Appeared in Symposia Mathematica XXII, Analisi armonica e spazi di funzioni su gruppi localmente compatti (1977), 21–31

The Orlicz–Paley–Sidon Phenomenon for Singular Measures*

Edwin Hewitt and Gunter Ritter

1. Introduction.

This note is concerned with the question "how small can the Fourier–Stieltjes transform of a singular measure be?," a question which we have also studied in another communication [5]. There is a companion question "how large can the Fourier transform of a continuous function with compact support be?." This question too we have studied in another communication [6].

Throughout this note, the symbol G will denote a nondiscrete locally compact abelian group and X will denote the character group of G. Haar measure on G is denoted by λ and on X by θ. The symbol $\mathbf{M}(G)$ will denote the space of all measures on G, defined as in [7], §19. The symbols $\mathbf{M}_d(G)$, $\mathbf{M}_c(G)$, $\mathbf{M}_s(G)$, and $\mathbf{M}_a(G)$ denote the discrete, continuous, continuous singular (with respect to Haar measure), and absolutely continuous (with respect to Haar measure) measures, respectively, in $\mathbf{M}(G)$. The nonnegative (real) measures in a set $\mathbf{B}$ of measures are denoted by $\mathbf{B}^+$ ($\mathbf{B}^r$). The nonnegative functions in a space $\mathfrak{F}$ of functions are similarly denoted by $\mathfrak{F}^+$.

The *Fourier–Stieltjes transform* of a measure $\mu \in \mathbf{M}(G)$ is the function $\widehat{\mu}$ on X such that

$$\widehat{\mu}(\chi) = \int_G \overline{\chi(t)} d\mu(t) \qquad \text{for } \chi \in X.$$

For $f \in \mathfrak{L}_1(G)$, we identify f and the measure $\mu \in \mathbf{M}(G)$ such that $d\mu = f d\lambda$, we write $\widehat{f}$ for the Fourier transform of f:

$$\widehat{f}(\chi) = \int_G \overline{\chi(t)} f(t) d\lambda(t) \qquad \text{for } \chi \in X.$$

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the positive integers, the integers, the real numbers, and the complex numbers, respectively.

For a topological space Y, the symbols $\mathfrak{C}(Y)$, $\mathfrak{C}_0(Y)$, and $\mathfrak{C}_k(Y)$ denote respectively: the space of all bounded complex–valued continuous functions on Y; the space of all $f \in \mathfrak{C}(Y)$ that are arbitrarily small in absolute value outside of compact sets; and the space of all $f \in \mathfrak{C}(Y)$ that vanish outside of compact sets. If Y is discrete, we write $\mathfrak{C}_0(Y)$ as $\mathfrak{c}_0(Y)$.

The symbol $\mathbf{S}(G)$ denotes the set of all $\sigma \in \mathbf{M}_s^+(G)$ with compact support such that $\sigma(G) = 1$ and $\widehat{\sigma} \in \mathfrak{C}_0^+(X)$.

2. Singular transforms that are pointwise small.

Wiener and Wintner [14] constructed a probability measure $\sigma \in \mathbf{M}_s^+(\mathbb{T})$ ($\mathbf{M}_s^+(\mathbb{R})$) such that for every $\varepsilon > 0$, the relation

$$\widehat{\sigma}(n) = O(|n|^{-\frac{1}{2}+\varepsilon}) \quad |n| \to \infty, \quad n \in \mathbb{Z}$$
$$\big(\widehat{\sigma}(t) = O(|t|^{-\frac{1}{2}+\varepsilon}) \quad |t| \to \infty, \quad t \in \mathbb{R}\big)$$

holds. They conjectured that if $\gamma_1, \gamma_2, \ldots$ are given positive numbers for which $\sum \gamma_n^2 = \infty$, then there exists a singular nonnegative measure σ for which $\widehat{\sigma}(n) = O(\gamma_n)$ or $\widehat{\sigma}(n) = o(\gamma_n)$ (*loc.cit.*, p. 514). This conjecture is incorrect, as the following construction shows.

