s. th. $\nu \ll \mu$, then $\nu \in B$. An L-subspace $\mathscr{M}$ of $\mathbf{M}(G)$ is called an *L-subalgebra*, if it is also an algebra with respect to convolution. Examples are $\mathbf{M}_c(G) + \mathbb{C}\,\delta_0$, where $\mathbf{M}_c$ stands for the continuous measures on G, and the discrete measures $\mathbf{M}_d(G)$ on G. Special L-subalgebras are the *elementary* L-subalgebras $\mathbb{L}^1(\mu)\,\mu$, where $\mu \in \mathbf{M}(G)$, $\delta_0 \ll \mu$, and $\mu * \mu \ll \mu$. If $\nu \in \mathbf{M}(G)$, we denote by $\mathscr{M}_\nu$ the elementary L-subalgebra generated by ν, i. e., $M_\nu = \mathbb{L}^1(\mu)\,\mu$ for $\mu = \sum_{n=0}^{\infty} 2^{-n} \nu_1^n$, $\nu_1 := |\nu|/\|\nu\|$. If $\mathbf{E}$ is an at most countable subset of $\mathbf{M}(G)$, then $\mathbf{E}$ is trivially contained in an elementary L-subalgebra of $\mathbf{M}(G)$.

In the sequel we will consider a fixed unital (i. e., $\delta_0 \in \mathscr{M}$) L-subalgebra $\mathscr{M}$ of $\mathbf{M}(G)$. We denote the Gelfand space of a commutative Banach algebra $\mathscr{A}$ by $\Delta(\mathscr{A})$, but, if $\mathscr{A}$ is our fixed $\mathscr{M}$, we will just write Δ. Šreider [16] has given a workable representation of the multiplicative, linear functionals on $\mathscr{M}$ as generalized characters on G: If $\mathscr{B}$ is an L-subspace of $\mathbf{M}(G)$, then $\mathscr{B}$ is the inductive limit of its closed subspaces $\mathbb{L}^1(\mu)\,\mu$, $\mu \in \mathscr{B}$. Therefore the dual space $\mathscr{B}'$ of $\mathscr{B}$ is the projective limit of the dual spaces $\mathbb{L}^\infty(\mu)$, i. e., each continuous, linear functional f on $\mathscr{B}$ is given via integration $f(\mu) := \int \bar{\chi}_\mu \, d\mu$ by a *generalized function* $\chi = (\chi_\mu) \in \prod_{\mu \in \mathscr{B}} \mathbb{L}^\infty(\mu)$ with the properties

(i) $\mu \ll \nu \Rightarrow \chi_\mu = \chi_\nu \pmod{\mu}$,

(ii) $\sup_{\mu \in \mathscr{B}} \|\chi_\mu\|_{\mathbb{L}^\infty(\mu)} = \|f\|$.

The multiplicative, linear functionals on the L-algebra $\mathscr{M}$ are exactly those generalized functions χ (called *geneeralized characters*) that satisfy

(iii) $\chi_{\mu * \nu}(x + y) = \chi_\mu(x)\,\chi_\nu(y) \quad \mu \otimes \nu$ — a. a. (x, y).

From this representation, the following properties of Δ can easily be derived.

α) Δ is a semi-group with coordinatewise and pointwise multiplication.

β) $\chi \in \Delta \Rightarrow \bar{\chi} := (\bar{\chi}_\mu) \in \Delta$.

γ) $\chi \in \Delta \Rightarrow |\chi|^\alpha := (|\chi_\mu|^\alpha) \in \Delta$, where $\operatorname{Re} \alpha \geqslant 0$.

δ) $\chi \in \Delta \Rightarrow \chi/|\chi| := (\chi_\mu/|\chi_\mu|) \in \Delta$ $(0/0 := 0)$.

ε) $\chi \in \Delta \Rightarrow \lim_{n\to\infty} |\chi|^n := (\lim |\chi_\mu|^n) \in \Delta$.

It is mostly by using these properties that one has tried to unravel the structure of $\mathcal{M}$ and $\Delta(\mathcal{M})$ ever since.

Δ^{-1} will denote the group of invertible elements $\chi \in \Delta$ for the semi-group structure of Δ, i. e., the generalized characters $\chi \in \Delta$ such that $|\chi| = 1 \pmod{\mathcal{M}}$. (We will say that a generalized function χ has a certain property mod $\mathcal{M}$, if χ_μ has this property mod μ for all μ.) The symbol H will stand for the generalized characters h of $\mathcal{M}$ with idempotent modulus: $|h|^2 = |h| \pmod{\mathcal{M}}$.

Note that $\chi/|\chi| \in H$ for all $\chi \in \Delta$. As a consequence of α), γ), and δ), each $\chi \in \Delta$ possesses a (not necessarily unique) polar decomposition $\chi = |\chi|(\chi/|\chi|) \in \Delta^+ \cdot H$.

An L-subalgebra $\mathcal{M}_1$ of $\mathcal{M}$ is called *prime*, if $\mathcal{M}_1^\perp := \{\mu \in \mathcal{M} \mid \mu \perp \nu$ for all $\nu \in \mathcal{M}_1\}$ is an ideal. There is a $1-1$ correspondence between prime L-subalgebras of $\mathcal{M}$ and elements in $H^+ := \{h \in \Delta \mid h^2 = h \bmod \mathcal{M}\}$ $(\subseteq H)$ (see [9], 5.2.2). We denote the Gelfand transform of $\mu \in \mathcal{M}$ by $\hat{\mu}$, $\hat{\mathcal{M}}$ stands for the set of all Gelfand transforms on Δ.

(2.2) *Boundaries*. Suppose that $\mathfrak{C}$ is a separating, additive semi-group of continuous functions $u: X \to [-\infty, \infty[$ on a compact space X that contains the (real) constants. The *Bishop boundary*, the *strong boundary*, the *Choquet boundary*, and the *Šilov boundary* of X with respect to $\mathfrak{C}$ are defined as follows:

$$\mathrm{Bi}^{\mathfrak{C}} X := \{x \in X \mid \exists\, u \in \mathfrak{C} \text{ s. th. } u \text{ is maximal exactly at } x\},$$

$$\mathrm{St}^{\mathfrak{C}} X := \{x \in X \mid \forall\, U \in \mathcal{U}(x) \,\exists\, u \in \mathfrak{C} \text{ s. th. } u \text{ is maximal at } x \text{ and } u(y) < u(x)\ (y \notin U)\},$$

$$\mathrm{Sh}^{\mathfrak{C}} X := \{x \in X \mid \forall\, U \in \mathcal{U}(x) \,\exists\, u \in \mathfrak{C} \text{ s. th. } u(y) < \max u\ (y \notin U)\}.$$

In order to define the Choquet boundary we need the notion of a *representing measure*: $\mu \in \mathbf{M}_1^+(X)$ is said to represent $x \in X$ with respect to $\mathfrak{C}$, if

$$u(x) \leqslant \int u \, d\mu$$

for all $u \in \mathfrak{C}$. Let $\mathbf{M}_x^{\mathfrak{C}}$ be the set of all measures that represent x.

x is said to be in the *Choquet boundary* of X with respect to $\mathfrak{E}$, if $\mathbf{M}_x^{\mathfrak{E}} = \{\delta_x\}$, i.e., the only representing measure is the trivial one.

