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ON DIFFUSION PROCESSES AND THEIR SEMIGROUPS IN HILBERT SPACES WITH AN APPLICATION TO INTERACTING STOCHASTIC SYSTEMS

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We study solutions to stochastic differential equations in Hilbert space. In particular we give sufficient conditions for nonexplosion and for the associated semigroup to be of Feller type. We also give applications to systems of stochastic differential equations.

1. Introduction. Diffusion processes play an important role, among others, in the following situations:

There is a well–known connection between elliptic and parabolic differential operators of second order and diffusion processes. It is possible to construct for certain of these operators $\mathfrak{L}$ a diffusion process ξ whose characteristic operator is an extension of $\mathfrak{L}$ (cf. e.g., Dynkin [9], for a survey). The potential theory of $\mathfrak{L}$ may then be studied with the aid of the associated process ξ: A function h on the state space is (sub)harmonic if $h(\xi)$ is a (sub)martingale, the solution to the Dirichlet problem is obtained through the first hitting distributions, and so on.

In statistical mechanics Gibbs measures μ may often be characterized as reversible measures with respect to a certain semigroup arising from a diffusion process (cf., e.g., Doss–Royer [8], and Holley–Stroock [15]).

Certain motions, such as physical Brownian motion are governed by the classical laws of particle mechanics together with a statistical perturbation. This leads to a stochastic integral equation and to a diffusion process which is under certain simplifying assumptions in the case of physical Brownian motion the Ornstein–Uhlenbeck process.

As in the last case all the processes mentioned above may be constructed as solutions to stochastic integral equations. This idea goes back essentially to K. Itô [16, 17], who treated the finite dimensional case. However, in view of the second and third examples above, infinite dimensional state spaces have drawn more attention in recent years. The case of compact state spaces as occurring in the study of the infinite dimensional Wright–Fisher genetic model or the infinite dimensional plane rotor model have been treated in Ethier–Kurtz [11] and Holley–Stroock [15].

We deal here with a finite or infinite dimensional vector space as state space, a situation that was already considered by Daletskii [4, 5, 6], Lang [20], Doss–Royer [8], Shiga–Shimizu [30], and Fritz [12]. The case of an infinite dimensional vector space has gained additional interest due to the so–called lattice approximation in quantum field theory (cf. Nelson [28], Guerra–Rosen–Simon [13]).

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As out results are most naturally formulated if the state space is a Hilbert space, we start with a stochastic integral equation (see 3.2) with diffusion operator $\mathfrak{A}$ and drift vector $\mathfrak{b}$ in a separable Hilbert space $\mathbb{H}$. This setup permits us to use systematically the theory of stochastic integration in Hilbert space as developed in Métivier [24, 25] and Métivier–Pellaumail [26].

In Section 2 we review the basic facts about Brownian motion and stochastic integration in Hilbert space needed in the sequel. In Section 3 we formulate and sketch a proof of a theorem of existence and uniqueness that fits in our framework. Contrary to other methods that yield weak solutions (cf. Doss–Royer [8], and Stroock–Varadhan [32]), we use a method familiar in the finite dimensional case (cf. McKean [22]), obtaining strong solutions.

Our main parts are Sections 4 and 5. In Section 4 we give a sufficient condition for nonexplosion (Theorem 4.5) and in Section 5 we give conditions for the associated transition kernels to be of Feller type (Theorem 5.19). These conditions are mainly growth conditions on the coefficients of the stochastic equation. Theorem 4.5 says essentially that there is no explosion if the diffusion grows at most linearly and if the outward drift is not too strong. Roughly speaking, Theorem 5.19 says that the transition kernels are of Feller type if the diffusion is bounded and the inward drift is not too strong. It seems that the later result has not been formulated before even in the finite dimensional case.

In Section 6 we compute the infinitesimal generator for sufficiently smooth functions and mention a connection with a martingale problem. In Section 7 we deal with applications to systems of stochastic integral equations that are, e.g., used to describe physical systems with a finite or infinite number of degrees of freedom. In previous work, infinite systems were treated by approximation from the finite dimensional case (cf. Doss–Royer [8], Fritz [12], Shiga–Shimizu [30]). By contrast, we work in Hilbert space including the infinite dimensional case without approximation. For our results we do not need any “symmetry”, “finite range”, or “stationarity” conditions in the drift (interaction) part of the stochastic equation. Also we do not need boundedness conditions on the derivative of the drift term as, e.g., the condition (H4) in Doss–Royer (loc. cit.). On the other hand our method requires that the “one site part” in the drift term is not too large. We will deal with large one site drifts in Leha–Ritter [21].

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2. Preliminaries on stochastic integrals in Hilbert spaces. We shall give here a brief account of Brownian motion and stochastic integrals in a real, separable Hilbert space $\mathbb{H}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. The main references are Métivier–Pellaumail [26] and Métivier [24, 25]. Our basic space is a probability space $(\Omega, \mathcal{F}, P)$ endowed with a growing family $(\mathcal{F}_t)_{t\geq 0}$ of sub–σ–algebras of $\mathcal{F}$. The symbol $E[\,]$ will stand for the expectation with respect to the probability P. We suppose that all processes $(\eta_t)_{0\leq t<T}$ (where T is a stopping time) on $(\Omega, \mathcal{F}, P)$ that occur in the sequel are adapted to the family $(\mathcal{F}_t)$. This means that the random variables $\xi_{x,t}1_{[t<T]}$ are $\mathcal{F}_t$–measurable.

(2.1) *Brownian motion on* $\mathbb{H}$. Let $\mathfrak{Q}$ be a positive definite, self–adjoint nuclear operator on $\mathbb{H}$. A continuous, $\mathbb{H}$–valued process $(\beta_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ is called a *Brownian motion* with *covariance operator* $\mathfrak{Q}$, if

(i) for every $s < t$, $\beta_t - \beta_s$ is independent of $\mathcal{F}_s$;

(ii) for every $s < t$ and $y \in \mathbb{H}$, the distribution of the real random variable $(\beta_t - \beta_s, y)$ is Gaussian centered with variance $(t-s)(\mathfrak{Q}y, y)$.

As all linear combinations $\sum \lambda_k(\beta_t - \beta_s, y_k) = (\beta_t - \beta_s, \sum \lambda_k y_k)$ are Gaussian, finitely many variables $(\beta_t - \beta_s, y_k)$ are jointly Gaussian.

The following example shows that Brownian motions exist. We shall need it in Section 7. Let I be an at most countable index set. Let β be the product on $\mathbb{R}^I$ of independent, one–dimensional normal Brownian motions. Then β is a Brownian motion on each Hilbert space $\mathbb{H} := \ell^2(\gamma) := \{y = (y_k) \in \mathbb{R}^I / \sum_{k\in I} \gamma_k y_k^2 < \infty\}$, where $(\gamma_k)_{k\in I}$ is a summable sequence of strictly positive real numbers. Its covariance operator $\mathfrak{Q}$ is given via multiplication with the diagonal matrix $D_\gamma := (\gamma_k \delta_{kl})_{k,l\in I}$.

Note that this example describes the most general Brownian motion. If $\mathfrak{Q}$ is a self–adjoint, positive definite, nuclear operator on a general real, separable Hilbert space $\mathbb{H}$ with eigenvalues γ_k and normalized eigenvectors c_k, then $\mathbb{H}$ may be identified in a natural way with the space $\ell^2(\gamma)$:

$$y = (y_k) \in \ell^2(\gamma) \sim \sum y_k \sqrt{\gamma_k} c_k \in \mathbb{H}.$$

The image of the process on $\ell^2(\gamma)$ constructed above is a Brownian motion on $\mathbb{H}$ with covariance operator $\mathfrak{D}$.

(2.2) *The isometric stochastic integral.* Let $\mathcal{B}$ be the Borel σ–algebra on the strictly positive real line $]0,\infty[$. Let $\mathcal{P}$ be the σ–algebra on $\Omega\times]0,\infty[$ of *predictable sets*, i.e., the sub–σ–algebra of $\mathcal{B}\times\mathcal{F}$ generated by the predictable rectangles, i.e., the sets of the form $]s,t[\times F$, where $0 \le s < t$ and $F \in \mathcal{F}_s$. Let μ be a continuous $\mathbb{H}$–valued $\mathbb{L}^2(P)$–martingale. Let α_μ be the Doléans measure of the real submartingale $\|\mu\|^2$. This is the measure α_μ on $\mathcal{P}$ such that

$$\alpha_\mu(]s,t[\times F) = E[1_F \|\mu_t - \mu_s\|^2] = E[1_F(\|\mu_t\|^2 - \|\mu_s\|^2)]. \tag{1}$$

There is one predictable process $\mathfrak{Q}_\mu$ with values in the set of positive, self–adjoint, nuclear operators on $\mathbb{H}$ such that the following equality obtains for all $y,z \in \mathbb{H}$ and all predictable rectangles $]s,t[\times F$:

$$\int_{]s,t[\times F} (\mathfrak{Q}_\mu y, z) d\alpha_\mu = E[1_F(\mu_t - \mu_s, y)(\mu_t - \mu_s, z)]. \tag{2}$$

It is straightforward to show that for Brownian motion β with covariance operator $\mathfrak{Q}$ we have

$$\alpha_\beta = (\mathrm{tr}\,\mathfrak{Q})\lambda \otimes P|\mathcal{P}, \tag{3}$$

where λ is Lebesgue measure on $]0,\infty[$ and $\mathrm{tr}\,\mathfrak{Q}$ is the trace of the nuclear operator $\mathfrak{Q}$. A similarly simple argument shows that

$$(\mathfrak{Q}_\beta)_t = \mathfrak{Q}/\mathrm{tr}\,\mathfrak{Q}. \tag{4}$$

Let $\mathbb{G}$ be another real separable Hilbert space. The set of integrands for the stochastic integral with respect to the martingale μ is denoted by $\mathbf{\Lambda}^2(\mathbb{H},\mathbb{G},\mathcal{P},\mu)$ (cf. Métivier–Pellaumail [26], 14.5). This is the completion of the vector space of $\mathcal{P}$–step functions $\mathbf{X}$ with values in $L(\mathbb{H},\mathbb{G})$ (the space of bounded linear operators $\mathbb{H}\to\mathbb{G}$) with respect to the norm

$$\|\mathbf{X}\|^2_{\mathbf{\Lambda}^2} = \int_{]0,\infty[\times\Omega} \mathrm{tr}\,\mathbf{X}_s \circ \mathfrak{Q}_\mu \circ \mathbf{X}_s^*\, d\alpha_\mu. \tag{5}$$

Note that $\mathbf{X}^*$ denotes the adjoint operator of $\mathbf{X}$. $\mathbf{\Lambda}^2(\mathbb{H},\mathbb{G},\mathcal{P},\mu)$ has a representation in the space of (not necessarily continuous) operators from $\mathbb{H}$ to $\mathbb{G}$. However, we will need only $L(\mathbb{H},\mathbb{G})$–valued integrands. We will repeatedly use the following sufficient condition for a process $\mathbf{X}$ to be an integrand (Métivier–Pellaumail [26], 14.5): $\mathbf{\Lambda}^2(\mathbb{H},\mathbb{G},\mathcal{P},\mu)$ contains all predictable (e.g., left continuous) $L(\mathbb{H},\mathbb{G})$–valued processes $\mathbf{X}$ such that

(6)
$$\int_{]0,\infty[\times\Omega} \operatorname{tr} \mathbf{X}_s \circ \mathfrak{Q}_\mu \circ \mathbf{X}_s^* \; d\alpha_\mu < \infty.$$

For a predictable step function $\mathbf{Y} = \sum_{j=1}^r Y_j 1_{]s_j,t_j]\times F_j}$ ($Y_j \in L(\mathbb{H},\mathbb{G})$), the stochastic integral of $\mathbf{Y}$ is the $\mathbb{G}$–valued random variable on $(\Omega,\mathcal{F},P)$ defined by

(7)
$$\int \mathbf{Y} d\mu = \sum_j Y_j(\mu_{t_j} - \mu_{s_j}) 1_{F_j}.$$

This definition establishes an isometry from the normed space of all $\mathcal{P}$–step functions with values in $L(\mathbb{H},\mathbb{G})$ into $\mathbb{L}^2_{\mathbb{G}}(P)$. The stochastic integral is the unique extension of this isometry to $\mathbf{\Lambda}^2(\mathbb{H},\mathbb{G},\mathcal{P},\mu)$. For $\mathbf{Y} \in \mathbf{\Lambda}^2(\mathbb{H},\mathbb{G},\mathcal{P},\mu)$, $\int_0^t \mathbf{Y}_s d\mu_s$ is defined as $\int 1_{[0,t]} \mathbf{Y} d\mu$; the process $(\int_0^t \mathbf{Y}_s d\mu_s)_{t\geq 0}$ is an $\mathbb{L}^2_{\mathbb{G}}(P)$–martingale. For an $\mathbb{H}$–valued process φ such that $\tilde{\varphi} := (\varphi,\cdot) \in \mathbf{\Lambda}^2(\mathbb{H},\mathbb{R},\mathcal{P},\mu)$ we will write $\int(\varphi_s, d\mu_s)$ instead of $\int \tilde{\varphi}_s, d\mu_s$.

We finish this paragraph by reproducing two formulae that will be important in the sequel (cf. Métivier–Pellaumail [26], 14.7.1 and 14.7.2). Let $\mathbf{X} \in \mathbf{\Lambda}^2(\mathbb{H},\mathbb{G},\mathcal{P},\beta)$ and let

$$\mu_t = \int_0^t \mathbf{X}_s d\beta_s.$$

Then the Doléans measure α_μ of $\|\mu\|^2_{\mathbb{G}}$ is

(8)
$$\alpha_\mu = \operatorname{tr}\mathbf{X} \circ \mathfrak{Q} \circ \mathbf{X}^* \lambda \otimes P|\mathcal{P}$$

and

(9)
$$\mathfrak{Q}_\mu = \mathbf{X} \circ \mathfrak{Q} \circ \mathbf{X}^* / \operatorname{tr}\mathbf{X} \circ \mathfrak{Q} \circ \mathbf{X}^*.$$

(2.3) *Itô's formula.* In order to formulate a version of Itô's formula suiting our purposes we need the tensor–quadratic variation of a continuous, $\mathbb{H}$–valued $\mathbb{L}^2(P)$–martingale μ. Let $\mathbf{X}$ be the $L(\mathbb{H},\mathbb{H})$–valued process $(\mu,\cdot)\mu$. Then the tensor–quadratic variation $\ll\mu\gg$ of μ is the predictable process of bounded variation such that $\mathbf{X}- \ll\mu\gg$ is a martingale. Since the process $\mathbf{X}$ consists of operators whose range is one–dimensional, $\ll\mu\gg$ has values in the space of nuclear operators. When μ is of the form

$$\mu_t = \int_0^t \mathbf{X}_s d\beta_s,$$

where

$$\mathbf{X} \in \mathbf{\Lambda}^2(\mathbb{H},\mathbb{G},\mathcal{P},\beta),$$

we have by 2.2(3), 2.2(4), and Métivier–Pellaumail [26] 14.7.5

(1)
$$\ll\mu\gg = \int_0^t \mathbf{X}_s \circ \mathfrak{Q} \circ \mathbf{X}_s^* ds.$$

In particular $\ll\beta\gg_t = t\mathfrak{Q}$.