*I risultati conseguiti in questo lavoro sono stati esposti da E. Hewitt nella conferenza tenuta il 25 marzo 1976.

(2.1) Theorem: There exists a strictly positive sequence $(\gamma_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty \gamma_n^2 = \infty$ and such that if $\mu \in \mathbf{M}^r(\mathbb{T})$ and $|\widehat{\mu}(n)| \le \gamma_n$ for all $n \in \mathbb{N}$, then μ is absolutely continuous (with Radon–Nikodym derivative in $\mathfrak{L}_2(\mathbb{T})$).

Proof: First, let F be a Sidon set contained in $\mathbb{Z}$ and suppose that ν is a measure in $\mathbf{M}(\mathbb{T})$ such that $\widehat{\nu}(n) = 0$ for $n \in \mathbb{Z} \setminus F$. As Professor Alessandro Figà–Talamanca has kindly pointed out to us, this implies that $\widehat{\nu} \in l_2(\mathbb{Z})$. For, let α be any function on $\mathbb{Z}$ of absolute value 1. Since F is a Sidon set, there is a measure $\varrho \in \mathbf{M}(\mathbb{T})$ such that $\widehat{\varrho}|_F = \alpha|_F$. Since $\widehat{\nu}$ vanishes on $\mathbb{Z} \setminus F$, we have

$$\widehat{\nu \star \varrho} = \alpha \widehat{\nu}.$$

That is, $\alpha\widehat{\nu}$ is a Fourier–Stieltjes transform for all choices of α. It is well known (see for example [7], (36.13)) that this property implies that $\widehat{\nu} \in l_2(\mathbb{Z})$ and so in particular ν is absolutely continuous.

To define $(\gamma_n)_{n=1}^\infty$, let D be any infinite Sidon set contained in $\mathbb{N}$, and let γ_n be 1 (say) for $n \in D$. On the set $\mathbb{N} \setminus D$, let γ_n be strictly positive and such that $\sum_{n \in \mathbb{N} \setminus D} \gamma_n^2 < \infty$. Let μ be any measure in $\mathbf{M}^r(\mathbb{T})$ such that $|\widehat{\mu}(n)| \le \gamma_n$ for all $n \in \mathbb{N}$. Write F for the set $D \cup (-D)$. By a theorem of Drury [2], F is a Sidon set. Since $\widehat{\mu}(-n) = \overline{\widehat{\mu}(n)}$, the function $\widehat{\mu} \cdot 1_{\mathbb{Z}\setminus F}$ is in $l_2(\mathbb{Z})$ and so has the form $\widehat{g}$ for some g in $\mathfrak{L}_2(\mathbb{T})$.

The function $\widehat{\mu} - \widehat{g}$ vanishes on $\mathbb{Z} \setminus F$ and as noted above is therefore in $l_2(\mathbb{Z})$. Thus $\widehat{\mu}$ is in $l_2(\mathbb{Z})$, so that μ is absolutely continuous with Radon–Nikodym derivative in $\mathfrak{L}_2(\mathbb{T})$.

Some additional conditions must be imposed on $(\gamma_n)_{n=1}^\infty$ in order to establish the conjecture of Wiener and Wintner. The most refined result in this direction that we know of is due to Ivašev–Musatov [9]. We state a typical case of his theorem for measures on $\mathbb{R}$. For every $p \in \mathbb{N}$, there is a probability measure $\sigma \in \mathbf{M}_s^+(\mathbb{R})$ with support a perfect set E of Lebesgue measure 0 (E may be contained in $[0, 2\pi]$ or it may be unbounded) such that

$$\widehat{\sigma}(y) = o\Big(\big(y \log(y) \log(\log(y)) \ldots \log_{(p)}(y)\big)^{-\frac{1}{2}}\Big) \qquad y \to \infty. \tag{1}$$

We know of no complete resolution of the Wiener–Wintner conjecture.