As the other three boundaries, the Choquet boundary too may be characterized by means of functions. Let $C(\mathfrak{E})$ be the convex cone generated by $\mathfrak{E}$. Then $\mathrm{Ch}^{C(\mathfrak{E})}X = \mathrm{Ch}^{\mathfrak{E}}(X)$ and $\mathrm{Ch}^{\mathfrak{E}}X = \{x \in X \mid \forall\, U \in \mathcal{U}(x)\, \exists\, u \in C(\mathfrak{E})$ s.th. $u \leqslant 3$, $u(x) \geqslant 2$, and $u(y) \leqslant 1$ $(y \notin U)\}$. (The numbers 1, 2, 3 may be replaced by any triplet $\alpha < \beta < \gamma$ of real numbers.) It is clear that $\mathrm{Bi} \subseteq \mathrm{St} \subseteq \mathrm{Ch}$ and that $x \in \mathrm{Bi}$ if and only if $x \in \mathrm{St}$ and x is G_δ. BAUER [2] has shown that each $u \in \mathfrak{E}$ attains its maximum at a point of the Choquet boundary (Bauer's Maximum Principle) and that Ch is a dense subset of Sh.

Given an L subalgebra $\mathcal{M}$ of $\mathbf{M}(G)$ (or a general unital Banach algebra), two semigroups of functions on the Gelfand space Δ arise in a natural way: $\log|\hat{\mathcal{M}}| := \{\log|\hat{\mu}| \mid \mu \in \mathcal{M}\}$ and $\mathrm{Re}\,\hat{\mathcal{M}} := \{\mathrm{Re}\,\hat{\mu} \mid \mu \in \mathcal{M}\}$. We would therefore obtain eight boundaries. However, using the identity $\mathrm{Re}\,u = \log|e^u|$, one easily verifies that all boundaries for $\mathrm{Re}\,\hat{\mathcal{M}}$ are contained in the respective boundaries for $\log|\hat{\mathcal{M}}|$. It is even easier to see that the converse inclusions hold for the Bishop, the strong, and the Šilov boundaries. We will mainly be interested in $\mathrm{Ch}^{\mathcal{M}}\Delta := \mathrm{Ch}^{\mathrm{Re}\,\hat{\mathcal{M}}}\Delta$. A similar reasoning shows that

$$\begin{aligned}\mathrm{Ch}^{\mathcal{M}}\Delta &= \{\chi \in \Delta \mid \forall\, U \in \mathcal{U}(\chi)\, \exists\, \mu \in \mathcal{M} \text{ s.th. } \mathrm{Re}\,\hat{\mu} \leqslant 3,\ \mathrm{Re}\,\hat{\mu}(\chi) \geqslant 2,\\ &\qquad \text{and } \mathrm{Re}\,\hat{\mu}(\psi) \leqslant 1\ (\psi \notin U)\}\\ &= \{\chi \in \Delta \mid \forall\, U \in \mathcal{U}(\chi)\, \exists\, \mu \in \mathcal{M} \text{ s.th. } |\hat{\mu}| \leqslant 3,\ |\hat{\mu}(\chi)| \geqslant 2,\\ &\qquad \text{and } |\hat{\mu}(\psi)| \leqslant 1\ (\psi \notin U)\}.\end{aligned}$$

(Again 1, 2, 3 may be replaced by $\alpha < \beta < \gamma$ ($\alpha > 0$ in the last expression).)

If $\mathfrak{B}$ is a separating, *uniformly closed* subalgebra of $\mathfrak{C}_{\mathbb{C}}(X)$ that contains the constants then BISHOP (see [17]) showed that $\mathrm{Ch}^{\mathfrak{B}}X = \mathrm{St}^{\mathfrak{B}}X$. (His original result is for metrizable X where $\mathrm{Bi} = \mathrm{St}$.) As the Choquet boundary does not vary when taking uniform closures, we obtain in our situation $\mathrm{Ch}^{\mathcal{M}}\Delta = \mathrm{St}^{\hat{\mathcal{M}}^-}\Delta$, where $\hat{\mathcal{M}}^-$ is the uniform closure of $\hat{\mathcal{M}}$.

(2.3) *Convexity*. Let now $\mathfrak{E}$ be a separating vector space of continuous real-valued functions on a compact space X that contains the constants (e.g. $\mathrm{Re}\,\hat{\mathcal{M}}$ on Δ). A continuous, injective mapping $\varphi: X \to E$, where E is a real, locally convex Hausdorff space, is called a *convex embedding* of $(X, \mathfrak{E})$, if

(i) the closed convex hull $\overline{\text{conv}}\,\varphi(X)$ of $\varphi(X)$ is compact, and if

(ii) $\mathfrak{E} = \{l \circ \varphi \mid l \in E'\}$.

Such an embedding always exists. Then there is a third (geometrical) characterization of the Choquet boundary of X: $x \in \text{Ch}^{\mathfrak{E}} X$ if and only if $\varphi(x)$ is an extreme point of $\overline{\text{conv}}\,\varphi(X)$. If $\mathscr{A}$ is a unital, commutative Banach algebra, two ways of embedding $(\Delta(\mathscr{A}), \text{Re}\,\hat{\mathscr{A}})$ are useful:

α) The natural mapping $\Delta(\mathscr{A}) \to \mathscr{A}'$ (we always endow dual spaces with their weak* topologies), and

β) the natural mapping $\delta: \Delta(\mathscr{A}) \to \hat{\mathscr{A}}'$, where $\hat{\mathscr{A}}$ carries the uniform norm.

It turns out that $\overline{\text{conv}}\,\delta(\Delta(\mathscr{A})) = \{L \in \hat{\mathscr{A}}' \mid \|L\| = 1 = L(1)\} =: S_{\mathscr{A}}$, the *state space* of $\mathscr{A}$. (The inclusion $S_{\mathscr{A}} \subseteq \overline{\text{conv}}\,\delta(\Delta(\mathscr{A}))$ follows from the fact that a state extends to a probability measure on $\Delta(\mathscr{A})$.) $S_{\mathscr{A}}$ is also characterized as the set of all positive, normalized linear forms on $\hat{\mathscr{A}}$ (*positive* means $\text{Re}\,L(\hat{a}) \geqslant 0$ for all $a \in \mathscr{A}$ such that $\text{Re}\,\hat{a} \geqslant 0$).

Let us call a generalized function χ of *positive type* for $\mathscr{M}$ if $\text{Re}\,\chi(\mu) \geqslant 0$ for all $\mu \in \mathscr{M}$ whose spectrum is contained in the right half plane. $S_{\mathscr{M}}$ may also be identified with the set of all generalized functions of positive type for $\mathscr{M}$. If $\mathscr{M}$ is elementary, we will use the term essentially bounded function of *positive type*.