Now let ξ be the semi–martingale $\xi = \mu + \varphi$, where φ is a continuous, $\mathbb{H}$–valued process of bounded variation on bounded intervals. Let the function $f : \mathbb{H} \to \mathbb{R}$ be twice continuously differentiable and suppose that the mapping $f'' : \mathbb{H} \to B(\mathbb{H},\mathbb{H})$ is uniformly continuous, where

$B(\mathbb{H},\mathbb{H})$ denotes the continuous bilinear forms on $\mathbb{H}$. In this situation, the following version of Itô's formula obtains (cf., e.g., Métivier [23], Section 7):

$$f(\xi_t) = f(\xi_0) + \int_0^t f'(\xi_s) d\mu_s + \int_0^t f'(\xi_s) d\varphi_s + \frac{1}{2}\int_0^t f''(\xi_s)\, d\ll\mu\gg_s \tag{2}$$

for all $t \geq 0$.

The second term on the right side is *not* an isometric stochastic integral if f' grows too fast. As f'' is bounded on bounded sets, so is f'. Therefore, denoting by $T^{(n)}$ the first exit time of ξ from the ball $\{y \in \mathbb{H}/\|y\| \leq n\}$,

$$T^{(n)} := \inf\{t \geq 0/\|\xi_t\| \geq n\},$$

we see that

$$\left(\int_0^{t\wedge T^{(n)}} f'(\xi_s) d\mu_s\right)_{t\geq 0}$$

is an $\mathbb{L}^2(P)$–martingale, i.e., the expression $(\int_0^t f'(\xi_s)d\mu_s)_{t\geq 0}$ is a local $\mathbb{L}^2(P)$–martingale. Thus equality (2) is to be understood in the sense that t is replaced by $t \wedge T^{(n)}$.

In the last term on the right side of formula (2) the continuous bilinear forms $f''(y)$ act on the nuclear operators $\mathfrak{R}$ via the duality

$$\langle f''(y), \mathfrak{R}\rangle \to \operatorname{tr} f''(y)(\mathfrak{R}\cdot,\cdot). \tag{3}$$

3. Existence and uniqueness of solutions. For the reader's convenience and for the sake of completeness we state and prove in this section a theorem of existence and uniqueness which fits in the framework of the following sections.

(3.1) *Historical remarks.* To our knowledge, Chantladze [3] and Daletskii [4] were the first to construct diffusion processes in infinite dimensions, using Itô's device of stochastic integration. Daletskii (loc. cit.) considers a certain Hilbert space $\mathbb{H}$ and two functions $\mathfrak{b} : \mathbb{H} \times \mathbb{R}^+ \to \mathbb{H}$ and $\mathfrak{A} : \mathbb{H}\times\mathbb{R}^+ \to L(\mathbb{H},\mathbb{H})$. Under the hypothesis that these two functions satisfy a uniform Lipschitz condition he asserts existence of a unique solution to the stochastic differential equation

$$\xi_{x,t} = x + \int_0^t \mathfrak{A}(\xi_{x,s}, s) d\beta_s + \int_0^t \mathfrak{b}(\xi_{x,s}, s) ds.$$

In a subsequent paper, Daletskii [5], he also considers unbounded operators $\mathfrak{A}$. A more recent reference in the continuous case is Yor [34]. The question was pursued by Doss and Royer [8], who had in mind an unbounded spin model in infinite dimensions of statistical mechanics. They consider a system of equations of the form

$$\xi_{x,t,k} = x_k + \beta_{t,k} + \int_0^t b_k(\xi_{x,s}) ds \qquad (k \in \mathbb{Z}^d),$$

where the functions $b_k : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ arise from a family of pair potentials that satisfy certain conditions that we will not reproduce here. Their method is an approximation from the finite dimensional case. The function b_k is supposed to be partially differentiable, so that, at least on finite dimensional subspaces, b_k satisfies a local Lipschitz condition.

We will now state a theorem on existence and uniqueness of solutions to stochastic differential equations in Hilbert spaces that is sufficient for our purposes. The proof below follows the lines of the well–known theorem in the finite dimensional case. We will therefore give only a sketch of the proof.

(3.2) *Notation and definition.* Throughout, the letter K will stand for a real constant that may vary from one line to another. As in Section 2, let $\mathbb{H}$ be a separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let β be a Brownian motion with covariance operator $\mathfrak{Q}$ on $\mathbb{H}$. We will first be interested in solutions ξ_x starting at $x \in \mathbb{H}$ to the stochastic differential equation in $\mathbb{H}$

$$\xi_{x,t} = x + \int_0^t \mathfrak{A}(\xi_{x,s})d\beta_s + \int_0^t \mathfrak{b}(\xi_{x,s})ds \tag{1}$$

on a stochastic interval $[0, T_x[$. To make this more precise, we consider two norm–continuous mappings $\mathfrak{A} : \mathbb{H} \to L(\mathbb{H},\mathbb{H})$ and $\mathfrak{b} : \mathbb{H} \to \mathbb{H}$. We will say that a continuous process $(\xi_{x,t})_{0\le t<T_x}$ on $(\Omega, \mathcal{F}, P)$ adapted to $(\mathcal{F}_t)$ is a (*strong*) *maximal solution to the stochastic integral equation* 3.2(1) starting at $x \in \mathbb{H}$, and with *explosion time* T_x, if

(i) $\limsup_{t\to\infty} \|\xi_{x,t}\| = \infty$ on the set $\{T_x < \infty\}$;

(ii) there exists a sequence $(T_m)_{m\in\mathbb{N}}$ of stopping times, growing to T_x, such that

  (α) $\mathfrak{A}(\xi_{x,s})1_{[0,T_m]} \in \mathbf{\Lambda}^2(\mathbb{H},\mathbb{H},\mathcal{P},\beta)$

  (β) $\xi_{x,t\wedge T_m} = x + \int_0^{t\wedge T_m} \mathfrak{A}(\xi_{x,s})d\beta_s + \int_0^{t\wedge T_m} \mathfrak{b}(\xi_{x,s})ds$
  for all $t \geq 0$, for all $m \in \mathbb{N}$, and for almost all paths.

The integral $\int_0^{t\wedge T_m} \mathfrak{b}(\xi_{x,s})ds$ is a Bochner (or also a weak) integral.

(3.3) Theorem. *Let $\mathfrak{A} : \mathbb{H} \to L(\mathbb{H},\mathbb{H})$ and $\mathfrak{b} : \mathbb{H} \to \mathbb{H}$ be functions that satisfy Lipschitz conditions on bounded sets, i.e., for each $n \in \mathbb{N}$ there exists a constant L_n such that*

$$\begin{aligned}\|\mathfrak{A}(x) - \mathfrak{A}(y)\| \le& L_n\|x-y\|\\ \|\mathfrak{b}(x) - \mathfrak{b}(y)\| \le& L_n\|x-y\|\end{aligned}$$

for all x, y such that $\|x\| \le n$, $\|y\| \le n$. Let $(\Omega, \mathcal{F}, P, \beta)$ be any Brownian motion starting at zero with covariance operator $\mathfrak{Q}$. Then, for each $x \in \mathbb{H}$, there exists exactly one strong maximal solution $(\xi_{x,t})_{0\le t<T_x}$ to the stochastic integral equation 3.2(1).

Proof. As usual, we divide our proof in two steps. We first suppose that the constants L_n $(= L)$ are independent of n (uniform Lipschitz condition) and show that in this case ξ_x has infinite lifetime, i.e., $T_x = \infty$. Let $x \in \mathbb{H}$ be fixed. We define by induction on $n \in \mathbb{Z}^+$ a sequence $\xi^n_{x,t} = \xi^n_t$ of (continuous) stochastic processes on $\Omega \times \mathbb{R}^+$ with the property

$$E\left[\int_0^t \|\xi^n_s\|^2 ds\right] \le \infty \qquad \text{for all } t \geq 0. \tag{1}$$

Let

$$\begin{aligned}\xi^0_t :=& x,\\ \xi^{n+1}_t :=& x + \int_0^t \mathfrak{A}(\xi^n_s)d\beta_s + \int_0^t \mathfrak{b}(\xi^n_s)ds \qquad (t \geq 0).\end{aligned}$$

We have to show that ξ^{n+1} exists and that (1) holds for $n+1$ if ξ^n exists and if (1) holds for n. We first show that the mapping

$$(s,\omega) \to \mathfrak{A}(\xi^n_s(\omega))1_{[0,t]}$$

is an element of $\mathbf{\Lambda}^2(\mathbb{H}, \mathbb{H}, \mathcal{P}, \beta)$ (see 2.2). Since $\mathfrak{A}(\xi_s^n)$ is continuous, our claim will follow from 2.2(6), i.e., we have to show

$$E\left[\int_0^t \operatorname{tr}(\mathfrak{A}(\xi_s^n) \circ \mathfrak{Q} \circ \mathfrak{A}^*(\xi_s^n))ds\right] < \infty. \tag{2}$$

However, since $\mathfrak{A}(x) = O(\|x\|)$ by our uniform Lipschitz condition, the quantity on the left side of (2) is majorized by

$$\operatorname{tr}\mathfrak{Q}E\left[\int_0^t \|\mathfrak{A}(\xi_s^n)\|^2 ds\right] \leq K \operatorname{tr}\mathfrak{Q}E\left[\int_0^t (1 + \|\xi_s^n\|^2)ds\right] < \infty.$$

It follows (cf. 2.2) that the martingale $(\int_0^t \mathfrak{A}(\xi_s^n)d\beta_s)_t$ is square integrable; we obtain

$$\begin{aligned} E\left[\int_0^t \|\int_0^u \mathfrak{A}(\xi_s^n)d\beta_s\|^2 du\right] &= \int_0^t E\left[\|\int_0^u \mathfrak{A}(\xi_s^n)d\beta_s\|^2\right] du \\ &\leq \int_0^t E\left[\|\int_0^t \mathfrak{A}(\xi_s^n)d\beta_s\|^2\right] du \\ &= t \cdot E\left[\|\int_0^t \mathfrak{A}(\xi_s^n)d\beta_s\|^2\right]. \end{aligned}$$

It remains to show finiteness of the expectation

$$E\left[\int_0^t \|\int_0^u \mathfrak{b}(\xi_s^n)ds\|^2 du\right].$$

This follows from

$$\left(\int_0^u \|\mathfrak{b}(\xi_s^n)\|ds\right)^2 \leq u\int_0^u \|\mathfrak{b}(\xi_s^n)\|^2 ds \leq Kt\int_0^t (1 + \|\xi_s^n\|^2)ds$$

by our uniform Lipschitz assumption.

We now show that the sequence $(\xi_t^n)_{n\geq 0}$ converges almost surely and locally uniformly to a process (ξ_t). We first use the isometric property of the stochastic integral and our Lipschitz condition to estimate

$$\begin{aligned} &E\left[\|\int_0^t (\mathfrak{A}(\xi_s^n) - \mathfrak{A}(\xi_s^{n-1}))d\beta_s\|^2\right] \\ &= E\left[\int_0^t \operatorname{tr}(\{\mathfrak{A}(\xi_s^n) - \mathfrak{A}(\xi_s^{n-1})\} \circ \mathfrak{Q} \circ \{\mathfrak{A}^*(\xi_s^n) - \mathfrak{A}^*(\xi_s^{n-1})\})ds\right] \\ &\leq \operatorname{tr}\mathfrak{Q}E\left[\int_0^t \|\mathfrak{A}(\xi_s^n) - \mathfrak{A}(\xi_s^{n-1})\|^2 ds\right] \\ &\leq \operatorname{tr}\mathfrak{Q}L^2\int_0^t E[\|\xi_s^n - \xi_s^{n-1}\|^2]ds. \end{aligned}$$

For the term $E[\|\int_0^t \mathfrak{b}(\xi_s^n) - \mathfrak{b}(\xi_s^{n-1})ds\|^2]$ one obtains a similar estimate (cf. the finite dimensional case). Putting $D_t^n := E[\|\xi_t^{n+1} - \xi_n^t\|^2]$ and using the inequality $\|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$ we obtain the estimate

$$\begin{aligned} D_t^n &\leq 2E\left[\|\int_0^t (\mathfrak{A}(\xi_s^n) - \mathfrak{A}(\xi_s^{n-1}))d\beta_s\|^2 + \|\int_0^t (\mathfrak{b}(\xi_x^n) - \mathfrak{b}(\xi_s^{n-1}))ds\|^2\right] \\ &\leq 2L^2(\operatorname{tr}\mathfrak{Q} + t)\int_0^t D_s^{n-1}ds. \end{aligned}$$

On the other hand we have again by isometry

$$\begin{aligned} D_t^0 &\leq 2E\left[\|\int_0^t \mathfrak{A}(x)d\beta_s\|^2\right] + 2\left[\|\int_0^t \mathfrak{b}(x)ds\|^2\right] \\ &= 2E\left[\int_0^t \mathrm{tr}\, \mathfrak{A}(x)\circ \mathfrak{Q} \circ \mathfrak{A}^*(x)ds\right] + 2\left[\|\int_0^t \mathfrak{b}(x)ds\|^2\right] \\ &\leq 2\mathrm{tr}\, \mathfrak{Q}\|\mathfrak{A}(x)\|^2 t + 2\|\mathfrak{b}(x)\|^2 t^2 =: K_0(t). \end{aligned}$$

Using monotonicity of $K_0(t)$ and $K(t) := 2L^2(\mathrm{tr}\, \mathfrak{Q}+t)$ we now see by induction that the quantity D_t^n satisfies the inequality

$$D_t^n \leq K_0(t)(K(t)t)^n/n!.$$

The proof of existence (in the uniform Lipschitz case) now continues as in McKean [22], page 53. A similar reasoning shows that the solution is unique (cf. McKean [22], page 54).

