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STABILITY OF SOLUTIONS TO SEMILINEAR STOCHASTIC EVOLUTION EQUATIONS

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ABSTRACT

We study global and local stabilities of the stationary zero solution to certain infinite-dimensional stochastic differential equations. The stabilities are in terms of fractional powers of the linear part of the drift. The abstract results are applied to semilinear stochastic partial differential equations with non-Lipschitzian drift terms and, in particular, to some specific models of population dynamics. We also expose the stabilizing effect of noise on the otherwise unstable zero solution.

As a basic tool we use the Forward Inequality, a generalization of Kolmogorov's forward equation; it is an appliation of Lyapunov's second method with a sequence of Lyapunov functionals.

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INTRODUCTION

The classical simple "logistic" model of population growth introduced by Verhulst (1836) has been modified or improved by numerous authors in various directions. One way is to add a "geographical structure" to the model and to consider a certain "migration in space" of the population, which basically leads to a parabolic partial differential equation (or a parabolic system, see for instance the monographs Smoller [33], Grindrod [9] or Murray [26] and the references there). Another way is to include random fluctuations and inaccuracies by adding a stochastic perturbation term, in which way a stochastic differential equation is obtained. The basic stochastic logistic model with bilinear noise term has been introduced by May [25] and a detailed analysis of the long-time behavior has been carried out, e.g., by Feldman and Roughgarden [6], Turelli [36], or Gard [8]; cf. also the monograph by Roughgarden [29] and the references there.

It is natural to consider both modifications in one model; this leads, however, to semilinear stochastic partial differential equations with non-Lipschitzian drift terms and solutions which can explode for initial values outside the cone of nonnegative functions. For such equations,

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results on existence and uniqueness of solutions have been proved in various settings in Viot [37] or Kotelenez [14], Manthey and Stiewe [21], Manthey and Zausinger [22]. In these works also suitable comparison techniques have been developed.

In the present paper (long-range) global and local stabilities of general infinite-dimensional stochastic differential equations are studied. These cover some stochastic PDE's arising in population dynamics as special cases. In the Lipschitzian case, such or similar stabilities for stochastic semilinear equations have been investigated, e.g., by Ichikawa [10], [11], Taniguchi [34], or Maslowski, Seidler, and Vrkoč [23], [24] in the semigroup framework, or by Chow [2], Khas'minskii and Mandrekar [13], Mandrekar [20], or Caraballo and Real [1] in the variational context.

The novelty of the present paper is twofold: First of all, we consider a non-Lipschitzian case, the solutions being defined only on a certain subset of the basic state space. This enables us to handle stochastic PDE's like those considered in Examples 4.14 and 4.17 – 4.19, where the nonlinear term in the drift is not always globally Lipschitzian and the solutions exist only for nonnegative functions as initial values. Secondly, we deal with stability in subspaces $\mathbf{X}$ of $\mathbf{H}$ endowed with separable norms stronger than that of the basic Hilbert space $\mathbf{H}$ (which, in applications, is usually a space of square-integrable functions). This enables us to draw conclusions on local stability of stationary solutions to stochastic PDE's from local behavior of their coefficients. This is not possible in general with L_2-norms because the local behavior of a function $F : \mathbf{R} \to \mathbf{R}$ does not determine the local behavior of the mapping $y \longmapsto F \circ y$, $y \in L_2$, in the L_2-norm.

Our basic tool is a version of Lyapunov's method; cf. Khas'minskii [12] in the finite-dimensional case. However, typical Lyapunov functions are not smooth and are only densely defined on the basic Hilbert space. Moreover, the drift term which appears in the Lyapunov operator is densely defined, only. To overcome these difficulties Leha and Ritter [16], [17] and Leha, Ritter, and Wakolbinger [18] developed a Forward Inequality. It is a generalization of Kolmogorov's Forward Equation for a *sequence* of Lyapunov functionals rather than one functional alone. In this paper, we employ a more general, localized, version of the Forward Inequality and prove general stability statements in terms of explicit characteristics of the coefficients of the equation. Note that the method described above can also be used in order to establish the required regularity of solutions in some cases; cf. [18] and Remark 3.4(b) of the present paper.

The main body of the paper is divided into four Sections. In Section 1, the abstract equation and the basic example, a stochastic parabolic partial differential equation, are introduced. We recall some basic facts about stochastic parabolic partial differential equations and introduce the concept of $\mathbf{X}$-stability as the Lyapunov stability in probability or the exponential stability in the norm of the space $\mathbf{X}$. Section 2 is of independent interest and devoted to two versions of the Forward Inequality, cf. [16], [17], [18], for nonsmooth functionals defined on a subset of the basic Hilbert space. Section 3 contains general Lyapunov-type theorems for various $\mathbf{X}$-stabilities formulated in terms of sequences of Lyapunov functionals (Theorems 3.1 – 3.3).

In Section 4, $\mathbf{X}$ is the domain of a fractional power of the linear drift part of the equation equipped with its graph norm. In our basic example, $\mathbf{X}$ is equivalent to a Sobolev-Slobodeckii space, cf. Kufner, John, and Fučik [15], Section 8.3. Proposition 4.1 is a reformulation of Theorems 3.1 – 3.3 in this case. Corollaries and more explicit sufficient conditions for stability follow. The nonlinear part of the drift is considered to be either Lipschitzian (Proposition 4.11) or dissipative in a certain sense (Proposition 4.12). If the Wiener process in the equation is just *one*-dimensional then sharper sufficient conditions for stability can be found: Proposition 4.15 and Remark 4.16 show that "sufficiently large" noise can stabilize the zero solution of the equation. At the end of the section, we present three examples: A stochastic parabolic equation with a general noise sufficiently regular in space (Example 4.14) and with scalar noise (Example

4.18). In the latter example we investigate again the stabilizing effect of noise. Example 4.19 is an application of the previous one to a model of population dynamics.

Notation. Let $(\mathbf{H}, \langle\cdot,\cdot\rangle)$ and $(\mathbf{K}, \langle\cdot,\cdot\rangle)$ be real, separable Hilbert spaces with norms $\|\cdot\|_{\mathbf{H}}$ and $\|\cdot\|_{\mathbf{K}}$, respectively. The symbols $\mathcal{L}$ and $\mathcal{L}_2$ denote the spaces of linear, bounded and Hilbert-Schmidt operators between the spaces indicated and S^* denotes the adjoint of an operator S in $\mathcal{L}$ or $\mathcal{L}_2$. The domain of an operator A defined in $\mathbf{H}$ is denoted by $\mathcal{D}(A)$. The restriction of a function g on a subset B of its domain is denoted by $g_{|B}$. The symbols α^+ and α^- stand for the positive and negative parts of a real number α, respectively. The symbols $\underline{\lim}$ and $\overline{\lim}$ denote the lower and upper limits, respectively. For any normed space $\mathbf{X}$ and $r > 0$ we let $B_{\mathbf{X}}(r) = \{y \in \mathbf{X}; \|y\|_{\mathbf{X}} < r\}$, the centered ball in $\mathbf{X}$ of radius r.

1 PRELIMINARIES

1.1 The Stochastic Initial Value Problem

Consider the initial-value problem

$$(1) \quad \begin{cases} dX(t) + AX(t)dt = f(t, X(t))dt + \sigma(t, X(t))dW(t), \\ X(0) = x, \end{cases}$$

in $\mathbf{H}$. Here, A is a linear, self-adjoint operator in $\mathbf{H}$ such that $-A$ generates an analytic semigroup $S(\cdot)$ on $\mathbf{H}$. Furthermore, $W(t)$ stands for a Wiener process with positive, nuclear incremental covariance operator Q on $\mathbf{K}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. Let $\mathbf{K}_0 = Q^{1/2}\mathbf{K}, \|y\|_{\mathbf{K}_0} = \|Q^{-1/2}y\|_{\mathbf{K}}$, where $Q^{-1/2}$ is the pseudo-inverse of $Q^{1/2}$. We assume that there exists a measurable subset $\mathbf{E} \subseteq \mathbf{H}$ such that the mappings $f : \mathbf{R}_+ \times \mathbf{E} \to \mathbf{H}$ and $\sigma : \mathbf{R}_+ \times \mathbf{E} \to \mathcal{L}_2(\mathbf{K}_0, \mathbf{H})$ are measurable (note that $\mathcal{L}(\mathbf{K}, \mathbf{H}) \subseteq \mathcal{L}_2(\mathbf{K}_0, \mathbf{H})$). Moreover, *we assume that, for every $x \in \mathbf{E}$, Equation (1) has a unique, $\mathbf{H}$-continuous, mild solution with infinite lifetime, that is, an $\mathbf{E}$-valued, $\mathbf{H}$-continuous, $(\mathcal{F}_t)$-adapted process $X(t) = X^x(t), t \geq 0$, satisfying P-a.s. for all $t > 0$*

$$(2) \quad \int_0^t \|f(r, X(r))\|_{\mathbf{H}} dr < \infty,$$

$$(3) \quad \int_0^t \|\sigma(r, X(r))\|^2_{\mathcal{L}_2(\mathbf{K}_0,\mathbf{H})} dr < \infty, \qquad \text{and}$$

$$(4) \quad X(t) = S(t)x + \int_0^t S(t-r)f(r, X(r))dr + \int_0^t S(t-r)\sigma(r, X(r))\, dW(r).$$

The latter integral is a stochastic Itô integral in $\mathbf{H}$ (cf. Da Prato and Zabczyk [3] for details). The mild solutions $(X^x)_{x\in\mathbf{E}}$ of Equation (1) are assumed to define a Markov semigroup on $\mathbf{E}$ in the usual way. Furthermore, we assume that there exists an $x_0 \in \mathcal{D}(A) \cap \mathbf{E}$ such that $Ax_0 = f(t, x_0) = 0$, $\sigma(t, x_0) = 0$ for $t \geq 0$; thus $X(t) \equiv x_0$ is a stationary solution to (1). For simplicity we take $x_0 = 0$. A simple class of functions f and σ satisfying the assumptions (2) and (3) with $\mathbf{E} = \mathbf{H}$ are the Lipschitzian ones. However, in our basic example of a stochastic reaction-diffusion equation described below, the drift coefficient f is not Lipschitzian on $\mathbf{H}$.

1.2 Example

Consider a formal stochastic partial differential equation

$$(5) \quad \frac{\partial u}{\partial t}(t, \xi) = Lu(t, \xi) + F(t, u(t, \xi)) + G(t, u(t, \xi))\eta(t, \xi), \quad (t, \xi) \in \mathbf{R}_+ \times \mathcal{O},$$

where $\mathcal{O} \subset \mathbf{R}^d$ is a bounded domain with a smooth boundary, η stands formally for a noise correlated in space and white in time, $F : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$ and $G : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$ are continuous, real-valued functions, and

$$(6) \quad Lu(t,\xi) = \sum_{i,j=1}^{d} \frac{\partial}{\partial \xi_i}(a_{ij}(\xi)\frac{\partial}{\partial \xi_j}u(t,\xi))$$

is a second-order, uniformly-elliptic operator, $a_{ij} \in \mathcal{C}^\infty(\bar{\mathcal{O}}), i,j = 1,2,\dots d$, $a_{ij} = a_{ji}$. Furthermore, assume $F(t,0) = G(t,0) = 0, t \in \mathbf{R}_+$, and

$$(7) \quad |G(t,u_1) - G(t,u_2)| \le k_G|u_1 - u_2|, \quad t \in \mathbf{R}_+,\ u_1,\ u_2 \in \mathbf{R}_+,$$

$$(8) \quad |F(t,u_1) - F(t,u_2)| \le k_{F,N}|u_1 - u_2|, \quad t \in \mathbf{R}_+,\ u_1,\ u_2 \in \mathbf{R}_+, |u_1| + |u_2| \le N,\ N \in \mathbf{N},$$

$$(9) \quad |F(t,u)| \le k_\nu(1 + u^\nu), \quad t,\ u \ge 0,\ \text{for some } \nu \ge 1,$$

$$(10) \quad F(t,u) \le k(1+u), \quad t,\ u \ge 0$$

for some constants $k_G > 0,\ k_{F,N} > 0,\ k_\nu > 0$, and $k > 0$.

We consider Equation (5) with initial condition

$$(11) \quad u(0,\cdot) = u_0 \in L_2(\mathcal{O})$$

and boundary conditions either of Dirichlet's type, i.e.,

$$(12) \quad u(t,\xi) = 0, \quad (t,\xi) \in \mathbf{R}_+ \times \partial\mathcal{O},$$

or of Neumann's type, i.e.,

$$(13) \quad \frac{\partial u}{\partial \nu_L}(t,\xi) = 0, \quad (t,\xi) \in \mathbf{R}_+ \times \partial\mathcal{O};$$

here $\frac{\partial}{\partial \nu_L}$ stands for the conormal derivative. The formal system (5), (11), and (12) or (13) is given a precise mathematical meaning in the sense of Equation (1) if we specify

$$(14) \quad \mathbf{H} = L_2(\mathcal{O}),\ \mathbf{E} = \{\varphi \in L_{2\nu}(\mathcal{O}); \varphi \ge 0\} \text{ with some } \nu \text{ as in (9)},$$

$$(15) \quad A = \text{the closed extension of} - L \text{ to } \mathcal{D}(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$$

(in the Dirichlet case) or

$$(16) \quad A = \text{ the closed extension of } -L \text{ to } \quad \mathcal{D}(A) = \{\varphi \in H^2(\mathcal{O}); \frac{\partial \varphi}{\partial \nu_L}(\xi) = 0, \xi \in \partial\mathcal{O}\}$$

(in the Neumann case), and

$$(17) \quad f(t,y)(\xi) := F(t,y(\xi)), \quad (t,\xi) \in \mathbf{R}_+ \times \mathcal{O}, y \in \mathbf{E},$$

$$(18) \quad [\sigma(t,y)h](\xi) := G(t,y(\xi))h(\xi), \quad (t,\xi) \in \mathbf{R}_+ \times \mathcal{O},\ y \in \mathbf{E},\ h \in \mathbf{K};$$

here, $\mathbf{K}$ is a Hilbert space continuously embedded into $\mathbf{H} = L_2(\mathcal{O})$. Since the covariance operator Q of the Wiener process $W(t)$ is assumed to be a self-adjoint, positive, nuclear operator on $\mathbf{K}$ there exists an orthonormal basis (ψ_n) of $\mathbf{K}$ such that

$$(19) \quad Q\psi_n = \alpha_n\psi_n, \quad n \in \mathbf{N},$$

where $\alpha_n \geq 0$, $\sum \alpha_n < \infty$. We shall assume that each ψ_n is in $L_\infty(\mathcal{O})$ and

$$(20) \quad \sup_n \|\psi_n\|_{L_\infty(\mathcal{O})} < \infty.$$

From (7) and (20) it easily follows that $\sigma : \mathbf{R}_+ \times \mathbf{E} \to \mathcal{L}_2(\mathbf{K}_0, \mathbf{H})$. Note that the Conditions (19) and (20) are obviously satisfied if $\mathbf{K}$ is continuously embedded into $L_\infty(\mathcal{O})$. Also note that the behavior of $F(t,u)$ and $G(t,u)$ for $u < 0$ is not controlled by Conditions (7) – (10). However, since $F(t,0) = G(t,0) = 0$ it can be shown by comparison techniques that Equation (1) with coefficients defined as above has a unique solution evolving in the cone of nonnegative functions provided that the initial point is also nonnegative, cf. Manthey and Zausinger [22], Kotelenez [14]. More precisely, there is the following proposition which is essentially due to Manthey and Zausinger [22].

1.3 Proposition

Let $\mathbf{H}, \mathbf{E}, A, f,$ and σ be as defined by (14), (15) or (16), (17), and (18), respectively. Let $\mathbf{K} \subseteq L_2(\mathcal{O})$ and let $W(t)$ be a $\mathbf{K}$-valued Wiener process with a trace-class covariance operator Q satisfying (19) and (20). Let F and G satisfy Conditions (7) – (10) and let $F(t,0) = G(t,0) = 0$ for all $t \geq 0$. Then, for each $x \in \mathbf{E}$, Equation (1) has a unique solution X in $\mathbf{E}$ starting from x. This solution is a Markov process satisfying

$$(21) \quad \sup_{r \in \,]0,T]} E\|X(r)\|_{L_{2\nu}}^{2\nu} < \infty$$

for all $T > 0$, where ν is as in (9).

Proof. See [22], Theorem 3.4.1 for the proof of existence of solutions. Nonnegativity of solutions is obtained from the construction of this proof together with the hypotheses present here. Uniqueness (which follows from local Lipschitzianity of the coefficients) and the Markov property can be shown in the standard way; cf. also [14] for a similar proof of existence and uniqueness of solutions. □

Note that the growth condition (9) is true for any $\nu' \geq \nu$ and the relative state space $\mathbf{E}' \subseteq \mathbf{E}$, cf. (14); in particular, if the process is started in $\mathbf{E}'$ then (21) is valid with ν' instead of ν.

1.4 Example

A nonlinearity F to which Proposition 1.3 applies is

$$(22) \quad F(t,u) = F(u) = au - cu^2, \quad u \geq 0,$$

where $a > 0$ and $c > 0$ are constants. The corresponding stochastic differential equation is a model of stochastic population dynamics of logistic type with geographical structure and migration.

