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Fourier Multipliers for Certain Spaces of Functions with Compact Supports

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§1. Introduction

(1.1) *History.* The investigations in this paper are based on a celebrated result proved independently, and published almost simultaneously, by Orlicz[10], Paley [11], and Sidon [16] in 1932, *viz.*: if $(w(n))_{n=-\infty}^{\infty}$ is a sequence of complex numbers such that

$$\sum_{n=-\infty}^{\infty} |w(n)\widehat{\phi}(n)| < \infty$$

for every continuous function ϕ on the line with period 2π, then

$$\sum_{n=-\infty}^{\infty} |w(n)|^2 < \infty.$$

This is perhaps the final word on how large overall the Fourier coefficients of continuous periodic functions can be. (See the remarks in Paley [11] on this point.) Helgason [5], Theorem 2, generalized the Orlicz–Paley–Sidon theorem to all compact groups, Abelian and non–Abelian. An independent treatment of the compact Abelian case appears in Edwards [3]. Hewitt and Ritter have given a slightly more general theorem for systems of functions in [6], §4.

(1.2) *The Present Problem.* It is natural to ask, and is for certain applications (see §7) useful to know, what the analogue of the Orlicz–Paley–Sidon theorem is for arbitrary locally compact Abelian groups. On the real line, for example, what are the measurable functions w such that

$$\int_{-\infty}^{\infty} |w(y)\widehat{\phi}(y)|\mathrm{d}y < \infty$$

for all continuous functions ϕ on the line with compact supports? We will answer this question for all locally compact Abelian groups and along with it some related ones.

(1.3) *Notation and Terminology.* All notation and terminology not explained here are as in [7]. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{T}$ denote respectively the nonnegative integers, the integers, the real numbers, the complex numbers, and the interval $[-\pi, \pi]$ (with addition modulo 2π and the topology identifying $-\pi$ and π). The symbol G denotes an arbitrary locally compact Abelian group. The character group of G will be denoted by X. We will constantly use the structure theorem for G and X (see e.g. [7], Vol. I, Theorems (24.29) and (24.30)). That is, G has the form $\mathbb{R}^a \times H$, where a is a nonnegative integer and

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where H is a locally compact Abelian group containing a compact open subgroup, say J. The group J may or may not be unique. In any case, we pick one such subgroup J of H and fix it once and for all.

Write Y for the character group of H, A for the annihilator of J in Y, and Z for the (discrete) character group of J. The character group X has the form $\mathbb{R}^a \times \mathsf{Y}$, every element of X being a function

$$(\mathbf{x}, t) \to \exp\Big(i\Big(\sum_{j=1}^{a} y_j x_j\Big)\Big)\chi(t) = \exp(i\langle \mathbf{y}, \mathbf{x}\rangle)\chi(t). \tag{1}$$

Here $\mathbf{x} = (x_1, x_2, \ldots, x_a)$ is a generic element of $\mathbb{R}^a$, t is a generic element of H, $\mathbf{y}$ is a fixed element of $\mathbb{R}^a$, and χ is a fixed element of Y. The annihilator A is a compact open subgroup of Y, and the group Z is isomorphic with the quotient group Y/A.

We take as Haar measure λ on $G = \mathbb{R}^a \times H$ the product of a–dimensional Lebesgue measure on $\mathbb{R}^a$ and the Haar measure on H that assigns measure 1 to the compact open subgroup J. We take as Haar measure θ on $\mathsf{X} = \mathbb{R}^{\mathsf{a}} \times \mathsf{Y}$ the product of a–dimensional Lebesgue measure on $\mathbb{R}^a$ and the Haar measure on Y that assigns measure 1 to the compact open subgroup A. For a function f in $\mathfrak{L}_1(G)$, we define the Fourier transform $\widehat{f}$ of f as the function on X

$$\begin{aligned}(\mathbf{y}, \chi) \to \widehat{f}(\mathbf{y}, \chi) &= \int_H \int_{\mathbb{R}^a} \exp(-i\langle \mathbf{y}, \mathbf{x}\rangle)\overline{\chi(t)} f(\mathbf{x}, t)\, \mathrm{d}\mathbf{x}\mathrm{d}t \\ &= \int_G \exp(-i\langle \mathbf{y}, \mathbf{x}\rangle)\overline{\chi(t)} f(\mathbf{x}, t)\, \mathrm{d}\lambda(\mathbf{x}, t).\end{aligned} \tag{2}$$

The symbol γ will denote a positive constant, whose exact value is unimportant and which may vary from one assertion to another.

(1.4) *Definition.* For a set $\mathfrak{F}(G)$ of complex–valued functions on G, let $\mathfrak{F}_k(G)$ denote the set of functions in $\mathfrak{F}(G)$ that vanish λ–almost everywhere outside of compact sets. We shall be mainly concerned with $\mathfrak{C}_k(G)$, which is the space of continuous functions on G with compact supports, and with $\mathfrak{L}_{p,k}(G)$ $(1 \le p < \infty)$. Where no confusion seems possible, we will write $\mathfrak{C}_k$ and $\mathfrak{L}_{p,k}$ for $\mathfrak{C}_k(G)$ and $\mathfrak{L}_{p,k}(G)$, respectively. For a fixed compact subset F of G, we write $\mathfrak{F}(F)$ for the set of all functions in $\mathfrak{F}(G)$ that vanish λ–almost everywhere outside of F. Thus $\mathfrak{C}(F)$ is the set of all continuous complex–valued functions on G with support contained in F.

(1.5) *Definition.* Let $\mathfrak{A}$ be a subset of $\mathfrak{L}_1(G)$. The set of all complex–valued, θ–measurable functions w on X such that $w\widehat{\phi}$ belongs to $\mathfrak{L}_1(\mathsf{X})$ for all ϕ in $\mathfrak{A}$ will be denoted by $\mathfrak{M}(\mathfrak{A})$. Functions in $\mathfrak{M}(\mathfrak{A})$ will be called *multipliers* of $\mathfrak{A}$ (strictly speaking they are *Fourier multipliers*). In the terminology of [7], Vol. II, (35.1), these functions are $(\widehat{\mathfrak{A}}, \mathfrak{L}_1(\mathsf{X}))$–multipliers.