3. **Singular transforms that are small on the average.**

For behavior "on the average" of singular Fourier–Stieltjes transforms, however, the situation is much clearer. First, in the communication [5], we have proved the following.

(3.1) Theorem: (a) Let r be a real number greater than or equal to 2. There is a measure $\sigma \in \mathbf{S}(G)$ such that $\widehat{\sigma} \in \mathfrak{L}_p(X)$ if and only if $p > r$.

(b) Let r be a real number strictly greater than 2. There is a measure $\sigma \in \mathbf{S}(G)$ such that $\widehat{\sigma} \in \mathfrak{L}_p(X)$ if and only if $p \ge r$.

(c) There is a measure $\sigma \in \mathbf{S}(G)$ such that $\widehat{\sigma}$ is in no space $\mathfrak{L}_p(X)$, $1 \le p < \infty$.

We also have proved in [5] the following improvement on (3.1) Part (a) for $r = 2$.

(3.2) Theorem: Let φ be a nonnegative, Borel measurable, locally bounded function defined on $]0, \infty[$ with the property that $\lim_{s\to\infty} \varphi(s) = 0$. Then there exists a measure σ in $\mathbf{S}(G)$ such that

(i) $\int_X \widehat{\sigma}(\chi)^2 \varphi(\widehat{\sigma}(\chi)) d\theta(\chi) < \infty$.

(3.3) Remarks: (a) Theorem (3.2) is a counterpart for singular measures of Gronwall's theorem [4] for Fourier transforms of continuous functions on $\mathbb{T}$, which we have extended in [6] to all groups G, using $\mathfrak{C}_k(G)$. Nothing like (3.2) appears in Gronwall *loc.cit.*, however.

(b) In (3.2) we need only set $\varphi(s) = \big(\log(1 + 1/s)\big)^{-1}$ to obtain (3.1) Part (a) with $r = 2$.

4. **Statement of present results.**

A classical result of Orlicz [11], Paley [12], and Sidon [13], later extended by R.E. Edwards [3], asserts that $l_2(X)$ is the multiplier set for $\mathfrak{C}(G)$ if G is compact. The exact result is as follows.

(4.1) THEOREM: Let G be a compact group, so that X is discrete. If w is any complex function on X such that $w\widehat{f} \in l_1(X)$ for all $f \in \mathfrak{C}(G)$, then w must be in $l_2(X)$.

The analogue of (4.1) for compact not necessarily abelian groups also holds, and can be found for example in [7], (36.12).

We may ask if any "inverse" phenomenon occurs with singular measures. Our first result is a somewhat surprising affirmative answer.

(4.2) THEOREM: Let G be a nondiscrete locally compact abelian group, and let w be any function in $\mathfrak{L}_2^+(X)$. There exists a measure $\sigma \in \mathbf{S}(G)$ such that

(i) $\int_X \widehat{\sigma}(\chi)w(\chi)d\theta(\chi) < \infty$.

For compact G, we can say even more.

(4.3) THEOREM: Let G be a compact abelian group and let w be any function in $\mathfrak{c}_0^+(X)$. There is a measure $\sigma \in \mathbf{S}(G)$ such that

(i) $\sum_{\chi \in X} \widehat{\sigma}(\chi)w(\chi) < \infty$.

5. **Proof of** (4.2).

We first prove a measure–theoretic lemma. In (5.1), let (Y, A, μ) be an arbitrary measure space, i.e., a set Y, a σ–algebra A of subsets of Y, and a countably additive nonnegative (possibly infinite) measure μ on A. Write $\mathfrak{L}_2$ for $\mathfrak{L}_2(Y, A, \mu)$.

(5.1) LEMMA: Let w be a function in $\mathfrak{L}_2^+$. There exists a function Ψ defined on $]0, \infty[$ with values in $]0, \infty[$ such that

(i) Ψ is continuous;

(ii) Ψ is strictly increasing;

(iii) $\lim_{y \to 0} \Psi(y) = 0$;

(iv) $\lim_{y \to 0} \Psi(y) \cdot y^{-1} = \infty$;

(v) $\lim_{y \to 0} \Psi(y) = \infty$;

(vi) $\int_Y w(t)\Psi(w(t))d\mu(t) < \infty$.