We now turn to the proof of the general case. We extend the restriction of $\mathfrak{A}$ and $\mathfrak{b}$ to the ball $\{\|y\| \leq n\} \subseteq \mathbb{H}$ to functions $\mathfrak{A}_n$ and $\mathfrak{b}_n$ that satisfy Lipschitz constants uniformly on $\mathbb{H}$, e.g.,

$$\mathfrak{A}_n(y) := \begin{cases} \mathfrak{A}(y) & \text{if } \|y\| \leq n \\ \mathfrak{A}(ny/\|y\|) & \text{elsewhere.} \end{cases}$$

According to our first step, we obtain stochastic processes $(\xi_t^{(n)})_{0\leq t<\infty}$ for the modified data $\mathfrak{A}_n$ and $\mathfrak{b}_n$. Let

$$T^{(n)} := \inf\{t \geq 0/\|\xi_t^{(n)}\| \geq n\}$$

be the first exit time of $\xi_t^{(n)}$ from the ball $\{\|y\| \leq n\} \subseteq \mathbb{H}$. By uniqueness,

$$\xi_t^{(n+1)} = \xi_t^{(n)} \text{ for all } t \leq T^{(n)}.$$

By continuity of $\xi^{(n)}$, the sequence $T^{(n)}$ of stopping times is eventually strictly ascending and eventually strictly positive. Let

$$T_x := \sup_n T^{(n)}.$$

If $T_x(\omega) < \infty$, then

$$\limsup_{0\leq t<T_x} \|\xi_t(\omega)\| \geq \limsup_n \|\xi_{T^{(n)}(\omega)}(\omega)\| = \infty.$$

(3.4) *Remark.* The proof shows that in the uniform Lipschitz case we may use the constant time m for the stopping time T_m in 3.2(ii).

4. Nonexplosion. The main question of this section is: What conditions on the data $\mathfrak{A}$ and $\mathfrak{b}$ ensure infinity of the lifetime T_x?

(4.1) *Historical remarks.* It is well–known that the solution to 3.2(1) has almost surely infinite life time T_x, if the functions $\mathfrak{A}$ and $\mathfrak{b}$ satisfy uniform Lipschitz conditions (see, e.g., Itô [18], and McKean [22], in the finite dimensional case, and Daletskii [4] in the infinite dimensional case.) Several authors have weakened this condition. In the finite dimensional case, Stroock–Varadhan (cf. [32], 10.2.2) prove infinity of the life time T_x if $\|\mathfrak{A}(y)\| = O(\|y\|)$ and if $(y, \mathfrak{b}(y)) \leq K(1 + \|y\|^2)$. Their theorem is couched in terms of the martingale problem. A fairly general condition is due to Has'minskii [14]. In some instances, however, his condition is not easy to verify. In the infinite dimensional case, if $\mathfrak{A}$ is identity, Doss and Royer [8] give some kind of an upper boundedness condition on $\mathfrak{b}$. Métiver–Pellaumail [26], 7.2, use the condition $\|\mathfrak{b}|(y)\| = O(\|y\|)$, although in a somewhat different context.

We will use a boundedness condition on $\mathfrak{A}$ and the same upper boundedness condition on $(y, \mathfrak{b}(y))$ as in Stroock–Varadhan (loc. cit.). We first prove two lemmas. The first lemma is a $\mathbf{\Lambda}^2$–version of (Métiver–Pellaumail [26], 4.2).

(4.2) LEMMA. *Let $(\mathbf{X}_t)_{t\geq 0}$ be a left–continuous process in $\mathbf{\Lambda}^2(\mathbb{H}, \mathbb{G}, \mathcal{P}, \beta)$ and let $(\mathbf{Y}_t)_{t\geq 0}$ be a left–continuous process in $L(\mathbb{G}, \mathbb{K})$, where $\mathbb{G}$ and $\mathbb{K}$ are Hilbert spaces. Suppose that the expectation*

$$(1) \qquad E\left[\int_0^\infty tr(\mathbf{Y}_s \circ \mathbf{X}_s \circ \mathfrak{Q} \circ \mathbf{X}_s^* \circ \mathbf{Y}_s^*) ds\right]$$

is finite. Define $\mu_t := \int_0^t \mathbf{X}_s d\beta_s$. Then the two stochastic integrals

$$\int_0^t \mathbf{Y}_s d\mu_s \text{ and } \int_0^t \mathbf{Y}_s \circ \mathbf{X}_s d\beta_s$$

exist and are P–equivalent.

PROOF. We apply 2.2(6) to the process $\mathbf{Y}$. Because of 2.2(8) and 2.2(9), 2.2(6) is nothing else than finiteness of the expectation (1), so that the first integral exists. This finiteness together with 2.2(6), 2.2(3), and 2.2(4) shows that also the second integral exists. At the same time we see that the mapping

$$\begin{array}{rcl} \mathbf{\Lambda}^2(\mathbb{G}, \mathbb{K}, \mathcal{P}, \mu) & \to & \mathbf{\Lambda}^2(\mathbb{H}, \mathbb{K}, \mathcal{P}, \beta) \\ \mathbf{Z} & \to & \mathbf{Z} \circ \mathbf{X} \end{array}$$

is an isometry. The two stochastic integrals are P–equivalent when $\mathbf{Z}$ is a $\mathcal{P}$–step function. The rest of the lemma now follows from the isometric property of the stochastic integral.

(4.3) *Remarks.* (1) Suppose that η_t is a left–continuous process in $\mathbb{H}$. Then $\mathbf{Y}_t := (\eta_t, \cdot)$ is a left–continuous process in $\mathbb{H}' = L(\mathbb{H}, \mathbb{R})$. If $\mathbb{G} = \mathbb{H}$, the expectation 4.2(1) is in this case

$$(1) \qquad E\left[\int_0^\infty (\mathbf{X}_s \circ \mathfrak{Q} \circ \mathbf{X}_s^* \eta_s, \eta_s) ds\right].$$

If this quantity is finite, then the stochastic integrals

$$\int_0^t (\eta_s, d\mu_s) \text{ and } \int_0^t (\mathbf{X}_s^* \eta_s, d\beta_s)$$

exist and are P–equivalent.

(2) The expectation 4.2(1) is finite, if Y is uniformly bounded.

Our next lemma is an application of Itô's formula.

(4.4) LEMMA. *Let $t > 0$, let $(\mathbf{X}_s)_{s\geq 0}$ be a process in $L(\mathbb{H}, \mathbb{H})$ s.th. $\mathbf{X}1_{[0,t]} \in \mathbf{\Lambda}^2(\mathbb{H}, \mathbb{H}, \mathcal{P}, \beta)$ and let $(\varphi_s)_{s\geq 0}$ be a process that is locally integrable with respect to Lebesgue measure and s.th. $E[\int_0^t \|\varphi_s\| ds] < \infty$. Define $\eta_s := x + \int_0^s \mathbf{X}_u d\beta_u + \int_0^s \varphi_u du$ and suppose that the expectation $E[\int_0^t (\mathbf{X}_s \circ \mathfrak{Q} \circ \mathbf{X}_s^* \eta_s, \eta_s) ds]$ is finite. Then we have*

$$\|\eta_t\|^2 = \|x\|^2 + 2\int_0^t (\mathbf{X}_s^* \eta_s, d\beta_s) + 2\int_0^t (\eta_s, \varphi_s) ds + \int_0^t \mathrm{tr}(\mathbf{X}_s \circ \mathfrak{Q} \circ \mathbf{X}_s^*) ds,$$

where the second term on the right is an isometric integral.

PROOF. We apply Itô's formula 2.3(2) to the function $y \to \|y\|^2$ on $\mathbb{H}$ and to the process

$$\xi_s := \eta_{s\wedge t} = x + \int_0^{s\wedge t} \mathbf{X}_u d\beta_u + \int_0^{s\wedge t} \varphi_u du.$$

By hypothesis on $\mathbf{X}$ the process $\mu_s := \int_0^{s\wedge t} \mathbf{X}_u d\beta_u$ is an $\mathbb{L}^2(P)$–martingale and $(\int_0^{s\wedge t} \varphi_u du)_{s\geq 0}$ is of bounded variation. Therefore we have for $s \leq t$

$$\|\xi_s\|^2 = \|x\|^2 + 2\int_0^s (\xi_u, d\mu_u) + 2\int_0^s (\xi_u, \varphi_u)du + \int_0^s d \ll\mu\gg_u .$$

By Remark 4.3.1 and by hypothesis, we have for these values of s

$$\int_0^s (\xi_u, d\mu_u) = \int_0^s (\mathbf{X}_s^* \xi_u, d\beta_u).$$

This equality together with 2.3(1) implies the lemma.

Our theorem reads as follows.

(4.5) THEOREM. *Let notation be as explained in paragraph* 3.2 *and let* $\mathfrak{b}$ *be bounded on bounded sets. Suppose that*

(I) $\mathrm{tr}(\mathfrak{A}(y) \circ \mathfrak{Q} \circ \mathfrak{A}^*(y)) \leq K(1 + \|y\|^2)$ (e.g., $\|\mathfrak{A}(y)\| = O(\|y\|)$)

(II) $(y, \mathfrak{b}(y)) \leq K(1 + \|y\|^2)$

for all $y \in \mathbb{H}$. *Let* $(\xi_{x,t})_{0\leq t<T_x}$ *be a maximal solution of equation* 3.2(1), *starting at* x. *Then, for any* $x \in \mathbb{H}$, *the explosion time* T_x *is infinite* (a.s.).

PROOF. We denote by $T^{(n)}$ the first exit time of ξ_x from the ball around the origin of radius $n \in \mathbb{N}$ in $\mathbb{H}$. Let (T_m) be a sequence of stopping times as in 3.2.

$$\mathbf{X}_s := \mathfrak{A}(\xi_{x,s})1_{[0,T^{(n)}]} \text{ and } \varphi_s := \mathfrak{b}(\xi_{x,s})1_{[0,T^{(n)}]}.$$

We show that we may apply Lemma 4.4 in the present situation. Since, by Condition I,

$$E\left[\int_0^t \mathrm{tr}\mathbf{X}_s \circ \mathfrak{Q} \circ \mathbf{X}_s^* ds\right] \leq KE\left[\int_0^{t\wedge T^{(n)}} (1 + \|\xi_{x,s}\|^2)ds\right] < \infty, \tag{1}$$

we have

$$\mathbf{X}1_{[0,t]} \in \mathbf{\Lambda}^2(\mathbb{H}, \mathbb{H}, \mathcal{P}, \beta) \tag{2}$$

for all $t \geq 0$. By the boundedness condition on $\mathfrak{b}$, the process φ_s satisfies $E[\int_0^t \|\varphi_s\|ds] < \infty$ for all $t \geq 0$. The process $\eta_t := \xi_{x,t\wedge T^{(n)}}$ is bounded. Therefore

$$E\left[\int_0^t (\mathbf{X}_s \circ \mathfrak{Q} \circ X_s^* \eta_s, \eta_s)ds\right] \leq E\left[\int_0^t \|\eta_s\|^2 \mathrm{tr}\mathbf{X}_s \circ \mathfrak{Q} \circ X_s^* ds\right] < \infty,$$

for all $t \geq 0$ because of (2). From (1) we infer that $1_{[0,t\wedge T_m \wedge T^{(n)}]}\mathfrak{A}(\xi_{x,s})$ converges to $1_{[0,t\wedge T^{(n)}]}\mathfrak{A}(\xi_{x,s})$ in $\mathbf{\Lambda}^2(\mathbb{H}, \mathbb{H}, \mathcal{P}, \beta)$ as $m \to \infty$. It follows that

$$\int_0^{t\wedge T_m \wedge T^{(n)}} \mathfrak{A}(\xi_{x,s})d\beta_s \to \int_0^{t\wedge T^{(n)}} \mathfrak{A}(\xi_{x,s})d\beta_s$$

in $\mathbb{L}^2_{\mathbb{H}}(P)$ as $m \to \infty$. A standard argument now shows that, without loss of generality, we may suppose $T_n = T^{(n)}$ for all $n \in \mathbb{N}$. In particular, we have

$$\eta_t = x + \int_0^t \mathbf{X}_s d\beta_s + \int_0^t \varphi_s ds.$$

Lemma 4.4 yields

$$\|\eta_t\|^2 = \|x\|^2 + 2\int_0^t (\mathfrak{A}^*(\eta_s)\eta_s, d\beta_s) + 2\int_0^t (\eta_s, \mathfrak{b}(\eta_s))ds$$
$$+ \int_0^t \operatorname{tr} \mathfrak{A}(\eta_s) \circ \mathfrak{Q} \circ \mathfrak{A}^*(\eta_s) ds.$$

Hence we have

$$\begin{aligned} E[\|\eta_t\|^2] &= \|x\|^2 + 2E\left[\int_0^t (\eta_s, \varphi_s)ds\right] + E\left[\int_0^t \operatorname{tr}(\mathbf{X}_s \circ \mathfrak{Q} \circ X_s^*)ds\right] \\ &= \|x\|^2 + 2E\left[\int_0^{t\wedge T^{(n)}} (\eta_s, \mathfrak{b}(\eta_s))ds\right] + E\left[\int_0^{t\wedge T^{(n)}} \operatorname{tr}(\mathfrak{A}(\eta_s) \circ \mathfrak{Q} \circ \mathfrak{A}^*(\eta_s))ds\right]. \end{aligned}$$

Using Hypotheses I and II we obtain

$$\begin{aligned} E[\|\eta_t\|^2] &\le \|x\|^2 + 3KE\left[\int_0^{t\wedge T^{(n)}} (1 + \|\eta_s\|^2)ds\right] \\ &\le \|x\|^2 + 3KE\left[\int_0^t (1 + \|\eta_s\|^2)ds\right] \\ &= \|x\|^2 + 3Kt + 3K\int_0^t E[\|\eta_s\|^2]ds. \end{aligned}$$

As in Doss–Royer [8], we now use Gronwall's lemma (cf. Dieudonné [7]) in order to obtain the following estimate:

(3) $$E[\|\xi_{x,t\wedge T^{(n)}}\|^2] = E[\|\eta_t\|^2] \le (\|x\|^2 + 1)(e^{3Kt} - 1) + \|x\|^2 =: g(x,t).$$

Note that g is independent of n. Since the set $\{T^{(n)} \le t\}$ is contained in the set $\{\|\xi_{x,t\wedge T^{(n)}}\| \ge n\}$ we have, using Tchebyshev's inequality,

(4) $$P[T^{(n)} \le t] \le P[\|\xi_{x,t\wedge T^{(n)}}\| \ge n] \le n^{-2}E[\|\xi_{x,t\wedge T^{(n)}}\|^2] \le n^{-2}g(x,t).$$

Therefore $P[T_x \le t] = 0$ for all $t > 0$, i.e., $T_x = \infty$ a.s.