(1.6) *Remark.* The spaces $\mathfrak{M}(\mathfrak{C}_k)$ and $\mathfrak{M}(\mathfrak{L}_{p,k})$ $(2 \le p < \infty)$ will be completely identified in this paper. They turn out to be function spaces that have been studied by a number of authors over the past two decades. They also bear a family resemblance to Wiener's well–known space $\mathfrak{M}_1(\mathbb{R})$ (see for example [7], Vol. II, (39.33)). A discussion of previous work is given in (1.16) *infra.*

(1.7) *Definition.* The following decomposition of the character group X will be used. For every sequence $\mathbf{n} = (n_1, n_2, \ldots, n_a)$ in $\mathbb{Z}^a$, let $C_{\mathbf{n}}$ be the cube in $\mathbb{R}^a$ consisting of all $\mathbf{y}$ such that $n_j \le y_j \le n_j + 1$ for all $j = 1, 2, \ldots, a$. For each coset $\chi\mathsf{A}$ of A in the group Y, we consider the "block"

$$C_{\mathbf{n}} \times (\chi\mathsf{A})$$

in X. Distinct blocks intersect in sets of θ–measure zero, their union is X, and each has θ–measure one. We index the family $\mathfrak{B}$ of all blocks in an arbitrary fashion, writing a block as B_ι, so that the family $\mathfrak{B}$ is $\{B_\iota : \iota \in I\}$ for some index class I. Plainly I is countable if and only if X is σ–compact, i.e., if and only if G is metrizable ([7], Vol. I, (24.48)). We make the following conventions. If G is discrete, then X is compact and we have only one block, X itself. If G is nondiscrete and contains a compact open subgroup, then $G = H$, there are no cubes $C_{\mathbf{n}}$, and the blocks that comprise the family $\mathfrak{B}$ are the cosets of A. If G contains no compact open subgroup, then the integer a is positive, and cubes $C_{\mathbf{n}}$ appear. If $G = \mathbb{R}^a$, then H and Y are one–element groups, and the family $\mathfrak{B}$ consists exactly of the cubes $C_{\mathbf{n}}$. We can now make our basic definition.

(1.8) *Definition.* Let p be a real number greater than or equal to 1. Let $\mathfrak{S}_p(\mathsf{X})$ be the set of all θ–measurable, complex–valued functions w on X for which

$$\|w\|_{[p]}^p := \sum_{\iota \in I} \Big(\int_{B_\iota} |w| \, \mathrm{d}\theta \Big)^p < \infty. \tag{1}$$

It is trivial that $\mathfrak{S}_1(\mathsf{X})$ is $\mathfrak{L}_1(\mathsf{X})$. For the relations between the spaces $\mathfrak{S}_p(\mathsf{X})$ and $\mathfrak{L}_p(\mathsf{X})$ in general, see (6.3) *infra*.

(1.9) *Remark.* For $1 \le p < \infty$, the set $\mathfrak{S}_p(\mathsf{X})$ is a Banach space with pointwise linear operations and with the norm $\|w\|_{[p]}$.

(1.10) *Definition.* For $1 < p < \infty$, let $\mathfrak{T}_p(\mathsf{X})$ be the set of all θ–measurable, complex–valued functions v on X such that

$$\|v\|_{(p)}^p := \sum_{\iota \in I} [\operatorname{ess\,sup}\{|v(\mathbf{y}, \chi)| : (\mathbf{y}, \chi) \in B_\iota\}]^p < \infty. \tag{1}$$

(1.11) *Remark.* The set $\mathfrak{T}_p(\mathsf{X})$ is a Banach space under pointwise linear operations and the norm $\|v\|_{(p)}$. For $1 < p < \infty$, write p' for $p/(p-1)$. Then the space $\mathfrak{T}_{p'}(\mathsf{X})$ can be identified in a natural way with the conjugate space of $\mathfrak{S}_p(\mathsf{X})$, the generic bounded linear functional $\mathfrak{S}_p(\mathsf{X})$ having the form

$$w \to \int_{\mathsf{X}} wv \, \mathrm{d}\theta$$

for some v in $\mathfrak{T}_{p'}(\mathsf{X})$.

The proofs of (1.9) and (1.11) are omitted.

We can now state our main theorems.

(1.12) **First Main Theorem.** *The spaces $\mathfrak{M}(\mathfrak{C}_k(G))$ and $\mathfrak{M}(\mathfrak{L}_{q,k}(G))$ for $2 \le q < \infty$ are all equal to $\mathfrak{S}_2(\mathsf{X})$.*

(1.13) **Second Main Theorem.** *Let G be nondiscrete. For $1 < p < 2$, the space $\mathfrak{M}(\mathfrak{L}_{p,k}(G))$ properly contains $\mathfrak{S}_p(\mathsf{X})$.*

(1.14) **Third Main Theorem.** *The space $\mathfrak{M}(\mathfrak{L}_{1,k}(G))$ is equal to $\mathfrak{L}_1(\mathsf{X})$, up to functions vanishing locally θ–almost everywhere.*

(1.15) *Remark.* For the classical case $G = \mathbb{R}$, (1.12) has the following simple form. If w is a measurable function on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |w(y)\widehat{\phi}(y)| \mathrm{d}y < \infty \tag{1}$$

for all $\phi \in \mathfrak{C}_k(\mathbb{R})$, then we have

$$\sum_{n=-\infty}^{\infty} \Big[\int_n^{n+1} |w(y)| \mathrm{d}y \Big]^2 < \infty. \tag{2}$$

If (2) holds, then (1) holds for all measurable functions ϕ on $\mathbb{R}$ that vanish almost everywhere outside of some compact set and are in $\mathfrak{L}_2(\mathbb{R})$.

We carry out the proofs of these three theorems in §§2–5. In §6, we give some examples and in §7, an application to hyperbolic differential equations.

(1.16) *Previous work on* $\mathfrak{S}_p$ *and* $\mathfrak{T}_p$. Our spaces $\mathfrak{S}_p(\mathsf{X})$ and $\mathfrak{T}_p(\mathsf{X})$ are special cases on the L^p spaces with mixed norm studied by Benedek and Panzone [1]: see also their references to yet earlier work. Benedek and Panzone do not consider applications to harmonic analysis. Liu, van Rooij, and Wang in [9] define function spaces on an arbitrary locally compact group L that are identical with the spaces $\mathfrak{S}_p(\mathsf{X})$ and $\mathfrak{T}_p(\mathsf{X})$ for $L = \mathsf{X}$. In their definition (*loc. cit.* p. 509), take their basic set K to be the set $C_{(0,\ldots,0)} \times \mathsf{A}$, in the notation of (1.7). Then their space $\mathcal{V}_p$ is our space $\mathfrak{S}_p(\mathsf{X})$ and their space $\mathcal{N}_p$ is our $\mathfrak{T}_p(\mathsf{X})$. The proof of this is a routine computation. Proofs of (1.9) and (1.11) along with much else, is found in [9]. We are indebted to Kenneth A. Ross for drawing our attention to the paper [9].

Finbarr Holland [8] has studied a class of function spaces on $\mathbb{R}$ that contain $\mathfrak{S}_p(\mathbb{R})$ and $\mathfrak{T}_p(\mathbb{R})$ as special cases, obtaining many interesting facts regarding harmonic analysis on these spaces, but not considering our theorems (1.12)–(1.14).