PROOF: For all $n \in \mathbb{N}$, let $A_n = \{y \in Y : (n+1)^{-1} \leq w(y) < n^{-1}\}$. It is clear that

$$\sum_{n=1}^{\infty}(n+1)^{-2}\mu(A_n) \leq \sum_{n=1}^{\infty} \int_{A_n} w^2(t)d\mu(t) \leq \int_Y w^2(t)d\mu(t) < \infty, \tag{1}$$

and since

$$\frac{1}{n^2} \leq \frac{4}{(n+1)^2},$$

(1) implies that

$$\sum_{n=1}^{\infty} n^{-2}\mu(A_n) < \infty. \tag{2}$$

A theorem of Dini (see e.g. [10], §39, paragraph 15, p. 302) shows that there is a nondecreasing sequence $(\gamma_n)_{n\in\mathbb{N}}$ of positive real numbers such that

$$\lim_{n\to\infty} \gamma_n = \infty \tag{3}$$

and

$$\sum_{n=1}^{\infty} \gamma_n n^{-2}\mu(A_n) < \infty. \tag{4}$$

Now define a sequence $(\beta_n)_{n\in\mathbb{N}}$ of positive real numbers by induction. Let $\beta_1 = \gamma_1$, and suppose that $\beta_1, \ldots, \beta_{n-1}$ have been defined. Then we define

$$\beta_n = \min(\gamma_n, n^{\frac{1}{2}}, (n + \frac{3}{2})(n+1)^{-1}\beta_{n-1}).$$

It is obvious that $(\beta_n)_{n\in\mathbb{N}}$ is nondecreasing, that

$$\beta_n \leq n^{\frac{1}{2}}, \tag{5}$$

$$\beta_n/\beta_{n-1} < (n+2)/(n+1), \tag{6}$$

and from (4) that

$$\sum_{n=1}^{\infty} \beta_n n^{-2}\mu(A_n) < \infty. \tag{7}$$

The only way for $(\beta_n)_{n\in\mathbb{N}}$ possibly to be bounded is for β_n to be $(n+\frac{3}{2})(n+1)^{-1}\beta_{n-1}$ for all $n \geq k$ for some fixed index k. Then we have

$$\begin{aligned}\beta_{k+l} =& \prod_{j=1}^{l+1}\left(1 + \frac{1}{2}\frac{1}{k+j}\right)\beta_{k-1} \\ >& \frac{1}{2}\left(\frac{1}{k+1} + \cdots + \frac{1}{k+l+1}\right)\beta_{k-1}.\end{aligned}$$

We thus have

$$\lim_{n\to\infty} \beta_n = \infty. \tag{8}$$

Now for all $n \in \mathbb{N}$, define

$$\psi\left(\frac{1}{n+1}\right) = \beta_n,$$

interpolate ψ as a linear function in each interval $]1/(n+1), 1/n[$ $(n \in \{2, 3, \ldots\})$, and for $y > \frac{1}{2}$ let $\psi(y) = \beta_1(2y)^{-\frac{1}{2}}$. Finally, define the function Ψ on $]0, \infty[$ by

$$\Psi(y) = y\psi(y). \tag{9}$$

Clearly Ψ is continuous. A short computation using (6) shows that Ψ is strictly increasing: we omit the details. It follows from (5) and the monotonicity of Ψ that $\lim_{y\to 0}\Psi(y) = 0$. Relation (iv) follows from (8) and the definitions of ψ and Ψ while (v) is immediate.