BROWN—MORAN [5] showed that every invertible element in $\Delta(\mathscr{M})$ is in the strong boundary. To give the reader a flavor of our method, we give a simple proof that it is in $\text{Ch}^{\mathscr{M}}\Delta$: If $|\chi_\mu| = 1 \pmod{\mathscr{M}}$ then it follows from the definition of an extreme point χ that χ is an extreme point of the closed unit ball of $\mathscr{M}'$. Hence χ is extreme in $\overline{\text{conv}}\,\Delta\ (\subseteq \mathscr{M}')$. Standard references for convexity and integral representation are [1], [3], and [12].

3. Necessary Conditions and Sufficient Conditions for $\chi \in \Delta$ to Be in the Choquet Boundary

We fix a unital L-subalgebra $\mathscr{M}$ of $\mathbf{M}(G)$ and deal first with necessary conditions for $\chi \in \Delta$ to be in the Choquet boundary. J. L. TAYLOR [18], p. 162, has shown that every Gelfand transform assumes its maximum modulus at a point of H. Moreover, he has shown that $\text{St}^{\mathscr{M}}\Delta \subseteq H$. In view of Bauer's Maximum Principle [2] (see also [3], Satz 1), the following theorem is a sharpening of these two results.

(3.1) **Theorem.** $\mathrm{Ch}^{\mathscr{M}}\Delta \subseteq \mathrm{Ch}^{\log|\hat{\mathscr{M}}|}\Delta \subseteq H$.

Proof. For the first inclusion see (2.2); for the second inclusion let $\chi \in \Delta$. As TAYLOR [18] we start with analyticity on $\mathrm{Re}\, z > 0$ of the function $z \to \hat{\mu}(|\chi|^z h)$ for all $\mu \in \mathscr{M}$, where $|\chi| h$ is the polar decomposition of χ. It follows that the function $z \to \log|\hat{\mu}(|\chi|^z h)|$ is subharmonic there and we have

$$\log|\hat{\mu}|(\chi) \leqslant \int \log|\hat{\mu}|(|\chi|^z h)\,\lambda(dz).$$

Here λ is normalized Lebesgue measure on a circle with center 1 and radius $r < 1$. This representation of χ with respect to $\log|\hat{\mathscr{M}}|$ is non-trivial except in the case $|\chi|^2 = |\chi|$ $\mathscr{M}$-a. e. This completes the proof.

(3.2) *Explanation.* There is no known characterization of the extreme points of the limit of a projective system S of convex, compact sets in terms of the extreme points of these sets. G. CHOQUET [7], Lemme 38 (see also D. A. EDWARDS [8]), gave a necessary condition which says roughly that the coordinates of an extreme point may be *approximated* by extreme points in the coordinates. In order to reformulate this result in our context we need some preliminaries.

Let $p_{\nu,\mu}: \mathscr{M}_\nu' \to \mathscr{M}_\mu'$ $(\nu, \mu \in \mathscr{M}, \mu \ll \nu)$ be the canonical restriction mapping and let $q_{\nu,\mu}: \overline{\mathrm{conv}}\,\Delta(\mathscr{M}_\nu) \to \overline{\mathrm{conv}}\,\Delta(\mathscr{M}_\mu)$ be its restriction to $\overline{\mathrm{conv}}\,\Delta(\mathscr{M}_\nu)$. The system $(q_{\nu,\mu})_{\mu \ll \nu}$ of affine, continuous mappings of convex, compact sets is projective. We show that its limit

$$\underset{\longleftarrow}{\lim}\,\overline{\mathrm{conv}}\,\Delta(\mathscr{M}_\mu) = \{(\chi_\mu) \in \prod_{\mu \in \mathscr{M}} \overline{\mathrm{conv}}\,\Delta(\mathscr{M}_\mu) \mid \chi_\mu = \chi_\nu \ \mu\text{-a. e. for } \mu \ll \nu\}$$

is equal to $\overline{\mathrm{conv}}\,\Delta$. It is clear that the canonical mapping $\pi: \overline{\mathrm{conv}}\,\Delta \to \underset{\longleftarrow}{\lim}\,\overline{\mathrm{conv}}\,\Delta(\mathscr{M}_\mu)$ is continuous and injective. Each extreme point in $\overline{\mathrm{conv}}\,\Delta(\mathscr{M}_\mu)$ is in the Šilov boundary of $\Delta(\mathscr{M}_\mu)$ and extends therefore to a multiplicative linear functional on $\mathscr{M}$.

It follows that the range of the canonical mapping

$$\pi_\mu: \overline{\mathrm{conv}}\,\Delta \to \overline{\mathrm{conv}}\,\Delta(\mathscr{M}_\mu)$$

contains all the extreme points of $\overline{\mathrm{conv}}\,\Delta(\mathscr{M}_\mu)$. By Krein-Milman's theorem, π_μ is onto. Hence π is onto.

We now obtain the following necessary condition.

(3.3) **Proposition.** *Let U be a neighborhood in* $\mathrm{Ch}^{\mathcal{M}}\Delta$ *of* $h \in \mathrm{Ch}^{\mathcal{M}}\Delta$. *Then for all sufficiently large* (*w.r.t.* $\preccurlyeq$) $\nu \in \mathcal{M}$ *s.th.* $\delta_0 \preccurlyeq \nu$, $\nu * \nu \preccurlyeq \nu$, *there exists* $h^\nu \in U$ *whose ν-th coordinate is in* $\mathrm{Ch}^{\mathcal{M}_\nu}(\Delta(\mathcal{M}_\nu))$.

Proof. We may suppose that U is of the form

$$U = \{\chi \in \mathrm{Ch}^{\mathcal{M}}\Delta \mid \chi_\mu \in V\}$$

where $\mu \in \mathcal{M}$ and V is a neighborhood of h_μ in $\overline{\mathrm{conv}}\,\Delta(\mathcal{M}_\mu)$. By Choquet's lemma cited in (3.2) there exists $\nu_0 \in \mathcal{M}$, $\nu_0 \succcurlyeq \mu$ such that for all $\nu \succcurlyeq \nu_0$ there exists an extreme point $\chi^\nu \in \overline{\mathrm{conv}}\,\Delta(\mathcal{M}_\nu)$ such that $\pi_{\nu,\mu}(\chi^\nu) \in V$. Since π_ν maps $\overline{\mathrm{conv}}\,\Delta$ onto $\overline{\mathrm{conv}}\,\Delta(\mathcal{M}_\nu)$ (see (3.2)), χ^ν extends to an extreme point h^ν of $\overline{\mathrm{conv}}\,\Delta$. If $\delta_0 \preccurlyeq \nu$, $\nu * \nu \preccurlyeq \nu$, then h^ν has all the desired properties.

(3.4) *Remark.* If $h \in \Delta$ is an isolated point of $\mathrm{Ch}^{\mathcal{M}}\Delta$, then $h_\nu \in \mathrm{Ch}^{\mathcal{M}_\nu}\Delta(\mathcal{M}_\nu)$ for all sufficiently large $\nu \in \mathcal{M}$ s.th. $\delta_0 \preccurlyeq \nu$, $\nu * \nu \preccurlyeq \nu$.