1.5 The Concept of Stability

We review next several concepts of stability of the zero solution to Equation (1). They correspond to their classical finite-dimensional counterparts as introduced, e.g., in Khas'minskii [12]; however, in order to cover some important examples, we need a smaller state space $\mathbf{X}$ with a norm stronger than that of $\mathbf{H}$, cf. Examples 4.14, 4.17, 4.18, and 4.19.

Let $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$ be a separable Banach space, continuously and densely embedded in $\mathbf{H}$. By a theorem of Lusin's, cf. Schwartz [30], p. 101, Theorem 5, $\mathbf{X}$ is a (Borel-)measurable subset of $\mathbf{H}$. In the following definitions we restrict ourselves to the initial time $t = 0$ and the trivial solution $X^0(t) \equiv 0$. We first need an assumption that relates the processes X^x to the space $\mathbf{X}$.

(A1) For every $x \in \mathbf{X} \cap \mathbf{E}$, we have P-a.s. $X^x(t) \in \mathbf{X} \cap \mathbf{E}$ for all $t \geq 0$ and X^x is right-continuous in the norm of $\mathbf{X}$.

1.5.1 Definition

The trivial solution $X^0(t) \equiv 0$ of Equation (1) is said to be

(i) $\mathbf{X}$-*stable in probability* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in \mathbf{X} \cap \mathbf{E}$, $\|x\|_{\mathbf{X}} < \delta$, we have

$$P[\sup_{t \geq 0} \|X^x(t)\|_{\mathbf{X}} \leq \varepsilon] \geq 1 - \varepsilon;$$

(ii) *asymptotically* $\mathbf{X}$-*stable in probability* if (i) holds and if, for every $\varepsilon > 0$, there exists a $\delta_0 > 0$ such that for all $x \in \mathbf{X} \cap \mathbf{E}$, $\|x\|_{\mathbf{X}} < \delta_0$, we have

$$P[\lim_{t \to \infty} \|X^x(t)\|_{\mathbf{X}} = 0] \geq 1 - \varepsilon;$$

(iii) *globally asymptotically* $\mathbf{X}$-*stable in probability* if (i) holds and if

$$P[\lim_{t \to \infty} \|X^x(t)\|_{\mathbf{X}} = 0] = 1$$

for all $x \in \mathbf{X} \cap \mathbf{E}$.

Besides these "pathwise" stabilities of the trivial solution there are various stabilities in the mean. We restrict ourselves to recalling *exponential* stability in the mean.

1.5.2 Definition

Let $p > 0$ be fixed. The solution $X^0(t) \equiv 0$ of Equation (1) ist said to be *exponentially* $\mathbf{X}$-*stable in the* pth mean (for $p = 2$ *in the mean square*) if there exist constants $K > 0$ and $c > 0$ such that

$$E\|X^x(t)\|_{\mathbf{X}}^p \leq K\|x\|_{\mathbf{X}}^p \, e^{-ct}, \qquad t \geq 0$$

holds for all $x \in \mathbf{X} \cap \mathbf{E}$.

1.6 The Operators π_k

For technical reasons we need, besides mild solutions, another concept of solution already introduced in Leha, Ritter, and Wakolbinger [18]: π-solutions. It turns out that, in the cases considered here, the two concepts are equivalent. Let $\pi_k \in \mathcal{L}(\mathbf{H})$, $k \in \mathbf{N}$, be such that

(α) the sequence (π_k) converges to the identity operator on $\mathbf{H}$ in the strong operator topology on $\mathcal{L}(\mathbf{H})$ as $k \longrightarrow \infty$,

(β) *Range* $\pi_k \subseteq \mathcal{D}(A)$ and $A\pi_k$ is bounded and extends $\pi_k A$ for all k.

From (β) it follows that

(γ) $\pi_k S_t = S_t \pi_k$ for all $k \in \mathbf{N}$ and $t \geq 0$.

A standard choice is $\pi_k = kR_k(-A) = k(kI + A)^{-1}$, the Yosida approximation of the identity operator associated with A. It follows from (α) and (β) that the sequence of vectors $(A\pi_k y)_k$ converges to Ay for every $y \in \mathcal{D}(A)$.

The proof of the following proposition is as in [18], Proposition 3.6, where the coefficients f and σ do not depend on time.

1.7 Proposition

Let $x \in \mathbf{E}$. Any mild solution $X = X^x$ of Equation (1) is a π-solution of Equation (1) in the sense that it satisfies the system of stochastic differential equations

$$(23) \quad \pi_k X(t) = \pi_k x + \int_0^t \{-A\pi_k X(s) + \pi_k f(s, X(s))\} ds + \int_0^t \pi_k \sigma(s, X(s)) dW(s)$$

for all $k \in \mathbf{N}$ and $t \geq 0$.

1.8 Notation

Let $G \subseteq \mathbf{R}_+ \times \mathbf{H}$ be open such that $(s, \pi_k y) \in G$ for all $(s, y) \in G$ and all $k \in \mathbf{N}$.

(a) The space of all functions $U : G \longrightarrow \mathbf{R}$, continuously differentiable in the first coordinate and twice continuously (Fréchet) differentiable in the second coordinate, will be denoted by $\mathcal{C}^{1,2}(G)$. If $U \in \mathcal{C}^{1,2}(G)$ and if $B \in \mathcal{L}(\mathbf{H})$ is nuclear then $tr \frac{\partial^2 U}{\partial y^2}(s, y)(B\cdot, \cdot)$ is the trace of the bilinear form $(u, v) \longmapsto \frac{\partial^2 U}{\partial y^2}(s, y)(Bu, v)$, $(s, y) \in G$, $u, v \in \mathbf{H}$. $\mathcal{C}_u^{1,2}(G)$ stands for the space of functions $U \in \mathcal{C}^{1,2}(G)$ such that $\frac{\partial^2 U}{\partial y^2}$ is uniformly continuous on bounded, closed subsets of G.

(b) The sequence of differential operators $\mathcal{L}^{(k)}$ associated with the system of equations (23) defined next plays a key role in the sequel. The idea of using a sequence rather than a single differential operator goes back to [18] where it appears in the autonomous case. For $U \in \mathcal{C}^{1,2}(G), k \in \mathbf{N}$, and $(s, y) \in G \cap (\mathbf{R}_+ \times \mathbf{E})$ let

$$\begin{aligned} \mathcal{L}^{(k)} U(s, y) &= \frac{\partial U}{\partial t}(s, \pi_k y) + \frac{\partial U}{\partial y}(s, \pi_k y)(-A\pi_k y + \pi_k f(s, y)) \\ &\quad + \frac{1}{2} tr \frac{\partial^2 U}{\partial y^2}(s, \pi_k y) \, (\pi_k \sigma(s, y) Q \sigma^*(s, y) \pi_k^* \cdot, \cdot). \end{aligned}$$

2 FORWARD INEQUALITIES

The material in this section is more general than actually needed in the sequel. Let G' be a subset of $\mathbf{R}_+ \times \mathbf{H}$, and let $G \subseteq G'$ be open in $\mathbf{R}_+ \times \mathbf{H}$. Let $v_n \in \mathcal{C}(G')$, $v_n \geq 0$, be a sequence of (Lyapunov) functions such that $v_{n|G} \in \mathcal{C}_u^{1,2}(G)$ for all n and define $v = \underline{\lim}_n v_n$. We assume throughout that the function v is lower semicontinuous on G'. For $u \geq 0$ and $x \in \mathbf{E}$, let $X = (X^{(u,x)}(t))_{t \geq u}$ be an $\mathbf{H}$-continuous, $\mathbf{E}$-valued π-solution to Equation 1(1) starting from (u, x), cf. Proposition 1.7, i.e.,

$$\pi_k X(t) = \pi_k x + \int_u^t \{-A\pi_k X(s) + \pi_k f(s, X(s))\} ds + \int_u^t \pi_k \sigma(s, X(s)) dW(s),$$

for all $k \in \mathbf{N}$ and all $t \geq u$. Fixing first $u \geq 0$, let $\tilde{\tau} \geq u$ be a bounded stopping time and suppose that $(u, x) \in G'$, $(s, X(s)) \in G$ for $u < s < \tilde{\tau}$, and $(\tilde{\tau}, X(\tilde{\tau})) \in G'$, P-a.s.. We denote the exit time of $(t, X(t))$ from G (after u) by τ_G. Note that $X^{(0,x)} = X^x$.

Let us note that, in our Forward Inequalities below, the process X need not be defined outside the stochastic interval $[u, \tilde{\tau}]$.

2.1 Lemma

Let $U \in \mathcal{C}(G')$ be lower bounded such that $U_{|G} \in \mathcal{C}_u^{1,2}(G)$. Suppose that

(i) there exists a P-a.s. locally Lebesgue integrable, random function $\varphi_0 : [u, \tilde{\tau}[\to [0, \infty]$ such that, P-a.s. for $u < s < \tilde{\tau}$, we have

$$\sup_k \mathcal{L}^{(k)} U(s, X(s)) \leq \varphi_0(s);$$

(ii) there exists a measurable function $\varphi : [u, \tilde{\tau}] \times \Omega \to [0, \infty]$ such that $E \int_u^{\tilde{\tau}} \varphi(s) ds < \infty$ and such that P-a.s. for all $u < s < \tilde{\tau}$ we have

$$\overline{\lim}_k \mathcal{L}^{(k)} U(s, X(s)) \leq \varphi(s).$$

Then we have

$$EU(\tilde{\tau}, X(\tilde{\tau})) \leq U(u, x) + E \int_u^{\tilde{\tau}} \overline{\lim}_k \mathcal{L}^{(k)} U(s, X(s)) ds.$$

Proof. Let (η_m) be an ascending sequence of stopping times such that $u \leq \eta_m < \tau_G$ and $sup\, \eta_m = \tau_G$ (the sequence (η_m) "announces" τ_G) and let

$$\tau_m := \inf\{t > u; \int_u^t \varphi_0(s) ds > m \text{ or } \|\frac{\partial U}{\partial y}(t, X(t))\|_{\mathbf{H}} > m\} \wedge \tilde{\tau} \wedge \eta_m.$$

Since $X(t) \in G$ for $u < t \leq \tau_m$, the same arguments as in the proof of [18], Lemma 4.2, show the estimate

$$(1) \quad EU(\tau_m, X(\tau_m)) \leq U(u, x) + E \int_u^{\tau_m} \overline{\lim}_k \mathcal{L}^{(k)} U(s, X(s)) ds.$$

(In [18], the coefficients do not depend explicitly on time and $G = \mathbf{H}$.) Due to local integrability of φ_0 and pathwise continuity of X τ_m converges to $\tilde{\tau}$ as $m \to \infty$. An application of Fatou's lemma to both sides of (1) yields the assertion. □

2.2 Forward Inequality, Elementary Version

(a) Suppose that

(i) for all n, there exists a P-a.s. locally Lebesgue integrable, random function $\varphi_n : [u, \tilde{\tau}[\to [0, \infty]$ such that, P-a.s. for $u < s < \tilde{\tau}$, we have

$$\sup_k \mathcal{L}^{(k)} v_n(s, X(s)) \leq \varphi_n(s);$$

(ii) there exists a P-a.s. locally Lebesgue integrable, random function $\varphi : [u, \tilde{\tau}[\to [0, \infty]$ such that, P-a.s. for all n and all $u < s < \tilde{\tau}$, we have

$$\overline{\lim}_k \mathcal{L}^{(k)} v_n(s, X(s)) \leq \varphi(s);$$

(iii) there exists a random function $\psi : [u, \tilde{\tau}] \to [0, \infty]$ such that $E \int_u^{\tilde{\tau}} \psi(s) ds < \infty$ and such that we have P-a.s. for all $u < s < \tilde{\tau}$

$$\overline{\lim}_n \overline{\lim}_k \mathcal{L}^{(k)} v_n(s, X(s)) \leq \psi(s).$$

If $v(u, x) < \infty$ then we have

$$Ev(\tilde{\tau}, X(\tilde{\tau})) \leq v(u, x) + E \int_u^{\tilde{\tau}} \overline{\lim}_n \overline{\lim}_k \mathcal{L}^{(k)} v_n(s, X(s)) ds.$$

(b) If (i) and (ii) are satisfied and if (iii) is strengthened to

(iii)′ P-a.s. we have for all $u < s < \tilde{\tau}$

$$\overline{\lim}_n \overline{\lim}_k \mathcal{L}^{(k)} v_n(s, X(s)) \leq 0$$

then we have $Ev(\tilde{\tau}, X(\tilde{\tau})) \leq v(u, x)$.

Proof. (a) Let

$$\tau_m := \inf\{t > u; \int_u^t \varphi(s) ds > m\} \wedge \tilde{\tau}, \quad m \in \mathbf{N}.$$

Using (i) and applying Lemma 2.1 with $U = v_n$ and $\tilde{\tau} = \tau_m$ we first obtain for all m and n

$$Ev_n(\tau_m, X(\tau_m)) \leq v_n(u, x) + E \int_u^{\tau_m} \overline{\lim}_k \mathcal{L}^{(k)} v_n(s, X(s)) ds. \tag{2}$$

Since $v_n \geq 0$ and $v = \underline{\lim}_n v_n$, and since $v(u, x) < \infty$, we infer from (2)

$$\begin{aligned} Ev(\tau_m, X(\tau_m)) &\leq \underline{\lim}_n Ev_n(\tau_m, X(\tau_m)) \\ &\leq v(u, x) + \overline{\lim}_n E \int_u^{\tau_m} \overline{\lim}_k \mathcal{L}^{(k)} v_n(s, X(s)) ds. \end{aligned} \tag{3}$$

By (ii) and the definition of τ_m we may interchange $\overline{\lim}_n$ and integration in (3) to obtain

$$Ev(\tau_m, X(\tau_m)) \leq v(u, x) + E \int_u^{\tau_m} \overline{\lim}_n \overline{\lim}_k \mathcal{L}^{(k)} v_n(s, X(s)) ds. \tag{4}$$

Since φ is locally integrable τ_m increases to $\tilde{\tau}$. Lower semi-continuity of v and (iii) finally enable passage to the limit in (4); hence Part (a).

If $v(u, x) = \infty$ nothing has to be shown in Part (b). Otherwise, (b) is a direct consequence of (a). □

2.3 Remark

Condition 2.2(i) is not critical in the cases of the Lyapunov functions v_n that we will use, cf. the proof of Proposition 4.1. If the function φ in 2.2(ii) is globally integrable on the stochastic interval $[u, \tilde{\tau}]$ then 2.2(iii) follows from 2.2(ii) with $\psi = \varphi$.

2.4 Corollary

Let $(0, x) \in G'$ and suppose that $X = X^{(0,x)}$ is Markovian. Let τ be a (not necessarily bounded) stopping time. Assume that

(i) for all n, there exists a P-a.s. locally Lebesgue integrable, random function $\varphi_n : [0, \tau[\to [0, \infty]$ such that, P-a.s. for $0 < s < \tau$, we have

$$\sup_k \mathcal{L}^{(k)} v_n(s, X(s)) \le \varphi_n(s);$$

(ii) there exists a P-a.s. locally Lebesgue integrable, random function $\varphi : [0, \tau[\to [0, \infty]$ such that, P-a.s. for all n and all $0 < s < \tau$, we have

$$\overline{\lim}_k \mathcal{L}^{(k)} v_n(s, X(s)) \le \varphi(s);$$

(iii)′ P-a.s. we have for all $0 < s < \tau$

$$\overline{\lim}_n \overline{\lim}_k \mathcal{L}^{(k)} v_n(s, X(s)) \le 0$$

(a) If τ satisfies P-a.s. $(t, X(t)) \in G$ for all $0 < t < \tau$ and $(\tau, X(\tau)) \in G'$ on $\{\tau < \infty\}$ then $v(t \wedge \tau, X(t \wedge \tau))$, $t \ge 0$, is a supermartingale ≥ 0.

(b) Suppose that Conditions (i), (ii), (iii)′ are satisfied up to $\tau \wedge \tau_G$ and that

(iv) $X(t)$ is P-a.s. constant for $t \ge \tau_G$;

(v) $v(t, y)$ is independent of t for $(t, y) \notin G$.

If we have $(\tau \wedge \tau_G, X(\tau \wedge \tau_G)) \in G'$ P-a.s. on $\{\tau \wedge \tau_G < \infty\}$ then $v(t \wedge \tau, X(t \wedge \tau))$ is a supermartingale.

Proof. (a) Letting $t > u \ge 0$ and writing $\tilde{\tau} = (t \wedge \tau) \vee u$ we have by the Markov property

$$\begin{aligned} & E[v(t \wedge \tau, X(t \wedge \tau) 1_{]u,\infty]}(\tau) \mid \mathcal{F}_u] \\ = \ & E[v(\tilde{\tau}, X(\tilde{\tau})) 1_{]u,\infty]}(\tau) \mid \mathcal{F}_u] \\ = \ & E[v(\tilde{\tau}, X(\tilde{\tau})) \mid \mathcal{F}_u] 1_{]u,\infty]}(\tau) \\ = \ & E[v(\tilde{\tau}, X^{(u,z)}(\tilde{\tau}))]_{|z=X(u)} \, 1_{]u,\infty]}(\tau). \end{aligned} \tag{5}$$

From Part (b) of the elementary version of the Forward Inequality we obtain next

$$E[v(\tilde{\tau}, X^{(u,z)}(\tilde{\tau}))]_{|z=X(u)} \le v(u, X(u)). \tag{6}$$

Now, for $0 \le u < t$, we have by (5) and (6)

$$\begin{aligned} & E[v(t \wedge \tau, X(t \wedge \tau)) \mid \mathcal{F}_u] \\ = \ & E[v(t \wedge \tau, X(t \wedge \tau)) 1_{]u,\infty]}(\tau) \mid \mathcal{F}_u] + E[v(t \wedge \tau, X(t \wedge \tau)) 1_{[0,u]}(\tau) \mid \mathcal{F}_u] \\ \le \ & v(u, X(u)) 1_{]u,\infty]}(\tau) + v(\tau, X(\tau)) 1_{[0,u]}(\tau) \\ = \ & v(u \wedge \tau, X(u \wedge \tau)) 1_{]u,\infty]}(\tau) + v(u \wedge \tau, X(u \wedge \tau)) 1_{[0,u]}(\tau) \\ = \ & v(u \wedge \tau, X(u \wedge \tau)). \end{aligned}$$

This is claim (a).