§2. Some Properties of Entire Functions of Exponential Type

(2.1) *Preliminaries.* We consider a fixed positive integer a and a–dimensional complex space $\mathbb{C}^a$. For $\mathbf{z} = (z_1, \ldots, z_a) \in \mathbb{C}^a$, we write $\|\mathbf{z}\| = \max\{|z_1|, \ldots, |z_a|\}$. We regard $\mathbb{R}^a$ as a subspace of $\mathbb{C}^a$. We write $z_j = x_j + iy_j$ with x_j and y_j real. We consider complex functions $f : \mathbb{C}^a \to \mathbb{C}$ that are entire (i.e., expansible in power series in the complex variables $z_1, \ldots, z_a$ that converge everywhere in $\mathbb{C}^a$) and that are also of exponential type $\le \tau$ ($\tau > 0$), i.e.,

$$|f(\mathbf{z})| \le C \exp(\tau(|z_1| + \cdots + |z_a|))$$

for all $\mathbf{z} \in \mathbb{C}^a$. (For other ways to define exponential type, see [14], Ch. 3, §1.)

Theorem (2.2) below is the basis for the proofs of our main theorems. A special case appears in [12], 2^e Partie, Théorème III, p. 149. We shall use the following notation. Let ψ be a nondecreasing convex function on $\mathbb{R} \cup \{-\infty\}$ with values in $[0, \infty[$, and let Φ be the function $\psi \circ \log$, defined on $[0, \infty[$ (we define $\log(0)$ as $-\infty$). Let $\{\mathbf{x}_\alpha\}_{\alpha \in A}$ be a subset of $\mathbb{R}^a \subset \mathbb{C}^a$ such that for some positive real number δ, we have $\|\mathbf{x}_\alpha - \mathbf{x}_\beta\| \ge 2\delta$ for distinct α and β.

(2.2) **Theorem.** *Notation is as in* (2.1). *Suppose that*

(i) $\int_{\mathbb{R}} \Phi[|f(z_1, \ldots, z_{j-1}, x_j, z_{j+1}, \ldots, z_a)|] \mathrm{d}x_j < \infty$

for each $j \in \{1, 2, \ldots, a\}$ *and each* $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_a) \in \mathbb{C}^{a-1}$. *Then we have*

(ii) $\sum_{\alpha \in A} \Phi[|f(\mathbf{x}_\alpha)| \exp(-\alpha\tau\delta)] \le \left(\frac{2}{\pi\delta}\right)^a \int_{\mathbb{R}^a} \Phi[|f(\mathbf{u})|] \, \mathrm{d}\mathbf{u}$.

Proof. Fixing $z_2, \ldots, z_a$, we obtain the function $z_1 \mapsto f(z_1, z_2, \ldots, z_a)$ carrying $\mathbb{C}$ into $\mathbb{C}$. It is shown in [2], page 100, line 7 from the bottom that

$$\Phi[|f(z_1, \ldots, z_a)| \exp(-\tau|y_1|)] \le \pi^{-1} \int_{\mathbb{R}} \Phi[|f(t_1, z_2, \ldots, z_a)|] \frac{|y_1|}{(x_1 - t_1)^2 + y_1^2} \mathrm{d}t_1. \tag{1}$$

(We have added the absolute value to y_1 on the right side of (1), which was plainly intended.) Now fix $z_1, z_3, \ldots, z_a$ and consider the function

$$z_2 \to f(z_1, z_2, \ldots, z_a) \exp(-\tau|y_1|).$$

Since Φ is nondecreasing, the hypothesis (6.7.5) in [2], p. 98, holds, and so as with (1) we find that

$$\begin{aligned} &\Phi[|f(z_1, z_2, \ldots, z_a)| \exp(-\tau(|y_1| + |y_2|))] \\ &\le \pi^{-1} \int_{\mathbb{R}} \Phi[|f(z_1, t_2, z_3, \ldots, z_a)| \exp(-\tau|y_1|)] \frac{|y_2|}{(x_2 - t_2)^2 + y_2^2} \mathrm{d}t_2. \end{aligned} \tag{2}$$

Combining (1) and (2), we find

$$\Phi[|f(z)| \exp(-\tau(|y_1| + |y_2|))] \le \pi^{-2} \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi[|f(t_1, t_2, z_3, \ldots, z_a)|] \prod_{j=1}^{2} \frac{|y_j|}{(x_j - t_j)^2 + y_j^2} \mathrm{d}t_1 \mathrm{d}t_2. \tag{3}$$

By induction, we find that

$$\Phi\Big[|f(\mathbf{z})|\exp\Big(-\tau\Big(\sum_{j=1}^{a}|y_j|\Big)\Big)\Big] \le \pi^{-a}\int_{\mathbb{R}^a}\Phi[|f(t_1,\dots,t_a)|]\prod_{j=1}^{a}\frac{|y_j|}{(x_j-t_j)^2+y_j^2}\,\mathrm{d}\mathbf{t}. \tag{4}$$

Integrate both sides of (4) over $\mathbb{R}^a$, as functions of $\mathbf{x}$, and use Fubini's theorem to find

$$\begin{aligned}&\int_{\mathbb{R}^a}\Phi\Big[|f(\mathbf{x}+i\mathbf{y})|\exp\Big(-\sum_{j=1}^{a}|y_j|\Big)\Big]\,\mathrm{d}\mathbf{x} \qquad (5)\\ \le&\pi^{-a}\int_{\mathbb{R}^a}\Phi[|f(\mathbf{t})|]\int_{\mathbb{R}^a}\prod_{j=1}^{a}\frac{|y_j|}{(x_j-t_j)^2+y_j^2}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{t}\\ =&\int_{\mathbb{R}^a}\Phi[|f(\mathbf{t})|]\,\mathrm{d}\mathbf{t}\end{aligned}$$

Suppose that all of the $|y_j|$ are less than or equal to δ. The hypothesis that Φ is nondecreasing and (5) imply that

$$\int_{\mathbb{R}^a}\Phi[|f(\mathbf{x}+i\mathbf{y})|\exp(-\tau a\delta)]\,\mathrm{d}\mathbf{x} \le \int_{\mathbb{R}^a}\Phi[|f(\mathbf{x})|]\,\mathrm{d}\mathbf{x}. \tag{6}$$