(3.5) *Explanation.* Nothing can be said *a priori* about the size of the Choquet boundary. It may be H or a proper subset of H and it may be Δ^{-1} or a proper superset of Δ^{-1}. This follows from known facts about $\mathrm{St}^{\mathcal{M}}\Delta$ and $\mathrm{Sh}^{\mathcal{M}}\Delta$ and is also illustrated by the following simple examples:

a) If $\mathcal{M} = \mathbb{L}^1(\mathbb{R})\lambda + \mathbb{C}\delta_0$, then Δ may be identified with $\mathbb{R} \cup \{\omega\}$ ($\omega = 1_{\{0\}}$ is the projection on $\mathbb{C}\delta_0$) and we have $H = \Delta$. As $\mathcal{M}$ is symmetric here, $\hat{\mathcal{M}}$ is dense in $\mathfrak{C}_{\mathbb{C}}(\Delta)$ by the theorem of Stone-Weierstraß. It follows that $\mathrm{Ch}^{\mathcal{M}}\Delta = \Delta = H$.

b) If $\mathcal{M} = \mathfrak{l}_1(\mathbb{N}_0) \subseteq \mathbf{M}(\mathbb{R})$ ($\mathbb{N}_0 = \{0, 1, 2, \ldots\}$), then Δ may be identified with the unit disk $\mathbb{D}$, where $\alpha \in \mathbb{D}$ is identified with the sequence $(1, \alpha^1, \alpha^2, \ldots) \in \mathfrak{l}^\infty(\mathbb{N}_0)$. Here $H = \mathbb{T} \cup \{0\}$. But 0 ($\approx \omega$) is not in the Choquet boundary of Δ, since all elements of $\hat{\mathcal{M}}$ are analytic in the interior of $\mathbb{D}$. Examples where the Choquet boundary is different from Δ^{-1} arise also from asymmetry of $\mathcal{M}$:

c) Let $\mathcal{M}$ be an asymmetric L-subalgebra of $\mathbf{M}(G)$ that is closed for conjugation $\sim$ ($\tilde{\mu}(E) := \overline{\mu(-E)}$) and s.th. $\mathbb{L}^1(G) \subseteq \mathcal{M}$. Then $\mathrm{Ch}^{\mathcal{M}}\Delta \supsetneqq \Delta^{-1}$. (Because of $\mathbb{L}^1(G) \subseteq \mathcal{M}$ we have $\Delta^{-1} = \hat{G}$. If the Choquet boundary were equal to $\hat{G}$, then the Šilov boundary would consist of symmetric functionals. Krein–Milman's theorem would then imply the symmetry of $\mathcal{M}$.)

(3.6) *Notations.* We will now deal with a necessary and sufficient criterion for $h \in H$ to be in $\mathrm{Ch}^{\mathcal{M}} \Delta$. Let $h \in H$ be fixed and let $\mathcal{M}_1$ be the prime L-subalgebra of $\mathcal{M}$ associated with h, i.e., $\mathcal{M}_1 := \{\mu \in \mathcal{M} \mid |h_\mu| \mu = \mu\}$. Clearly we have $\delta_0 \in \mathcal{M}_1$. Moreover, let $\Phi_1 : \overline{\mathrm{conv}}\, \Delta \to \overline{\mathrm{conv}}\, \Delta(\mathcal{M}_1)$ be the canonical restriction mapping $\chi \to \chi \mid \mathcal{M}_1$. Since every functional in the Šilov boundary of $\Delta(\mathcal{M}_1)$ extends to an element of Δ, φ_1 is onto. We put

$$\Delta_h := \{\chi \in \Delta \mid \chi \mid \mathcal{M}_1 = h \mid \mathcal{M}_1\} = \Phi_1^{-1}(\Phi_1(h)) \cap \Delta .$$

Δ_h is a compact subset of Δ that contains h. This set plays an important role in Brown—Moran [6], where the set $A_h := \Delta_h \setminus \{h\}$ is used to exhibit strong boundary points of Δ. However, their methods are different from ours.

In order to obtain the criterion mentioned above, we need some lemmas. First note that $\Phi_1(h) \in \Delta^{-1}(\mathcal{M}_1)$. Consequently (see (2.3)), $\Phi_1(h)$ is an extreme point in $\overline{\mathrm{conv}}\, \Delta(\mathcal{M}_1)$ and $\Phi_1^{-1}(\Phi_1(h))$ is a face in $\overline{\mathrm{conv}}\, \Delta$.

(3.7) **Lemma.** $\overline{\mathrm{conv}}\, \Delta_h$ *is a face in* $\overline{\mathrm{conv}}\, \Delta$.

Proof. By our last remark, we have only to show $\overline{\mathrm{conv}}\, \Delta_h = \Phi_1^{-1}(\Phi_1(h))$. Each extreme point $\varepsilon \in \Phi_1^{-1}(\Phi_1(h))$ is also extreme in $\overline{\mathrm{conv}}\, \Delta$. By Milman's theorem we have $\varepsilon \in \Delta$, i.e., $\varepsilon \in \Delta_h$. Therefore, by Krein–Milman's theorem, the right side is contained in the left side. The opposite inclusion is evident.

(3.8) **Lemma.** a) Δ_h *is closed for the hull-kernel operation.*

b) Δ_h *is canonically the Gelfand space of the factor algebra of* $\mathcal{M}$ *with respect to the kernel of* Δ_h.

Proof. Let $\chi(\mu) = 0$ for all μ in the kernel $k(\Delta_h)$ of Δ_h. If $\mu_1 \in \mathcal{M}_1$, $h(\mu_1) = 0$, then we have $\mu_1 \in k(\Delta_h)$. It follows that $\chi(\mu_1) = 0$. This shows that $\ker h \cap \mathcal{M}_1 \subseteq \ker \chi \cap \mathcal{M}_1$ and $\chi \mid \mathcal{M}_1 = c\, h \mid \mathcal{M}_1$, where c is a complex constant. But as $\delta_0 \in \mathcal{M}_1$, we obtain $c = 1$, i.e., $\chi \in \Delta_h$. This proves that $h\, k(\Delta_h) \subseteq \Delta_h$ and equality follows. Part b) is an immediate consequence of a).

We formulate our next lemma in the setting of a commutative semisimple Banach algebra $\mathcal{A}$ with identity. $C_h(X)$ stands for the connected component of an element h in a topological space X.

(3.9) **Lemma.** *For each multiplicative linear functional* h *on* $\mathcal{A}$, $\overline{\mathrm{conv}}\, C_h(\Delta(\mathcal{A}))$ *is a face in* $\overline{\mathrm{conv}}\, \Delta(\mathcal{A})$.

Proof. Let $K \neq \emptyset$ be an open and compact subset of $\Delta(\mathscr{A})$. We first show that $\overline{\operatorname{conv}} K$ is a face in $\overline{\operatorname{conv}} \Delta(\mathscr{A})$. It is well known that there exists an $x \in \mathscr{A}$ such that $\hat{x} = 1_K$. The element $x \in \mathscr{A}$ induces a continuous, affine functional f: $\overline{\operatorname{conv}} \Delta(\mathscr{A}) \to \mathbb{R}$. As f vanishes on $\Delta(\mathscr{A}) \setminus K$ and as it maps K on 1, its range is the unit interval. Our claim is proved if we show $\overline{\operatorname{conv}} K = f^{-1}(1)$. But if χ is an extreme point in the face $f^{-1}(1)$, then $\chi \in \Delta(\mathscr{A})$ by Milman's theorem and it follows $\chi \in K$, i. e., $f^{-1}(1) \subseteq \overline{\operatorname{conv}} K$ by Krein–Milman's theorem. The opposite inclusion is evident.