(b) The conditions imply that the processes $v(t \wedge \tau \wedge \tau_G, X(t \wedge \tau \wedge \tau_G))$ and $v(t \wedge \tau, X(t \wedge \tau))$ are indistinguishable, i.e., P-a.s. pathwise equal. An application of Part (a) with $\tau = \tau \wedge \tau_G$ shows that $v(t \wedge \tau \wedge \tau_G, X(t \wedge \tau \wedge \tau_G))$ is a supermartingale. Hence, Part (b) follows. □

We next formulate an advanced version of the Forward Inequality in which the functions v_n and v take over the role of φ; the main tool in order to obtain this version is Gronwall's lemma.

2.5 Lemma

Let U be a function as in Lemma 2.1 such that

(i) 2.1(i) is satisfied and

(ii) there exists a constant C such that P-a.s., for all $u < s < \tilde{\tau}$,

$$\overline{\lim}_k \mathcal{L}^{(k)} U(s, X(s)) \leq C(1 + U(s, X(s))).$$

Then, for all $t \geq u$, $EU(t \wedge \tilde{\tau}, X(t \wedge \tilde{\tau})) \leq (U(u,x) + Ct)e^{Ct}$.

Proof. Let

$$(7) \quad \tau_m := \inf\{u < t \leq \tilde{\tau}; \int_u^t U(s, X(s))ds \geq m\} \wedge \tilde{\tau},$$

a bounded stopping time for all $m \in \mathbf{N}$. Let us abbreviate

$$w(t) = U(t \wedge \tau_m, X(t \wedge \tau_m)), \quad t \geq u.$$

By (i), (ii), and (7), we may use Lemma 2.1 with τ_m instead of $\tilde{\tau}$ to obtain finiteness of $Ew(\tau_m)$. We now estimate

$$\begin{aligned} \int_u^t Ew(s)ds &\leq E\int_u^{\tau_m} U(s, X(s))ds + E[\int_{\tau_m}^t U(\tau_m, X(\tau_m))ds, \tau_m < t] \\ &\leq E\int_u^{\tau_m} U(s, X(s))ds + tEw(\tau_m) < \infty, \end{aligned}$$

i.e., Ew is locally integrable on $[u, \infty[$. Invoking again Lemma 2.1, this time with $t \wedge \tau_m$ instead of $\tilde{\tau}$, we next use (ii) to estimate for $t \geq u$

$$\begin{aligned} Ew(t) &\leq U(u,x) + E\int_u^{t\wedge\tau_m} \overline{\lim}_k \mathcal{L}^{(k)} U(s, X(s))ds \\ &\leq U(u,x) + CE\int_u^{t\wedge\tau_m} (1 + U(s, X(s))ds \\ &\leq (U(u,x) + Ct) + C\int_u^t Ew(s)ds. \end{aligned}$$

An application of Gronwall's lemma to the locally integrable function Ew now yields

$$(8) \quad EU(t \wedge \tau_m, X(t \wedge \tau_m)) \leq (U(u,x) + Ct)e^{Ct},$$

a bound independent of m. From $\mathbf{H}$-continuity of X and continuity of U we infer $\tau_m \uparrow \tilde{\tau}$ and letting $m \to \infty$ in (8) we obtain the claim from Fatou's lemma since U and X are continuous.□

2.6 Proposition

Suppose that

(i) *the sequence* (v_n) *satisfies 2.2(i) and*

(ii) *there exists a constant* C *such that* P*-a.s., for all* n *and all* $u < s < \tilde{\tau}$,

$$\overline{\lim}_k \mathcal{L}^{(k)} v_n(s, X(s)) \le C(1 + v_n(s, X(s))).$$

Then, for all $t \ge u$, *we have*

$$E \int_u^{t \wedge \tilde{\tau}} v(s, X(s)) ds \le g(x, t),$$

where $g(x,t) = (v(u,x) + C\,t)e^{Ct}$ *is independent of* $\tilde{\tau}$.

Proof. If $v(u, x) = \infty$ nothing has to be shown. In the opposite case it is sufficient to apply the preceding lemma with $U = v_n$ and to use Fatou's lemma. □

2.7 Forward Inequality, Advanced Version

Let $\tilde{D}$ *be a measurable subset of* $G \cap (\mathbf{R}_+ \times \mathbf{E})$ *and suppose that,* P*-a.s.,* $(t, X(t)) \in \tilde{D}$ *for all* $u < t < \tilde{\tau}$.

(a) Suppose that

(i) *for all* n, *there exists a* P*-a.s. locally Lebesgue integrable, random function* $\varphi_n : [u, \tilde{\tau}[\to [0, \infty]$ *such that,* P*-a.s. for* $u < s < \tilde{\tau}$, *we have*

$$\sup_k \mathcal{L}^{(k)} v_n(s, X(s)) \le \varphi_n(s);$$

(ii) *there exists a constant* C *such that, for all* n *and for all* $(s, y) \in \tilde{D}$, *we have*

$$v_n(s, y) \le C(1 + v(s, y))$$

and

$$\overline{\lim}_k \mathcal{L}^{(k)} v_n(s, y) \le C(1 + v_n(s, y)).$$

If $v(u, x) < \infty$ *then*

(α) *for all* $t \ge u$ *we have*

$$E \int_u^{t \wedge \tilde{\tau}} v(s, X(s)) ds \le g(x, t),$$

where $g(x,t) = (v(u,x) + C\,t)e^{Ct}$ *is independent of* $\tilde{\tau}$;

(β) $$Ev(\tilde{\tau}, X(\tilde{\tau})) \le v(u, x) + E \int_u^{\tilde{\tau}} \overline{\lim}_n \overline{\lim}_k \mathcal{L}^{(k)} v_n(s, X(s)) ds.$$

(b) If (i) and (ii) are fulfilled and if

(iii)′ for all $(s,y)\in\tilde D$

$$\overline{\lim}_n\overline{\lim}_k\,\mathcal{L}^{(k)}v_n(s,y)\le 0$$

then we have $Ev(\tilde\tau, X(\tilde\tau))\le v(u,x)$.

Proof. (a) We verify the conditions of the elementary version of the Forward Inequality 2.2. Condition 2.2(i) is just (i). Since Condition 2.6(ii) follows from (ii) we obtain Part (α) from 2.6. Conditions 2.2(ii),(iii) now follow from (ii) if we put $\psi(s)=\varphi(s)=C(1+v(s,X(s)),\ u\le s\le\tilde\tau$. □

The next corollary follows in the same way from the advanced version of the Forward Inequality 2.7 as Corollary 2.4 followed from its elementary version. One has to put $\tilde D=D\cap([0,t]\times\mathbf{E})$.

2.8 Corollary

Let $D\subseteq G\cap(\mathbf{R}_+\times\mathbf{E})$ *be measurable, let* $(0,x)\in G'$, *and suppose that* $X=X^{(0,x)}$ *is Markovian. Let* τ *be a (not necessaryly bounded) stopping time. Assume that*

(i) for all n, *there exists a P-a.s. locally Lebesgue integrable, random function* $\varphi_n:[0,\tau[\to[0,\infty]$ *such that, P-a.s. for* $u<s<\tau$, *we have*

$$\sup_k\mathcal{L}^{(k)}v_n(s,X(s))\le\varphi_n(s);$$

(ii) for all $t>0$ *there exists* C_t *such that, for all* n *and all* $(s,y)\in D\cap([0,t]\times\mathbf{E})$, *we have*

$$v_n(s,y)\le C_t(1+v(s,y))\qquad\text{and}$$
$$\overline{\lim_k}\,\mathcal{L}^{(k)}v_n(s,y)\le C_t(1+v_n(s,y));$$

(iii)′ for all $(s,y)\in D$,

$$\overline{\lim_n}\,\overline{\lim_k}\,\mathcal{L}^{(k)}v_n(s,y)\le 0.$$

(a) If τ *satisfies P-a.s.* $(t,X(t))\in D$ *for all* $0<t<\tau$ *and* $(\tau,X(\tau))\in G'$ *on* $\{\tau<\infty\}$ *then* $v(t\wedge\tau,X(t\wedge\tau))$, $t\ge0$, *is a supermartingale* ≥0.

(b) Suppose that Conditions (i), (ii), (iii)′ are satisfied up to $\tau\wedge\tau_G$ *and that*

(iv) $X(t)$ *is P-a.s. constant for* $t\ge\tau_G$;

(v) $v(t,y)$ *is independent of* t *for* $(t,y)\notin G$.

If we have P-a.s. $(t,X(t))\in D$ *for all* $0<t<\tau\wedge\tau_G$ *and* $(\tau\wedge\tau_G,X(\tau\wedge\tau_G))\in G'$ *on* $\{\tau\wedge\tau_G<\infty\}$ *then* $v(t\wedge\tau,X(t\wedge\tau))$ *is a supermartingale.*

2.9 Quadratic Forms

Let $G_0\subseteq\mathbf{H}$ be open and let $G=\mathbf{R}_+\times G_0$, let $S_n\in\mathcal{L}(\mathbf{H})$ be self-adjoint, and let $p>0$. Let $v_n(s,y)=\langle y,S_ny\rangle^p$, $(s,y)\in G\cap(\mathbf{R}_+\times\mathbf{E})$, a function independent of s. Here we have for all $(s,y)\in G\cap\mathbf{R}_+\times\mathbf{E}$

$$\begin{aligned}&\mathcal{L}^{(k)}v_n(s,y)\\ =\ &2p\langle S_n\pi_ky,\pi_ky\rangle^{p-1}\{-\langle S_n\pi_ky,A\pi_ky\rangle+\langle S_n\pi_ky,\pi_kf(s,y)\rangle\\ &\qquad+\frac12 tr\,S_n\mathcal{A}_k(s,y)-(1-p)\frac{\langle\mathcal{A}_k(s,y)S_n\pi_ky,S_n\pi_ky\rangle}{\langle S_n\pi_ky,\pi_ky\rangle}\},\end{aligned}$$

where $\mathcal{A}_k(s,y)=\pi_k\sigma(s,y)Q\sigma^*(s,y)\pi_k^*$.

2.10 Remark: Continuous Operator A and Quadratic Forms

In the case $A \in \mathcal{L}(\mathbf{H})$ we do not need the sequence (π_k) as introduced in 1.6. Putting $S_n = I$ for all n (cf. 2.9) we have for all $(s, y) \in D \cap (\mathbf{R}_+ \times \mathbf{E})$

$$\begin{aligned} v_n(s,y) &= v(y) = \|y\|_{\mathbf{H}}^{2p}, \\ \mathcal{A}_k(s,y) &= \mathcal{A}(s,y) = \sigma(s,y)Q\sigma^*(s,y), \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}^{(k)} v_n(s,y) &= \mathcal{L}v(s,y) \\ &= 2p\|y\|_{\mathbf{H}}^{2p-2}\{-\langle y, Ay\rangle + \langle y, f(s,y)\rangle + \frac{1}{2} tr\,\mathcal{A}(s,y) \\ &\qquad -(1-p)\frac{\langle \mathcal{A}(s,y)y, y\rangle}{\|y\|_{\mathbf{H}}^2}\}. \end{aligned}$$

3 GENERAL THEOREMS ON STABILITY

In the present section we formulate some sufficient conditions for the various stabilities defined in Section 1 by means of sequences of Lyapunov functionals. Note that balls in $\mathbf{X}$ are Borel measurable subsets of $\mathbf{H}$, cf. 1.5. Let $G' = \mathbf{R}_+ \times \mathbf{H}$, $G = \mathbf{R}_+ \times (\mathbf{H} \setminus \{0\})$ and let $\pi_k, k \in \mathbf{N}$, be as defined in 1.6 and injective. As in Section 2, let $v_n \in \mathcal{C}(G')$, $v_n \geq 0$, be a sequence of (Lyapunov) functions such that $v_{n|G} \in \mathcal{C}_u^{1,2}(G)$ for all n and define $v = \underline{\lim}_n v_n$. We assume again that the function v is lower semicontinuous on G'. As before, X^x stands for the process $X^{(0,x)}$ introduced in Section 2.

3.1 Theorem

Let $r > 0$, let $D = \mathbf{R}_+ \times (B_{\mathbf{X}}(r) \cap \mathbf{E} \setminus \{0\})$, and assume that, for all $x \in B_{\mathbf{X}}(r) \cap \mathbf{E}$, Conditions (i) and (ii) of Corollary 2.4(b) or of Corollary 2.8(b) are satisfied for $\tau = \tau_r$, the first exit time of $X = X^x$ from $B_{\mathbf{X}}(r) \setminus \{0\}$. Moreover, assume (A1), cf. 1.5, and

(A2) $\overline{\lim}_n \overline{\lim}_k\, \mathcal{L}^{(k)} v_n(s,y) \leq 0$
for all $(s,y) \in D$ (cf. 2.4(iii)' and 2.8(iii)');

(A3) $v(t, 0) = 0$ for all $t \geq 0$ and the function $v(0, \cdot)$ is finite on $\mathbf{X}$ and $\|\cdot\|_{\mathbf{X}}$-continuous at the origin;

(A4) $b(\varepsilon) := \inf\{v(t,y); t \geq 0, y \in B_{\mathbf{X}}(r) \backslash B_{\mathbf{X}}(\varepsilon)\} > 0$
for all $0 < \varepsilon < r$.

Then the trivial solution $X^0(t) \equiv 0$ of Equation 1(1) is $\mathbf{X}$-stable in probability.

Proof. Let $t > 0$ and let $0 < \varepsilon < r$, $x \in B_{\mathbf{X}}(\varepsilon) \cap \mathbf{E} \backslash \{0\}$. From $\mathbf{H}$-continuity of X^x and $\mathcal{B}(\mathbf{H})$-measurability of $B_{\mathbf{X}}(\varepsilon)$ it follows that the exit time τ_ε of X^x from $B_{\mathbf{X}}(\varepsilon) \setminus \{0\}$ is a stopping time (cf. Dellacherie and Meyer [4], Theorem 3.44). Thus, we can apply Part (b) of either Corollary 2.4 or Corollary 2.8 with $\tau = \tau_\varepsilon$ to obtain that $v(\tau_\varepsilon \wedge t, X^x(\tau_\varepsilon \wedge t))$ is a (nonnegative) supermartingale. Hence,

(1) $\quad Ev(\tau_\varepsilon \wedge t, X^x(\tau_\varepsilon \wedge t)) \leq v(0, x), \quad t \in \mathbf{R}_+,$

for all $x \in B_{\mathbf{X}}(\varepsilon) \cap \mathbf{E}$ since, for $x = 0$, (1) follows from (A3). By right continuity (A1) of X^x in the norm of $\mathbf{X}$ and by (A4) we have

$$\{\sup_{0\leq s\leq t} \|X^x(s)\|_{\mathbf{X}} > \varepsilon\} \subseteq \{v(\tau_\varepsilon \wedge t, X^x(\tau_\varepsilon \wedge t)) \geq b(\varepsilon)\},$$

and hence, by Chebyshev's inequality and (1),

$$(2) \quad P[\sup_{0\leq s\leq t} \|X^x(s)\|_{\mathbf{X}} > \varepsilon] \leq \frac{v(0,x)}{b(\varepsilon)}.$$

Letting $t \longrightarrow \infty$ we arrive at

$$(3) \quad P[\sup_{s\geq 0} \|X^x(s)\|_{\mathbf{X}} > \varepsilon] \leq \frac{v(0,x)}{b(\varepsilon)}, \quad x \in B_{\mathbf{X}}(\varepsilon) \cap \mathbf{E}.$$

The conclusion now follows from (A3). □

3.2 Theorem

Let all assumptions of Theorem 3.1 be satisfied with (A2) and (A3) strengthened to

(A2)′ For every $0 < \varepsilon < r$ there exists a number $\alpha(\varepsilon) > 0$ such that for all $(s,y) \in \mathbf{R}_+ \times ((B_{\mathbf{X}}(r) \setminus B_{\mathbf{X}}(\varepsilon)) \cap \mathbf{E})$

$$\overline{\lim}_n \overline{\lim}_k \mathcal{L}^{(k)} v_n(s,y) \leq -\alpha(\varepsilon);$$

(A3)′ $v(t,0) = 0$ for all $t \geq 0$ and the function v is finite and $\|\cdot\|_{\mathbf{X}}$-continuous on $\mathbf{R}_+ \times B_{\mathbf{X}}(r)$; moreover, $\sup_{t\geq 0} v(t,y) \longrightarrow 0$ as $\|y\|_{\mathbf{X}} \longrightarrow 0$.

Then the trivial solution $X^0(t) \equiv 0$ of Equation 1(1) is asymptotically $\mathbf{X}$-stable in probability.