As a nondecreasing convex function of a subharmonic function, each function

$$z_j \to \Phi[|f(z_1,\dots,z_j,\dots,z_a)|\exp(-\tau a\delta)]$$

is subharmonic (the z_k's for $k \ne j$ being fixed). A form of the minimum principle for subharmonic functions asserts that the value of a subharmonic function at the center of a disc in $\mathbb{C}$ does not exceed its integral mean value over the disc. Applying this version of the minimum principle repeatedly, we obtain

$$\begin{aligned}&\Phi[|f(\mathbf{x})|\exp(-\tau a\delta)] \qquad (7)\\ \le&(\pi\delta^2)^{-1}\int_{|w_1|\le\delta}\Phi[|f(x_1+w_1,x_2,\dots,x_a)|\exp(-\tau a\delta)]\,\mathrm{d}w_1\\ &\dots\\ \le&(\pi\delta^2)^{-a}\int_{|w_a|\le\delta}\dots\int_{|w_1|\le\delta}\Phi[|f(x_1+w_1,\dots,x_a+w_a)|\exp(-\tau a\delta)]\,\mathrm{d}w_1\dots\mathrm{d}w_a.\end{aligned}$$

In the integrals appearing in (7), the variables w_j are complex, and we integrate over discs in the plane. Replacing these discs by circumscribing squares, and writing $w_j = u_j + iv_j$, we infer from (7) that

$$\begin{aligned}&\Phi[|f(\mathbf{x})|\exp(-\tau a\delta)] \qquad (8)\\ \le&(\pi\delta^2)^{-a}\underbrace{\int_{-\delta}^{\delta}\dots\int_{-\delta}^{\delta}}_{2a-\text{fold integral}}\Phi[|f(\mathbf{x}+\mathbf{u}+i\mathbf{v})|\exp(-\tau a\delta)]\,\mathrm{d}u_1\,\mathrm{d}v_1\dots\mathrm{d}u_a\,\mathrm{d}v_a.\end{aligned}$$

We next set $\mathbf{x} = \mathbf{x}_\alpha$ in (8) and sum over all α. After a linear change of variables in (8), we find that the domains of integration for distinct α and β intersect in sets of Lebesgue measure 0, since $\|\mathbf{x}_\alpha - \mathbf{x}_\beta\| \ge 2\delta$. It follows from (8) that

$$\begin{aligned}&\sum_{\alpha\in A}\Phi[|f(\mathbf{x}_\alpha)|\exp(-\tau a\delta)] \qquad (9)\\ \le&(\pi\delta^2)^{-a}\underbrace{\int_{-\delta}^{\delta}\dots\int_{-\delta}^{\delta}}_{a-\text{fold integral}}\Big[\int_{\mathbb{R}^a}\Phi[|f(\mathbf{u}+i\mathbf{v})|\exp(-\tau a\delta)]\,\mathrm{d}\mathbf{u}\Big]\,\mathrm{d}\mathbf{v}.\end{aligned}$$

Combining (6) and (9), we find

$$\begin{aligned}\sum_{\alpha\in A}\Phi[|f(\mathbf{x}_\alpha)|\exp(-\tau a\delta)] \le&(\pi\delta^2)^{-a}\underbrace{\int_{-\delta}^{\delta}\dots\int_{-\delta}^{\delta}}_{a-\text{fold integral}}\int_{\mathbb{R}^a}\Phi[|f(\mathbf{u})|]\,\mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{v}\\ =&2^a(\pi\delta)^{-a}\int_{\mathbb{R}^a}\Phi[|f(\mathbf{u})|]\,\mathrm{d}\mathbf{u},\end{aligned}$$

and this is (ii). $\square$

(2.3) **Corollary.** *Let f and $\{\mathbf{x}_\alpha\}_{\alpha\in A}$ be as in* (2.1) *and let r be a positive real number. Suppose that*

(i) $\int_{\mathbb{R}} |f(z_1,\dots,z_{j-1},x_j,z_{j+1},\dots,z_a)|^r \,\mathrm{d}x_j < \infty$

for all j and for all $(z_1,\dots,z_{j-1},z_{j+1},\dots,z_a)\in\mathbb{C}^{a-1}$. Then we have
(ii) $\sum_{\alpha\in A}|f(\mathbf{x}_\alpha)|^r \le \exp(a\tau\delta r)2^\alpha(\pi\delta)^{-a}\int_{\mathbb{R}^a}|f(\mathbf{u})|^r\,\mathrm{d}\mathbf{u}$.

Proof. In (2.2), take $\Phi(t)=t^r=\exp(r\log(t))$. $\square$

§3. Proof that $\mathfrak{S}_p(\mathsf{X})\subset\mathfrak{M}(\mathfrak{L}_{p,k}(G))$

(3.1) *Explanation.* We will first apply (2.3) to Fourier transforms of functions in $\mathfrak{L}_{p,k}(\mathbb{R}^a)$. Let ϕ be any function in $\mathfrak{L}_{1,k}(\mathbb{R}^a)$, vanishing almost everywhere outside of the compact subset K of $\mathbb{R}^a$. The Fourier transform $\widehat{\phi}$ of ϕ is defined as in (1.3)(2) only on $\mathbb{R}^a$, but admits an immediate extension over $\mathbb{C}^a$:

$$\widehat{\phi}(\mathbf{z})=\int_K \exp(-i\langle\mathbf{z},\mathbf{x}\rangle)\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{1}$$

It is easy to see that $\widehat{\phi}$ is entire of exponential type not exceeding a number τ that depends only on the compact set K.

(3.2) **Lemma.** *Let p be a real number such that $1<p\le 2$, and write q for the number $p/(p-1)$. Let K be a compact subset of $\mathbb{R}^a$, and let $\{\mathbf{x}_\alpha\}_{\alpha\in A}$ be as in* (2.1). *There is a constant γ depending only upon p, δ, and K such that for all $\phi\in\mathfrak{L}_p(K)$, the inequality*
(i) $\sum_{\alpha\in A}|\widehat{\phi}(\mathbf{x}_\alpha)|^q\le\gamma\|\widehat{\phi}\|_q^q<\infty$
holds.