(4.6) *Remarks.* (1) Letting $n \to \infty$ we deduce from 4.5(3) that $\|\xi_{x,t}\|$ has finite variance:

$$E[\|\xi_{x,s}\|^2] \le \liminf_{n\to\infty} E[\|\xi_{x,t\wedge T^{(n)}}\|^2] \le g(x,t).$$

(2) In the situation of Theorem 4.5, localization in the definition of a solution to equation 3.2(1) is not necessary, i.e., we have

$$\mathfrak{A}(\xi_{x,s})1_{[0,t]} \in \mathbf{\Lambda}^2(\mathbb{H}, \mathbb{H}, \mathcal{P}, \beta)$$

and

$$\xi_{x,s} = x + \int_0^t \mathfrak{A}(\xi_{x,s})d\beta_s + \int_0^t \mathfrak{b}(\xi_{x,s})ds$$

for all $t \geq 0$. Indeed, by Condition I and Remark 1,

$$E\left[\int_0^t \operatorname{tr} \mathfrak{A}(\xi_{x,s}) \circ \mathfrak{Q} \circ \mathfrak{A}^*(\xi_{x,s}) ds\right] \leq K \int_0^t E[1 + \|\xi_{x,s}\|^2] ds$$
$$\leq Kt + K \int_0^t g(x,s) ds$$
$$< Kt(1 + g(x,t)).$$

Therefore, the function $\mathfrak{A}(\xi_{x,s}) 1_{[0,t\wedge T_m]}$ on $\mathbb{R}^+ \times \Omega$ converges in $\mathbf{\Lambda}^2(\mathbb{H}, \mathbb{H}, \mathcal{P}, \beta)$ to the function $\mathfrak{A}(\xi_{x,s}) 1_{[0,t]}$ as $m \to \infty$. By isometry, the process $\int_0^{t\wedge T_m} \mathfrak{A}(\xi_{x,s}) d\beta_s$ converges to the process $\int_0^t \mathfrak{A}(\xi_{x,s}) d\beta_s$ in $\mathbb{L}^2_{\mathbb{H}}(P)$ as $m \to \infty$ for all $t \geq 0$. From here it is plain that we can go to the limit in 3.2.ii.β.

(3) In the proof of Theorem 4.5 we have shown the equality

$$\|\xi_{x,t\wedge T^{(n)}}\|^2 = \|x\|^2 + 2\int_0^{t\wedge T^{(n)}} (\mathfrak{A}^*(\xi_{x,s})\xi_{x,s}, d\beta_s) + 2\int_0^{t\wedge T^{(n)}} (\xi_{x,s}, \mathfrak{b}(\xi_{x,s})) ds$$
$$+ \int_0^{t\wedge T^{(n)}} \operatorname{tr} \mathfrak{A}(\xi_{x,s}) \circ \mathfrak{Q} \circ \mathfrak{A}^*(\xi_{x,s}) ds.$$

We will need this equality in Section 5. Under an additional hypothesis on $\mathfrak{A}$ (Condition III) we may go to the limit as $n \to \infty$ (cf. 5.10).

(4.7) *Counterexample.* The following deterministic example shows that Condition II is sharp. We put $\mathbb{H} = \mathbb{R}$, $\mathfrak{A} = 0$, $\mathfrak{b}(y) = \operatorname{sgn} y|y|^\alpha$ for $\alpha > 1$. The solution to 3.2(1) is

$$\xi_{x,t} = x[1 - (\alpha - 1)|x|^{\alpha-1}t]^{-1/(\alpha-1)}$$

for $0 \leq t < T_x := 1/[(\alpha-1)|x|^{\alpha-1}]$, $x \neq 0$.

5. Semigroups and Feller semigroups.

(5.1) *Notation.* We are interested here in three spaces of continuous functions on $\mathbb{H}$ where $\xi_{x,t}$ induces semigroups of operators (under certain conditions on $\mathfrak{A}$ and $\mathfrak{b}$). The first one is the space $C_b(\mathbb{H})$ of bounded functions on $\mathbb{H}$ that are uniformly continuous on bounded sets, the second one is the subspace $C_{ub}(\mathbb{H}) \subseteq C_b(\mathbb{H})$ of bounded, uniformly continuous functions on $\mathbb{H}$ and the third one is the subspace $C_0(\mathbb{H}) \subseteq C_{ub}(\mathbb{H})$ of uniformly continuous functions on $\mathbb{H}$ that are small outside of bounded sets. These spaces are Banach spaces with the norm of uniform convergence.

(5.2) *Historical remarks.* Dynkin [10], 5.25, showed that to each partial differential operator

$$\mathfrak{L} := \sum_{k,l=1}^n a_{kl}(x) \frac{\partial^2}{\partial x_k \partial x_l} + \sum_{k=l}^n b_k(x) \frac{\partial}{\partial x_k}$$

on $\mathbb{R}^n$ with bounded, Hölder continuous coefficients there exists a continuous Markov process ξ whose generator is an extension of $\mathfrak{L}$ and whose transition kernels $k_t(x, A) = E[1_A(\xi_{x,t})]$ (A a Borel subset of $\mathbb{R}^n$) induce a semigroup (P_t) of operators on $C_0(\mathbb{R}^n)$. By Lebesgue's convergence theorem this semigroup is weakly continuous and hence strongly continuous (Yosida [35]) (for a direct proof see also Meyer [27], p. 25); i.e., the transition kernels k_t form a *Feller semigroup* on $\mathbb{R}^n$.

The question under what conditions on the generator of a diffusion process on a finite dimensional manifold its transition kernels form a Feller semigroup, was pursued by Azencott [1]. His method and conditions are inspired by Has'minskii's [14] work on nonexplosion. As in the latter case, Azencott's conditions have the disadvantage of not being easy to verify in some instances. We will instead use an estimate that goes back to Paley and Wiener (cf. Kahane [19], page 6). We first prove a result on stability of the solutions to Equation 3.2(1).

(5.3) PROPOSITION. Suppose that the functions $\mathfrak{A}$ and $\mathfrak{b}$ satisfy Lipschitz conditions on bounded sets (cf. Theorem 3.3) and the conditions

(I) $\operatorname{tr}(\mathfrak{A}(y) \circ \mathfrak{Q} \circ \mathfrak{A}^*(y)) \le K(1 + \|y\|^2)$

(II) $(y, \mathfrak{b}(y)) \le K(1 + \|y\|^2)$

for all $y \in \mathbb{H}$ (cf. Theorem 4.5). Let $(\xi_{x,t})_{t\ge 0}$ be the maximal solution (with infinite lifetime according to Theorem 4.5) to equation 3.2(1). Then for all bounded sets $B \subseteq \mathbb{H}$ and for all real numbers $t \ge 0$, $\varepsilon > 0$, and $\eta > 0$ there exists a real number $\delta > 0$ such that

$$P[\|\xi_{x,t} - \xi_{x',t}\| \ge \eta] \le \varepsilon$$

for all $x, x' \in B$ such that $\|x - x'\| \le \delta$.

PROOF. By 4.5(4) we have for all $x \in \mathbb{H}$ and $n \in \mathbb{N}$

$$P[T_x^{(n)} \le t] \le n^{-2} g(x,t).$$

Let $n \in \mathbb{N}$ be chosen in such a way that

$$P[T_x^{(n)} \le t] \le \varepsilon/4$$

for all $x \in B$. Let L be a Lipschitz constant for $\mathfrak{A}$ and $\mathfrak{b}$ on the ball $\{\|y\| \le n\} \subseteq \mathbb{H}$. Put

$$\delta^2 := (\varepsilon\eta^2/6)\exp(-3 + L^2(\operatorname{tr}\mathfrak{Q} + t)).$$

Now let $x, x' \in B$ such that $\|x - x'\| \le \delta$. Define

$$T : \min\{T_x^{(n)}, T_{x'}^{(n)}\} \text{ and } A := \{T \ge t\}.$$

Note that, as T is a predictable stopping time, A is predictable. Hence $1_A\mathbf{X} \in \mathbf{\Lambda}^2(\mathbb{H}, \mathbb{H}, \mathcal{P}, \beta)$ for every process $\mathbf{X} \in \mathbf{\Lambda}^2(\mathbb{H}, \mathbb{H}, \mathcal{P}, \beta)$. We obtain the equality

$$1_A \int_0^t \mathbf{X}_s d\beta_s = \int_0^t 1_A \mathbf{X}_s d\beta_s.$$

Applying this remark to the process $\mathbf{X}_s := (\mathfrak{A}(\xi_{x,s}) - \mathfrak{A}(\xi_{x',s}))1_{[0,T]}$ we may estimate as follows:

$$\begin{aligned} & E\left[\|\int_0^t \mathbf{X}_s d\beta_s\|^2; A\right] = E\left[\|\int_0^t 1_A \mathbf{X}_s d\beta_s\|^2\right] = E\left[\int_0^t \operatorname{tr} 1_A \mathbf{X}_s \circ \mathfrak{Q} \circ 1_A \mathbf{X}_s^* ds\right] \\ (1) \qquad & = E\left[\int_0^t 1_A \operatorname{tr} \mathbf{X}_s \circ \mathfrak{Q} \circ \mathbf{X}_s^* ds\right] \le \operatorname{tr}\mathfrak{Q} \int_0^t E[\|\mathbf{X}_s\|^2; A] ds \\ & = \operatorname{tr}\mathfrak{Q} \int_0^t E[\|\mathfrak{A}(\xi_{x,s}) - \mathfrak{A}(\xi_{x',s})\|^2; A] ds \le L^2 \operatorname{tr}\mathfrak{A} \int_0^t E[\|\xi_{x,s} - \xi_{x',s}\|^2; A] ds. \end{aligned}$$

On the other hand we have

$$E\left[\|\int_0^{t\wedge T}(\mathfrak{b}(\xi_{x,s})-\mathfrak{b}(\xi_{x',s}))ds\|^2;A\right] \le tE\left[\int_0^t \|\mathfrak{b}(\xi_{x,s})-\mathfrak{b}(\xi_{x',s})\|^2 ds;A\right]$$

$$\le tL^2\int_0^t E[\|\xi_{x,s}-\xi_{x',s}\|^2;A]ds. \tag{2}$$

Using (1) and (2) we now obtain

$$\begin{aligned} & E[\|\xi_{x,t}-\xi_{x',t}\|^2;A] \\ =\ & E[\|\xi_{x,t\wedge T}-\xi_{x',t\wedge T}\|^2;A] \\ \le\ & 3\|x-x'\|^2+3E\left[\|\int_0^t \mathbf{X}_s d\beta_s\|^2;A\right]+3E\left[\|\int_0^{t\wedge T}(\mathfrak{b}(\xi_{x,s})-\mathfrak{b}(\xi_{x',s}))ds\|^2;A\right] \\ \le\ & 3\|x-x'\|^2+3L^2(\mathrm{tr}\,\mathfrak{Q}+t)\int_0^t E[\|\xi_{x,s}-\xi_{x',s}\|^2;A]ds. \end{aligned}$$

Gronwall's lemma yields

$$E[\|\xi_{x,t}-\xi_{x',t}\|^2;A]\le 3\|x-x'\|^2\exp(3tL^2(\mathrm{tr}\,\mathfrak{Q}+t)).$$

By Tschebyshev's inequality, we have

$$\begin{aligned} P[\{\|\xi_{x,t}-\xi_{x',t}\|\ge\eta\}\cap A] &\le \eta^{-2}E[\|\xi_{x,t}-\xi_{x',t}\|^2;A] \\ &\le 3\eta^{-2}\|x-x'\|^2\exp(3tL^2(\mathrm{tr}\,\mathfrak{Q}+t))\le\varepsilon/2. \end{aligned}$$

As $P[A^C]\le P[T_x^{(n)}\le t]+P[T_{x'}^{(n)}\le t]\le\varepsilon/2$, the proof is finished.

An even simpler argument (which also uses Gronwall's lemma) shows the following propositon. We omit its proof.

(5.4) Proposition. *Suppose that the functions $\mathfrak{A}$ and $\mathfrak{b}$ satisfy uniform Lipschitz conditions. Let $(\xi_{x,t})_{t\ge0}$ be the solution to equation* 3.2(1). *Then, for all real numbers $t\ge0$, $\varepsilon>0$, and $\eta>0$ there exists a real number $\delta>0$ such that*

$$P[\|\xi_{x,t}-\xi_{x',t}\|\ge\eta]\le\varepsilon$$

for all $x,x'\in\mathbb{H}$ such that $\|x-x'\|\le\delta$.