Proof. By Theorem 3.1, the trivial solution is $\mathbf{X}$-stable in probability. We preserve the notation of the proof of Theorem 3.1.

Let $0 < \varepsilon < r$ and $x \in B_{\mathbf{X}}(\varepsilon) \cap \mathbf{E} \setminus \{0\}$. By either Corollary 2.4 or Corollary 2.8 the process $v(\tau_\varepsilon \wedge t, X^x(\tau_\varepsilon \wedge t))$ is a (nonnegative) supermartingale, which, by assumption and (A3)′, is (right-)continuous. Therefore, there exists P–a.s. $\lim_{t\to\infty} v(\tau_\varepsilon \wedge t, X^x(\tau_\varepsilon \wedge t)) =: \xi^x$. We show that

$$(4) \quad \lim_{\|x\|_{\mathbf{X}}\to 0} P[\xi^x = 0] = 1.$$

For $n > 1/\|x\|_{\mathbf{X}}$, denote the first time that X^x hits $B_{\mathbf{X}}(1/n)$ by τ^n. By the Forward Inequalities 2.2 or 2.7 and by (A2)′ we have

$$(5) \quad 0 \leq Ev(\tau_\varepsilon \wedge t \wedge \tau^n, X^x(\tau_\varepsilon \wedge t \wedge \tau^n)) \leq v(0,x) - E\int_0^{\tau_\varepsilon\wedge t\wedge\tau^n} \alpha(1/n)ds,$$

$t \in \mathbf{R}_+$. This implies $E(\tau_\varepsilon \wedge t \wedge \tau^n) \leq \frac{v(0,x)}{\alpha(1/n)}$ for these t and, letting $t \longrightarrow \infty$, we arrive at

$$(6) \quad P[\tau_\varepsilon \wedge \tau^n < \infty] = 1.$$

By $\mathbf{X}$-stability in probability there exists $\delta > 0$ such that, for all $y \in \mathbf{X} \cap \mathbf{E}$, $0 < \|y\|_{\mathbf{X}} < \delta$, we have $\tau_\varepsilon = \infty$ with probability exceeding $1 - \varepsilon$. By (6), τ^n is finite on $\{\tau_\varepsilon = \infty\}$ for all n. Since $\mathbf{X}$ is continuously embedded in $\mathbf{H}$ it follows from continuity of X^x in $\mathbf{H}$ that $X^x(\sup_n \tau^n) = 0$ and hence $X^x(t) = 0$ for all $t \geq \sup_n \tau^n$ on $\{sup_n \tau^n < \infty\}$. From right continuity of X^x we infer $\lim_{n\to\infty} X^x(\tau^n) = 0$ and by equicontinuity (A3)′ we have $\lim_{n\to\infty} v(X^x(\tau^n)) = 0$ and $\xi^x = 0$ on $\{\tau_\varepsilon = \infty, \sup_n \tau^n = \infty\}$ for all x, $\|x\|_{\mathbf{X}} < \delta$. This is (4). Asymptotic $\mathbf{X}$-stability now follows from (A4). □

3.3 Theorem

(a) Let $D = \mathbf{R}_+ \times (\mathbf{X} \cap \mathbf{E} \setminus \{0\})$ *and let* $\tau = \infty$. *Suppose that the function* $s \longmapsto v(s, X(s))$ *is locally integrable on* $[0, \infty[$ *and that Conditions (i) and (ii) of Corollary 2.4(b) or of Corollary 2.8(b) are satisfied. Moreover, assume (A1), (A3), (A4), and*

(A2)″ there exists a constant $c > 0$ *such that, for all* $(s, y) \in D$,

$$\overline{\lim}_n \overline{\lim}_k \left[\mathcal{L}^{(k)} v_n(s, y) + c v_n(s, y)\right] \leq 0.$$

Then $X^0(t) \equiv 0$ *is globally asymptotically* $\mathbf{X}$*-stable in probability.*

(b) If, moreover,

(A5) there exist constants $d_1 > 0$, $d_2 > 0$, *and* $p > 0$ *such that, for all* $(t, y) \in \mathbf{R}_+ \times (\mathbf{X} \cap \mathbf{E})$,

$$d_1 \|y\|_{\mathbf{X}}^p \leq v(t, y) \leq d_2 \|y\|_{\mathbf{X}}^p$$

then the solution $X^0(t) \equiv 0$ *is exponentially* $\mathbf{X}$*-stable in the* p*th mean.*

Proof. $\mathbf{X}$-stability in probability following from Theorem 3.1, in order to prove (a), we have to show

(7) $\quad P[\lim_{t\to\infty} \|X^x(t)\|_{\mathbf{X}} = 0] = 1, \quad x \in \mathbf{X} \cap \mathbf{E}.$

Let c be as in (A2)″, and let $w_n(s, y) := e^{cs} v_n(s, y)$ for all $n \in \mathbf{N}$, $(s, y) \in \mathbf{R}_+ \times (\mathbf{X} \cap \mathbf{E})$. We show that the hypotheses of Corollary 2.4(b) or 2.8(b), respectively, are fulfilled for w_n instead of v_n, $w := \underline{\lim}_n w_n$ instead of v, and $D = \mathbf{R}_+ \times (\mathbf{X} \cap \mathbf{E} \setminus \{0\})$. Since v_n is bounded on bounded sets in $\mathbf{R}_+ \times \mathbf{H}$, since $\mathbf{X}$ is continuously embedded into $\mathbf{H}$, and since $\mathcal{L}^{(k)} w_n(s, y) = e^{cs}(c v_n(s, y) + \mathcal{L}^{(k)} v_n(s, y))$ we have Condition (i) of both Corollaries. Since 2.4(ii) is valid for v_n and from local integrability of $v(s, X(s))$ we obtain 2.4(ii) also with w_n instead of v_n. Validity of Condition 2.8(ii) for w_n follows directly from that for v_n. Condition (iii) of both corollaries just follows from (A2)″. It now follows from Part (b) of either Corollary that the process $\xi^x(t) := v(t, X^x(t))$ is a nonnegative, right-continuous supermartingale, in particular,

(8) $\quad Ev(t, X^x(t)) \leq v(0, x) e^{-ct}, \quad (t, x) \in \mathbf{R}_+ \times (\mathbf{X} \cap \mathbf{E})$

and that there exists the limit

$$\xi^x := \lim_{t\to\infty} v(t, X^x(t)), \quad P - a.s..$$

From Fatou's lemma and (8) it follows that $\xi^x \equiv 0$ P-a.s.. Now, (A4) yields $\lim_{t\to\infty} \|X^x(t)\|_{\mathbf{X}} = 0$ P-a.s. which completes the proof of global asymptotical $\mathbf{X}$-stability.

If also (A5) is fulfilled then it follows from (8) that

$$E\|X^x(t)\|_{\mathbf{X}}^p \leq \frac{d_2}{d_1} e^{-ct} \|x\|_{\mathbf{X}}^p, \quad (t, x) \in \mathbf{R}_+ \times (\mathbf{X} \cap \mathbf{E})$$

which is exponential stability in the pth mean. □

3.4 Remarks

(a) For the proof of *exponential* stability in the pth mean in Theorem 3.3, $\mathbf{X}$-right-continuity of the solutions of Equation 1(1) is not necessary.

(b) If the Conditions (i),(ii), and (iii)′ of Corollary 2.8 and (A5) are satisfied for all $(t, y) \in \mathbf{R}_+ \times (\mathbf{H} \cap \mathbf{E} \setminus \{0\})$ then we need not assume in advance that the solution of 1(1) starting from $x \in \mathbf{X}$ has an $\mathbf{X}$-valued version. Indeed, we can apply these corollaries with $D = \mathbf{R}_+ \times (\mathbf{H} \cap \mathbf{E} \setminus \{0\})$ to arrive at (8) and then we can use (A5) to obtain exponential $\mathbf{X}$-stability in the pth mean.

4 APPLICATIONS

Note that under the assumptions stated at the beginning of Section 1 the operator A is *lower semi-bounded*, i.e.,

$$(1) \quad \langle Ay, y\rangle \geq \beta\|y\|_{\mathbf{H}}^2, \quad y \in \mathcal{D}(A),$$

for some $\beta \in \mathbf{R}$. It is called *uniformly positive* if (1) is satisfied with a constant $\beta > 0$. Since β plays a role in the Lyapunov inequalities (31), (34), and (56), we fix β as large as possible. Let us put $\hat{\beta} = 0$ if $\beta > 0$ and $\hat{\beta} = -\beta + \varepsilon$ for some small number $\varepsilon > 0$, otherwise. Then $\hat{A} = A + \hat{\beta} I$ is uniformly positive. Now introduce the spaces $\mathbf{H}_\alpha = \mathcal{D}(\hat{A}^\alpha)$, $\alpha \geq 0$, equipped with the norm $\|y\|_\alpha = \|\hat{A}^\alpha y\|$, $y \in \mathbf{H}_\alpha$; this norm is equivalent to the $\hat{A}^\alpha$-graph norm.

In the present section, the general results from Section 3 will be applied to the case $\mathbf{X} = \mathbf{H}_\alpha$, for some suitably chosen $\alpha \geq 0$. The section is organized as follows. At the beginning results of the preceding section, applied to the present case, are summarized in Proposition 4.1, followed by a series of lemmas which state some useful estimates for verifying the Lyapunov inequalities appearing in Corollaries 2.4 and 2.8. The main results of the section, stability results formulated in terms of explicit characteristics of the coefficients, are stated in Propositions 4.11, 4.12 and 4.15. More concrete examples are given subsequently (Examples 4.14, 4.18, 4.19).

Denote by $R_n(-\hat{A}^{2\alpha}) = (nI + \hat{A}^{2\alpha})^{-1}$ the resolvent operator of $-\hat{A}^{2\alpha}$ and by $J_{2\alpha}(n), n \in \mathbf{N}$, the Yosida-type operators

$$(2) \quad J_{2\alpha}(n) := nR_n(-\hat{A}^{2\alpha}),$$

and let

$$(3) \quad A_{2\alpha}(n) := \hat{A}^{2\alpha} J_{2\alpha}(n), \quad n \in \mathbf{N}.$$

In this section, we shall use the sequence of time-independent Lyapunov functions

$$(4) \quad v_n(t, y) = v_n(y) = \langle A_{2\alpha}(n)y, y\rangle^p, \quad n \in \mathbf{N}, y \in \mathbf{H},$$

for a suitable exponent $p > 0$. By self-adjointness of A, the functions v_n and v have the form

$$(5) \quad v_n(y) = \|\hat{A}^\alpha J_{2\alpha}^{\frac{1}{2}}(n)y\|_{\mathbf{H}}^{2p} = \|J_{2\alpha}^{\frac{1}{2}}(n)y\|_\alpha^{2p}$$

and

$$(6) \quad v(y) = \lim_n v_n(y) = \begin{cases} \|y\|_\alpha^{2p}, & y \in \mathbf{H}_\alpha, \\ +\infty, & \text{otherwise.} \end{cases}$$

For $\alpha = 0$ we have $v_n(y) = \frac{n}{n+1}\|y\|^{2p}$ and $v(y) = \|y\|^{2p}$, $y \in \mathbf{H}$. A possible choice of π_k enjoying all properties required in Section 3 is $\pi_k = J_1(k) = kR_k(-\hat{A})$.

Theorems 3.1 – 3.3, applied to the case described above, are summarized in the following proposition.

4.1 Proposition

Let $r \in]0, +\infty]$ and let $p > 0$. Assume that for every $x \in \mathbf{H}_\alpha \cap \mathbf{E}$ the solution X^x of Equation 1(1) is a right-continuous, $\mathbf{H}_\alpha$-valued process. Furthermore, for all $n \in \mathbf{N}$ and all $T > 0$, assume either

$$(7) \quad \overline{\lim}_k \mathcal{L}^{(k)} v_n(s, y) \leq C_{\varepsilon,r,T}(1 + v(y))$$

for all $\varepsilon > 0$, all $(s, y) \in [0, T] \times (B_{\mathbf{H}\alpha}(r) \cap \mathbf{E})$, $\|y\|_{\mathbf{H}} > \varepsilon$, and some $C_{\varepsilon,r,T} < \infty$ and, P-a.s.,

$$(8) \qquad \int_0^T \|X^x(s)\|_\alpha^{2p} ds < \infty$$

or

$$(9) \qquad \overline{\lim}_k \, \mathcal{L}^{(k)} v_n(s, y) \leq C_{r,T}(1 + v_n(y))$$

for all $(s, y) \in [0, T] \times (B_{\mathbf{H}\alpha}(r) \cap \mathbf{E} \setminus \{0\})$ and some $C_{r,T} < \infty$. Then the following assertions hold true:

(a) If

$$(10) \quad \overline{\lim}_n \overline{\lim}_k \, \mathcal{L}^{(k)} v_n(s, y) \leq 0, \quad (s, y) \in \mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E} \setminus \{0\}),$$

then the solution $X^0(t) \equiv 0$ is $\mathbf{H}_\alpha$-stable in probability.

(b) If, for any $0 < \varepsilon < r$, there exists $\alpha(\varepsilon) > 0$ such that

$$(11) \quad \overline{\lim}_n \overline{\lim}_k \, \mathcal{L}^{(k)} v_n(s, y) \leq -\alpha(\varepsilon)$$

for all $(s, y) \in \mathbf{R}_+ \times (\{y \in \mathbf{H}_\alpha; 0 < \varepsilon < \|y\|_\alpha < r\} \cap \mathbf{E})$ then $X^0(t) \equiv 0$ is asymptotically $\mathbf{H}_\alpha$-stable in probability.

(c) If there exists $c > 0$ such that

$$(12) \quad \overline{\lim}_n \overline{\lim}_k \, \mathcal{L}^{(k)} v_n(s, y) \leq -c\, v(y)$$

for all $(s, y) \in \mathbf{R}_+ \times (\mathbf{H}_\alpha \cap \mathbf{E} \setminus \{0\})$ and if one of the alternative Hypotheses (7) (together with (8)) or (9) is satisfied globally (i.e., with $r = +\infty$) then the solution $X^0(t) \equiv 0$ is globally asymptotically $\mathbf{H}_\alpha$-stable in probability and exponentially $\mathbf{H}_\alpha$-stable in the $2p$th mean.

Proof: We verify the conditions of the theorems in Section 3 and begin with Condition (i) of 2.4(b) (which equals (i) of 2.8(b)). Let τ_r be as in Theorem 3.1 and let $x \in B_{\mathbf{X}}(r) \cap \mathbf{E}$. Note that τ_G is the first hitting time τ^0 of the origin of the process $X = X^x$. By 1.8(b) we have

$$(13) \quad \begin{aligned} \mathcal{L}^{(k)} v_n(s, X(s)) = {} & \frac{\partial v_n}{\partial y}(\pi_k X(s))(-A\pi_k X(s)) \\ & + \frac{\partial v_n}{\partial y}(\pi_k X(s))(\pi_k f(s, X(s)) \\ & + \frac{1}{2}\, tr \frac{\partial^2 v_n}{\partial y^2}(\pi_k X(s))(\pi_k \sigma(s, X(s)) Q \sigma^*(s, X(s)) \pi_k^* \cdot, \cdot), \end{aligned}$$

$0 \leq s < \tau^0$. Since

$$\frac{\partial v_n}{\partial y}(\pi_k y)(\cdot) = 2p \langle A_{2\alpha}(n) \pi_k y, \pi_k y \rangle^{p-1} \, \langle \cdot, A_{2\alpha}(n) \pi_k y \rangle$$

for all $y \in \mathbf{H} \setminus \{0\}$, the first term on the right side of (13) is ≤ 0 and the second term is bounded by $C_n^{(1)}(s) \|f(s, X(s))\|_{\mathbf{H}}$, where

$$C_n^{(1)}(s) = \sup_k \{ \| \frac{\partial v_n}{\partial y}(\pi_k(X(s))\|_{\mathbf{H}'} \, \|\pi_k\|_{\mathcal{L}(\mathbf{H})} \}.$$

In a similar way the third term of (13) is bounded by $C_n^{(2)}(s) \|\sigma(s, X(s))\|^2_{\mathcal{L}_2(\mathbf{K}_0, \mathbf{H})}$, where

$$C_n^{(2)}(s) = \frac{1}{2} \sup_k \{ \| \frac{\partial^2 v_n}{\partial y^2}(\pi_k X(s))\|_{\mathcal{L}(\mathbf{H})} \|\pi_k\|^2_{\mathcal{L}(\mathbf{H})} \}.$$

Since v_n is a power of a continuous quadratic form, since X is $\mathbf{H}$-continuous, and since the sequence (π_k) converges strongly to the identity operator the functions $C_n^{(1)}$ and $C_n^{(2)}$ are locally bounded and hence locally integrable on $[0, \tau_0[$. Thus, by 1(2) and 1(3), Conditions 2.4(b)(i) and 2.8(b)(i) are satisfied with $\varphi_n(s) = C_n^{(1)}(s)\|f(s, X(s))\|_{\mathbf{H}} + C_n^{(2)}(s)\|\sigma(s, X(s))\|^2_{\mathcal{L}^2(\mathbf{K}_0,\mathbf{H})}$.

We next deal with Condition 2.4(b)(ii). By (7) we have

$$\overline{\lim}_k \mathcal{L}^{(k)} v_n(s, X(s)) \le C_{\varepsilon,r,T}(1 + \|X(s)\|_\alpha^{2p}),$$

$0 < s \le \tau^\varepsilon \wedge \tau_r$, where τ^ε is the first hitting time of $B_{\mathbf{X}}(\varepsilon)$ and the required condition follows from (8). Condition 2.8(b)(ii) with D as in Theorem 3.1 follows directly from (9) and the fact, that the sequence (v_n) is increasing. Condition (A1) is just the assumption at the beginning of the proposition.