Proof. We use (2.3) with $r=q$ and $f=\widehat{\phi}$. We need only to verify (2.3)(i). We write $\mathbf{z}'$ for the element $(z_1,\dots,z_{j-1},z_{j+1},\dots,z_a)$ of $\mathbb{C}^{a-1}$ and $\mathbf{t}'$ similarly for the element $(t_1,\dots,t_{j-1},t_{j+1},\dots,t_a)$ of $\mathbb{R}^{a-1}$. With a slightly inaccurate but useful notation, we may write

$$\begin{aligned}\widehat{\phi}(z_1,\dots,z_{j-1},x_j,z_{j+1},\dots,z_a)&=\widehat{\phi}(x_j,\mathbf{z}')\\ &=\int_{\mathbb{R}}\Big[\int_{\mathbb{R}^{a-1}}\exp(-i\langle\mathbf{z}',\mathbf{t}'\rangle)\phi(t_j,\mathbf{t}')\,\mathrm{d}\mathbf{t}'\Big]\exp(-ix_jt_j)\,\mathrm{d}t_j.\end{aligned} \tag{1}$$

The function $\mathbb{R}\to\mathbb{C}$:

$$t_j\to\int_{\mathbb{R}^{a-1}}\exp(-i\langle\mathbf{z}',\mathbf{t}'\rangle)\phi(t_j,\mathbf{t}')\,\mathrm{d}\mathbf{t}'$$

appearing as the inner integral in (1) plainly vanishes almost everywhere outside of a compact set, as this is true of the function $\phi(t_1,\dots,t_a)$. For every choice of $\mathbf{z}'$, this function is in $\mathfrak{L}_p(\mathbb{R})$. To see this, estimate as follows:

$$\begin{aligned}\Big|\int_{\mathbb{R}^{a-1}}\exp(-i\langle\mathbf{z}',\mathbf{t}'\rangle)\phi(t_j,\mathbf{t}')\,\mathrm{d}\mathbf{t}'\Big| &\le\Gamma(\mathbf{z}')\int_{\mathbb{R}^{a-1}}|\phi(t_j,\mathbf{t}')|\,\mathrm{d}\mathbf{t}'\\ &\le\Gamma(\mathbf{z}')\gamma\Big[\int_{\mathbb{R}^{a-1}}|\phi(t_j,\mathbf{t}')|^p\,\mathrm{d}\mathbf{t}'\Big]^{1/p},\end{aligned} \tag{2}$$

Γ being a positive function of $\mathbf{z}'$ and γ being a constant determined by Hölder's inequality. Thus (2) yields

$$\int_{\mathbb{R}}\Big|\int_{\mathbb{R}^{a-1}}\exp(-i\langle\mathbf{z}',\mathbf{t}'\rangle)\phi(t_j,\mathbf{t}')\,\mathrm{d}\mathbf{t}'\Big|^p\,\mathrm{d}t_j\le\gamma\int_{\mathbb{R}}\int_{\mathbb{R}^{a-1}}|\phi(t_j,\mathbf{t}')|^p\,\mathrm{d}\mathbf{t}'\,\mathrm{d}t_j<\infty. \tag{3}$$

The Hausdorff–Young theorem now gives (2.3)(i) for the present case. The inequality (i) is just (2.3)(ii). $\square$

(3.3) **Theorem.** *Let p, q, K and ϕ be as in* (3.2). *For $\mathbf{n} \in \mathbb{Z}^a$, let $C_{\mathbf{n}}$ be the cube defined in* (1.7). *There is a positive constant γ (depending upon p and K) such that*

(i) $\sum_{\mathbf{n}\in\mathbb{Z}^a} \left[\max_{y\in C_{\mathbf{n}}} |\widehat{\phi}(\mathbf{y})|\right]^q \leq \gamma \|\widehat{\phi}\|_q^q < \infty.$

Proof. For each $\mathbf{n}$ in $\mathbb{Z}^a$, let $\mathbf{x_n}$ be a point in $C_{\mathbf{n}}$ such that

$$|\widehat{\phi}(\mathbf{x_n})| = \max_{y\in C_{\mathbf{n}}} \{|\widehat{\phi}(\mathbf{y})|\}.$$

Let $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_a)$ be a sequence of length a consisting only of 0's and 1's. Let I_ε be the subset of $\mathbb{Z}^a$ consisting of exactly those sequences $\mathbf{n} = (n_1, \ldots, n_a)$ for which n_j is even if ε_j is 0 and n_j is odd if ε_j is 1. If $\mathbf{m}$ and $\mathbf{n}$ are distinct elements of I_ε, it is clear that $\|\mathbf{x_n} - \mathbf{x_m}\| \geq 1$. By (3.2)(i) with $\delta = \frac{1}{2}$, we have

$$\sum_{\mathbf{n}\in I_\varepsilon} |\widehat{\phi}(\mathbf{x_n})|^q \leq \gamma \|\widehat{\phi}\|_q^q < \infty. \tag{1}$$

Since $\mathrm{card}(I_\varepsilon) = 2^a$, we may add (1) over all $\boldsymbol{\varepsilon}$ to find (i) with γ multiplied by 2^a. $\square$

(3.4) *Explanation.* We now consider a group $G = \mathbb{R}^a \times J$, where J as in (1.1) is a compact Abelian group with (discrete) character group Z. Let L be a compact subset of $\mathbb{R}^a \times J$, and K the (obviously compact) projection of L into $\mathbb{R}^a$. As before, p is a real number such that $1 < p \leq 2$, and q is equal to $p/(p-1)$. The sets $C_{\mathbf{n}}$ are as in (1.7).

(3.5) **Theorem.** *There is a constant γ depending only upon L and p such that for all ϕ in the space $\mathfrak{L}_p(L)$, we have*

(i) $\sum_{\mathbf{n}\in\mathbb{Z}^a} \sum_{\chi\in\mathsf{Z}} \left[\max_{y\in C_{\mathbf{n}}} \{|\widehat{\phi}(\mathbf{y}, \chi)|\}\right]^q \leq \gamma \|\widehat{\phi}\|_q^q < \infty.$

Proof. For each $\mathbf{x} \in \mathbb{R}^a$ and $\chi \in \mathsf{Z}$, write

$$\psi_\chi(\mathbf{x}) = \int_J \phi(\mathbf{x}, t)\overline{\chi(t)}\mathrm{d}t, \tag{1}$$

where $\mathrm{d}t$ denotes integration with respect to normalized Haar measure on J. For each $\mathbf{y} \in \mathbb{R}^a$, we define $\widehat{\psi}_\chi(\mathbf{y})$ by

$$\widehat{\psi}_\chi(\mathbf{y}) = \int_{\mathbb{R}^a} \exp(-i\langle \mathbf{y}, \mathbf{x}\rangle)\psi_\chi(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{2}$$

From (1), (2), and (1.3)(2), it is clear that

$$\widehat{\psi}_\chi(\mathbf{y}) = \widehat{\phi}(\mathbf{y}, \chi) \quad \text{for all } (\mathbf{y}, \chi) \in \mathbb{R}^a \times \mathsf{Z}. \tag{3}$$

It is plain that $\psi_\chi(\mathbf{x})$ vanishes almost everywhere outside of K. Also we have

$$|\psi_\chi(\mathbf{x})| \leq \int_J |\phi(\mathbf{x}, t)|\mathrm{d}t = \int_J \mathbf{1}\cdot|\phi(\mathbf{x}, t)|\mathrm{d}t \leq \Big[\int_J |\phi(\mathbf{x}, t)|^p \mathrm{d}t\Big]^{1/p} \cdot (\mathbf{1})^{1/q}. \tag{4}$$