(5.5) Proposition. *Hypotheses are as in Proposition* 5.3. *Then $P_tf(x):=E[f(\xi_{x,t})]$ defines a semigroup of continuous, linear operators on $C_b(\mathbb{H})$.*

Proof. The semigroup property follows from the Markov property of the process $(\xi_{x,t})$, which itself is a consequence of uniqueness in Theorem 3.3 (cf. McKean [22], page 56). We have to show that $P_tf\in C_b(\mathbb{H})$ if $f\in C_b(\mathbb{H})$. Let $B\subseteq\mathbb{H}$ be bounded, $t\ge0$, $\varepsilon>0$. Choose $n\in\mathbb{N}$ so large that

$$P[T_x^{(n)}\le t]\le\varepsilon \tag{1}$$

for all $x\in B$ (4.5(4)). Let $\eta>0$ be so small that

$$|f(y)-f(y')|\le\varepsilon$$

for all y, y' such that $\|y\| \le n$, $\|y'\| \le n$, $\|y - y'\| \le \eta$. Choose $\delta > 0$ according to Proposition 5.3 so that

$$P[\|\xi_{x,t} - \xi_{x',t}\| \ge \eta] \le \varepsilon$$

for all $x, x' \in B$ such that $\|x - x'\| \le \delta$. For $x, x' \in B$, $\|x - x'\| \le \delta$ we may then use (1)

$$\begin{aligned}
&|P_t f(x) - P_t f(x')| \\
&\le E[|f(\xi_{x,t}) - f(\xi_{x',t})|] \\
&\le E[|f(\xi_{x,t}) - f(\xi_{x',t})|; T_x^{(n)} \le t] + E[|f(\xi_{x,t}) - f(\xi_{x',t})|; T_{x'}^{(n)} \le t] \\
&\quad + E[|f(\xi_{x,t}) - f(\xi_{x',t})|; \|\xi_{x,t} - \xi_{x',t}\| \ge \eta] \\
&\quad + E[|f(\xi_{x,t}) - f(\xi_{x',t})|; T_x^{(n)}, T_{x'}^{(n)} > t, \|\xi_{x,t} - \xi_{x',t}\| < \eta] \\
&\le 6\|f\|_u \varepsilon + \varepsilon.
\end{aligned}$$

(5.6) *Explanation.* Proposition (5.5) together with the monotone class theorem implies that the mapping

$$\begin{aligned}
\mathbb{H} \times \mathcal{B}(\mathbb{H}) &\to \mathbb{R} \\
(x, F) &\to E[1_F(\xi_{x,t})]
\end{aligned}$$

is a kernel k_t if the hypotheses of Proposition 5.3 are satisfied. (Here $\mathcal{B}(\mathbb{H})$ is the σ–algebra of Borel subsets of $\mathbb{H}$ with respect to the norm topology). If moreover (ξ_x) has infinite lifetime for all x, then $(k_t)_{t\ge 0}$ is a semigroup of Markovian transition kernels. Proposition 5.5 says that (k_t) induces a semigroup on $C_b(\mathbb{H})$. We now give conditions which ensure that these kernels induce strongly continuous semigroups on the spaces $C_{ub}(\mathbb{H})$ and $C_0(\mathbb{H})$.

(5.7) PROPOSITION. *Suppose that the functions $\mathfrak{A}$ and $\mathfrak{b}$ are bounded and satisfy uniform Lipschitz conditions. Then $P_t f(x) := E[f(\xi_{x,t})]$ defines a strongly continuous semigroup of continuous, linear operators on $C_{ub}(\mathbb{H})$.*

PROOF. The fact that P_t operates on $C_{ub}(\mathbb{H})$ follows from Proposition 5.4. Let us prove strong continuity of $(P_t)_{t\ge 0}$ as $t \downarrow 0$. For the expectation of $\|\xi_{x,t} - x\|^2$ we have by isometry

$$\begin{aligned}
&E[\|\xi_{x,t} - x\|^2] \\
&\le 2E\left[\|\int_0^t \mathfrak{A}(\xi_{x,s}) d\beta_s\|^2\right] + 2E\left[\|\int_0^t \mathfrak{b}(\xi_{x,s}) ds\|^2\right] \\
&\le 2\mathrm{tr}\,\mathfrak{Q} \int_0^t E[\|\mathfrak{A}(\xi_{x,s})\|^2] ds + 2E\left[\left(\int_0^t \|\mathfrak{b}(\xi_{x,s})\| ds\right)^2\right] \\
&\le 2K^2 t \mathrm{tr}\,\mathfrak{Q} + 2K^2 t^2,
\end{aligned}$$

where the constant K is independent of x.

Let $\varepsilon > 0$ and choose $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ when $\|x - y\| < \delta$. Using Tchebyshev's inequality we may estimate

$$\begin{aligned}
&|P_t f(x) - f(x)| \\
&\le E[|f(\xi_{x,t}) - f(x)|; \|\xi_{x,t} - x\| < \delta] + E[|f(\xi_{x,t}) - f(x)|; \|\xi_{x,t} - x\| \ge \delta] \\
&\le \varepsilon + 2\|f\|_u P[\|\xi_{x,t} - x\| \ge \delta] \\
&\le \varepsilon + 4K^2 t \|f\|_u \delta^{-2} (\mathrm{tr}\,\mathfrak{Q} + t).
\end{aligned}$$

(5.8) *Counterexample.* The semigroup (P_t) is in general not strongly continuous if the coefficients $\mathfrak{A}$ and $\mathfrak{b}$ are not bounded. It is sufficient to consider Counterexample 4.7 with $0 < \alpha < 1$. Here

$$\xi_{x,t} = \operatorname{sgn} x[(1-\alpha)t + |x|^{1-\alpha}]^{1/1-\alpha}.$$

It is plain that $P_t f(x) = f(\xi_{x,t})$ does not converge uniformly to f as $t \downarrow 0$ if, e.g., f is the sine function.

We now deal with the question: Under what conditions on $\mathfrak{A}$ and $\mathfrak{b}$ does $\xi_{x,t}$ induce a "Feller semigroup", i.e., a strongly continuous semigroup of operators on $C_0(\mathbb{H})$?

(5.9) *Hypotheses.* For the rest of this section we suppose that the diffusion operators $\mathfrak{A}(y)$ and the drift vectors $\mathfrak{b}(y)$ satisfy the conditions of Theorem 4.5. Furthermore we suppose that, for each $x \in \mathbb{H}$, we are given a solution $(\xi_{x,t})_{t\geq 0}$ to equation 3.2(1) (according to Theorem 4.5, this solution has infinite lifetime).

In the sequel we wish to control the size of $\|\xi_{x,t}\|^2$ and $\|\xi_{x,t}\|^4$. To this end we will from now on use two more growth conditions on the diffusion operators $\mathfrak{A}(y)$ and the drift vectors $\mathfrak{b}(y)$:

(III) $(\mathfrak{A}(y) \circ \mathfrak{Q} \circ \mathfrak{A}^*(y)y, y) \leq K(1 + \|y\|^2)$

(IV) $|(y, \mathfrak{b}(y))| \leq K(1 + \|y\|^2)$

for all $y \in \mathbb{H}$.

Note that Condition III is trivially satisfied when the function $y \to \mathfrak{A}(y) \circ \mathfrak{Q}^{1/2}$ is norm–bounded. Heuristically, if P_t is to map C_0 into C_0, then the drift towards the origin must not be too strong. This is part of Condition IV.

(5.10) *Notations and explanations.* First consider the martingale part on the right side of 4.6.3, namely

$$\int_0^{t\wedge T^{(n)}} (\mathfrak{A}^*(\xi_{x,s})\xi_{x,s}, d\beta_s).$$

As

$$E\left[\int_0^t (\mathfrak{A}(\xi_{x,s}) \circ \mathfrak{Q} \circ \mathfrak{A}^*(\xi_{x,s})\xi_{x,s}, \xi_{x,s})ds\right]$$
$$\leq KE\left[\int_0^t (1 + \|\xi_{x,s}\|^2)ds\right] \leq Kt + K\int_0^t g(x,s)ds < \infty$$

by Condition III and Remark 4.6.1, a similar reasoning as in Remark 4.6.2 shows that the above martingale converges in $\mathbb{L}^2(P)$ to the $\mathbb{L}^2(P)$–martingale

$$\mu_{x,t} := \int_0^t (\mathfrak{A}^*(\xi_{x,s})\xi_{x,s}, d\beta_s).$$

For abbreviation we put

$$\psi_{x,s} := 2(\xi_{x,s}, \mathfrak{b}(\xi_{x,s})) + \operatorname{tr} \mathfrak{A}(\xi_{x,s}) \circ \mathfrak{Q} \circ \mathfrak{A}^*(\xi_{x,s}).$$

Going to the limit as $n \to \infty$ in 4.6.3 we have

(1) $$\|\xi_{x,t}\|^2 = \|x\|^2 + 2\mu_{x,t} + \int_0^t \psi_{x,s}ds$$

for all $t \geq 0$. Itô's formula in the one dimensional case, applied to the process $\|\xi_{x,t}\|^2$ and the function $a \to a^2$, yields by equality (1)

$$\begin{aligned} \|\xi_{x,t\wedge T^{(n)}}\|^4 = \|x\|^4 + 4\int_0^{t\wedge T^{(n)}} \|\xi_{x,s}\|^2 d\mu_{x,s} \\ + 2\int_0^{t\wedge T^{(n)}} \|\xi_{x,s}\|^2 \psi_{x,s} ds + 4\langle \mu_x \rangle_{t\wedge T^{(n)}}, \end{aligned} \tag{2}$$

where $\langle \mu_x \rangle$ stands for the quadratic variation of the real martingale μ_x.

According to 2.3(1),

$$\langle \mu_x \rangle_t = \int_0^t (\mathfrak{A}(\xi_{x,s}) \circ \mathfrak{Q} \circ \mathfrak{A}^*(\xi_{x,s})\xi_{x,s}, \xi_{x,s}) ds \tag{3}$$

for all $t \geq 0$. Taking expectations in (2) we obtain

$$E[\|\xi_{x,t\wedge T^{(n)}}\|^4] = \|x\|^4 + 2E\left[\int_0^{t\wedge T^{(n)}} \|\xi_{x,s}\|^2 \psi_{x,s} ds\right] + 4E[\langle \mu_x \rangle_{t\wedge T^{(n)}}] \tag{4}$$

for all $n \in \mathbb{N}$.

(5.11) *Consequences.* From Hypotheses 5.9 III and IV we infer two growth properties of $\langle \mu_x \rangle$ and ψ_x that are essential in what follows. Formula 5.10(3) and Condition III together imply

$$\langle \mu_x \rangle_t \leq K \int_0^t (1 + \|\xi_{x,s}\|^2) ds, \tag{1}$$

whereas Conditions I and IV imply

$$|\psi_{x,s}| \leq K(1 + \|\xi_{x,s}\|^2). \tag{2}$$

We break up the proof of the main theorem (5.19) of this section in several steps which we formulate as lemmas.

(5.12) Lemma. *There exists a constant K such that*

$$|E[\|\xi_{x,t}\|^2 - \|x\|^2]| \leq Kt(1 + \|x\|^2) \tag{3}$$

for all $t \leq 1$ and all $x \in \mathbb{H}$.

Proof. Using 5.10(1), 5.11(2), and Remark 4.6.1 we estimate

$$\begin{aligned} |E[\|\xi_{x,t}\|^2 - \|x\|^2]| \leq E\left[\int_0^t |\psi_{x,s}| ds\right] \leq KE\left[\int_0^t (1 + \|\xi_{x,s}\|^2) ds\right] \\ \leq Kt + \int_0^t g(x,s) ds \leq Kt + tg(x,t) \leq Kt + Kt\|x\|^2 \end{aligned}$$

for $t \geq 1$.

(5.13) Lemma. *For all $x \in \mathbb{H}$ and $t \geq 0$ we have*

$$E[\|\xi_{x,t}\|^4] \leq (\|x\|^4 + 5/4)(e^{Kt} - 1) + \|x\|^4.$$

PROOF. Going to the limit in 5.10(4) as $n \to \infty$ and using 5.11 we estimate

$$\begin{aligned} E[\|\xi_{x,t}\|^4] &\le \|x\|^4 + 2K\int_0^t E[\|\xi_{x,s}\|^2(1+\|\xi_{x,s}\|^2)]ds + 4Kt + 4K\int_0^t E[\|\xi_{x,s}\|^2]ds \\ &= \|x\|^4 + 4Kt + 6K\int_0^t E[\|\xi_{x,s}\|^2]ds + 2K\int_0^t E[\|\xi_{x,s}\|^4]ds \\ &\le \|x\|^4 + 10Kt + 8K\int_0^t E[\|\xi_{x,s}\|^4]ds. \end{aligned}$$

The lemma now follows from Gronwall's lemma.

(5.14) LEMMA. *There exists a constant* K *such that*

(α) $|E[\|\xi_{x,t}\|^4 - \|x\|^4]| \le Kt(1+\|x\|^4)$

(β) $E[(\|\xi_{x,t}\|^2 - \|x\|^2)^2] \le Kt(1+\|x\|^4)$

for all $t \le 1$ *and all* $x \in \mathbb{H}$.

PROOF. Lemma 5.13 shows that equality holds in 5.10(4) in the limit as $n \to \infty$ (use 5.11(2) and Lebesgue's theorem). Using 5.11 and 5.13 we then estimate

$$\begin{aligned} |E[\|\xi_{x,t}\|^4 - \|x\|^4]| &\le 2E\left[\int_0^t \|\xi_{x,s}\|^2|\psi_{x,s}|ds\right] + 4E[\langle\mu_x\rangle_t] \\ &\le 2K\int_0^t E[\|\xi_{x,t}\|^2(1+\|\xi_{x,s}\|^2)]ds + 4K\int_0^t E[1+\|\xi_{x,s}\|^2]ds \\ &\le 4Kt + 6K\int_0^t g(x,s)ds + 2K\int_0^t h(x,s)ds \end{aligned}$$

where g is as in 4.5(3) and h is the right side of 5.13. For $s \le 1$ we have

$$g(x,s) \le K(1+\|x\|^4)$$

and

$$h(x,s) \le K(1+\|x\|^4)$$

The estimate α now follows.