It remains to establish (vi). Write B for the set $\{t \in Y : w(t) \geq 1\}$, and then estimate as follows:

$$\begin{aligned}\int_Y w\Psi(w)d\mu &= \int_Y w^2\psi(w)d\mu \\ &= \int_B w^2\psi(w)d\mu + \sum_{n=1}^{\infty}\int_{A_n} w^2\psi(w)d\mu \\ &\leq \psi(1)\int_Y w^2 d\mu + \sum_{n=1}^{\infty}\frac{1}{n^2}\mu(A_n)\beta_n \\ &< \infty.\end{aligned}$$

(5.2) Completion of Proof: For the measure space of Lemma (5.1), take the character group X of G, the σ–algebra of Borel sets, and Haar measure θ on X. Given a function w in $\mathfrak{L}_2^+(X)$, construct the function Ψ as in (5.1), and further define $\Psi(0) = 0$. Thus Ψ is continuous on $[0, \infty[$. Let Φ be the inverse function to $\Psi : \Psi(\Phi(x)) = x$ for all $x \in [0, \infty[$. For $x \in]0, \infty[$, let $\varphi(x) = \Phi(x)/x$. For $x \in]0, \infty[$, let y be the (unique) number such that $x = \Psi(y)$. Then we have

$$\varphi(x) = \Phi(x)/x = y/\Psi(y)$$

and (5.1.iv) implies that

$$\lim_{x \to 0} \varphi(x) = 0. \tag{1}$$

Note next that for arbitrary numbers a, b in $[0, \infty[$, the inequality

$$ab \leq a\Psi(a) + b\Phi(b) \tag{2}$$

obtains (a simple sketch suffices to verify this).

Finally, we cite Theorem (3.2) to produce a measure $\sigma \in \mathbf{S}(G)$ such that

$$\int_X \widehat{\sigma}(\chi)^2\varphi(\widehat{\sigma}(\chi))d\theta(\chi) < \infty, \tag{3}$$

where φ is the function defined in the preceding paragraph. For every $\chi \in X$, (2) shows that

$$\begin{aligned} w(\chi)\widehat{\sigma}(\chi) &\leq w(\chi)\Psi(w(\chi)) + \widehat{\sigma}(\chi)\Phi(\widehat{\sigma}(\chi)) \\ &= w(\chi)\Psi(w(\chi)) + \widehat{\sigma}(\chi)^2\varphi(\widehat{\sigma}(\chi)). \end{aligned} \tag{4}$$

Integrate the inequality (4) over X and apply (3) and (5.1.vi). This shows that

$$\int_X w(\chi)\widehat{\sigma}(\chi)d\theta(\chi) < \infty,$$

as we wished to prove.

6. **Proof of** (4.3).

Throughout this section, we take G to be compact (and infinite), so that X is discrete. We prove first a group–theoretic fact.

(6.1) Lemma: Let X be an infinite abelian group and $(\Phi_n)_{n=1}^{\infty}$ a sequence of finite subsets of X. There is a countably infinite dissociate subset $(\chi_n)_{n=1}^{\infty}$ of X such that for all $n \in \mathbb{N}$,

(i) no product $\chi_1^{\varepsilon_1}\chi_2^{\varepsilon_2}\ldots\chi_{n-1}^{\varepsilon_{n-1}}\chi_n^{\pm 1}$ lies in Φ_n,

where $\varepsilon_j \in \{-1, 0, 1\}$ for $1 \leq j \leq n-1$.

PROOF: *Case I*: the set $\{\chi^2 : \chi \in X\}$ is infinite. The proof is by induction. Let χ_1 be any element not in $\{1\} \cup \Phi_1 \cup \Phi_1^{-1}$. Suppose that $\chi_1, \dots, \chi_{n-1}$ have been chosen so as to be dissociate and to satisfy (i) with n replaced by $n-1$. Let A_n be the set of all products

$$\psi_1^{\delta_1}\psi_2^{\delta_2}\dots\psi_n^{\delta_n} \tag{1}$$

where $\delta_j \in \{-4,-3,-2,-1,0,1,2,3,4\}$, and $\{\psi_1, \psi_2, \dots, \psi_n\}$ is an arbitrary subset of $\{\chi_1, \chi_2, \dots, \chi_{n-1}\} \cup \Phi_n$. The set A_n is finite. Under our hypothesis, there is an element $\chi_n \in X$ such that $\chi_n^2 \notin A_n$. If