We now turn to the proof of (a). Condition (A2) is just (10) and Conditions (A3) and (A4) follow from (6). The claim now follows from Theorem 3.1. Parts (b) and (c) follow in a similar way from Theorem 3.2 and (11) and Theorem 3.3 and (12), respectively. □

In the remainder of the section we apply Proposition 4.1 to various situations. To this end we first give an explicit representation of $\overline{\lim}_k \mathcal{L}^{(k)} v_n$. From 2.9, we have

$$\begin{aligned}
&\mathcal{L}^{(k)} v_n(t, y)\\
(14)\quad =\;& 2p\langle A_{2\alpha}(n)\pi_k y, \pi_k y\rangle^{p-1}\\
&\quad \{-\langle A\pi_k y, A_{2\alpha}(n)\pi_k y\rangle + \langle \pi_k f(t, y), A_{2\alpha}(n)\pi_k y\rangle\\
&\quad + \frac{1}{2}\, tr A_{2\alpha}(n)\pi_k\sigma(t, y)Q\sigma^*(t, y)\pi_k^*\\
&\quad + (p-1)\frac{\langle \pi_k\sigma(t, y)Q\sigma^*(t, y)\pi_k^* A_{2\alpha}(n)\pi_k y, A_{2\alpha}(n)\pi_k y\rangle}{\langle A_{2\alpha}(n)\pi_k y, \pi_k y\rangle}\},
\end{aligned}$$

$(t, y) \in \mathbf{R}_+ \times (\mathbf{E} \setminus \{0\})$. Since $\pi_k \to id$ (cf. 1.6(α)), all terms but one converge as $k \to \infty$. In particular, we have $\sup_k \|\pi_k\|_{\mathcal{L}(\mathbf{H})} < \infty$, so it is easy to see that $\pi_k A_{2\alpha}(n)\pi_k^* \to A_{2\alpha}(n)$ as $k \to \infty$ in the strong operator topology and hence

$$\|A_{2\alpha}(n)\pi_k^*\sigma(t, y)Q\sigma^*(t, y)\pi_k^* - A_{2\alpha}(n)\sigma(t, y)Q\sigma^*(t, y)\|_{\mathcal{L}_1(\mathbf{H})} \to 0$$

as $k \to \infty$. Thus, the trace term in (14) converges to $tr A_{2\alpha}(n)\sigma(t, y)Q\sigma^*(t, y)$ and we obtain

$$\begin{aligned}
&\overline{\lim}_k \mathcal{L}^k v_n(t, y)\\
(15)\quad =\;& 2p\langle A_{2\alpha}(n)y, y\rangle^{p-1}\{-\underline{\lim}_k\langle A\pi_k y, A_{2\alpha}(n)\pi_k y\rangle + \langle f(t, y), A_{2\alpha}(n)y\rangle\\
&\quad + \frac{1}{2}\, tr A_{2\alpha}(n)\sigma(t, y)Q\sigma^*(t, y)\\
&\quad + (p-1)\frac{\langle \sigma(t, y)Q\sigma^*(t, y)A_{2\alpha}(n)y, A_{2\alpha}(n)y\rangle}{\langle A_{2\alpha}(n)y, y\rangle}\}\\
=:\;& 2p\, I_0\{-I_1 + I_2 + \frac{1}{2}\, I_3 + (p-1)I_4\},
\end{aligned}$$

$(t, y) \in \mathbf{R}_+ \times (\mathbf{E} \setminus \{0\})$. Note that $I_0 = v_n(y)^{1-\frac{1}{p}}$, $y \in \mathbf{H}_\alpha$. In the series of Lemmas below we give some useful estimates on the particular terms $I_0, \cdots, I_4$ which will allow us to formulate the Conditions (7) – (12) more explicitly. We first deal with the terms I_1 and $I_0 I_1$ in (15). The next lemma follows directly from (15) and (1).

4.2 Lemma

With β as in (1) we have for $y \in \mathbf{H}_\alpha$

(16) $I_1 = \underline{\lim}_k\langle A\pi_k y, A_{2\alpha}(n)\pi_k y\rangle \ge \beta\|\hat{A}^\alpha J_{2\alpha}^{\frac{1}{2}}(n)y\|^2_{\mathbf{H}} = \beta v_n(y)^{1/p}$ and

(17) $I_0 I_1 = \langle A_{2\alpha}(n)y, y\rangle^{p-1} \underline{\lim}_k \langle A\pi_k y, A_{2\alpha}(n)\pi_k y\rangle \geq \beta v_n(y)$. □

4.3 Lemma

Assume that A is positive and let $\eta > 1$.

(a) There exists an $n_0 \in \mathbf{N}$ such that, for all $\rho \geq 0$, all $n \geq n_0$, and all $y \in \mathbf{H}$, we have

(18) $\|A^{\alpha+\rho} J_{2\alpha}^{\frac{1}{2}}(n)y\|_{\mathbf{H}} \geq \eta^{-1}\beta^{\alpha+\rho}\|y\|_{\mathbf{H}}$.

(b) If $0 \leq \alpha \leq \frac{1}{2}$ then we have for all n and all $y \in \mathbf{H}$

(19) $\|A^{\alpha+\frac{1}{2}} J_{2\alpha}^{\frac{1}{2}}(n)y\|_{\mathbf{H}} \geq \beta^{\frac{1}{2}-\alpha}\|A^{2\alpha} J_{2\alpha}(n)y\|_{\mathbf{H}}$

and there exists n_0 such that, for all $n \geq n_0$ and all $y \in \mathbf{H}$, we have

(20) $\|A^{\alpha+\frac{1}{2}} J_{2\alpha}^{\frac{1}{2}}(n)y\|_{\mathbf{H}} \geq \eta^{-1}\beta^{\frac{1}{2}}\|A^{\alpha}y\|_{\mathbf{H}}$.

Proof: The function $z \longmapsto \frac{nz}{n+z}$ being nondecreasing for $z \geq 0$, we have

$$\lambda^{2\rho}\frac{n\lambda^{2\alpha}}{n+\lambda^{2\alpha}} \geq \lambda^{2\rho}\frac{n\beta^{2\alpha}}{n+\beta^{2\alpha}} \geq \beta^{2\rho+2\alpha}\frac{n}{n+\beta^{2\alpha}} \geq \beta^{2\alpha+2\rho}/\eta^2$$

for all $\rho \geq 0$ and all $\lambda \geq \beta$ (≥ 0) if n is large enough. Therefore, we obtain (18) from the spectral representation of $A^{2\alpha+2\rho}J_{2\alpha}(n)$ with respect to A. The inequalities (19) and (20) are proved analogously. □

Our next aim is to estimate the term I_2 in (15). We will distinguish between two cases: The function f will be assumed to be either (locally) Lipschitzian on $\mathbf{H}$ or dissipative in a certain sense. At first, assume

(F1) There exists $r > 0$ and a locally-bounded function $k_f : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\|f(t,y)\|_{\mathbf{H}} \leq k_f(t)\|y\|_{\mathbf{H}}$ for all $(t,y) \in \mathbf{R}_+ \times (B_{\mathbf{H}\alpha}(r) \cap \mathbf{E})$.

4.4 Lemma

Let A be uniformly positive and assume Condition (F1), let $\alpha \in [0, \frac{1}{2}]$, and let $\eta > 1$. Then there exists an $n_0 = n_0(\eta)$ such that, for all $n \geq n_0$, all $t \in \mathbf{R}_+$, and all $y \in B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E}$, we have

(21) $I_2 = \langle f(t,y), A_{2\alpha}(n)y\rangle \leq \dfrac{\eta k_f(t)}{\beta}\|A^{\alpha+\frac{1}{2}} J_{2\alpha}^{\frac{1}{2}}(n)y\|_{\mathbf{H}}^2$.

Proof: From (F1), (18) with $\rho = 1/2$, and (19) we infer for all $n \geq n_0$ and all $(t,y) \in \mathbf{R}_+ \times (B_{\mathbf{H}\alpha}(r) \cap \mathbf{E})$

$$\langle f(t,y), A_{2\alpha}(n)y\rangle \leq k_f(t)\|y\|_{\mathbf{H}} \cdot \|A^{2\alpha}J_{2\alpha}(n)y\|_{\mathbf{H}} \leq \frac{\eta k_f(t)}{\beta}\|A^{\alpha+\frac{1}{2}} J_{2\alpha}^{\frac{1}{2}}(n)y\|_{\mathbf{H}}^2.$$ □

4.5 Definition

We call a function $f : \mathbf{R}_+ \times \mathbf{E} \longrightarrow \mathbf{H}$ *α-dissipative* (on $\mathbf{R}_+ \times (B_{\mathbf{H}\alpha}(r) \cap \mathbf{E})$ if there exists a locally-bounded function $a_f : \mathbf{R}_+ \longrightarrow \mathbf{R}$ such that the function $\tilde{f}(t,y) := f(t,y) - a_f(t)y$ satisfies

(22) $\langle \tilde{f}(t,y), A_{2\alpha}(n)y\rangle \leq 0$

for all $n \in \mathbf{N}$, all $(t,y) \in \mathbf{R}_+ \times (B_{\mathbf{H}\alpha}(r) \cap \mathbf{E})$, and some $r \in]0, +\infty]$.

4.6 Example

Let $\mathbf{H} = L_2(\mathcal{O})$, $\mathcal{O} \subset \mathbf{R}^d$ being a bounded domain with a smooth boundary, let $-A$ be defined by the second order uniformly elliptic operator 1(6) and the boundary conditions 1(12) or 1(13), let f be defined by 1(17), where F satisfies the Conditions 1(8) – 1(10). Set

$$(23) \quad \mathbf{E} = \{y \in L_{2\nu}(\mathcal{O}); y \geq 0\},$$

where the number ν is defined in 1(9). If F has the form

$$(24) \quad F(t,u) = a_f(t)u + \tilde{F}(t,u), \quad (t,u) \in \mathbf{R}_+^2,$$

with a_f as in Definition 4.5 and $\tilde{F} \leq 0$ then, since $u \geq 0$ and $A_{2\alpha}(n) = \frac{n}{n+1}I$, f is α-dissipative for $\alpha = 0$ on $\mathbf{R}_+ \times (\mathbf{H} \cap \mathbf{E}) = \mathbf{R}_+ \times \mathbf{E}$. If, moreover, $\tilde{F}$ has the form

$$(25) \quad \tilde{F}(t,u) = -c(t)u^q, \quad (t,u) \in \mathbf{R}_+ \times [0,R[,$$

for some $c : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$, $1 \leq q \leq \nu$, $R \in \,]0,\infty]$, and if $\alpha > \frac{d}{4}$ then f is α-dissipative on $\mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E})$ for some $r > 0$. If we have only $\alpha > \frac{d}{8}$, but (25) is satisfied globally (with $R = \infty$) then f is α-dissipative on the whole space $\mathbf{R}_+ \times (\mathbf{H}_\alpha \cap \mathbf{E})$.

To prove these two statements note that, for $\alpha > \frac{d}{8}$, the resolvent $R_n(-\hat{A}^{2\alpha}) = (nI + \hat{A}^{2\alpha})^{-1}$ and hence $J_{2\alpha}$ are Hilbert-Schmidt operators (cf. Edmunds and Triebel [5]); therefore, for fixed n and α, there exists a kernel $k \in L_2(\mathcal{O} \times \mathcal{O})$ such that

$$(26) \quad J_{2\alpha}(n)y(\xi) = \int_{\mathcal{O}} k(\xi,\rho)y(\rho)d\rho, \quad y \in L_2(\mathcal{O}), \quad \xi \in \mathcal{O},$$

(see e.g. Reed and Simon [27]). Since the operator A is self-adjoint and positive the operators $J_{2\alpha}(n)$ are contractions and R_n is a sub-Markovian contraction resolvent in the sense of Ma and Röckner [19], Definition I.4.1 and Theorem I.4.4. This means in particular that the operators J_n are sub-Markovian in the usual sense. Thus, the kernel k is nonnegative and symmetric and we have $\int_{\mathcal{O}} k(\xi,\rho)d\rho \leq 1$ for all $\xi \in \mathcal{O}$. Now, if (25) is satisfied globally we can apply Example 6.4 and Remark 6.5 of Leha, Ritter, and Wakolbinger [18] to obtain

$$\langle \tilde{f}(t,y), A_{2\alpha}(n)y \rangle \leq 0, \quad y \in \mathbf{E}, \quad n \in \mathbf{N},\ t \in \mathbf{R}_+,$$

where $\tilde{f}(t,y) = \tilde{F}(t,y(\cdot))$. If, moreover, $\alpha > \frac{d}{4}$ then $\mathbf{H}_\alpha$ is continuously embedded into $\mathcal{C}(\bar{\mathcal{O}})$ by Sobolev's theorem. Thus, proceeding as above, we obtain α-dissipativity of f on $\mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E})$ for some $r > 0$.

It is clear that f is α-dissipative also if the right side in (25) is a positive linear combination of terms of the form $-c(t)u^q$. The following lemma is elementary.

4.7 Lemma

Let f be α-dissipative on $\mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E})$. Then we have

$$(27) \quad I_2 = \langle f(t,y), A_{2\alpha}(n)y \rangle \leq a_f(t)\langle y, A_{2\alpha}(n)y \rangle = a_f(t)v_n(y)^{1/p}$$

for $(t,y) \in \mathbf{R}_+ \times (\mathbf{E} \cap B_{\mathbf{H}_\alpha}(r) \setminus \{0\})$, $n \in \mathbf{N}$, $\alpha \geq 0$, $p > 0$, where r and a_f are specified in Definition 4.5. □

In the sequel, we estimate the terms I_3 and I_4 of (15) which come from the stochastic term of Equation 1(1). We formulate two more conditions.

($\Sigma 1$) For all $(t,y) \in \mathbf{R}_+ \times (\mathbf{H}_\alpha \cap \mathbf{E})$ we have $\sigma(t,y) \in \mathcal{L}(\mathbf{K},\mathbf{H})$ and there exists $r \in]0,\infty]$ and a locally-bounded function $k_\sigma : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$\|\sigma(t,y)\|_{\mathcal{L}(\mathbf{K},\mathbf{H})} \leq k_\sigma(t)\|y\|_{\mathbf{H}}$$

for all $(t,y) \in \mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E})$.

($\Sigma 2$) For all $(t,y) \in \mathbf{R}_+ \times (\mathbf{H}_\alpha \cap \mathbf{E})$ we have $\sigma(t,y) \in \mathcal{L}_2(\mathbf{K}_0,\mathbf{H}_\alpha)$ and there exist $r \in]0,\infty]$ and a locally-bounded function $b_\sigma : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$\|\sigma(t,y)\|_{\mathcal{L}_2(\mathbf{K}_0,\mathbf{H}_\alpha)} \leq b_\sigma(t)\|y\|_\alpha$$

for all $(t,y) \in \mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E})$.

Note that, for $\alpha = 0$, ($\Sigma 1$) implies ($\Sigma 2$) with $b_\sigma = \|Q\|^{1/2}_{\mathcal{L}_1(\mathbf{K})} k_\sigma$.

4.8 Lemma

If ($\Sigma 2$) *holds then we have for all* $(t,y) \in \mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E})$

(28) $\quad I_3 = tr A_{2\alpha}(n)\sigma(t,y)Q\sigma^*(t,y) \leq b^2_\sigma(t)\|y\|^2_\alpha = b^2_\sigma(t)v(y)^{1/p}.$

Proof: We estimate

$$\begin{aligned} tr\, A_{2\alpha}(n)\sigma(t,y)Q\sigma^*(t,y) &\leq \|J_{2\alpha}(n)\|_{\mathcal{L}(\mathbf{H})} tr\, \hat{A}^{2\alpha}\sigma(t,y)Q\sigma^*(t,y) \\ &\leq \|\hat{A}^\alpha \sigma(t,y)Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathbf{K},\mathbf{H})} \\ &= \|\sigma(t,y)\|^2_{\mathcal{L}_2(\mathbf{K}_0,\mathbf{H}_\alpha)} \end{aligned}$$

and (28) follows from ($\Sigma 2$). □

The next lemma follows from (28) and (20).

4.9 Lemma

Let $\alpha \in [0,\frac{1}{2}]$, *let* A *be uniformly positive and assume Condition* ($\Sigma 2$). *Then, for any* $\eta > 1$ *there exists an* $n_0 = n_0(\eta) \in \mathbf{N}$ *such that*

(29) $\quad I_3 = tr A_{2\alpha}(n)\sigma(t,y)Q\sigma^*(t,y) \leq \dfrac{\eta b^2_\sigma(t)}{\beta}\|A^{\alpha+\frac{1}{2}}J^{\frac{1}{2}}_{2\alpha}(n)y\|^2_{\mathbf{H}}$

for all $n \geq n_0$ *and all* $(t,y) \in \mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E})$. □

4.10 Lemma

If A *is uniformly positive and if* ($\Sigma 1$) *holds then, for any* $\eta > 1$, *there exists* $n_0 = n_0(\eta)$ *such that*

$$\begin{aligned} (30) \quad I_4 &= \frac{\langle \sigma(t,y)Q\sigma^*(t,y)A_{2\alpha}(n)y, A_{2\alpha}(n)y\rangle}{\langle A_{2\alpha}(n)y,y\rangle} \\ &\leq \frac{\eta k^2_\sigma(t)}{\beta}\|Q\|_{\mathcal{L}_1(\mathbf{K})}\|A^{\alpha+\frac{1}{2}}J^{\frac{1}{2}}_{2\alpha}(n)y\|^2_{\mathbf{H}} \end{aligned}$$

for all $n \geq n_0, (t,y) \in \mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E})$.