In order to derive estimate β we again use Itô's formula and 5.10(1):

$$(\|\xi_{x,t}\|^2 - \|x\|^2)^2 = 4\int_0^t (\|\xi_{x,s}\|^2 - \|x\|^2)d\mu_{x,s} + 2\int_0^t (\|\xi_{x,s}\|^2 - \|x\|^2)\psi_{x,s}ds + 4\langle\mu_x\rangle_t.$$

Localizing and taking expectations we obtain

$$E[(\|\xi_{x,t\wedge T^{(n)}}\|^2 - \|x\|^2)^2] = 2\int_0^{t\wedge T^{(n)}} (\|\xi_{x,s}\|^2 - \|x\|^2)\psi_{x,s}ds + 4\langle\mu_x\rangle_{t\wedge T^{(n)}}.$$

Going to the limit and using 5.11, we have

$$\begin{aligned} &E[(\|\xi_{x,t}\|^2 - \|x\|^2)^2] \\ &\le 2E\left[\int_0^t (\|\xi_{x,s}\|^2 + \|x\|^2)|\psi_{x,s}|\right] ds + 4E[\langle\mu_x\rangle_t] \\ &\le 2K\|x\|^2 t + 4Kt + (6K + 2K\|x\|^2)\int_0^t E[\|\xi_{x,s}\|^2]ds + 2K\int_0^t E[\|\xi_{x,s}\|^4]ds. \end{aligned}$$

We now use 5.12 and α to obtain for $t \leq 1$

$$\begin{aligned} &E[(\|\xi_{x,t}\|^2 - \|x\|^2)^2] \\ &\leq 2K\|x\|^2 t + 4Kt + K(6K + 2K\|x\|^2)(1 + \|x\|^2)t + 2K^2(1 + \|x\|^4)t. \end{aligned}$$

(5.15) Lemma. *For all $t \leq 1$ and all $x \in \mathbb{H}$ we have*

$$E[\|\xi_{x,t}\|^4] - (E[\|\xi_{x,t}\|^2])^2 \leq K(1 + \|x\|^4)t^{3/2} + K(1 + \|x\|^2).$$

Proof. According to 5.10(1) we have

$$(E[\|\xi_{x,t}\|^2])^2 = \left(\|x\|^2 + E\left[\int_0^t \psi_{x,s} ds\right]\right)^2 \geq \|x\|^4 + 2E\left[\int_0^t \|x\|^2 \psi_{x,s}\right] ds.$$

Together with 5.10(4) we obtain

$$\begin{aligned} &E[\|\xi_{x,t}\|^4] - (E[\|\xi_{x,t}\|^2])^2 \\ &\leq 2E\left[\int_0^t (\|\xi_{x,s}\|^2 - \|x\|^2)\psi_{x,s} ds\right] + 4E[\langle\mu_x\rangle_t] \\ &=: 2A + 4E[\langle\mu_x\rangle_t]. \end{aligned}$$

We now estimate the number A using 5.14.β, 5.11(2), and 5.13:

$$\begin{aligned} A &= \int_0^t E[(\|\xi_{x,s}\|^2 - \|x\|^2)\psi_{x,s}] ds \\ &\leq \int_0^t E[(\|\xi_{x,s}\|^2 - \|x\|^2)^2]^{1/2} E[\psi_{x,s}^2]^{1/2} ds \\ &\leq \int_0^t \{Ks(1 + \|x\|^4)\}^{1/2} \{KE[1 + \|\xi_{x,s}\|^4]\}^{1/2} ds \\ &\leq K(1 + \|x\|^4) \int_0^t s^{1/2} ds = \frac{2}{3} K(1 + \|x\|^4)t^{3/2} \end{aligned}$$

for $t \leq 1$. Using 5.11(1) and Lemma 5.12 we obtain

$$E[\langle\mu_s\rangle_t] \leq K(1 + \|x\|^2)$$

for $t \geq 1$. The lemma now follows.

(5.16) Proposition. *There exist constants K, $t_0 > 0$ such that*

$$P[\|\xi_{x,t}\|^2 < \|x\|] \leq K(t^{3/2} + 1/\|x\|)$$

for all $t \leq t_0$ and all $x, \|x\| \geq 3$.

Proof. A well–known inequality in analysis (cf. Kahane [19], page 6) says that for a random variable $X \geq 0$ with finite variance we have

$$P[X \geq a] \geq \frac{(E[X] - a)^2}{E[X^2]}$$

for all real numbers a such that $0 \leq a \leq E[X]$. From Lemma 5.12 we derive

$$E[\|\xi_{x,t}\|^2] \geq \|x\|^2 - Kt(1 + \|x\|^2) \geq \|x\|$$

for all $t \le t_1$ and $\|x\| \ge 3$. For these t and x we may apply the above mentioned inequality to the random variable $X = \|\xi_{x,t}\|^2$ and the real number $a = \|x\|$ obtaining

$$\begin{aligned} &P[\|\xi_{x,t}\|^2 < \|x\|] \\ &\le \frac{E[\|\xi_{x,t}\|^4] - (E[\|\xi_{x,t}\|^2])^2 + 2\|x\| E[\|\xi_{x,t}\|^2] - \|x\|^2}{E[\|\xi_{x,t}\|^4]}. \end{aligned}$$

For sufficiently small t and for $\|x\| \ge 1$ the denominator exceeds $\|x\|^4/2$ (Lemma 5.14.α). Applying Lemma 5.15 and 4.6.1 we see that, for $t \ge 1$, the numerator is majorized by the number

$$K(1 + \|x\|^4)t^{3/2} + K(1 + \|x\|^2) + 2K\|x\|(1 + \|x\|^2).$$

Collecting these estimates we obtain the proposition.

Our following lemma states a general condition for a contraction semigroup $(P_t)_{t\ge 0}$ on $C_b(\mathbb{H})$ to map $C_0(\mathbb{H})$ into itself.

(5.17) Lemma. *Let $(P_t)_{t\ge 0}$ be a contraction real number $r_f > 0$ such that*

$$|P_t f(x)| \le K t^\alpha + \varepsilon$$

for all x such that $\|x\| \ge r_f$ and all $t \le t_0$. Then (P_t) induces a contraction semigroup on $C_0(\mathbb{H})$.

Proof. Let f be in the unit ball of $C_0(\mathbb{H})$, and let $\varepsilon > 0$ be given. For $\|x\| \ge r_f$ and for $t \le t_0$ we have

$$|P_{t/2} f(x)| \le K t^\alpha / 2^\alpha + \varepsilon/2.$$

We write

$$P_{t/2} f = g + h$$

with $g \in C_0(\mathbb{H})$, $\|g\| \le 1$, and $h \in C_b(\mathbb{H})$, $\|h\| \le K t^\alpha/2^\alpha + \varepsilon/2$. Applying our hypotheses to the function g, we see that

$$|P_{t/2} g(x)| \le K t^\alpha/2^\alpha + \varepsilon/2$$

for $\|x\| \ge r_g$ and $t \le t_0$. Using the semigroup property, we now estimate

$$|P_t f(x)| = |P_{t/2}(P_{t/2} f)(x)| \le |P_{t/2} g(x)| + |P_{t/2} h(x)| \le K t^\alpha/2^{\alpha-1} + \varepsilon$$

for $\|x\| \ge \max\{r_f, r_g\}$ and $t \le t_0$. Hence our hypotheses are satisfied with the smaller constant $K/2^{\alpha-1}$ instead of K. Finite iteration of this procedure shows that $P_t f$ is small outside of bounded sets for $t \le t_0$. Another application of the semigroup property finishes the proof.

(5.18) Proposition. *Suppose that $\mathfrak{A}$ and $\mathfrak{b}$ satisfy the hypotheses of Theorem* 4.5 *and Conditions* III *and* IV *of* 5.9. *Let for all $x \in \mathbb{H}$ $(\xi_{x,t})_{t\ge 0}$ be a maximal solution to equation* 3.2(1) *starting at x and let $f \in C_0(\mathbb{H})$. Then*

$$E[f(\xi_{x,t})] \to f(x)$$

as $t \downarrow 0$, uniformly for $x \in \mathbb{H}$.

Proof. Without loss of generality we suppose $\|f\|_u \le 1$. Let $\varepsilon > 0$. Let $r \ge 2$ be so large that $|f(y)| \le \varepsilon$ for all y, $\|y\| \ge r$. By Proposition 5.16 there are constants $k > 0$ and $t_0 > 0$ such that for all $x \in \mathbb{H}$, $\|x\| \ge r_f := r^2$ and $t \le t_0$ we have

$$\begin{aligned} |E[f(\xi_{x,t})]| &\le E[|f(\xi_{x,t})|; \|\xi_{x,t}\| \ge r] + E[|f(\xi_{x,t})|; \|\xi_{x,t}\| < r] \\ &\le \varepsilon + P[\|\xi_{x,t}\| < r] \le \varepsilon + P[\|\xi_{x,t}\|^2 < \|x\|] \qquad (1) \\ &\le \varepsilon + K(t^{3/2} + 1/\|x\|). \end{aligned}$$

It follows that for large x, $\|x\| \geq m$ say, and for small t we have

$$|E[f(\xi_{x,t})] - f(x)| \leq 3\varepsilon.$$

For $\|x\| \leq m$ we proceed as follows. According to 4.5(4) there exists $n \in \mathbb{N}$ such that

$$P[T_x^{(n)} \leq 1] \leq \varepsilon$$

for all such x. (Recall that $T_x^{(n)}$ is the first exit time of ξ_x from the ball of radius n.) Let A be the subset

$$A := \{T_x^{(n)} > 1\} \subseteq \Omega.$$

Note that $\operatorname{tr} \mathfrak{A}(\xi_{x,s}) \circ \mathfrak{D} \circ \mathfrak{A}^*(\xi_{x,s})$ and $\mathfrak{b}(\xi_{x,s})$ are bounded on A for $0 \leq s \leq 1$ since $\|\xi_{x,s}\| \leq n$ there. As in the proof of Proposition 5.3 we have

$$1_{A\times[0,1]}\mathfrak{A}(\xi_{x,s}) \in \mathbf{\Lambda}^2(\mathbb{H}, \mathbb{H}, \mathcal{P}, \beta)$$

and

$$1_A \int_0^1 \mathfrak{A}(\xi_{x,s}) d\beta_s = \int_0^1 1_A \mathfrak{A}(\xi_{x,s}) d\beta_s$$

for all $t \leq 1$.

We now estimate the quantity $E[\|\xi_{x,s} - x\|^2; A]$.

$$\begin{aligned}
&E[\|\xi_{x,s} - x\|^2; A] \\
&\leq 2E\left[\| \int_0^t \mathfrak{A}(\xi_{x,s}) d\beta_s\|^2; A\right] + 2E\left[\| \int_0^t \mathfrak{b}(\xi_{x,s}) ds\|^2; A\right] \\
&= 2E\left[\| \int_0^t 1_A \mathfrak{A}(\xi_{x,s}) d\beta_s\|^2\right] + 2E\left[\| \int_0^t \mathfrak{b}(\xi_{x,s}) ds\|^2; A\right] \\
&\leq 2E\left[\int_0^t \operatorname{tr} \mathfrak{A}(\xi_{x,s}) \circ \mathfrak{Q} \circ \mathfrak{A}^*(\xi_{x,s}) ds; A\right] + 2tE\left[\int_0^t \|\mathfrak{b}(\xi_{x,s})\|^2 ds; A\right] \\
&\leq 2Kt + 2K^2t^2.
\end{aligned}$$

To conclude the proof in the case $\|x\| < m$ choose $\delta > 0$ so that $|f(y) - f(z)| \leq \varepsilon$ for $\|y - z\| \leq \delta$. Then

$$\begin{aligned}
&|E[f(\xi_{x,t})] - f(x)| \\
&\leq E[|f(\xi_{x,t}) - f(x)|; \|\xi_{x,t} - x\| < \delta] + E[|\ldots|; A^C] + E[|\ldots|; \{\|\xi_{x,t} - x\| \geq \delta\} \cap A] \\
&\leq \varepsilon + 2\|f\|_u\varepsilon + 2\|f\|P[\{\|\xi_{x,t} - x\| \geq \delta\} \cap A] \\
&\leq \varepsilon + 2\|f\|_u\varepsilon + 2\|f\|\delta^{-2}E[\|\xi_{x,t} - x\|^2; A] \\
&\leq \varepsilon + 3\|f\|_u\varepsilon
\end{aligned}$$

for t sufficiently small. The proof is finished.

(5.19) THEOREM. *Let the functions* $\mathfrak{A}$ *and* $\mathfrak{b}$ *and the process* $(\xi_{x,t})_{t\geq 0}$ $(x \in \mathbb{H})$ *be as in Theorem* 4.5. *Suppose that* $\mathfrak{A}$ *and* $\mathfrak{b}$ *satisfy Conditions* III *and* IV *of* 5.9. *Furthermore suppose that the process* ξ_x *is Markovian and that the semigroup* $k_t(x, F) = E[1_F(\xi_{x,t})]$ *defined by* $(\xi_{x,t})$ *operates on* $C_b(\mathbb{H})$. *Then* (k_t) *is a "Feller semigroup", i.e.,* (k_t) *induces a strongly continuous semigroup of operators on* $C_0(\mathbb{H})$.

PROOF. Strong continuity is proved in Proposition 5.18. We use 5.17 to show that P_t operates on $C_0(\mathbb{H})$. In 5.18(1) we have shown that there exist constants $K > 0$ and $t_0 > 0$ such that

$$|P_t f(x)| \leq \varepsilon + K(t^{3/2} + 1/\|x\|)$$

for all $f \in C_0(\mathbb{H})$, all $t \le t_0$, and $\|x\| \ge r_f$, i.e., (P_t) satisfies the hypotheses of Lemma 5.17.

(5.20) *Counterexample.* An example similar to Counterexample 5.8 shows that Condition IV is sharp. To illustrate this let us again consider the case $\mathbb{H} = \mathbb{R}$, $\mathfrak{A} = 0$, and choose $\mathfrak{b}(y) = -\mathrm{sgn}\, y|y|^\alpha$ with $\alpha > 0$. Here, the solution to equation 3.2(1) is

$$\xi_{x,t} = \begin{cases} \mathrm{sgn}\, x[|x|^{1-\alpha} - (1-\alpha)t]^{1/(1-\alpha)} 1_{[0,|x|^{1-\alpha}/(1-\alpha)[}(t) & (0 < \alpha < 1) \\ xe^{-t} & (\alpha = 1) \\ x[1 + (\alpha - 1)|x|^{\alpha-1}t]^{1/(1-\alpha)} & (\alpha > 1). \end{cases}$$

For $\alpha > 1$, at time $t = 1$, the position is in the ball of radius $[1/(\alpha-1)]^{1/(1-\alpha)}$, no matter where the starting point is. Therefore the associated semigroup of operators does not operate on $C_0(\mathbb{R})$.