$$\chi_1^{\varepsilon_1}\dots\chi_{n-1}^{\varepsilon_{n-1}}\chi_n^{\pm 1} \in \Phi_n,$$

with $\varepsilon_j \in \{-1,0,1\}$, then χ_n^2 is in A_n. Thus (i) holds with this choice of χ_n. To prove that $\{\chi_1, \dots, \chi_n\}$ is dissociate, suppose that

$$\chi_1^{m_1}, \chi_2^{m_2}\dots\chi_n^{m_n} = 1$$

with $m_j \in \{-2,-1,0,1,2\}$. If $m_n \neq 0$, then χ_n^2 is obviously a product (1). Hence $m_n = 0$ and $\chi_1^{m_1} = \dots = \chi_{n-1}^{m_{n-1}} = 1$ by our inductive hypothesis.

Case II: the set $\{\chi^2 : \chi \in X\}$ is finite. In this case X contains a subgroup Γ isomorphic with the direct sum $P^{*\infty}_{n=1}\mathbf{Z}(2)$, which we realize as the group D of all sequences $\mathbf{x} = (x_1, x_2, \dots)$ with $x_n \in \{0,1\}$ and x_n eventually zero. (See [8].) Replacing Φ_n by $\Phi_n \cap \Gamma$, we may suppose that $X = D$. For $n \in \mathbb{N}$, let $e^{(n)}$ be the element of D such that $e_k^{(n)} = \delta_{nk}$ (Kronecker's δ–function). For $n \in \mathbb{N}$, let

$$l_n = \max\{l \in \mathbb{N} : x_l = 1 \text{ for some } x \in \Phi_n\}.$$

We define a sequence $(s_n)_{n=1}^{\infty}$ of positive integers by induction. Let $s_1 = l_1 + 1$. When $s_1, \dots, s_{n-1}$ have been defined, let

$$s_n = \max\{s_1, \dots, s_{n-1}, l_1, \dots, l_n\} + 1.$$

It is clear that the set $\{e^{(s_n)}\}_{n\in\mathbb{N}} \subset D$ is dissociate and satisfies (i).

(6.2) COMPLETION OF PROOF. Let w be any function in $\mathfrak{c}_0^+(X)$. For $n \in \mathbb{N}$, let

$$\Phi_n = \{\chi \in X : w(\chi) \geq 4^{-n}\}.$$

Let $\Delta = \{\chi_n\}_{n\in\mathbb{N}}$ be a dissociate set in X satisfying (6.1.i) for these sets Φ_n. Now form the Riesz product $\mu_{\Delta,\beta}$ for this set Δ and the function β on Δ such that $\beta(\chi_n) = \frac{1}{2}n^{-\frac{1}{2}}$ for all $n \in \mathbb{N}$. The measure $\mu_{\Delta,\beta}$ is defined by the following relations:

$$\widehat{\mu}(1) = 1; \tag{1}$$

$$\widehat{\mu}(\chi_1^{\delta_1}\dots\chi_m^{\delta_m}) = \left(\frac{1}{2}\right)^{|\delta|} 1^{-\frac{1}{2}|\delta_1|}\dots m^{-\frac{1}{2}|\delta_m|} \tag{2}$$

for all sequences $\delta = (\delta_1, \dots, \delta_m)$ with $\delta_j \in \{-1,0,1\}$, where we write $|\delta| = \sum_{j=1}^m |\delta_j|$;

$\widehat{\mu}(\psi) = 0$ for all $\psi \in X$ not of the form appearing in (2).

It is known that $\mu_{\Delta,\beta}$ is in $\mathbf{M}^+(G)$, $\mu(G) = 1$, and that $\mu_{\Delta,\beta}$ is uniquely determined by (1)–(3). See for example [7], (37.14). We write σ for the measure $\mu_{\Delta,\beta}$. It is clear that $\widehat{\sigma} \in \mathfrak{c}_0(X)$ and hence $\sigma \in \mathbf{M}_c(G)$. Since $\sum_{n=1}^{\infty}\beta_n^2 = \infty$, we cite a theorem of Brown and Moran [1] to prove that σ is purely singular: $\sigma \in \mathbf{M}_s(G)$.