We now show that $\overline{\operatorname{conv}}\, C_h(\Delta(\mathscr{A})) = \bigcap \overline{\operatorname{conv}} K$, where the intersection ranges over all open and compact K that contain h. As the intersection of the faces $\overline{\operatorname{conv}} K$ is again a face, our lemma will follow. If χ is an extreme point of the right side, then it is extreme in $\overline{\operatorname{conv}} K$ for all such K, hence $\chi \in K$ by Milman's theorem. But, $\Delta(\mathscr{A})$ being compact, $C_h(\Delta(\mathscr{A}))$ is the intersection of all open and compact subsets of $\Delta(\mathscr{A})$ that contain h, i. e., $\chi \in C_h(\Delta(\mathscr{A}))$. Again by Krein-Milman's theorem we obtain $\bigcap \overline{\operatorname{conv}} K \subseteq \overline{\operatorname{conv}}\, C_h(\Delta(\mathscr{A}))$. The opposite inclusion is obvious.

(3.10) **Lemma.** a) *$C_h(\Delta(\mathscr{A}))$ is closed for the hull-kernel operation.*

b) *$C_h(\Delta(\mathscr{A}))$ is canonically the Gelfand space of the factor algebra of $\mathscr{A}$ with respect to the kernel of $C_h(\Delta(\mathscr{A}))$.*

Proof. Every open and closed subset K of $\Delta(\mathscr{A})$ is the hull of its kernel. This is a consequence of the existence of $x \in \mathscr{A}$ s. th. $\hat{x} = 1_K$. Since $C_h(\Delta(\mathscr{A}))$ is the intersection of all such K that contain h, Part a follows. Part b is an immediate consequence thereof.

We are now in the position of reducing the problem of deciding whether $h \in H$ is in the Choquet boundary of Δ to deciding the same question for $C_h(\Delta_h)$.

(3.11) **Theorem.** *A multiplicative linear functional $h \in H$ belongs to* $\mathrm{Ch}^{\mathscr{M}}\Delta$ *if and only if it belongs to* $\mathrm{Ch}^{\mathscr{M}} C_h(\Delta_h)$.

Proof. If h is in the Choquet boundary of Δ, it is trivially in the Choquet boundary of each compact subset of Δ. The natural injection $C_h(\Delta_h) \to \mathscr{M}'$ is a convex embedding. If h is extreme in $\overline{\operatorname{conv}}\, C_h(\Delta_h)$, then h is extreme in $\overline{\operatorname{conv}}\, \Delta_h$ by (3.8.b) and (3.9) applied to the factor algebra $\mathscr{A} = \mathscr{M}/k(\Delta_h)$. The theorem now follows from (3.7).

We now derive some corollaries from this theorem. It is sometimes possible to decide the question if h is a Choquet boundary point by looking at the elementary L-algebras contained in $\mathcal{M}$. This is the content of our first corollary. Unfortunately, the condition that we obtain seems only to be sufficient (cf. (3.2)).

(3.12) **Corollary**. *Let $h \in H$. Suppose that there is a system* $\mathbf{F}$ *of measures $\mu \in \mathcal{M}$ such that*

(i) $\mathbf{F}$ *is cofinal in $\mathcal{M}$ with respect to $\ll$, and*

(ii) $h_\mu \in \mathrm{Ch}^{\mathcal{M}_\mu} C_{h_\mu}(\Delta_{h_\mu}(\mathcal{M}_\mu))$ *for all $\mu \in \mathbf{F}$.*

Then $h \in \mathrm{Ch}^{\mathcal{M}}\Delta$.

Proof. Hypothesis ii) and Theorem (3.11) imply that h_μ is an extreme point of $\overline{\mathrm{conv}}\,\Delta\,(M_\mu) \subseteq M'_\mu$ for all $\mu \in \mathbf{F}$. Then h is extreme in $\overline{\mathrm{conv}}\,\Delta$ by the very definition of an extreme point.

(3.13) **Corollary.** *Suppose that for all $\chi \in C_h(\Delta_h), \chi \neq h$, there exists a measure $\mu \in \mathcal{M}$ (depending on χ) such that*

(i) $\mathrm{Re}\,\hat{\mu} \,|\, C_h(\Delta_h) \geqslant 0$,

(ii) $\mathrm{Re}\,\hat{\mu}(h) = 0$,

(iii) $\mathrm{Re}\,\hat{\mu}(\chi) > 0$.

Then $h \in \mathrm{Ch}^{\mathcal{M}}(\Delta)$.

Proof. A standard compactness argument shows that $h \in \mathrm{St}^{\mathcal{M}} C_h(\Delta_h)$.

(3.14) *Remark.* Let $\mathbb{L}^1(G) \subseteq \mathcal{M}$. If $K \neq \emptyset$ is open and compact in Δ, then there exists $\mu \in \mathcal{M}$ s. th. $\hat{\mu} = 1_K$. It follows that $K \cap \hat{G} \neq \emptyset$. This argument shows that Δ has no isolated points outside of $\hat{G}$, and if the connected component of $\chi \in \Delta$ is $\{\chi\}$, then χ must be in the closure of $\hat{G}$. (We thank Dr. W. Arendt for a discussion on this point.) On the other hand, if G is nondiscrete, if $\mathcal{M} = \mathbf{M}(G)$, and if $h \in \Delta$ is *proper maximal* (see Brown–Moran [6], see also [9], (8.5.5)), then h is isolated in Δ_h and $h \notin \hat{G}$. Another standard example is furnished by $G = \mathbb{T}$, $\mathcal{M} = \mathbb{L}^1(\delta_0 + \mu + \lambda)$, where $\mu \neq 0$ is a continuous singular measure on $\mathbb{T}$ such that $\mu * \mu \ll \mu$. If $h \in H$ is the generalized character induced by the prime L-subalgebra $\mathbb{L}^1(\delta_0 + \mu)$, then $\Delta_h = \{1, h\}$.

Our next corollary gives a purely topological condition for $h \in \mathrm{Ch}$. It should be compared with Brown and Moran's lemma [6], Section 3. They show that h is a strong boundary point of Δ if it is isolated in Δ_h.

(3.15) **Corollary.** *If* $C_h(\Delta_h) = \{h\}$, *then* $h \in \mathrm{Ch}^{\mathscr{M}}\Delta$.

(3.16) **Corollary.** *Suppose that* $\hat{\mathscr{M}}$ *is symmetric on* $C_h(\Delta_h)$. *Then* $h \in \mathrm{Ch}^{\mathscr{M}}\Delta$.

Proof. By hypothesis, $\mathrm{Re}\,\hat{\mathscr{M}}$ is dense in the set of all real-valued, continuous functions on $C_h(\Delta_h)$. Hence every element of $C_h(\Delta_h)$ is in $\mathrm{Ch}^{\mathscr{M}}(C_h(\Delta_h))$.