(5.21) *Remark.* A process (ξ_x) that satisfies the conditions of Theorem 5.19 has the strong Markov property with respect to the family $(\mathcal{F}_{t+})_{t\ge0}$ of sub–σ–algebras of $\mathcal{F}$. Here

$$\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s.$$

This is a consequence of the Feller property of (k_t) (cf. Blumenthal–Getoor [2], Theorem 8.11). However, the strong Markov property may also be proved in more general situations (cf. McKean [22], page 56).

6. Generators. A usual way to reconstruct the original data $\mathfrak{A}$ and $\mathfrak{b}$ from a solution ξ of equation 3.2(1) is to compute the generator of its transition semigroup. We show that for sufficiently smooth functions the generator coincides with the operator $\mathfrak{L}$ below.

(6.1) *Historical Remarks.* A first treatment of the above problem in the case of finite dimensional manifolds appears in Itô [16, 17], cf. also McKean [22]. The infinite dimensional case was treated by Daletskii [5] for twice continuously differentiable data $\mathfrak{A}$ and $\mathfrak{b}$. The connection with the martingale problem goes back to Stroock and Varadhan [31]; cf. also Stroock–Varadhan [32] and Varadhan [33], pages 92, 103, and 230 ff).

(6.2) *Explanation.* Let the function $f : \mathbb{H} \to \mathbb{R}$ be twice differentiable at a point $y \in \mathbb{H}$, so that $f''(y)$ is a continuous bilinear form $\mathbb{H} \times \mathbb{H} \to \mathbb{R}$. For a nuclear operator $\mathfrak{R}$ on $\mathbb{H}$, the bilinear form $f''(y)(\mathfrak{R}\cdot, \cdot)$ on $\mathbb{H} \times \mathbb{H}$ is nuclear. Again, let $\mathfrak{A}$ and $\mathfrak{b}$ be continuous mappings from $\mathbb{H}$ to $L(\mathbb{H}, \mathbb{H})$ and $\mathbb{H} \to \mathbb{H}$, respectively. We define $f''(y)\mathfrak{R} := \mathrm{tr}\, f''(\mathfrak{R}\cdot, \cdot)$ and

$$\mathfrak{L}f(y) := \frac{1}{2} f''(y)\mathfrak{A}(y) \circ \mathfrak{Q} \circ \mathfrak{A}^*(y) + f'(y)\mathfrak{b}(y).$$

We first show that a process $(\xi_{x,t})$ that satisfies our stochastic integral equation 3.2(1) solves a certain martingale problem. Let for this section f be a twice continuously differentiable function $\mathbb{H} \to \mathbb{R}$. $B(\mathbb{H}, \mathbb{H})$ stands for the set of continuous bilinear forms on $\mathbb{H} \to \mathbb{H}$.

(6.3) PROPOSITION. *Suppose that the functions $\mathfrak{A}$ and $\mathfrak{b}$ satisfy the hypotheses of Theorem 4.5 and let $(\xi_{x,t})_{t\ge0}$ be a maximal solution to equation* 3.2(1) *starting at $x \in \mathbb{H}$. Further suppose that $f'' : \mathbb{H} \to B(\mathbb{H}, \mathbb{H})$ is uniformly continuous.*

(a) *Then the process $f(\xi_{x,t}) - f(x) - \int_0^t \mathfrak{L}f(\xi_{x,s})ds$ is equivalent to the local $\mathbb{L}^2(P)$–martingale $\int_0^t f'(\xi_{x,s}) \circ \mathfrak{A}(\xi_{x,ts})d\beta_s$.*

(b) *If moreover f' is bounded then the two processes in* (a) *are P–equivalent $\mathbb{L}^2(P)$–martingales.*

PROOF. By Remark 4.6.2, the process $\mu_t := \int_0^t \mathfrak{A}(\xi_{x,s})d\beta_s$ is an $\mathbb{L}^2_{\mathbb{H}}(P)$–martingale. We may therefore apply Itô's formula 2.3(2) to the process ξ_x and the function f, obtaining the formal equality

$$
\begin{aligned}
& f(\xi_{x,t}) - f(x) - \int_0^t f'(\xi_{x,s})\mathfrak{b}(\xi_{x,s})ds - \frac{1}{2}\int_0^t f''(\xi_{x,s})\mathfrak{A}(\xi_{x,s}) \circ \mathfrak{Q} \circ \mathfrak{A}^*(\xi_{x,s})ds \\
(1) \qquad & = \int_0^t f'(\xi_{x,s})d\mu_s,
\end{aligned}
$$

where the right side is a local $\mathbb{L}^2(P)$–martingale. To finish the proof of Part (a) we use Lemma 4.2 to show that this local martingale is P–equivalent to the local martingale $\int_0^t f'(\xi_{x,s}) \circ \mathfrak{A}(\xi_{x,s})d\beta_s$. Again, let $T^{(n)}$ be the first exit time from the centered n–ball in $\mathbb{H}$ and let ρ_s be the vector of $\mathbb{H}$ such that $(\rho_s, \cdot) = f'(\xi_{x,s})$. We have

$$
\begin{aligned}
& E\left[\int_0^{t\wedge T^{(n)}} (\rho_s, \mathfrak{A}(\xi_{x,s}) \circ \mathfrak{Q} \circ \mathfrak{A}^*(\xi_{x,s})\rho_s)ds\right] \\
& \leq \sup_{\|y\|\leq n} \|f'(y)\|^2 E\left[\int_0^t \mathrm{tr}\mathfrak{A}(\xi_{x,s}) \circ \mathfrak{Q} \circ \mathfrak{A}^*(\xi_{x,s})ds\right].
\end{aligned}
$$

The sup on the right side is finite as f'' is uniformly continuous and the expectation is finite by Remark 4.6.2.

The same estimate shows that $\int_0^t f'(\xi_{x,s}) \circ \mathfrak{A}(\xi_{x,s})d\beta_s$ is an $\mathbb{L}^2(P)$–martingale if f' is bounded. As in Remark 4.6.2, the process $\int_0^{t\wedge T^{(n)}} f'(\xi_{x,s}) \circ \mathfrak{A}(\xi_{x,s})d\beta_s$ converges in $\mathbb{L}^2(P)$ to this martingale as $n \to \infty$. This proves Part (b) of this proposition.

(6.4) COROLLARY. *Let hypotheses be as in Proposition* 6.3b *and suppose that f is bounded. We then have for all $t \geq 0$*

$$
E[f(\xi_{x,t})] = f(x) + E\left[\int_0^t \mathfrak{L}f(\xi_{x,s})ds\right].
$$

PROOF. The expectation of the martingale $\int_0^t f'(\xi_{x,s}) \circ \mathfrak{A}(\xi_{x,s})d\beta_s$ is zero.

Our next proposition deals with one of the properties of the weak generator of $(\xi_{x,t})$.

(6.5) PROPOSITION. *Suppose that the functions $\mathfrak{A}$ and $\mathfrak{b}$ satisfy the hypotheses of Theorem* 4.5 *and Conditions* III *and* IV (cf. 5.9), *and that $\|\mathfrak{b}(y)\| \leq K(1 + \|y\|^2)$ for all $y \in \mathbb{H}$. Let $(\xi_{x,t})_{t\geq 0}$ be a maximal solution to equation* 3.2(1). *Suppose that $f : \mathbb{H} \to \mathbb{R}$ is a twice continuously differentiable function such that f, f', and f'' are bounded and f'' is uniformly continuous. Then $(E[f(\xi_{x,t})] - f(x))/t$ converges to $\mathfrak{L}f(x)$ as $t \downarrow 0$ for all $x \in \mathbb{H}$.*

PROOF. According to Corollary 6.4 we have to show

$$
E\left[\frac{1}{t}\int_0^t \mathfrak{L}f(\xi_{x,s})ds\right] \to \mathfrak{L}f(x).
$$

In order to apply Lebesgue's convergence theorem, we need bounds for

$$
(1) \qquad |\frac{1}{t}\int_0^t \mathrm{tr} f''(\xi_{x,s})(\mathfrak{A}(\xi_{x,s}) \circ \mathfrak{Q} \circ \mathfrak{A}^*(\xi_{x,s})\cdot, \cdot)ds|
$$

and

$$(2) \qquad |\frac{1}{t}\int_0^t f'(\xi_{x,s})\mathfrak{b}(\xi_{x,s})ds|.$$

By condition I, quantity (1) is majorized by

$$\begin{aligned}\|f''\|_u\frac{1}{t}\int_0^t \operatorname{tr}\mathfrak{A}(\xi_{x,s})\circ\mathfrak{Q}\circ\mathfrak{A}^*(\xi_{x,s})ds &\le K\|f''\|_u\frac{1}{t}\int_0^t(1+\|\xi_{x,s}\|^2)ds\\ &\le K\|f''\|_u\left(1+\frac{1}{t}\int_0^t\|\xi_{x,s}\|^2ds\right)\end{aligned}$$

whereas, by hypothesis on $\mathfrak{b}$, Quantity (2) is dominated by

$$\|f'\|_u\frac{1}{t}\int_0^t\|\mathfrak{b}(\xi_{x,s})\|ds \le K\|f'\|_u\left(1+\frac{1}{t}\int_0^t\|\xi_{x,s}\|^2ds\right).$$

It is therefore sufficient to find an upper bound for $\|\xi_{x,s}\|^2$ ($0\le s\le 1$, x fixed). By 5.10(1) it is sufficient to find upper bounds for $|\mu_{x,s}|$ and $|\int_0^s\psi_{x,u}du|$ ($0\le s\le 1$). As to $\mu_{x,s}$, we have

$$|\mu_{x,s}|\le\max_{0\le u\le 1}|\mu_{x,u}|\in\mathbb{L}^2(P)$$

according to Doob's inequality. On the other hand, by 5.11(2),

$$|\int_0^s\psi_{x,u}du|\le\int_0^1|\psi_{x,u}\|du\le K\left(1+\int_0^1\|\xi_{x,u}\|^2du\right)$$

which is integrable by 4.6.1. The proof is finished.

We now reformulate Proposition 6.3 under slightly modified hypotheses: We replace boundedness of f' by boundedness of $\mathfrak{L}f$.

(6.6) Lemma. *Suppose that the functions $\mathfrak{A}$ and $\mathfrak{b}$ satisfy the hypotheses of Theorem 4.5 and let $(\xi_{x,t})_{t\ge0}$ be a maximal solution to equation 3.2(1), starting at $x\in\mathbb{H}$. Further suppose that f and $\mathfrak{L}f$ are uniformly bounded and that $f'':\mathbb{H}\to B(\mathbb{H},\mathbb{H})$ is uniformly continuous. Then the process*

$$f(\xi_{x,t})-f(x)-\int_0^t\mathfrak{L}f(\xi_{x,s})ds$$

is a P–martingale.

Proof. By 6.3(1), the process

$$f(\xi_{x,t\wedge T^{(n)}}-f(x)-\int_0^{t\wedge T^{(n)}}\mathfrak{L}f(\xi_{x,s})ds$$

is an $\mathbb{L}^2(P)$–martingale. Boundedness of f and $\mathfrak{L}f$ allows us to go to the limit as $n\to\infty$.

Our following proposition deals with the strong generator of $(\xi_{x,t})$.

(6.7) Proposition. *Suppose that the functions $\mathfrak{A}$ and $\mathfrak{b}$ satisfy the hypotheses of Theorem 4.5 and Conditions* III *and* IV. *Let for $x\in\mathbb{H}$ $(\xi_{x,t})_{t\ge0}$ be a maximal solution to equation 3.2(1) starting at x. Further let $f\in C_0(\mathbb{H})$ be twice continuously differentiable such that $\mathfrak{L}f\in C_0(\mathbb{H})$ and $f'':\mathbb{H}\to B(\mathbb{H},\mathbb{H})$ is uniformly continuous. Then*

$$\frac{E[f(\xi_{x,t})]-f(x)}{t}\to\mathfrak{L}f(x)$$

as $t \downarrow 0$ *uniformly in* $x \in \mathbb{H}$.

Proof. By Lemma 6.6 we have to show that

$$E\left[\frac{1}{t}\int_0^t (\mathfrak{L}f(\xi_{x,s}) - \mathfrak{L}f(x))ds\right] \to 0$$

uniformly for $x \in \mathbb{H}$. By Fubini's theorem and Proposition 5.18 we have

$$\begin{aligned}
&|E\left[\frac{1}{t}\int_0^t \mathfrak{L}f(\xi_{x,s}) - \mathfrak{L}f(x)ds\right]| \\
\leq\ & \frac{1}{t}\int_0^t |E[\mathfrak{L}f(\xi_{x,s})] - \mathfrak{L}f(x)|ds \\
\leq\ & \sup_{0\leq s\leq t} \|E[\mathfrak{L}f(\xi_{\cdot,s})] - \mathfrak{L}f\|_u \to 0 \qquad \text{as } t \downarrow 0.
\end{aligned}$$

As a resumé we collect our main theorems in one result. It is self–contained in as much as it only uses conditions on the diffusion operators $\mathfrak{A}(y)$ and the drift vectors $\mathfrak{b}(y)$ rather than on the diffusion process ξ_x and the semigroup (P_t).

(6.8) Theorem. *Let* $\mathfrak{A}$ *and* $\mathfrak{b}$ *satisfy Lipschitz conditions on bounded sets and Conditions* I, II, *and* IV (cf. 5.3 *and* 5.9). *Then:*

(a) *For each* $x \in \mathbb{H}$ *there exists exactly one maximal solution* ξ_x *to equation* 3.2(1).

(b) *The lifetime of* ξ_x *is infinite.*

(c) *The family* $(\xi_x)_{x\in\mathbb{H}}$ *of processes is Markovian and strongly Markovian with respect to* $(\mathcal{F}_{t+})$.

(d) *The semigroup* (k_t) *of transition kernels defined by the family* $(\xi_x)_{x\in\mathbb{H}}$ *operates on* $C_b(\mathbb{H})$ *and induces a strongly continuous semigroup* (P_t) *of operators on* $C_0(\mathbb{H})$.