Raising both ends of (4) to the p^{th} power and integrating over $\mathbb{R}^a$, we find that each function ψ_χ is in $\mathfrak{L}_p(\mathbb{R}^a)$. We may thus apply (3.3)(i) to each function ψ_χ, obtaining

$$\sum_{\mathbf{n}\in\mathbb{Z}^a} [\max_{y\in C_{\mathbf{n}}} \{|\widehat{\psi}(\mathbf{y})|\}]^q \leq \gamma \int_{\mathbb{R}^a} |\widehat{\psi}_\chi(\mathbf{y})|^q \,\mathrm{d}\mathbf{y}. \tag{5}$$

Applying (3) and summing (5) over all $\chi \in \mathsf{Z}$, we obtain (i). $\square$

(3.6) *Explanation.* We now consider an arbitrary nondiscrete locally compact Abelian group G, having the form $\mathbb{R}^a \times H$, as in (1.3), H being a locally compact Abelian group containing a compact open subgroup J. Let K be a compact subset of $\mathbb{R}^a$. The symbols p, q, and $C_{\mathbf{n}}$ are as in (3.4). Blocks B_ι in X are defined as in (1.7).

(3.7) **Theorem.** *Let ϕ be a function in $\mathfrak{L}_p(K \times (u_1 J))$ for some $u_1 \in H$. There is a constant γ depending only upon K and p such that*

(i) $\|\widehat{\phi}\|_{(q)} \leq \gamma \|\widehat{\phi}\|_q$.

Proof. The translate $T_{(\mathbf{0},u_1)}\phi$, whose value at $(\mathbf{x}, t)$ is $\phi(\mathbf{x}, u_1 t)$, vanishes almost everywhere outside of the compact set $K \times J$. Furthermore, we have

$$\begin{aligned}(T_{(\mathbf{0},u_1)}\phi)\widehat{\ }(\mathbf{y}, \omega) &= \int_{\mathbb{R}^a} \int_H \phi(\mathbf{x}, u_1 t)\overline{\omega(t)} \mathrm{d}t \exp(-i\langle \mathbf{y}, \mathbf{x}\rangle)\, \mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^a} \int_H \phi(\mathbf{x}, t)\overline{\omega(u_1^{-1} t)} \mathrm{d}t \exp(-i\langle \mathbf{y}, \mathbf{x}\rangle)\, \mathrm{d}\mathbf{x} \\ &= \omega(u_1)\widehat{\phi}(\mathbf{y}, \omega) \end{aligned} \tag{1}$$

for all $(\mathbf{y}, \omega) \in \mathsf{X} = \mathbb{R}^a \times \mathsf{Y}$. We apply (3.5)(i) to the function $T_{(\mathbf{0},u_1)}\phi$, noting that Z can be identified with Y/A. We thus obtain

$$\sum_{\mathbf{n}\in\mathbb{Z}^a} \sum_{\chi\mathsf{A}\in\mathsf{Y}/\mathsf{A}} [\max_{y\in C_{\mathbf{n}}}\{|(T_{(\mathbf{0},u_1)}\phi)\widehat{\ }(\mathbf{y}, \chi\mathsf{A})|\}]^q \leq \gamma \|(T_{(\mathbf{0},u_1)}\phi)\widehat{\ }\|_q^q. \tag{2}$$

By (1), we have

$$|\widehat{\phi}(\mathbf{y}, \omega)| = |(T_{(\mathbf{0},u_1)}\phi)\widehat{\ }(\mathbf{y}, \omega)| \quad \text{for all } (\mathbf{y}, \omega) \in \mathsf{X}.$$

As each block B_ι has the form $C_{\mathbf{n}} \times (\chi\mathsf{A})$, (2) immediately yields (i). $\square$

We can now prove part of Theorem (1.12).

(3.8) **Theorem.** *Let p be a real number such that $1 < p \leq 2$, and let G be an arbitrary locally compact Abelian group. Then we have*

(i) $\mathfrak{S}_p(\mathsf{X}) \subset \mathfrak{M}(\mathfrak{L}_{p,k}(G))$.

Proof. Suppose first that $\phi \in \mathfrak{L}_{p,k}(G)$ and that ϕ vanishes almost everywhere outside of some set $\mathbb{R}^a \times (u_1 J)$. For $w \in \mathfrak{S}_p(\mathsf{X})$, we have

$$\begin{aligned}\int_{\mathsf{X}} |w\widehat{\phi}|\, \mathrm{d}\theta &= \sum_{\iota\in I} \int_{B_\iota} |w\widehat{\phi}|\, \mathrm{d}\theta \\ &\leq \sum_{\iota\in I} \max\{|\widehat{\phi}(y, \omega)| : (y, \omega) \in B_\iota\} \int_{B_\iota} |w|\, \mathrm{d}\theta \\ &\leq \Big[\sum_{\iota\in I} [\max\{|\widehat{\phi}(y, \omega)| : (y, \omega) \in B_\iota\}]^q\Big]^{1/q} \Big[\sum_{\iota\in I} [\int_{B_\iota} |w|\, \mathrm{d}\theta]^p\Big]^{1/p}. \end{aligned} \tag{1}$$

Applying (3.7)(i) to (1), we find that

$$\int_{\mathsf{X}} |w\widehat{\phi}|\, \mathrm{d}\theta \leq \gamma \|w\|_{[p]} \|\widehat{\phi}\|_q \leq \gamma \|w\|_{[p]} \|\phi\|_p < \infty. \tag{2}$$

Every function ψ in $\mathfrak{L}_{p,k}(G)$ is a finite linear combination of functions ϕ of the sort already considered, and so we have proved that $w\widehat{\psi} \in \mathfrak{L}_1(\mathsf{X})$ for all $w \in \mathfrak{S}_p(\mathsf{X})$. □

(3.9) *Remark.* Theorem (3.7) holds for all functions ϕ in $\mathfrak{L}_{1,k}(G)$ such that $\widehat{\phi} \in \mathfrak{L}_q(\mathsf{X})$ for $1 \leq q < \infty$. As we do not need this fact for the present paper, and as the proof would lead us somewhat afield, we omit it.

§4. The Inclusion $\mathfrak{M}(\mathfrak{C}_k(G)) \subset \mathfrak{S}_2(\mathsf{X})$

In this section we prove that $\mathfrak{M}(\mathfrak{C}_k(G)) \subset \mathfrak{S}_2(\mathsf{X})$. The proof is carried out by reducing it to the compact case, which we have already cited in §1.