We now use lemma (3.10) to obtain

(3.17) **Corollary.** *Suppose that for all* $\chi \in C_h(\Delta_h)$, $\chi \neq h$, *there exists a measure* $\nu \in \mathscr{M}$ *(depending on* χ*) so that*

(i) $\hat{\nu}(\chi) = \hat{\nu}(h)$ *and that*

(ii) *the complex number* $\hat{\nu}(h)$ *can be touched with a closed disk* $D \subseteq \mathbb{C}$ *whose interior points lie outside of* $\hat{\nu}(C_h(\Delta_h))$. *Then* $h \in \mathrm{Ch}^{\mathscr{M}}\Delta$.

Proof. $\hat{\nu}(C_h(\Delta_h))$ is the spectrum of the class $\dot{\nu}$ of ν in the factor algebra of $\mathscr{M}$ with respect to the ideal $I := k(C_h(\Delta_h))$ (Lemma 3.10). Without loss of generality $D \cap \hat{\nu}(C_h(\Delta_h)) = \{\hat{\nu}(h)\}$. Let a be the center of D. We use the linear transformation $T: z \to (z - \hat{\nu}(h))/(z - a)$ of the Riemann sphere. For $|z - a| > |\hat{\nu}(h) - a|$ we have $|1 - Tz| < 1$, in particular $\mathrm{Re}\, Tz > 0$. The functional calculus applied to the algebra $\mathscr{M}/I$ and $\dot{\nu}$ now shows that there exists an element $\mu \in \mathscr{M}$ such that $\hat{\mu} = T \circ \hat{\nu}$ on $C_h(\Delta_h)$. Therefore μ satisfies the hypotheses of Corollary (3.13). This proves the corollary.

The question whether h is in the Choquet boundary may sometimes be reduced to the question if ω (the indicator function of the neutral element in G) is in the Choquet boundary of a certain subalgebra of $\mathscr{M}$. We will describe this now.

(3.18) *Notations.* Let $h \in H$ be fixed and let I_0 be the prime L-ideal of $\mathscr{M}$ associated with h, i.e., $I_0 := \{\mu \in \mathscr{M} \mid (1 - |h_\mu|)\mu = \mu\}$.

Clearly we have $\mathscr{M} = I_0 \oplus \mathscr{M}_1$, $\mathscr{M}_1 \simeq \mathscr{M}/I_0$, and $\delta_0 \notin I_0$. We put $\mathscr{M}_0 := I_0 \oplus \mathbb{C}\,\delta_0$. Moreover, let $\Phi_0 : \overline{\mathrm{conv}}\,\Delta \to \overline{\mathrm{conv}}\,\Delta(\mathscr{M}_0) \subseteq \mathscr{M}_0'$ be the canonical restriction mapping $\chi \to \chi \mid \mathscr{M}_0$. Like Φ_1, Φ_0 is onto. It is clear that Φ_0 induces a homeomorphism $\Delta_h \to \Phi_0(\Delta_h)$ as well as an isomorphism $\overline{\mathrm{conv}}\,\Delta_h \to \Phi_0(\overline{\mathrm{conv}}\,\Delta_h)$ of compact, convex sets. Clearly $\Phi_0(h) = \omega$.

(3.19) **Proposition.** *Let* $h \in H$. *We have* $h \in \mathrm{Ch}^{\mathscr{M}}\Delta$ *if and only if* $\omega \in \mathrm{Ch}^{\mathscr{M}_0} C_\omega(\Phi_0(\Delta_h))$.

Proof. By the above remarks on Φ_0, h is extreme in $\overline{\text{conv}}\, C_h(\Delta_h)$ if and only if ω is extreme in $\Phi_0(\overline{\text{conv}}\, C_h(\Delta_h))$ and

$$\Phi_0(\overline{\text{conv}}\, C_h(\Delta_h)) = \overline{\text{conv}}\, C_\omega(\Phi_0(\Delta_h)).$$

The proposition now follows from Theorem (3.11).

(3.20) *Remark.* It follows immediately that $h \in \text{Ch}^{\mathcal{M}} \Delta$ if

$$\omega \in \text{Ch}^{\mathcal{M}_0} C_\omega(\Delta(\mathcal{M}_0)) \text{ or } \omega \in \text{Ch}^{\mathcal{M}_0} \Delta(\mathcal{M}_0).$$

There are also corollaries analogous to (3.12), (3.13) and (3.15)—(3.17). We omit the details.

4. Examples of Choquet Boundaries

The Šilov boundaries of certain L-algebras have been determined by various authors. We now determine the Choquet boundaries in certain cases.

a) *The Discrete Case.*

(4.1) **Proposition.** *If* $\mathcal{M}$ *consists of discrete measures, then* $\text{Sh}^{\mathcal{M}} \Delta = \text{Ch}^{\mathcal{M}} \Delta = \Delta^{-1}$.

Proof. Any functional χ in $\text{Sh}^{\mathcal{M}} \Delta$ extends to a functional in $\Delta(\mathbf{M}_d(G)) = \hat{G}_d = \Delta^{-1}(\mathbf{M}_d(G))$. ($G_d$ is the group G endowed with the discrete topology.) Hence $\chi \in \Delta^{-1}$.

b) *Hewitt—Kakutani's Example.*

(4.2) *Explanation.* A subset $P \subseteq G$ is called *independent*, if $\sum_{k=1}^{l} n_k \chi_k = 0$ implies $n_1 \chi_1 = \ldots = n_l \chi_l = 0$ ($n_k \in \mathbb{Z}$, $\chi_k \in P$ distinct), and a subset $Q \subseteq G$ is called *algebraically scattered*, if for each $x \in \langle Q \rangle$, $x \neq 0$, there exists a countable subset $Y \subseteq Q$ such that $x \notin \langle Q \setminus Y \rangle$. (Here $\langle X \rangle$ is the subgroup of G generated by $X \subseteq G$.) If G is non-discrete, then it contains independent, compact, perfect sets P, and for each independent P, $Q := \bigcup_{n \in \mathbb{Z}} \{nx \mid x \in P\}$ is algebraically scattered.

Let Q be algebraically scattered and σ-compact, and let $\mathbf{M}_c(Q)$ be the L-subspace of $\mathbf{M}(G)$ consisting of all continuous measures $\mu \in \mathbf{M}(G)$ that live on Q. Let $\mathcal{M}$ be the L-algebra generated by $\mathbf{M}_c(Q)$ and δ_0. Hewitt–Kakutani showed in the case $Q = P \cup (-P)$, P

independent, that *every* generalized function of $\mathbf{M}_c(Q)$ of modulus $\leqslant 1$ extends to a generalized character of $\mathbf{M}(G)$. The extension to algebraically scattered Q is mainly due to SAEKI [13] (see [9], 6.2.8).

The measures that are absolutely continuous with respect to a measure of the form $\delta_0 + \sum_{j=1}^{n} \mu_{j1} * \ldots * \mu_{j,l_j} (\mu_{j,l} \in \mathbf{M}_c^+(Q))$ are dense in $\mathcal{M}$. It follows (cf. [18], p. 155 (3)) that these sums of convolutions themselves are dense in $\mathcal{M}$. Hence the extension is unique, i. e., Δ is homeomorphic with the unit ball of $\mathbf{M}_c(Q)$.