For $m \in \mathbb{N}$, let Q_m be the set of all sequences δ as in (2) for which $\delta_m = \pm 1$. If $\delta \in Q_m$, the character

$$\chi_1^{\delta_1}\chi_2^{\delta_2}\dots\chi_m^{\delta_m} = \psi_\delta$$

is not in Φ_m (Lemma (6.1)). The cardinal number of Q_m is $2 \cdot 3^{m-1}$. The number $\widehat{\sigma}(\psi_\delta)$ does not exceed $\frac{1}{2}m^{-\frac{1}{2}}$. We have

$$\sum_{\chi\in X} w(\chi)\widehat{\sigma}(\chi) \leq w(1) + \sum_{m=1}^{\infty}\sum_{\delta\in Q_m} w(\psi_\delta)\widehat{\sigma}(\psi_\delta) \leq w(1) + \sum_{m=1}^{\infty} 4^{-m}2\cdot 3^{m-1}\frac{1}{2}m^{-\frac{1}{2}} < \infty.$$

Testo pervenuto il 30 aprile 1976
Bozze licenziate il 10 maggio 1977.

References

[1] Gavin Brown and William Moran. On orthogonality of Riesz products. *Math. Proc. Camb. Phil. Soc.*, 76:173–181, 1974.

[2] S.W. Drury. Sur les ensembles de Sidon. *C.R. Acad. Sci. Paris, Sér. A-B*, 271:A 162–A 163, 1970.

[3] R.E. Edwards. Changing signs of Fourier coefficients. *Pacific J. Math.*, 15:463–475, 1965.

[4] T.H. Gronwall. On the Fourier coefficients of a continuous function. *Bull. Amer. Math. Soc.*, 27:320–321, 1921.

[5] Edwin Hewitt and Gunter Ritter. On the integrability of Fourier transforms on groups, Part II: Fourier-Stieltjes transforms of singular measures. *Proc. Royal Irish Acad.*, 77, Sec. A:265–287, 1976.

[6] Edwin Hewitt and Gunter Ritter. Über die Integrierbarkeit von Fourier–Transformierten auf Gruppen. Teil I. Stetige Funktionen mit kompaktem Träger und eine Bemerkung über hyperbolische Differentialoperatoren. *Math. Ann.*, 224:77–96, 1976.

[7] Edwin Hewitt and Kenneth A. Ross. *Abstract Harmonic Analysis*, volume I, II. Springer, Berlin, Heidelberg, New York, 1963 and 1970.

[8] Edwin Hewitt and H.S. Zuckerman. Singular measures with absolutely continuous convolution squares. *Proc. Camb. Phil. Soc.*, 62:399–420, 1966. Corrigendum ibid. 63, 367-368 (1967).

[9] O.S. Ivašev-Musatov. On coefficients of trigonometric null–series. *Izv. Akad. Nauk SSSR, Ser. mat.*, 21:559–578, 1957.

[10] K. Knopp. *Theorie und Anwendungen der unendlichen Reihen.* Springer, Berlin-Heidelberg, 4. edition, 1947.

[11] W. Orlicz. Beiträge zur Theorie der Orthogonalentwicklungen (III). *Bull. Int. Acad. Polon., Série A*, 8/9:229–238, 1932.

[12] R.E.A.C. Paley. A note on power series. *J. London Math. Soc.*, 7:122–130, 1932.

[13] S. Sidon. Ein Satz über die Fourierschen Reihen stetiger Funktionen. *Math. Z.*, 34:485–486, 1932.

[14] N. Wiener and A. Wintner. Fourier–Stieltjes transforms and singular infinite convolutions. *Amer. J. Math.*, 60:513–522, 1938.