(4.1) *Definition.* We will choose once and for all a certain compact subset F of G. If G is compact, we set $F = G$. If G contains a compact open subgroup, so that G has the form H as in (1.3), we take F to be the compact open subgroup J that we selected in (1.3). Suppose finally that G contains no compact open subgroup, so that $G = \mathbb{R}^a \times H$ with $a > 0$. In this case, we write $I = [-\pi, \pi]^a$ and we define F as $I \times J$.

(4.2) **Lemma.** *Let w be a function in $\mathfrak{M}(\mathfrak{C}_k(G))$. There is a positive constant γ such that the inequality*

(i) $\int_{\mathsf{X}} |w\widehat{\phi}| \, \mathrm{d}\theta \leq \gamma \|\phi\|_\infty$

holds for all $\phi \in \mathfrak{C}(F)$. The infimum of all numbers γ for which (i) *holds will be written as $\|w\|$.*

Proof. We use the closed graph theorem, regarding the mapping

$$\phi \to w\widehat{\phi} \tag{1}$$

as a linear mapping of $\mathfrak{C}(F)$ into $\mathfrak{L}_1(\mathsf{X})$. Thus suppose that $(\phi_n)_{n=1}^\infty$ is a sequence in $\mathfrak{C}(F)$ converging in the uniform norm to a function $\phi \in \mathfrak{C}(F)$. Suppose that the sequence $(w\widehat{\phi_n})_{n=1}^\infty$ converges to some function $f \in \mathfrak{L}_1(\mathsf{X})$. It is clear that $(w\widehat{\phi_n})_{n=1}^\infty$ converges pointwise on X to the function $w\widehat{\phi}$. A subsequence of the sequence $(w\widehat{\phi_n})_{n=1}^\infty$ converges θ–almost everywhere to the function f. Hence f is equal in $\mathfrak{L}_1(\mathsf{X})$ to $w\widehat{\phi}$. That is, the mapping (1) has a closed graph and so is continuous. This is just (i). □

An essential part of our argument is the following lemma, which is simple enough although rather unwieldy to state.

(4.3) *Definition.* Let K be the set of all a–tuples $\mathbf{k} = (k_1, \ldots, k_a)$ of nonnegative integers. For $\mathbf{x} \in \mathbb{R}^a$ and $\mathbf{k} \in K$, write $\mathbf{x}^{\mathbf{k}}$ for the number

$$x_1^{k_1} x_2^{k_2} \ldots x_a^{k_a}.$$

Write $\mathbf{k}!$ for the number

$$k_1! k_2! \ldots k_a!.$$

Write $|\mathbf{k}|$ for the number

$$k_1 + k_2 + \cdots + k_a.$$

Recall that G is a general group $\mathbb{R}^a \times H$. For $\phi \in \mathfrak{C}(F)$, let $\phi_{\mathbf{k}}$ be the function on G defined by

(i) $(\mathbf{x}, t) \to (-i)^{|\mathbf{k}|} \mathbf{x}^{\mathbf{k}} \phi(\mathbf{x}, t)$.

It is obvious that $\phi_{\mathbf{k}}$ belongs to $\mathfrak{C}(F)$ and that

(ii) $\|\phi_{\mathbf{k}}\|_\infty \leq \pi^{|\mathbf{k}|} \|\phi\|_\infty$.

In what follows, all sums $\sum_{\mathbf{k}}$ mean summation in any order over all $\mathbf{k} \in K$.

(4.4) **Lemma.** *Let G be a general group of the form $\mathbb{R}^a \times H$ with $a > 0$. For $\phi \in \mathfrak{C}(F)$, for $\mathbf{y}$ and $\mathbf{u}$ elements of $\mathbb{R}^a$, and $\chi \in \mathsf{Y}$, we have*

(i) $\widehat{\phi}(\mathbf{y}+\mathbf{u},\chi) = \sum_{\mathbf{k}} \widehat{\phi}_{\mathbf{k}}(\mathbf{y},\chi)\frac{\mathbf{u}^{\mathbf{k}}}{\mathbf{k}!}$,

the series on the right side of (i) *converging absolutely and uniformly over all compact subsets of $\mathbb{R}^a \times \mathbb{R}^a \times \mathsf{Y}$.*

Proof. We compute as follows:

$$\widehat{\phi}(\mathbf{y}+\mathbf{u},\chi) = \int_F \exp(-i\langle \mathbf{y}+\mathbf{u},\mathbf{x}\rangle)\overline{\chi(t)}\phi(\mathbf{x},t)\,\mathrm{d}\lambda(\mathbf{x},t) \tag{1}$$
$$= \int_J\int_I \exp(-i\langle \mathbf{y},\mathbf{x}\rangle)\left(\sum_{\mathbf{k}} \frac{(-i)^{|\mathbf{k}|}\mathbf{u}^{\mathbf{k}}\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}\right)\phi(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\overline{\chi(t)}\mathrm{d}t.$$

The multiple series appearing in (1) converges uniformly for $\mathbf{x} \in I$ for each $\mathbf{u} \in \mathbb{R}^a$. Hence we have

$$\int_I \exp(-i\langle \mathbf{y},\mathbf{x}\rangle)\left(\sum_{\mathbf{k}} (-i)^{|\mathbf{k}|}\frac{\mathbf{u}^{\mathbf{k}}\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}\right)\phi(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \tag{2}$$
$$= \sum_{\mathbf{k}} \frac{\mathbf{u}^{\mathbf{k}}}{\mathbf{k}!}\int_I \exp(-i\langle \mathbf{y},\mathbf{x}\rangle)(-i)^{|\mathbf{k}|}\mathbf{x}^{\mathbf{k}}\phi(\mathbf{x},t)\,\mathrm{d}\mathbf{x}$$
$$= \sum_{\mathbf{k}} \frac{\mathbf{u}^{\mathbf{k}}}{\mathbf{k}!}\int_I \exp(-i\langle \mathbf{y},\mathbf{x}\rangle)\phi_{\mathbf{k}}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}.$$

Since

$$\left|\int_I \exp(-i\langle \mathbf{y},\mathbf{x}\rangle)\phi_{\mathbf{k}}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\right| \le \phi^{|\mathbf{k}|}\|\phi\|_\infty$$

for each $t \in J$, we can insert (2) in (1), interchange $\int_J$ and $\sum_{\mathbf{k}}$, and thus obtain (i). The uniform and absolute convergence of the series in (i) is evident. □

(4.5) **Theorem.** *Let G be a general group of the form $\mathbb{R}^a \times H$ with $a > 0$. Let w be a function in $\mathfrak{M}(\mathfrak{C}_k(G))$. Then w is in $\mathfrak{S}_2(\mathsf{X})$.*