Proof: For sufficiently large n, we have

$$\begin{aligned} \frac{\langle \sigma(t,y) Q \sigma^*(t,y) A_{2\alpha}(n) y, A_{2\alpha}(n) y\rangle}{\|A^\alpha J_{2\alpha}^{\frac{1}{2}}(n) y\|_{\mathbf{H}}^2} &\leq \frac{\|\sigma(t,y)\|_{\mathcal{L}(\mathbf{K},\mathbf{H})}^2 \|Q\|_{\mathcal{L}_1(\mathbf{K})} \ \|A^{2\alpha} J_{2\alpha}(n) y\|_{\mathbf{H}}^2}{\|A^\alpha J_{2\alpha}^{\frac{1}{2}}(n) y\|_{\mathbf{H}}^2} \\ &\leq \frac{\eta k_\sigma^2(t)}{\beta} \|Q\|_{\mathcal{L}_1(\mathbf{K})} \|A^{\alpha+\frac{1}{2}} J_{2\alpha}^{\frac{1}{2}}(n) y\|_{\mathbf{H}}^2 \end{aligned}$$

by $(\Sigma 1)$, (19), and (18) with $\rho = 0$. □

4.11 Proposition

Let $\alpha \in [0, \frac{1}{2}]$ *and let* $p > 0$. *Assume that the solutions of Equation 1(1) starting from initial points in* $\mathbf{H}_\alpha \cap \mathbf{E}$ *have* $\mathbf{H}_\alpha$*-valued and* $\mathbf{H}_\alpha$*-right-continuous versions. Assume also that* A *is uniformly positive and that the Conditions (F1),* $(\Sigma 1)$, $(\Sigma 2)$ *and*

$$(31) \quad -\beta + \sup_{t\geq 0}\{k_f(t) + \frac{1}{2}\, b_\sigma^2(t) + (p-1)^+ \|Q\|_{\mathcal{L}_1(\mathbf{K})} k_\sigma^2(t)\} < 0$$

are satisfied. Then the trivial solution $X^0(t) \equiv 0$ *of Equation 1(1) is asymptotically* $\mathbf{H}_\alpha$*-stable in probability. Moreover, if (F1),* $(\Sigma 1)$, *and* $(\Sigma 2)$ *are satisfied with* $r = \infty$ *then the trivial solution is globally asymptotically* $\mathbf{H}_\alpha$*-stable in probability and exponentially stable in the* $2p$*th mean.*

Proof: From (15) and Lemmas 4.4, 4.9, and 4.10 we obtain

$$\begin{aligned} & \overline{\lim}_k\, \mathcal{L}^{(k)} v_n(t,y) \\ = \quad & 2pI_0\{-I_1 + I_2 + \frac{1}{2} I_3 + (p-1) I_4\} \\ \leq \quad & 2pI_0\, \|A^{\alpha+\frac{1}{2}} J_{2\alpha}^{\frac{1}{2}}(n) y\|_{\mathbf{H}}^2 \{-1 + \frac{\eta k_f(t)}{\beta} + \frac{\eta b_\sigma^2(t)}{2\beta} + (p-1)^+ \frac{\eta k_\sigma^2(t)}{\beta} \|Q\|_{\mathcal{L}_1(\mathbf{K})}\} \end{aligned}$$

for $(t,y) \in \mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E} \setminus \{0\}), n \geq n_0(\eta)$, where $n_0(\eta)$ is a natural number independent of t and y. If we choose $\eta > 1$ small enough then the sum in curly brackets becomes negative, $\leq -c$ say, uniformly in $t \geq 0$ by Assumption (31). For this reason, by uniform positivity of A and by (17), we see that (32) is majorized by $-2pcI_0 I_1 \leq -2pc\beta v_n(y)$. This is (9). Moreover,

$$(32) \quad \overline{\lim}_n \overline{\lim}_k\, \mathcal{L}^{(k)} v_n(t,y) \leq -2pc\beta v(y),$$

$(t,y) \in \mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E} \setminus \{0\})$; hence (11) follows from (6). Furthermore, if (F1), $(\Sigma 1)$, and $(\Sigma 2)$ are valid globally then we also have (12). Therefore, we can use Proposition 4.1 to complete the proof. □

In the following statement, we avoid uniform positivity of A and the restriction on α and we replace Lipschitzianity of f by its α-dissipativity in the sense of Definition 4.5. We apply again Proposition 4.1, using this time (7) and (8) instead of (9).

4.12 Proposition

Let $\alpha \geq 0$ *and let* $p \in]0,1]$. *Assume that the solutions* X^x *of Equation 1(1) starting from initial points* $x \in \mathbf{H}_\alpha \cap \mathbf{E}$ *have* $\mathbf{H}_\alpha$*-valued, right-continuous versions satisfying P-a.s., for any* $T > 0$,

$$(33) \quad \int_0^T \|X^x(t)\|_\alpha^{2p} dt < \infty.$$

Let f be α-dissipative and assume that the Conditions $(\Sigma 2)$ *and*

$$(34)\quad -\beta + \sup_{t\ge 0}\{a_f(t) + \frac{1}{2}b_\sigma^2(t)\} < 0$$

hold. Then the trivial solution $X^0(t) \equiv 0$ of Equation 1(1) is asymptotically $\mathbf{H}_\alpha$-stable in probability. If, moreover, f is α-dissipative on $\mathbf{R}_+ \times (\mathbf{H}_\alpha \cap \mathbf{E})$ and $(\Sigma 2)$ is satisfied with $r = \infty$ then the trivial solution is globally $\mathbf{H}_\alpha$-stable in probability and exponentially $\mathbf{H}_\alpha$-stable in the 2pth mean.

Proof: By (15), $p \le 1$, (16), (27), and Lemma 4.8, we have for $(t,y) \in \mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E})$ and $n \in \mathbf{N}$

$$\begin{aligned}
&\overline{\lim}_k\, \mathcal{L}^{(k)} v_n(t,y)\\
&\le\ 2pv_n(y)^{1-1/p}\{-I_1 + I_2 + \frac{1}{2}I_3\}\\
(35)\qquad &\le\ 2pv_n(y)^{1-1/p}\{-\beta v_n(y)^{1/p} + a_f(t)v_n(y)^{1/p} + \frac{1}{2}b_\sigma^2(t)v(y)^{1/p}\}\\
&\le\ 2p\{-\beta v_n(y) + a_f(t)v_n(y) + \frac{1}{2}b_\sigma^2(t)v(y)\}\\
&\le\ 2p\{\beta^- + a_f^+(t) + \frac{1}{2}b_\sigma^2(t)\}v(y).
\end{aligned}$$

Now (7) follows from local boundedness of a_f and b_σ. From (35) we also deduce

$$(36)\quad \overline{\lim}_n\overline{\lim}_k\, \mathcal{L}^{(k)} v_n(t,y) \le 2p\{-\beta + a_f(t) + \frac{1}{2}b_\sigma^2(t)\}v(y),$$

$(t,y) \in \mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E})$. Together with Condition (34) this yields (11) and, if the global assumptions stated in the proposition are satisfied, also (12). The two claims now follow from Proposition 4.1. □

4.13 Remark

If Equation 1(1) is *autonomous*, that is, if $f(t,y) = f(y)$ and $\sigma(t,y) = \sigma(y)$ do not depend on t then we may choose $k_f(t) \equiv k_f$, $a_f(t) \equiv a_f$, $b_\sigma(t) \equiv b_\sigma$, and $k_\sigma(t) \equiv k_\sigma$ independent of $t \ge 0$ and Conditions (31) and (34) become

$$-\beta + k_f + \frac{1}{2}b_\sigma^2 + (p-1)^+\|Q\|_{\mathcal{L}_1(\mathbf{K})}k_\sigma^2 < 0 \qquad \text{and}$$
$$-\beta + a_f + \frac{1}{2}b_\sigma^2 < 0,$$

respectively.

4.14 Example

Resuming the basic Example 1.2 we next discuss in detail the Stochastic Parabolic Equation 1(5) with the Initial Condition 1(11) and boundary conditions of either Dirichlet's 1(12) or Neumann's type 1(13) in the case of a noise η sufficiently "regular in the space variable." We assume that the coefficients F, G are measurable satisfying $F(t,0) = G(t,0) \equiv 0$, $t \ge 0$, and that the Conditions 1(7) – 1(10) hold true. As described in Section 1, the formal system 1(5), 1(11) with boundary conditions 1(12) or 1(13) can be considered as an infinite-dimensional equation of the form 1(1), where A is defined by 1(15) or 1(16) and $\mathbf{H}, \mathbf{E}$, and f are defined by 1(14) and 1(17), respectively. Since $\beta > 0$ we have $\hat{\beta} = 0$.

We assume that $\mathbf{K}$ is continuously embedded into $L_\infty(\mathcal{O})$ so that

$$(37) \quad \|h\|_{L_\infty(\mathcal{O})} \le \rho \|h\|_{\mathbf{K}}, \qquad h \in \mathbf{K},$$

for some $\rho > 0$. For example, if $\mathbf{K} = \mathbf{H}_\delta$ with $\delta > \frac{d}{4}$ then $\mathbf{K}$ is continuously embedded in $\mathcal{C}(\bar{\mathcal{O}})$ by Sobolev's embedding theorem. In view of the basic example this can be interpreted as a certain "spatial regularity" of the noise.

Now define $\sigma : \mathbf{R}_+ \times \mathbf{E} \to \mathcal{L}(\mathbf{K}, \mathbf{H})$ by 1(18), i. e.,

$$(38) \quad [\sigma(t,y)h](\xi) := G(t, y(\xi))h(\xi), \qquad (t,\xi) \in \mathbf{R}_+ \times \mathcal{O},\ y \in \mathbf{E},\ h \in \mathbf{K}.$$

By Proposition 1.3, Equation 1(1) specified in this way has a unique solution in $\mathbf{E}$ which is a Markov process in $\mathbf{E}$ in the usual way. We have

$$(39) \quad \begin{aligned} \|\sigma(t,y)\|^2_{\mathcal{L}(\mathbf{K},\mathbf{H})} &= \sup_{\|h\|_{\mathbf{K}} \le 1} \int_{\mathcal{O}} |G(t,y(\xi))h(\xi)|^2 d\xi \\ &\le \sup_{\|h\|_{\mathbf{K}} \le 1} \|h\|^2_{L_\infty(\mathcal{O})} k_G^2 \int_{\mathcal{O}} |y(\xi)|^2 d\xi \\ &\le \rho^2 k_G^2 \|y\|^2_{\mathbf{H}}, \end{aligned}$$

so we get $(\Sigma 1)$ with $k_\sigma = \rho k_G$ and, by the remark following the definition of $(\Sigma 2)$, we also get $(\Sigma 2)$ with $\alpha = 0$.

Assume that F is of the form

$$(40) \quad F(t,u) = a_f(t)u + \tilde{F}(t,u),$$

with $\tilde{F}(t,u) \le 0$. Note that, by 1(10), F is always this form with some $a_f : \mathbf{R}_+ \to \mathbf{R}$, but in order to exploit the Lyapunov inequality a_f should be chosen as small as possible. Since X has $\mathbf{H}$-continuous paths we have (33) with $\alpha = 0$ and $p = 1$. Now, by Proposition 4.12 applied with these parameters α and p, the trivial solution is globally asymptotically $\mathbf{H}$-stable in probability and exponentially $\mathbf{H}$-stable in the mean square provided

$$(41) \quad -\beta + \sup_{t \ge 0} a_f(t) + \frac{1}{2} M \rho^2 k_G^2 < 0;$$

here $\beta > 0$ is a lower bound of the spectrum of A and $M = \|Q\|_{\mathcal{L}_1(\mathbf{K})} < \infty$ with the covariance operator Q of the Wiener process $W(t)$ in $\mathbf{K}$.

Our next aim is to establish some $\mathbf{H}_\alpha$-stability results with $\alpha > 0$. In order to apply Proposition 4.12 we need first sufficient regularity (33) of the solutions to the equation under consideration. Let F satisfy the growth condition 1(9) for some $\nu > 1$ and let $\alpha \in [0, 1 - \frac{1}{2\nu}[$; note that ν in 1(9) can always be chosen large enough so that we can basically consider $\alpha \in [0,1[$. Let $x \in \mathbf{H}_\alpha \cap L_{2\nu^2}(\mathcal{O})$, $x \ge 0$. By definition, a mild solution $X(t) = X^x(t)$ satisfies

$$(42) \quad \begin{aligned} X(t) &= S(t)x + \int_0^t S(t-r) f(r, X(r))dr + \int_0^t S(t-r)\sigma(r, X(r))dW(r) \\ &=: S(t)x + J_1(t) + J_2(t). \end{aligned}$$

Since $x \in \mathbf{H}_\alpha$, the function $t \longmapsto S(t)x$ belongs to $\mathcal{C}([0,T], \mathbf{H}_\alpha)$. A standard estimate for analytic semigroups, cf. [3], A.32, 1(9), and Hölder's inequality show

$$
\begin{aligned}
& E \int_0^T \|J_1(t)\|_\alpha^{2\nu} dt \\
\leq\ & c \int_0^T E \left[\int_0^t \frac{\|f(r, X(r))\|_{\mathbf{H}}}{(t-r)^\alpha} dr \right]^{2\nu} dt \\
\leq\ & c + c \int_0^T E \left[\int_0^t \frac{\|X(r)\|_{L_{2\nu}}^{\nu}}{(t-r)^\alpha} dr \right]^{2\nu} dt \qquad (43) \\
\leq\ & c + c \int_0^T \left(\int_0^t \frac{dr}{r^{\alpha q}} \right)^{\frac{2\nu}{q}} E \left(\int_0^t \|X(r)\|_{L_{2\nu}}^{2\nu^2} dr \right) dt,
\end{aligned}
$$

for $q = \frac{2\nu}{2\nu - 1}$ (note that $\alpha q < 1$). In this and the following estimates, c stands for a constant that may vary from line to line. Since the embedding of $L_{2\nu^2}(\mathcal{O})$ into $L_{2\nu}(\mathcal{O})$ is continuous we have by 1(21), applied with ν^2 instead of ν,

$$
E \int_0^T \|J_1(t)\|_\alpha^{2\nu} dt < \infty. \qquad (44)
$$

Similarly, we obtain that $J_1(\cdot) \in \mathcal{C}([0,T], \mathbf{H}_\alpha)$, P-a.s..

We next prove an analogous assertion for J_2. The equivalence of the δ-norm and the Sobolev-Slobodeckii norm (cf. Seeley [31], Triebel [35], or Kufner, John, and Fučik [15], p. 386), 1(7), (39), and (37) show

$$
\begin{aligned}
& \|\sigma(t,y)\|_{\mathcal{L}(\mathbf{K}, \mathbf{H}_\delta)}^2 \\
=\ & \sup_{\|h\|_{\mathbf{K}} \leq 1} \|\sigma(t,y)h\|_\delta^2 \\
\leq\ & c \sup_{\|h\|_{\mathbf{K}} \leq 1} \left[\int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|G(t,y(\xi))h(\xi) - G(t,y(\eta))h(\eta)|^2}{|\xi - \eta|^{d+2\delta}} d\xi d\eta + \|\sigma(t,y)h\|_{\mathbf{H}}^2 \right] \\
\leq\ & c \sup_{\|h\|_{\mathbf{K}} \leq 1} \left[\int_{\mathcal{O}} \int_{\mathcal{O}} \left(\frac{|G(t,y(\xi)) - G(t,y(\eta))|^2 \ |h(\xi)|^2}{|\xi - \eta|^{d+2\delta}} \right. \right. \qquad (45) \\
& \qquad \left. \left. + \frac{|G(t,y(\eta))|^2 \ |h(\xi) - h(\eta)|^2}{|\xi - \eta|^{d+2\delta}} \right) d\xi d\eta + \|y\|_{\mathbf{H}}^2 \right] \\
\leq\ & c \sup_{\|h\|_{\mathbf{K}} \leq 1} \left[\|h\|_{L_\infty}^2 \|y\|_\delta^2 + \|G(t,\cdot) \circ y\|_{L_\infty}^2 \|h\|_\delta^2 + \|y\|_{\mathbf{H}}^2 \right]. \\
\leq\ & c \sup_{\|h\|_{\mathbf{K}} \leq 1} \left[\|y\|_\delta^2 + \|G(t,\cdot) \circ y\|_{L_\infty}^2 \|h\|_\delta^2 + \|y\|_{\mathbf{H}}^2 \right].
\end{aligned}
$$

Now, suppose that $\delta \in [0, \frac{1}{2} - \frac{1}{2\nu}[$ is such that $\mathbf{K}$ is continuously embedded into $\mathbf{H}_\delta$. Then $\|h\|_\delta \leq c\|h\|_{\mathbf{K}}$. If $\mathbf{H}_\delta$ is continuously embedded into $L_\infty(\mathcal{O})$ then, by formula 1(7), $\|G(t,\cdot) \circ y\|_{L_\infty} \leq k_G \|y\|_{L_\infty} \leq c\|y\|_\delta$. Hence,

$$
\|\sigma(t,y)\|_{\mathcal{L}(\mathbf{K}, \mathbf{H}_\delta)}^2 \leq c\|y\|_\delta^2, \quad (t,y) \in \mathbf{R}_+ \times \mathbf{H}_\delta, \qquad (46)
$$

in this case. Alternatively, if G is bounded then $\|G(t,\cdot) \circ y\|_{L_\infty} \leq \|G\|_\infty$ and hence

$$
\|\sigma(t,y)\|_{\mathcal{L}(\mathbf{K}, \mathbf{H}_\delta)}^2 \leq c(\|y\|_\delta^2 + 1), \quad (t,y) \in \mathbf{R}_+ \times \mathbf{H}_\delta.
$$

Furthermore, we can use analyticity of the semigroup $S(t)$ (similarly as in (43)) and Seidler's [32], Theorem 1.1, regularity result to obtain pathwise $\mathbf{H}_\delta$-continuity of J_2 and

$$(47)\quad E\int_0^T ||J_2(t)||_\delta^{2\nu}dt < \infty.$$

Thus, if $\delta \geq \alpha$ we conclude from (42), (44), and (47) that the solution X^x is continuous in $\mathbf{H}_\alpha$ and satisfies the required regularity (33) for $p = 1$ in this case. If $\delta < \alpha$ then (44) and (47) yield

$$E\int_0^T ||X^x(t)||_\delta^{2\nu}dt < \infty.$$

Taking into account (46) we use again [32], Theorem 1.1, obtaining continuity of J_2 in $\mathbf{H}_{\delta+\lambda}$ and

$$(48)\quad E\int_0^T ||J_2(t)||_{\delta+\lambda}^{2\nu}dt < \infty$$

for $\lambda < 1/2 - 1/2\nu$. Similarly as above, we obtain (33) for $\alpha < \delta + \frac{1}{2} - \frac{1}{2\nu}$.