The invertible elements of Δ correspond to the generalized functions ψ for $\mathbf{M}_c(Q)$ s. th. $|\psi| = 1 \pmod{\mathbf{M}_c(Q)}$: Let $\chi \in \Delta$ and suppose that $\psi := \chi | \mathbf{M}_c(Q)$ has this property. As the measures absolutely continuous with respect to a measure of the form

$$\delta_0 + \sum_{j=1}^{n} \mu_{j,1} * \ldots * \mu_{j,l_j} \quad (\mu_{j,l} \in \mathbf{M}_c^+(Q))$$

are dense in $\mathcal{M}$, and as $|\chi| \leqslant 1$ $\mathcal{M}$-a. e., it is sufficient to show $|\chi|(\mu_1 * \ldots * \mu_l) = \|\mu_1 * \ldots * \mu_l\|$ for $\mu_k \in \mathbf{M}_c^+(Q)$:

$$|\chi|(\mu_1 * \ldots * \mu_k) = |\chi|(\mu_1) \ldots |\chi|(\mu_k) = |\psi|(\mu_1) \ldots |\psi|(\mu_k) =$$
$$= \|\mu_1\| \ldots \|\mu_k\| = \|\mu_1 * \ldots * \mu_k\|.$$

(4.3) **Proposition.** *Let notation be as in* (4.2). *Then* $\mathrm{Ch}^{\mathcal{M}}\Delta = \Delta^{-1}$.

Proof. By Theorem (3.1), we have to show that each $h \in H \setminus \Delta^{-1}$ has a nontrivial representation. Let $\psi_t (|t| = 1)$ be the generalized function for $\mathbf{M}_c(Q)$ obtained by „inserting t for zero“:

$$\psi_{t,\mu} := \begin{cases} h_\mu \text{ where } |h_\mu| = 1, \\ t \quad \text{else}. \end{cases}$$

Let χ_t be the extension of ψ_t to a generalized character of $\mathcal{M}$. By (4.2), $\chi_t \in \Delta^{-1}$. It remains to prove that

$$\hat{\mu}(h) = \int_{|t|=1} \hat{\mu}(\chi_t)\, \lambda(dt),$$

where λ is normalized Lebesque measure on the unit circle. For the same reason as above it is sufficient to prove this claim for μ of the form $\mu_1 * \ldots * \mu_l$ $(\mu_k \in \mathbf{M}_c(Q))$:

$$\int \chi_t(\mu_1 * \ldots * \mu_l)\,\lambda(dt) = \int [\int \bar{\psi}_{t,\mu_1}\, d\mu_1] \cdot \ldots \cdot [\int \bar{\psi}_{t,\mu_l} d\mu_l]\,\lambda(dt) =$$
$$= \int [\int_{|h_{\mu_1}|=1} \bar{\psi}_{t,\mu_1}\, d\mu_1 + \int_{|h_{\mu_1}|=0} \bar{\psi}_{t,\mu_1}\, d\mu_1] \cdot \ldots \cdot$$
$$\cdot [\int_{|h_{\mu_l}|=1} \bar{\psi}_{t,\mu_l} d\mu_l + \int_{|h_{\mu_l}|=0} \bar{\psi}_{t,\mu_l} d\mu_l]\,\lambda(dt) =$$
$$= \int [\int \bar{h}_{\mu_1} d\mu_1 + t\,\mu_1\{h_{\mu_1} = 0\}] \ldots [\int \bar{h}_{\mu_l} d\mu_l + t\,\mu_l\{h_{\mu_l} = 0\}]\,\lambda(dt) =$$
$$= \int \bar{h}_{\mu_1} d\mu_1 \ldots \int \bar{h}_{\mu_l} d\mu_l = h(\mu_1) \ldots h(\mu_l) = h(\mu_1 * \ldots * \mu_l).$$

(4.4) *Remark.* If $P \subseteq G$ is $\mathbb{Z}$-independent and compact, it is even easier to see by pointwise considerations that $\mathrm{Ch}^{\mathcal{M}} \Delta = \Delta^{-1}$, where $\mathcal{M}$ is the unital L-algebra generated by *any* L-subspace of $\mathbf{M}(P)$.

c) *Simon's Example.*

(4.5) *Explanation.* Simon [14] showed that the Šilov boundary of the smallest translation invariant L-subalgebra $\mathcal{N}$ of $\mathbf{M}(G)$ (G nondiscrete) that contains the L-algebra $\mathcal{M}$ of the Hewitt–Kakutani example is homeomorphic with $\hat{G}_d \times U$, where G_d is the group G with the discrete topology and U is the unit ball in $\mathbf{M}_c(Q)'$.

(4.6) **Proposition.** *Let G be nondiscrete, let $\mathcal{M}$ be as in* (4.2), *and let $\mathcal{N}$ be the L-subalgebra of* $\mathbf{M}(G)$ *generated by* $\{\delta_x \mid x \in G\}$ *and* $\mathcal{M}$. *Then* $\mathrm{Ch}^{\mathcal{N}} \Delta(\mathcal{N}) \simeq \hat{G}_d \times \Delta^{-1}(\mathcal{M})\ (\simeq \Delta^{-1}(\mathcal{N}))$.

Proof. Note that $\mathrm{Ch}^{\mathbf{M}_d(G)} \Delta(\mathbf{M}_d(G)) = \Delta(\mathbf{M}_d(G)) = \hat{G}_d$, where $\mathbf{M}_d(G)$ denotes the discrete measures in $\mathbf{M}(G)$. We have to show that

$$\mathrm{Ch}^{\mathcal{N}} \Delta(\mathcal{N}) \simeq \Delta(\mathbf{M}_d(G)) \times \mathrm{Ch}^{\mathcal{M}} \Delta(\mathcal{M}).$$

By [9], 6.2.9 and its proof, $\Delta(\mathcal{N}) \simeq \Delta(\mathbf{M}_d(G)) \times \Delta(\mathcal{M})$ and the measures of the form $\sum_{j=1}^{n} \nu_j * \mu_j$, where $\nu_j \in \mathbf{M}_d(G)$ and $\mu_j \in \mathcal{M}$ are dense in $\mathcal{N}$. These measures appear on $\Delta(\mathbf{M}_d(G)) \times \Delta(\mathcal{M})$ as tensor products $\sum_{j=1}^{n} \hat{\nu}_j \otimes \hat{\mu}_j$; they are uniformly dense in $\hat{\mathcal{N}}$. The inclusion $\mathrm{Ch}\,\Delta(\mathbf{M}_d(G)) \times \Delta(\mathcal{M}) \subseteq \Delta(\mathbf{M}_d(G)) \times \mathrm{Ch}^{\mathcal{M}} \Delta(\mathcal{M})$ can now be seen by using representing measures, whereas the converse sense is most easily seen by applying the "function characterization" of the Choquet boundary (see 2.2).