(e) *Let* $\mathfrak{G}$ *be the (strong) infinitesimal generator of the semigroup* $(P_t)_{t\geq 0}$. *All twice continuously differentiable functions* $f \in C_0(\mathbb{H})$ *such that* $\mathfrak{L}f \in C_0(\mathbb{H})$ *and such that* $f'' : \mathbb{H} \to B(\mathbb{H},\mathbb{H})$ *is uniformly continuous are contained in its domain and we have for these functions*

$$\mathfrak{G}f = \mathfrak{L}f.$$

Proof. Combine Theorem 3.3, Theorem 4.5, Proposition 5.6, Theorem 5.19, and Proposition 6.7. The Markov property of ξ_x follows from uniqueness in Theorem 3.3. For the strong Markov property cf. Remark 5.21.

7. Systems of stochastic differential equations.

(7.1) *Explanation.* In the preceding sections we dealt with diffusions on Hilbert spaces. We now describe how this theory can be applied in situations where systems of stochastic differential equations indexed by an at most countable set I are given. We shall indicate how one can construct a Hilbert space $\mathbb{H}$ adapted to the system in one instance in 7.4–7.6. Natural candidates for $\mathbb{H}$ are spaces $\ell^2(\gamma)$, where $\gamma = (\gamma_k)_{k\in I}$ is a summable sequence of strictly positive real numbers, as the product of independent one–dimensional normal Brownian motions on $\mathbb{R}^I$ induces a Brownian motion with covariance matrix $D_\gamma = (\gamma_k\delta_{kl})_{k,l\in I}$ on each $\ell^2(\gamma)$ (cf. Section 2).

We start with the following system of stochastic integral equations

$$\xi_{x,t,k} = x_k + \sum_{j\in I}\int_0^t a_{kj}(\xi_{x,s})d\beta_s + \int_0^t b_k(\xi_{x,s})ds, \qquad k \in I \tag{1}$$

and we suppose first that we are given a sequence γ as described above such that a_{kl} and b_k are real functions on $\ell^2(\gamma)$.

In order to apply our theory we suppose that

(α) $(x_k) \in \ell^2(\gamma)$,

(β) $\mathfrak{b}(y) := (b_k(y))_{k\in I} \in \ell^2(\gamma) \qquad (y \in \ell^2(\gamma))$

and that the matrix $A(y) : (a_{kl}(y))_{k,l\in I}$ induces via multiplication $(z_l) \to (\sum_{l\in I} a_{kl}(y)z_l)_k$ a continuous linear operator $\mathfrak{A}(y)$ on $\ell^2(\gamma)$. According to Schur's test (cf. Schur [29]) this is the case, e.g., if

(γ) for all $y \in \ell^2(\gamma)$ there exists a constant M_y such that

$$\sum_l |a_{kl}(y)| \le M_y \qquad (k \in I)$$
$$\sum_k |a_{kl}(y)|\gamma_k \le M_y\gamma_l \qquad (l \in I).$$

In this case, the norm of the induced operator $\mathfrak{A}(y)$ defined by

$$(\mathfrak{A}(y)z)_k := \sum_l a_{kl}(y)z_l$$

is less than or equal to M_y.

Note that $\mathfrak{A}^*(y)$ is multiplication by the matrix $D_\gamma^{-1}A^t(y)D_\gamma$. In fact, since $(z, y) = z^tD_\gamma y$, we have

$$(D_\gamma^{-1}A^tD_\gamma x, y) = x^tD_\gamma AD_\gamma^{-1}D_\gamma y = x^tD_\gamma Ay = (x, \mathfrak{A}y) = (\mathfrak{A}^*x, y).$$

The trace of a nuclear operator $\mathcal{B}$ on $\ell^2(\gamma)$ induced by multiplication with a matrix $B = (b_{k,l})$ is $\mathrm{tr}(B) = \sum_{k\in I} b_{kk}$. Hence

$$\mathrm{tr}\, \mathfrak{A}(y) \circ \mathfrak{Q} \circ \mathfrak{A}^*(y) = \sum_{k\in I} \gamma_k \sum_{l\in I} a_{kl}^2(y).$$

(7.2) *The notion of a solution to* 7.1(1). We now suppose that the functions $\mathfrak{A} : \ell^2(\gamma) \to L(\ell^2(\gamma), \ell^2(\gamma))$ and $\mathfrak{b} : \ell^2(\gamma) \to \ell^2(\gamma)$ are norm–continuous and that we have a solution $(\xi_{x,t})_{0\le t<T_x}$ to equation 3.2(1) (cf. 3.2). From $\xi_{x,t\wedge T_n} = x + \int_0^{t\wedge T_n} \mathfrak{A}(\xi_{x,s})d\beta_s + \int_0^{t\wedge T_n} \mathfrak{b}(\xi_{x,s})ds$ we obtain, putting $e_k := (\delta_{kl})_{l\in I}$ $(k \in I)$,

$$\xi_{x,t\wedge T_n,k} = x_k + \frac{1}{\gamma_k}\left(e_k, \int_0^{t\wedge T_n} \mathfrak{A}(\xi_{x,s})d\beta_s\right) + \int_0^{t\wedge T_n} b_k(\xi_{x,s})ds. \tag{1}$$

Writing $\mu_t := \int_0^{t\wedge T_n} \mathfrak{A}(\xi_{x,s})d\beta_s$ and applying Remark 4.3.1 we have

$$\left(e_k, \int_0^{t\wedge T_n} \mathfrak{A}(\xi_{x,s})d\beta_s\right) = (e_k, \mu_t) = \int_0^t (e_k, d\mu_s) = \int_0^{t\wedge T_n} (\mathfrak{A}^*(\xi_{x,s})e_k, d\beta_s). \tag{2}$$

(Note that, by 3.2.ii.α, 4.3.1(1) is finite for $\mathbf{X}_s = \mathfrak{A}(\xi_{x,s})1_{[0,T_n]}$ and $\eta_s = e_k$). Now

$$\begin{aligned}\int_0^{t\wedge T_n} (\mathfrak{A}^*(\xi_{x,s})e_k, d\beta_s) &= \int_0^{t\wedge T_n} (D_\gamma^{-1}A^t(\xi_{x,s})D_\gamma e_k, d\beta_s)\\ &= \gamma_k \int_0^{t\wedge T_n} (D_\gamma^{-1}A^t(\xi_{x,s})e_k, d\beta_s) = \gamma_k \int_0^{t\wedge T_n} \left(\sum_l \frac{a_{kl}(\xi_{x,s})}{\gamma_l} e_l, d\beta_s\right)\\ &= \gamma_k \sum_l \int_0^{t\wedge T_n} \left(\frac{a_{kl}(\xi_{x,s})}{\gamma_l} e_l, d\beta_s\right)\end{aligned} \tag{3}$$

where the sum in the last term converges in $\mathbb{L}^2(P)$, if we can show that

$$\sum_l \left(\frac{a_{kl}(\xi_{x,s})}{\gamma_l} e_l, \cdot \right) 1_{[0,T_n]}$$

converges in $\mathbf{\Lambda}^2(\mathbb{H}, \mathbb{H}, \mathcal{P}, \beta)$. But by 2.2(5) we have

$$\begin{aligned} & \sum_l \| \left(\frac{a_{kl}(\xi_{x,s})}{\gamma_l} e_l, \cdot \right) 1_{[0,T_n]} \|^2_{\mathbf{\Lambda}^2(\mathbb{H},\mathbb{H},\mathcal{P},\beta)} \\ (4) \qquad &= \sum_l E\left[\int_0^{t\wedge T_n} \left(\frac{a_{kl}(\xi_{x,s})}{\gamma_l} e_l, \frac{a_{kl}(\xi_{x,s})}{\gamma_l} D_\gamma e_l \right) ds \right] \\ &= \sum_l E\left[\int_0^{t\wedge T_n} a_{kl}^2(\xi_{x,s}) ds \right] \\ &\le \frac{1}{\gamma_k} E\left[\int_0^{t\wedge T_n} \operatorname{tr} \mathfrak{A}(\xi_{x,s}) \circ \mathfrak{Q} \circ \mathfrak{A}^*(\xi_{x,s}) ds \right] < \infty. \end{aligned}$$

Again applying Remark 4.3.1 we see that

$$(5) \qquad \int_0^{t\wedge T_n} (a_{kl}(\xi_{x,s}) e_l, d\beta_s) = \gamma_l \int_0^{t\wedge T_n} a_{kl}(\xi_{x,s}) d\beta_{s,l}.$$

Collecting (1) – (5) we obtain

$$\xi_{x,t\wedge T_n,k} = x_k + \sum_l \int_0^{t\wedge T_n} a_{kl}(\xi_{x,s}) d\beta_{s,l} + \int_0^{t\wedge T_n} b_k(\xi_{x,s}) ds$$

where the sum converges in $\mathbb{L}^2(P)$. This makes it precise what we mean by a solution to the system 7.1(1) of equations.

(7.3) *Reformulation of conditions.* Condition I reads in this context

(I′) $\sum_k \gamma_k \sum_l a_{kl}^2(y) \le K(1 + \sum_k \gamma_k y_k^2)$.

This is the case, e.g., if

$$\sum_l a_{kl}^2(y) \le K(1 + y_k^2)$$

for all $k \in I$. Conditions II and IV read in this context

(II′) $\sum_k \gamma_k y_k b_k(y) \le K(1 + \sum \gamma_k y_k^2)$

(IV′) $|\sum_k \gamma_k y_k b_k(y)| \le K(1 + \sum \gamma_k y_k^2)$,

respectively. Reformulation of Condition III is not very instructive and we omit it. Of course this condition is satisfied if the constants M_y in 7.1.γ may be chosen independent of y.

The operator $\mathcal{L}$ (cf. 6.2) takes the form

$$\mathcal{L} f(y) = \frac{1}{2} \sum_{k,l} \mathcal{A}_{kl}(y) \frac{\partial^2 f(y)}{\partial y_k \partial y_l} + \sum_k b_k(y) \frac{\partial f(y)}{\partial y_k}$$

where $(\mathcal{A}_{kl}(y))_{kl} = A(y) A^t(y)$.

(7.4) *Explanation.* We now indicate in one particular case how one can find a sequence γ if a system 7.1(1) of stochastic integral equations is given. We restrict matters to the case $a_{kl}(y) = \delta_{kl}$, i.e., $\mathfrak{A}(y)$ is the identity operator. Moreover we suppose that all coefficients $b_k(y)$ are linear in y (for configurations $y \in \mathbb{R}^{(I)}$ with finite supports, say), that the index set I is an at most countable group (e.g., $I = \mathbb{Z}^d$), and that the sequence $(b_k)_{k\in I}$ is stationary with respect to I. In this situation, $\mathfrak{b}$ is given on $\mathbb{R}^{(I)}$ by convolution with a sequence $h \in \mathbb{R}^I$:

$$\mathfrak{b}(y) = h * y.$$

Finally we suppose that $h \in \ell^1(I)$.

(7.5) Lemma. *For each sequence $h \in \ell^1(I)$ there exists a sequence $\gamma \in \ell^1(I)$, $\gamma_k > 0$ for all k, such that h induces by convolution a continuous, linear operator on $\ell^2(\gamma)$.*

Proof. Without loss of generality $\|h\|_{\ell^1(I)} < 1$. Let p_k be strictly positive real numbers $\geq q_k := |h_{-k}|$, such that $\sum p_k < 1$. Put

$$\gamma = \sum_{n=1}^{\infty} p^{n*}.$$

Then $\gamma * p = \gamma - p \leq \gamma$. (This trick was already used in Doss–Royer [8], page 109). Since $\sum_l |h_{k-l}| = \|h\|_{\ell^1(I)} < \infty$ and since

$$\sum_k |h_{k-l}|\gamma_k = (q * \gamma)_l \leq (p * \gamma)_l \leq \gamma_l,$$

we see by Schur's test that convolution with the sequence h is a continuous (linear) operation on $\ell^2(\gamma)$.

(7.6) *Remark.* Suppose that there exists a real constant c and a sequence $h \in \ell^1(I)$ of positive numbers h_k such that

$$|b_k(y)| \leq c + (h * |y|)_k$$

for all $k \in I$ and all $y \in \mathbb{R}^{(I)}$. Then we may choose γ as in Lemma 7.5 for h. The correspondence $y \to (b_k(y))_k$ defines a mapping from the dense subset $\mathbb{R}^{(I)}$ of $\ell^2(\gamma)$ into $\ell^2(\gamma)$ which may be extended to $\ell^2(\gamma)$ if it is uniformly continuous with respect to the norm in $\ell^2(\gamma)$.

(7.7) *Remark.* In statistical mechanics the coefficients b_k are usually given via a family $\Phi = (\varphi_V)_{V \text{ finite subset of } I}$ of *interaction potentials* $\varphi_V : \mathbb{R}^V \to \mathbb{R}$. To this family Φ one associates the *energy function* at site k

$$H_k(y) = \sum_{V \ni k} \varphi_V(y_V) \qquad (k \in I),$$

where y_V is the restriction of the "configuration" y to the subset $V \subseteq I$. The drift coefficient b_k is then obtained by differentiation

$$b_k(y) = -(\partial/\partial y_k) H_k(y).$$

Typical examples of interaction potentials are the following pair potentials $\varphi_{\{k,l\}} : \mathbb{R}^2 \to \mathbb{R}$

$$\varphi_{\{k,l\}}(u, v) = -J_{k,l}uv \qquad (k \neq l)$$

where $J_{k,l}$ is small if k and l are "far away" from each other, together with a family of "one site energy functions" $\varphi_k : \mathbb{R} \to \mathbb{R}$. Here

$$H_k(y) = \varphi_k(y_k) - \sum_{l \neq k} J_{k,l} y_k y_l$$

and

$$b_k(y) = -\varphi_k'(y_k) + \sum_{l \neq k} J_{k,l} y_l.$$

Particular examples occur with continuous spin models which serve as lattice approximations of Euclidean quantum field theory (cf., e.g., Nelson [28], page 117). Here $I = \mathbb{Z}^d$. In Nelson [28],

$$\varphi_k(u) = (d + m^2/2)u^2 + P(u)$$

and

$$J_{k,l} = \begin{cases} 1 & \text{for } \sum_{j=1}^d |k_j - l_j| = 1 \\ 0 & \text{else,} \end{cases}$$

where $P(u) = a_n u^n + a_{n-2}u^{n-2} + \cdots + a_2 u^2 - a_0$ is an even, real polynomial with $a_n > 0$. Here our theory is applicable only in the case $n \leq 2$, as otherwise there exists no sequence γ as above such that b_k is defined on all of $\ell^2(\gamma)$.

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