In conclusion, if $\delta \in [0, \frac{1}{2} - \frac{1}{2\nu}[$ is such that $\mathbf{K}$ is continuously embedded into $\mathbf{H}_\delta$ and such that $\mathbf{H}_\delta$ is continuously embedded into $L_\infty(\mathcal{O})$ unless G is bounded then we have continuity of X^x in $\mathbf{H}_\alpha$, $x \in \mathbf{H}_\alpha \cap L_{2\nu^2}(\mathcal{O})$, and (33) with $p = 1$ for each

$$(49)\quad \alpha < \min(1 - \frac{1}{2\nu}, \delta + \frac{1}{2} - \frac{1}{2\nu}) = \delta + \frac{1}{2} - \frac{1}{2\nu}.$$

The assumption ($\Sigma 2$) can be verified in various situations. For example, if we have continuous embeddings $\mathbf{K} \hookrightarrow \mathbf{H}_\alpha \hookrightarrow L_\infty(\mathcal{O})$ then, similarly as in (45), we obtain from (46)

$$\|\sigma(t,y)\|_{\mathcal{L}_2(\mathbf{K}_0,\mathbf{H}_\alpha)} \leq \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathbf{K})}\|\sigma(t,y)\|_{\mathcal{L}(\mathbf{K},\mathbf{H}_\alpha)} \leq b_\sigma(t)\|y\|_\alpha,$$

$(t,y) \in \mathbf{R}_+ \times \mathbf{H}_\alpha$, where $b_\sigma(t) = \sqrt{c}\|\sqrt{Q}^{1/2}\|_{\mathcal{L}_2(\mathbf{K})}$ and c is as in (46).

Another interesting class of examples satisfying ($\Sigma 2$) flows from assuming $Q^{\frac{1}{2}} \in \mathcal{L}_2(\mathbf{K}, \mathcal{H})$ and

$$(50)\quad \|\sigma(t,y)\|_{\mathcal{L}(\mathcal{H},\mathbf{H}_\alpha)} \leq \tilde{b}_\sigma\|y\|_\alpha, \quad (t,y) \in \mathbf{R}_+ \times \mathbf{H}_\alpha,$$

where $\mathcal{H}$ is a suitable Hilbert space, usually a space of sufficiently smooth functions. For example, if $\alpha = \frac{1}{2}$, $G(t,\cdot) \in \mathcal{C}^1(\mathbf{R}_+)$, and if $\mathcal{H}$ is a Hilbert space continuously embedded into $\mathcal{C}^1(\bar{\mathcal{O}})$ it is easy to verify (50) with some $\tilde{b}_\sigma < \infty$ involving the constants k_G, and the norm of the embedding $\mathcal{H} \hookrightarrow \mathcal{C}^1(\bar{\mathcal{O}})$. Now we can use Proposition 4.12 in order to establish $\mathbf{H}_\alpha$-stability for the zero solution of our equation.

If F is of the form

$$(51)\quad F(t,u) = a_f(t)u - c(t)u^q, \quad (t,u) \in \mathbf{R}_+^2,$$

with $1 \leq q \leq \nu$, $a_f : \mathbf{R}_+ \to \mathbf{R}$, $c : \mathbf{R}_+ \to \mathbf{R}_+$, we obtain for $\frac{d}{8} < \alpha \leq \frac{1}{2}$ that the zero solution is globally asymptotically $\mathbf{H}_\alpha$-stable in probability and exponentially $\mathbf{H}_\alpha$-stable in the mean square provided

$$(52)\quad -\beta + \sup_{t\geq 0}\{a_f(t) + \frac{1}{2}\tilde{b}_\sigma^2(t)\} < 0.$$

If F has the form (51) only locally on a neighborhood of zero (i.e., for $(t,\xi) \in \mathbf{R}_+ \times [0,R[$ for some $R > 0$) and k_G is a Lipschitz constant of G just on a neighborhood of zero but, on the other hand, $\alpha > \frac{d}{4}$ (and, thus, $\mathbf{H}_\alpha$ is continuously embedded into $\mathcal{C}(\bar{\mathcal{O}})$) then (52) implies the (local) asymptotical $\mathbf{H}_\alpha$-stability in probability of the zero solution (cf. Example 4.6). □

In the rest of this section we consider the important particular case when the driving Wiener process is one-dimensional. More precisely, let W be a standard scalar Wiener process (put $\mathbf{K} = \mathbf{R}$ and $Q = 1$) and identify the operator σ in the usual way with the functional $\tilde{\sigma} : \mathbf{R}_+ \times \mathbf{E} \to \mathbf{H}$, $\tilde{\sigma}(t,y) := \sigma(t,y)(1)$. In this case, the terms I_3 and I_4 of (15) have the form

$$(53)\quad I_3 = tr A_{2\alpha}(n)\sigma(t,y)Q\sigma^*(t,y) = \langle A_{2\alpha}(n)\sigma(t,y), \sigma(t,y)\rangle \qquad \text{and}$$

$$(54)\quad I_4 = \frac{\langle \sigma(t,y)Q\sigma^*(t,y)A_{2\alpha}(n)y, A_{2\alpha}(n)y\rangle}{\langle A_{2\alpha}(n)y, y\rangle} = \frac{\langle A_{2\alpha}(n)y, \sigma(t,y)\rangle^2}{\langle A_{2\alpha}(n)y, y\rangle},$$

$n \in \mathbf{N}$, $\alpha \geq 0$, $(t,y) \in \mathbf{R}_+ \times (\mathbf{E} \setminus \{0\})$. We formulate now a sample result in the case of one-dimensional noise. Similarly to the finite-dimensional case, an unstable trivial solution of a deterministic equation can be "stabilized" in probability by adding sufficiently large noise. For simplicity we restrict ourselves to the case $p \in]0,1]$ which is the most interesting one for results on stability in probability. Note that Condition $(\Sigma 2)'$ below is just a reformulation of $(\Sigma 2)$ in the case of a scalar Wiener process.

4.15 Proposition

Let $\mathbf{K} = \mathbf{R}$, let $Q = 1$, let $\alpha \geq 0$, and let $p \in]0,1]$. Assume that the solutions X^x to Equation 1(1) starting from initial points $x \in \mathbf{H}_\alpha \cap \mathbf{E}$ have $\mathbf{H}_\alpha$-valued, right-continuous versions satisfying P-a.s., for any $T > 0$,

$$(55)\quad \int_0^T \|X^x(t)\|_\alpha^{2p}\, dt < \infty.$$

Let f be α-dissipative and suppose that there exists a number $r \in]0,\infty]$ and two locally-bounded functions $b_\sigma, \kappa_\sigma : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\sigma(\mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E})) \subseteq \mathbf{H}_\alpha$,

$$(\Sigma 2)'\quad \|\sigma(t,y)\|_\alpha \leq b_\sigma(t)\|y\|_\alpha,$$

$$(\Sigma 3)'\quad \langle \hat{A}^\alpha y, \hat{A}^\alpha \sigma(t,y)\rangle^2 \geq \kappa_\sigma(t)\|\hat{A}^\alpha y\|_{\mathbf{H}}^4, \qquad (t,y) \in \mathbf{R}_+ \times (B_{\mathbf{H}_\alpha}(r) \cap \mathbf{E}),$$

and

$$(56)\quad -\beta + \sup_{t \geq 0}\{a_f(t) + \frac{1}{2}b_\sigma^2(t) - (1-p)\kappa_\sigma(t)\} < 0$$

hold. Then the trivial solution $X^0(t) \equiv 0$ of Equation 1(1) is asymptotically $\mathbf{H}_\alpha$-stable in probability. If, moreover, f is α-dissipative on $\mathbf{R}_+ \times (\mathbf{H}_\alpha \cap \mathbf{E})$ and Conditions $(\Sigma 2)'$ and $(\Sigma 3)'$ are satisfied with $r = \infty$ then the trivial solution is globally $\mathbf{H}_\alpha$-stable in probability and exponentially $\mathbf{H}_\alpha$-stable in the $2p$th mean.

Proof: The claim follows easily from Propositon 4.1, similarly as Proposition 4.12. Note that, for $(t,y) \in \mathbf{R}_+ \times (\mathbf{E} \cap B_{\mathbf{H}_\alpha}(r) \setminus \{0\})$, we have

$$(57)\quad \overline{\lim}_n \overline{\lim}_k\, \mathcal{L}^{(k)} v_n(t,y) \leq 2p\{-\beta + a_f(t) + \frac{1}{2}b_\sigma^2(t) - (1-p)\kappa_\sigma(t)\}v(y). \qquad □$$

4.16 Remarks

(a) In the case of an *autonomous* equation we may choose $a_f(t) \equiv a_f$, $b_\sigma(t) \equiv b_\sigma$, and $\kappa_\sigma(t) \equiv \kappa_\sigma$ independent of $t \geq 0$ so that Condition (56) reads

$$-\beta + a_f + \frac{1}{2} b_\sigma^2 - (1-p)\kappa_\sigma < 0.$$

(b) The Conditions $(\Sigma 2)'$ and $(\Sigma 3)'$ are obviously satisfied if σ is *linear*, that is, if $\sigma(t,y) = b(t)y$, $t \in \mathbf{R}_+$, and $y \in \mathbf{E}$. In particular, we may choose $b_\sigma(t) = |b(t)|$ and $\kappa_\sigma(t) = b^2(t)$. Then the Lyapunov inequality (56) has the form

$$(58) \quad -\beta + \sup_{t \geq 0} \{a_f(t) - (\frac{1}{2} - p) b^2(t)\} < 0.$$

Since p may be chosen arbitrarily close to zero, the stabilizing effect of noise becomes apparent here.

4.17 Example

Besides linearity there is another important case when $(\Sigma 2)'$ can be verified. Let $\mathbf{H} = L_2(\mathcal{O})$ and A be as in Example 4.6, let $\mathbf{E}$ be a Borel subset of $\{\varphi \in \mathbf{H}; \varphi \geq 0\}$, and let

$$(59) \quad \sigma(t,y)(\xi) = G(t, y(\xi)), \qquad t \in \mathbf{R}_+,\ y \in \mathbf{H},\ \xi \in \mathcal{O},$$

where $G : \mathbf{R}_+^2 \to \mathbf{R}$ is a measurable function satisfying

$$(60) \quad |G(t,u)| \leq b_\sigma(t) u, \qquad (t,u) \in \mathbf{R}_+ \times [0, R[,$$

for some $R \in]0, \infty]$. If $R = \infty$ then $(\Sigma 2)'$ is clearly satisfied with $\alpha = 0$ and $r = \infty$. Assume, moreover, that $G(t, \cdot) \in \mathcal{C}^2([0,R[)$ and

$$(61) \quad \sup_{u \in]0,R[} |G'(t,u)| \leq b_\sigma(t), \qquad t \in \mathbf{R}_+,$$

where G' denotes the derivative with respect to the second variable u and $b : \mathbf{R}_+ \to \mathbf{R}_+$ is locally bounded. It follows from Fujiwara [7] that $\mathbf{H}_{1/2} = H^1(\mathcal{O})$, $\mathbf{H}_1 = \{\varphi \in H^2(\mathcal{O}); \frac{\partial \varphi}{\partial \nu_A} = 0 \text{ on } \partial\mathcal{O}\}$, in the Neumann case and $\mathbf{H}_{1/2} = \{\varphi \in H^1(\mathcal{O}); \varphi_{|\partial\mathcal{O}} = 0\}$, $\mathbf{H}_1 = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$, in the Dirichlet case. Hence, it is easy to see that in both cases the mapping $y \longmapsto G(t, y(\cdot))$ maps $\mathbf{H}_{1/2} \cap \mathbf{E}$ into $\mathbf{H}_{1/2}$ and $\mathbf{H}_1 \cap \mathbf{E}$ into $\mathbf{H}_1$ (note that $G(t,0) = 0$). For any $y \in \mathbf{H}_1 \cap \mathbf{E}$ such that $\sup_\xi |y(\xi)| < R$, we have

$$\begin{aligned}
& \langle A\sigma(t,y), \sigma(t,y) \rangle \\
= & \int_{\mathcal{O}} -G(t,y(\xi)) \sum_{i,j} \frac{\partial}{\partial \xi_i} (a_{ij}(\xi) \frac{\partial}{\partial \xi_j} G(t, y(\xi)))\, d\xi \\
= & -\int_{\partial\mathcal{O}} G(t,y(\xi))\, G'(t,y(\xi)) \frac{\partial}{\partial \nu_A} y(\xi) d_\xi S \\
(62) \qquad & + \int_{\mathcal{O}} \sum_{i,j} a_{ij}(\xi) \frac{\partial}{\partial \xi_i} G(t,y(\xi)) \frac{\partial}{\partial \xi_j} (G(t,y(\xi))\, d\xi \\
= & \int_{\mathcal{O}} (G'(t,y(\xi)))^2 \sum_{i,j} a_{ij}(\xi) \frac{\partial y(\xi)}{\partial \xi_i} \frac{\partial y(\xi)}{\partial \xi_j}\, d\xi \\
\leq & b_\sigma^2(t) \int_{\mathcal{O}} \sum_{i,j} a_{ij}(\xi) \frac{\partial y(\xi)}{\partial \xi_i} \frac{\partial y(\xi)}{\partial \xi_j} d\xi \\
= & b_\sigma^2(t) \langle Ay, y \rangle.
\end{aligned}$$

Note that the surface integral equals zero under both the Dirichlet and the Neumann conditions. Therefore, for each suitable $\hat\beta > 0$, we have

$$\begin{aligned}\|\hat A^{\frac12}\sigma(t,y)\|^2_{\mathbf H} &= \langle \hat A\sigma(t,y),\sigma(t,y)\rangle = \langle (A+\hat\beta I)\sigma(t,y),\sigma(t,y)\rangle \\ &\le b^2_\sigma(t)\langle Ay,y\rangle + \hat\beta b^2_\sigma(t)\|y\|^2_{\mathbf H} = b^2_\sigma(t)\langle \hat Ay,y\rangle = b^2_\sigma(t)\|\hat A^{\frac12}y\|^2_{\mathbf H}\end{aligned}$$

for $t \in \mathbf{R}_+$ and $y \in \mathbf{H}_1 \cap \mathbf{E}$ such that $\sup_\xi |y(\xi)| < R$. It is easy to see that the mapping $y \longmapsto \|\hat A^{\frac12}\sigma(t,y)\|^2_{\mathbf H}$ is continuous on $\mathbf{H}_{1/2}\cap\mathbf{E}$. Since $\mathbf{H}_1\cap\mathbf{E}$ is dense in $\mathbf{H}_{1/2}\cap\mathbf{E}$ (endowed with the $\mathbf{H}_{1/2}$-norm) we obtain $(\Sigma2)'$ with $\alpha = \frac12$ and either $r=\infty$ (if $R=\infty$ in (60)) or some $r \in\,]0,\infty[$ (if $R<\infty$ in (60) and the space dimension d equals 1, so that $\mathbf{H}_{1/2}$ is continuously embedded into $\mathcal{C}(\bar{\mathcal{O}})$ and the Nemytskii operator σ can be localized).