5. Integral Representation for Elementary L-Algebras

(5.1) *Explanation.* Bochner's theorem on the representation of normalized, positive definite functions as Fourier-Stieltjes transforms of probability measures may be obtained by applying Choquet's integral representation theorem to the algebra $\mathbb{L}^1(G)$ (see [12]). In fact, since the algebra $\mathcal{M} := \mathbb{L}^1(G) + \mathbb{C}\,\delta_0$ is symmetric, $\mathrm{Ch}\,\Delta = \Delta$ is closed here and the theorem is an application of Krein–Milman's theorem. It remains to identify the state space with (essentially) the set of normalized, positive definite functions.

It is only in the *asymmetric* case that the need for Choquet's theorem becomes apparent. We want to restrict matters to the metrizable case. However, $1 \in \hat{G}$ is G_δ if and only if G is σ-compact (see [11], (24.48)), and if G is non-discrete then no $\chi \in \Delta(\mathbf{M}(G)) \setminus \hat{G}$ is G_δ ([4], Cor. 3.4). On the other hand, if G has a countable base and if $\mathcal{M}$ is an elementary L-subalgebra of $\mathbf{M}(G)$, then $\mathcal{M}$ is separable. Hence the unit ball in $\mathcal{M}'$ and Δ are metrizable in the weak* topology. Choquet's theorem, Section (2.3), and Theorem (3.1) now combine to show

(5.2) **Theorem.** *Let G have a countable base and let $\mathcal{M}$ be an elementary L-subalgebra of* $\mathbf{M}(G)$. *Then:*

a) $\mathrm{Ch}^{\mathcal{M}}\Delta$ *is a G_δ-subset of Δ such that*

$$\Delta^{-1} \subseteq \mathrm{Ch}^{\mathcal{M}}\Delta \subseteq H.$$

b) *For every essentially bounded function χ of positive type for $\mathcal{M}$ there exists a probability measure $\varrho_\chi \in \mathbf{M}(\Delta)$ such that*

(i) $\varrho_\chi(\mathrm{Ch}^{\mathcal{M}}\Delta) = 1$ *and*

(ii) $\chi(\mu) = \int h(\mu)\,\varrho_\chi(dh)$ *for all* $\mu \in \mathcal{M}$.

We now show that if G and $\mathcal{M}$ are as in (5.2) then H is Baire measurable.

(5.3) **Lemma.** *Let X be a locally compact space with a countable base, μ a finite, positive, regular measure on X. $\mathbb{L}^\infty(\mu)$ is canonically embedded in $\mathbb{L}^1(\mu)$. Let τ_1 be the topology on $\mathbb{L}^\infty(\mu)$ induced by the $\mathbb{L}^1$-norm. Then we have $B(\tau_1) = B(\sigma(\mathbb{L}^\infty(\mu), \mathbb{L}^1(\mu)))$ (where $B(\tau)$ denotes the Borel σ-algebra generated by a topology τ).*

Proof. Since $\mathbb{L}^1(\mu)$ is separable, $B(\tau_1)$ is generated by the sets of the form $B_\varepsilon = \{f \in \mathbb{L}^\infty(\mu) \mid \int |f|\,d\mu < \varepsilon\}$ and their translates. Let $\mathfrak{C}_k$ be the set of continuous functions on X with compact supports and let C be a countable subset of $\{\psi \in \mathfrak{C}_k \mid \|\psi\|_u \leqslant 1\}$ that is uniformly dense there. Then we have $B_\varepsilon = \bigcap_{\psi \in C} \{f \in \mathbb{L}^\infty(\mu) \mid \int f\psi\, d\mu < \varepsilon\}$. This shows that B_ε belongs to $B(\sigma(\mathbb{L}^\infty(\mu), \mathbb{L}^1(\mu)))$.

Let U be the closed unit ball in $\mathbb{L}^\infty(\mu)$ (with respect to the $\mathbb{L}^\infty$-norm). The topology τ_1 is stronger than $\sigma(\mathbb{L}^\infty(\mu), \mathbb{L}^1(\mu))$ on U. Indeed, let (f_n) be a sequence in U such that $f_n \xrightarrow{\tau_1} 0$ and $g \in \mathbb{L}_1$. For given $\varepsilon > 0$ there is $M \in \mathbb{R}^+$ such that: $\int_{\{|g| \geqslant M\}} |g|\, d\mu \leqslant \varepsilon$.

Then we have

$$\int_{\{|g| \leqslant M\}} f_n g\, d\mu \to 0, \quad \Big| \int_{\{|g| \geqslant M\}} f_n g\, d\mu \Big| \leqslant \varepsilon .$$

It follows that the two considered Borel algebras coincide on U.

As U belongs to both Borel algebras and as $\mathbb{L}^\infty(\mu)$ is the union of countably many multiples of U, the assertion of the lemma follows.

(5.4) **Lemma.** *Let G and $\mathcal{M}$ be as in* (5.2). *Then H is Baire measurable.*

Proof. As Δ is metrizable it suffices to prove that H is Borel measurable. It is clear that H is closed for τ_1. The assertion now follows from Lemma (5.3).

(5.5) *Remark*. The mapping $\Delta \to \Delta^+$, $\chi \to |\chi|$ does not have nice properties for general $\mathcal{M}$ (see [18], p. 162). However, for G and $\mathcal{M}$ as in (5.2), as this map is continuous for τ_1, it is Baire measurable for $\sigma(\mathcal{M}', \mathcal{M})$ by Lemma (5.3).

The following corollary is an immediate consequence of (5.2) and (5.4).

(5.6) **Corollary.** *Let G and $\mathcal{M}$ be as in* (5.2). *Then:*

a) *H is Baire measurable.*

b) *For every essentially bounded function χ of positive type (in particular for every generalized character χ) for $\mathcal{M}$ there exists a probability measure $\varrho_\chi \in \mathbf{M}(\Delta)$ such that*

(i) $\varrho_\chi(H) = 1$ *and*

(ii) $\chi(\mu) = \int h(\mu)\, \varrho_\chi(dh)$ *for all* $\mu \in \mathcal{M}$.

(5.7) *Remarks.* a) The integral representation is not unique in general. Let G be non-discrete, let $\mu \in \mathbf{M}(G)$ be continuous and singular with independent powers (i. e., $\mu^n \perp \mu^m$ if $0 \leqslant n < m$), and let $\gamma \in \hat{G}$ be different from a constant $(\mathrm{mod}\, \mu)$. Define two families $(\chi_t)_{|t|=1}$, and $(\psi_t)_{|t|=1}$ of generalized characters for $\mathcal{M}_\mu$ by putting

$$\chi_t := t^n (\mathrm{mod}\, \mu^n),\ n \geqslant 0 \quad \text{and} \quad \psi_t := \chi_t \gamma .$$

It is now a simple calculation using the mean-value property to show

$$\int \chi_t \lambda(dt) = \omega = \int \psi_t \lambda(dt)$$

(λ = normalized Lebesgue measure on the unit circle).

b) If $H \neq \Delta$, each $\chi \in \Delta \setminus H$ has a representation over H. The possibility of representing multiplicative functionals (additively) by means of other multiplicative functionals is of course again a consequence of asymmetry of $\mathcal{M}$.

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