4.18 Example

Let us discuss again the Stochastic Parabolic Equation 1(5) with the Initial Condition 1(11) and boundary conditions of either Dirichlet's 1(12) or Neumann's 1(13) type, but with a "scalar" noise η. Again, we assume that the coefficients F and G are measurable and satisfy $F(t,0) = G(t,0) = 0$, $t\ge 0$, and that the Conditions 1(7) – 1(10) are satisfied. This stochastic parabolic problem can be rewritten in the form 1(1) with the spaces $\mathbf H$, $\mathbf E$, the operator A, and the functional f as in Example 4.14. The Wiener process W is standard one-dimensional, so we put $\mathbf K = \mathbf R$ and

$$(63)\quad \sigma : \mathbf{R}_+\times\mathbf{E}\to\mathbf{H},\ \sigma(t,y)(\xi) := G(t,y(\xi)),\qquad (t,y)\in\mathbf{R}_+\times\mathbf{E},\ \xi\in\mathcal{O}.$$

It is obvious that for $\alpha = 0$ the Conditions $(\Sigma1)$ and $(\Sigma2)'$ are satisfied with $k_\sigma(t) = b_\sigma(t) \equiv k_G$. Thus, it is easy to establish $\mathbf H$-stability results: for example, if F has the form (40) and

$$(64)\quad -\beta + \sup_{t\ge0} a_f(t) + \frac12 k_G^2 < 0$$

holds then, by Proposition 4.12, the trivial solution $X^0(t)\equiv 0$ of the equation under consideration is globally asymptotically $\mathbf H$-stable in probability and exponentially $\mathbf H$-stable in the $2p$th mean for each $p\in\,]0,1]$.

However, it is desirable to find some stability in a norm stronger than the $\mathbf H$-norm. At first we must verify the corresponding regularity. We proceed as in Example 4.14 to show that $J_1\in\mathcal{C}([0,T],\mathbf{H}_\alpha)$ and

$$E\int_0^T \|J_1(t)\|^{2\nu}_\alpha\,dt < \infty$$

for $\alpha\in[0,1-\frac{1}{2\nu}[$. Note that, in the present case,

$$(65)\quad \|\sigma(t,y)\|^\nu_\delta \le k_G\|y\|^\nu_\delta,\quad (t,y)\in\mathbf{R}_+\times(\mathbf{H}_\delta\cap\mathbf{E}),\ 0\le\delta<1/2,$$

so, mimicking the arguments in Example 4.14, we see that X^x has continuous trajectories in $\mathbf H_\alpha$ and

$$(66)\quad E\int_0^T \|X^x(t)\|^2_\alpha\,dt < \infty$$

for $x\in\mathbf{H}_\alpha\cap\mathbf{L}_{2\nu^2}(\mathcal{O})$ and $0\le\alpha<1-\frac1\nu$, cf. (49). We aim at showing $\mathbf{H}_{\frac12}$-stability (or, equivalently, $H^1(\mathcal{O})$-stability) of the trivial solution in the present case. Assume that $\nu>2$ and $d\le 3$ (so that $\alpha=\frac12>\frac d8$) and that F is of the form

$$(67)\quad F(t,u) = a_f(t)u - c(t)u^q,\quad (t,u)\in\mathbf{R}_+\times[0,R[,$$

for some $R \in]0,\infty]$, $a_f : \mathbf{R}_+ \to \mathbf{R}$, $c : \mathbf{R}_+ \to \mathbf{R}_+$, $1 \le q \le \nu$. Assume, moreover, that $G(t,\cdot) \in \mathcal{C}^2([0,R[)$ and

$$(68)\quad \sqrt{\kappa_\sigma(t)} \le |G'(t,u)| \le b_\sigma(t), \quad (t,u) \in \mathbf{R}_+ \times [0,R[,$$

for two locally-bounded functions κ_σ, $b_\sigma : \mathbf{R}_+ \to \mathbf{R}_+$.

We first deal with the case $R = \infty$ in (67), (68). By Example 4.6, the function f is $\frac{1}{2}$-dissipative on $\mathbf{R}_+ \times (\mathbf{H}_{\frac{1}{2}} \cap \mathbf{E})$ and, by Example 4.17, Condition $(\Sigma 2)'$ is satisfied with $\alpha = \frac{1}{2}$, $r = \infty$. In view of $(\Sigma 3)'$, for any suitable $\hat{\beta} \ge 0$ and $\hat{A} = A + \hat{\beta} I$, we have

$$(69)\quad \begin{aligned} &\langle \hat{A}^{\frac{1}{2}} y,\ \hat{A}^{\frac{1}{2}} \sigma(t,y)\rangle^2 \\ =\ & \langle (A+\hat{\beta} I)y, \sigma(t,y)\rangle^2 \\ =\ & \langle Ay, \sigma(t,y)\rangle^2 + |\hat{\beta}|^2 \langle y, \sigma(t,y)\rangle^2 + 2\hat{\beta}\langle Ay, \sigma(t,y)\rangle \langle y, \sigma(t,y)\rangle, \end{aligned}$$

$y \in \mathbf{H}_1 \cap \mathbf{E} = \mathcal{D}(A) \cap \mathbf{E}$. Moreover,

$$\begin{aligned} \langle Ay, \sigma(t,y)\rangle^2 &= \left(\int_{\mathcal{O}} -G(t,y(\xi)) \sum_{i,j} \frac{\partial}{\partial \xi_i} (a_{ij}(\xi) \frac{\partial}{\partial \xi_j} y(\xi)) d\xi \right)^2 \\ &= \left(\int_{\mathcal{O}} G'(t,y(\xi)) \sum_{i,j} a_{ij}(\xi) \frac{\partial}{\partial \xi_i} y(\xi) \frac{\partial}{\partial \xi_j} y(\xi) d\xi \right)^2 \\ &\ge \kappa_\sigma(t) \|A^{\frac{1}{2}} y\|^4_{\mathbf{H}}, \end{aligned}$$

cf. (62), and clearly

$$\begin{aligned} &2\hat{\beta}\langle Ay, \sigma(t,y)\rangle\langle y, \sigma(t,y)\rangle \ge 2\hat{\beta}\kappa_\sigma(t)\langle Ay, y\rangle \|y\|^2_{\mathbf{H}} \qquad \text{and} \\ &\hat{\beta}^2 \langle y, \sigma(t,y)\rangle^2 \ge \hat{\beta}^2 \kappa_\sigma(t) \|y\|^4_{\mathbf{H}}, \qquad (t,y) \in \mathbf{R}_+ \times \mathbf{H}_1. \end{aligned}$$

Hence, by a similar continuity argument as in Example 4.17, we arrive at

$$\langle \hat{A}^{\frac{1}{2}} y, \hat{A}^{\frac{1}{2}} \sigma(t,y)\rangle^2 \ge \kappa_\sigma(t) \|\hat{A}^{\frac{1}{2}} y\|^4_{\mathbf{H}}, \quad (t,y) \in \mathbf{R}_+ \times (\mathbf{H}_{\frac{1}{2}} \cap \mathbf{E}),$$

i.e., $(\Sigma 3)'$ with $\alpha = \frac{1}{2}$. It now follows from Proposition 4.15 that the trivial solution is globally asymptotically $\mathbf{H}_{\frac{1}{2}}$–stable (or, equivalently, $H^1(\mathcal{O})$-stable) in probability and exponentially $\mathbf{H}_{\frac{1}{2}}$-stable in the $2p$th mean, $0 < p < 1$, if

$$(70)\quad -\beta + \sup_{t \ge 0} \{a_f(t) + \frac{1}{2} b^2_\sigma(t) - (1-p)\kappa_\sigma(t)\} < 0.$$

If the Conditions (67), (68) are satisfied with some $R \in]0,\infty[$, only, and if the dimension d is 1 (so that $\mathbf{H}_{\frac{1}{2}}$ is continuously embedded into $\mathcal{C}(\bar{\mathcal{O}})$) then we have 1/2-dissipativity, $(\Sigma 2)'$, and $(\Sigma 3)'$ on the set $\mathbf{R}_+ \times (B_{\mathbf{H}_{\frac{1}{2}}}(r) \cap \mathbf{E})$ with some $r \in]0,\infty[$, only. In this case (70) yields (local) $\mathbf{H}_{\frac{1}{2}}$-stability in probability. In particular, if the coefficients F and G do not depend on $t \in \mathbf{R}_+$ (so that $F(u) = a_f u - cu^q$ for some $a_f \in \mathbf{R}, c \in \mathbf{R}_+$) then the Condition (70) for asymptotical $\mathbf{H}_{\frac{1}{2}}$-stability in probability has the simple form

$$(71)\quad -\beta + a_f - \frac{1}{2}(G'(0))^2 < 0$$

and (68) is valid with a possibly different number R.

4.19 A Stochastic Logistic Equation of Population Dynamics

Consider the formal equation of the form 1(5), 1(6), 1(22) (cf. also Example 4.14)

$$\frac{\partial u}{\partial t}(t,\xi) = \frac{\partial}{\partial \xi}(a(\xi)\frac{\partial}{\partial \xi}u(t,\xi)) + a_f u(t,\xi) - cu^2(t,\xi) + G(t,u(t,\xi))\eta(t,\xi), \tag{72}$$

$(t,\xi) \in \mathbf{R}_+ \times]0,1[$, with the initial and boundary conditions

$$u(0,\xi) = u_0(\xi) \geq 0, \quad \frac{\partial u}{\partial \xi}(t,0) = \frac{\partial u}{\partial \xi}(t,1) = 0. \tag{73}$$

Here, $a \in \mathcal{C}^\infty([0,1])$, $a \geq a_0 > 0$, $a_f > 0$, $c > 0$, $G : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$ satisfies $G(t,0) = 0$, and η stands for a scalar noise. The formal problem (72), (73) is a stochastic counterpart of a logistic population-growth model with migration (cf., e.g., Murray [26]). It is a particular case of Example 4.18. Assume that $G(t,\cdot) \in \mathcal{C}^2([0,R[)$ and that

$$\sqrt{\kappa_\sigma(t)} \leq |G'(t,u)| \leq b_\sigma(t), \quad (t,u) \in \mathbf{R}_+ \times]0,R[, \tag{74}$$

for some $0 < R < \infty$, where $\kappa_\sigma : \mathbf{R}_+ \to \mathbf{R}_+$ and $b_\sigma : \mathbf{R}_+ \to \mathbf{R}_+$ are locally bounded and

$$\sup_{t \geq 0}\{\frac{1}{2}b_\sigma^2(t) - (1-p)\kappa_\sigma(t)\} < -a_f \tag{75}$$

for some $0 < p \leq 1$. Then Example 4.18 shows that the zero solution is asymptotically $H^1([0,1])$-stable in probability. If $G(t,u) = G(u)$ does not depend on $t \in \mathbf{R}_+$ then Condition (75) becomes

$$a_f < \frac{1}{2}(G'(0))^2; \tag{76}$$

cf. (71). Thus, if $|G'(0)|$ is large enough, that is, if the "diffusion" is large in a certain sense then the zero solution can become stable, although it is unstable for Equation (72) without the stochastic term.

Now assume also $G(t, a_f/c) \equiv 0$, $t \in \mathbf{R}_+$; from the viewpoint of applications it is sometimes interesting when the diffusion stabilizes near the level of saturation $x_1 = a_f/c$. Then the formal equation (72) has another trivial stationary solution, namely the constant $X^{x_1} \equiv x_1$. Stability of this trivial solution can be analyzed by the method used above for the zero solution: After the simple transformation of variables $w = x_1 - u$, Equation (72) becomes the equation

$$\frac{\partial w}{\partial t}(t,\xi) = \frac{\partial}{\partial \xi}(a(\xi)\frac{\partial}{\partial \xi}w(t,\xi)) + a_f w(t,\xi) + cw^2(t,\xi) - G(t,x_1 - w(t,\xi))\eta(t,\xi),$$

$(t,\xi) \in \mathbf{R}_+ \times]0,1[$, and X^{x_1} is transformed into the zero solution. Let us, e.g., consider the autonomous case $G(t,u) \equiv G(u)$. Since

$$-a_f < \frac{1}{2}(G'(x_1))^2 \tag{77}$$

is always true another application of Formula (71) shows, that the trivial solution $X^{x_1}(t) \equiv x_1$ is always asymptotically $H^1(0,1)$-stable in probability. The behaviour of (72) at saturation is, however, different from that at 0 since, in the former case, the drift itself is stabilizing whereas in the latter case it is destabilizing.

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REFERENCES

[1] T. Caraballo and J. Real, On the pathwise exponential stability of nonlinear stochastic PDE's, Stoch. Anal. Appl. 12 (1994), 517–529

[2] P.L. Chow, Stability of nonlinear stochastic evolution equations, J. Math. Anal. Appl. 89 (1982), 400–419

[3] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge Univ. Press, Cambridge, 1992

[4] C. Dellacherie and P.-A. Meyer, Probabilities and Potential, North-Holland, Amsterdam, 1978

[5] D.E. Edmunds and H. Triebel, Entropy numbers and approximation numbers in function spaces, Proc. London Math. Soc. 58 (1989), 137–152

[6] M.W. Feldman and J. Roughgarden, A population stationary distribution and chance of extinction in a stochastic environment with remarks on the theory of species packing, Theoret. Popul. Biol. 7 (1975), 197–207

[7] D. Fujiwara, Concrete characterizations of the domains of fractional powers of some elliptic differential operators of second order, Proc. Japan Acad. Ser. A. Math. Sci. 43 (1967), 82–86

[8] T.C. Gard, Stochastic population models on a bounded domain: exit probabilities and persistence, Stochastics Stochastics Rep. 33 (1990), 63–73

[9] P. Grindrod, Patterns and Waves, The theory and applications of reaction-diffusion equations, Clarendon Press, Oxford, 1991

[10] A. Ichikawa, Absolute stability of a stochastic evolution equation, Stochastics 11 (1983), 143–158

[11] A. Ichikawa, Semilinear stochastic evolution equations: boundedness, stability and invariant measures, Stochastics 12 (1984), 1–39

[12] R.Z. Khas'minskii, Stochastic Stability of Differential Equations, Nauka, Moscow, 1969 (in Russian, English translation: Sijthoff and Noordhoff, Alphen aan den Rijn, 1980)

[13] R.Z. Khas'minskii, V. Mandrekar, On the stability of solutions of stochastic evolution equations. The Dynkin Festschrift, 185–195, Birkhäuser, Boston 1994

[14] P. Kotelenez, Comparison methods for a class of function valued stochastic partial differential equations, Probab. Theory Relat. Fields 93 (1992), 1–19

[15] A. Kufner, O. John, and S. Fučik, Function Spaces, Academia, Prague, 1977

[16] G. Leha, G. Ritter, Lyapunov-type conditions for stationary distributions of diffusion processes on Hilbert spaces, Stochastics Stochastics Rep. 48 (1994), 195–225

[17] G. Leha, G. Ritter, Lyapunov Functions and Stationary Distributions of Stochastic Evolution Equations, in preparation. (Appeared in Stochastic Analysis and Applications 21, 763–799 (2003))

[18] G. Leha, G. Ritter, and A. Wakolbinger, An improved Lyapunov-function approach to the behavior of diffusion process in Hilbert spaces, Stoch. Anal. Appl. 15 (1997), 59–89

[19] Z.-M. Ma and M. Röckner, Introduction to the Theory of (Non-Symmetric) Dirichlet Forms, Springer, Berlin, 1992

[20] V. Mandrekar, On Lyapunov stability theorems for stochastic (deterministic) evolution equations, in Stoch. Anal. Appl. in Physics, 219–237, Kluwer Acad. Publ., Dordrecht 1994

[21] R. Manthey and C. Stiewe, Existence and uniqueness of solutions of Volterra's population equation with diffusion and noise, Stochastics Stochastics Rep. 41 (1992), 135–161

[22] R. Manthey and T. Zausinger, Stochastic evolution equations in $L_p^{2\nu}$, to appear in Stochastics Stochastics Rep.

[23] B. Maslowski, J. Seidler, and I. Vrkoč, An averaging principle for stochastic evolution equations II, Mathematica Bohemica 116 (1991), 191–224

[24] B. Maslowski, J. Seidler and I. Vrkoč, Integral continuity for stochastic hyperbolic equations, Differential Integral Equations 6 (1993), 355–382

[25] R.M. May, Stability in randomly fluctuating versus deterministic environment, Amer. Natur. 107 (1973), 621–650

[26] J.D. Murray, Mathematical Biology, Biomathematics 19, Springer, 1989

[27] M. Reed and B. Simon, Methods of Modern Mathematical Physics I: Functional Analysis, Academic Press, New York, 1980

[28] F.S. Riesz and B.Nagy, Leçons d'analyse fonctionelle, Akad. Kiado, Budapest, 1952

[29] J. Roughgarden, Theory of Population Genetics and Evolutionary Ecology: An Introduction, Macmillan Publishing Inc., New York, 1970

[30] L. Schwartz, Radon measures on arbitrary topological spaces and cylindrical measures, Oxford University Press, 1973

[31] R. Seeley, Interpolation in L^p with boundary conditions, Studia Math. 44 (1972), 47–60

[32] J. Seidler, Da Prato-Zabczyk's maximal inequality revisited I, Mathematica Bohemica 118 (1993), 67–106

[33] J.A. Smoller, Shock Waves and Reaction Diffusion Equations, Springer-Verlag, New York, 1983

[34] T. Taniguchi, Asymptotic stability theorems of semilinear stochastic evolution equations in Hilbert spaces, Stochastics Stochastics Rep. 53 (1995), 41–52

[35] H. Triebel, Interpolation theory, function spaces, differential operators, Deutscher Verlag der Wissenschaften, Berlin, 1978

[36] M. Turelli, Random environments and stochastic calculus, Theoret. Popul. Biol. 12 (1977), 140–178

[37] M. Viot, Solution faibles d'équations aux derivées partielles non linéaires, Thèse, Université Pierre et Marie Curie, Paris, 1976