Proof. Consider an arbitrary block $B_\iota = C_{\mathbf{n}} \times (\chi\mathsf{A})$ in the character group X of G. Let ϕ be a function in $\mathfrak{C}(F)$. By (4.4)(i), we have

$$\widehat{\phi}(\mathbf{n},\omega) = \sum_{\mathbf{k}} \frac{1}{\mathbf{k}!}\widehat{\phi}_{\mathbf{k}}(\mathbf{y},\omega)(\mathbf{n}-\mathbf{y})^{\mathbf{k}}$$

for all $\mathbf{y} \in \mathbb{R}^a$ and all $\omega \in \mathsf{Y}$. Note that $\widehat{\phi}_{\mathbf{k}}(\mathbf{y},\omega) = \widehat{\phi}_{\mathbf{k}}(\mathbf{y},\omega')$ if ω and ω' lie in the same coset of A. Since each block B_ι has the form $C_{\mathbf{n}} \times (\chi\mathsf{A})$, we may compute as follows:

$$\left[\int_{B_\iota} |w(\mathbf{y},\omega)|\,\mathrm{d}\theta(\mathbf{y},\omega)\right]|\widehat{\phi}(\mathbf{n},\chi)|$$
$$\le \sum_{\mathbf{k}} \frac{1}{\mathbf{k}!}\int_{\chi\mathsf{A}}\int_{C_{\mathbf{n}}} |w(\mathbf{y},\omega)|\,|\widehat{\phi}_{\mathbf{k}}(\mathbf{y},\omega)||(\mathbf{n}-\mathbf{y})^{\mathbf{k}}|\,\mathrm{d}\mathbf{y}\,\mathrm{d}\omega$$
$$\le \sum_{\mathbf{k}} \frac{1}{\mathbf{k}!}\int_{\chi\mathsf{A}}\int_{C_{\mathbf{n}}} |w(\mathbf{y},\omega)|\,|\widehat{\phi}_{\mathbf{k}}(\mathbf{y},\omega)|\,\mathrm{d}\mathbf{y}\,\mathrm{d}\omega. \tag{1}$$

Now sum both ends of (1) over all blocks B_ι (since $\widehat{\phi}$ vanishes outside of a σ–compact subset of X, only countably many ι can yield a nonzero contribution). We obtain

$$\sum_{\iota\in I}\Big(\int_{B_\iota}|w(\mathbf{y},\omega)|\,\mathrm{d}\theta(\mathbf{y},\omega)\Big)|\widehat{\phi}(\mathbf{n},\chi)| \leq \sum_{\mathbf{k}}\frac{1}{\mathbf{k}!}\int_{\mathsf{X}}|w(\mathbf{y},\omega)|\,|\widehat{\phi}_{\mathbf{k}}(\mathbf{y},\omega)|\,\mathrm{d}\theta(\mathbf{y},\omega). \tag{2}$$

Applying (4.2)(i) and (4.3)(ii) to (2), we obtain

$$\sum_{\iota\in I}\Big(\int_{B_\iota}|w(\mathbf{y},\omega)|\,\mathrm{d}\theta(\mathbf{y},\omega)\Big)|\widehat{\phi}(\mathbf{n},\chi)| \leq \sum_{\mathbf{k}}\frac{1}{\mathbf{k}!}\|w\|\pi^{|\mathbf{k}|}\|\phi\|_\infty = \|w\|\|\phi\|_\infty\exp(a\pi) < \infty. \tag{3}$$

We now regard the function ϕ as being in $\mathfrak{C}(\mathbb{T}^a\times J)$, vanishing outside of the open subset $A =]-\pi,\pi[^a\times J$. The inequalities (3) show that the function

$$(\mathbf{n},\chi\mathsf{A}) \to \int_{C_{\mathbf{n}}\times(\chi\mathsf{A})}|w|\,\mathrm{d}\theta \tag{4}$$

is a multiplier for the function space $\mathfrak{C}(F)$ considered as a linear subspace of $\mathfrak{C}(\mathbb{T}^a\times J)$.

The general form of the Orlicz–Paley–Sidon theorem proved in [6], §4, admits the following corollary. Let L be a compact Abelian group and A any nonvoid open subset of L. If w is a function on the character group $\mathsf{\Lambda}$ of L that acts as a multiplier for all continuous functions on L vanishing outside of A, then w is in $l_2(\mathsf{\Lambda})$. Hence the function (4) is in l_2, that is, the function w is in $\mathfrak{S}_2(\mathsf{X})$. □

(4.6) **Theorem.** *Let G be a group of the form H, containing a compact open subgroup J. If w belongs to $\mathfrak{M}(\mathfrak{C}_k(G))$, then w also belongs to $\mathfrak{S}_2(\mathsf{X})$.*

Proof. Consider any continuous ϕ on G with support contained in J. The function $\widehat{\phi}$ is constant on cosets $\chi\mathsf{A}$ of A in Y. We have

$$\infty > \int_{\mathsf{Y}}|w(\omega)\widehat{\phi}(\omega)|\,\mathrm{d}\omega = \sum_{\chi\mathsf{A}\in(\mathsf{Y}/\mathsf{A})}\Big(\int_{\chi\mathsf{A}}|w(\omega)|\,\mathrm{d}\omega\Big)|\widehat{\phi}(\chi\mathsf{A})|,$$

and so the function $\chi\mathsf{A} \to \int_{\chi\mathsf{A}}|w|\,\mathrm{d}\theta$ belongs to $\mathfrak{M}(\mathfrak{C}(J))$. The generalized Orlicz–Paley–Sidon theorem [6] shows that $w \in \mathfrak{S}_2(\mathsf{X})$.

(4.7) *Note.* Theorem (4.6) contains as a trivial subcase the case in which G discrete. Here $\mathfrak{S}_2(\mathsf{X})$ is $\mathfrak{L}_1(\mathsf{X})$, which is obviously equal to $\mathfrak{M}(\mathfrak{C}_k(G))$.

§5. Completion of the Proofs of the Main Theorems

(5.1) We have

$$\mathfrak{C}_k(G) \subset \mathfrak{L}_{q,k}(G) \subset \mathfrak{L}_{2,k}(G) \tag{1}$$

for $2 < q \leq \infty$, and so the inclusions

$$\mathfrak{M}(\mathfrak{L}_{2,k}(G)) \subset \mathfrak{M}(\mathfrak{L}_{q,k}(G)) \subset \mathfrak{M}(\mathfrak{C}_k(G)) \tag{2}$$

obtain. By Theorem (3.8), $\mathfrak{S}_2(\mathsf{X})$ is a subset of the set on the left of (2). By Theorem (4.5), it is a superset of the set on the right of (2). It follows that all of the sets appearing in (2) are equal to $\mathfrak{S}_2(\mathsf{X})$. Thus Theorem (1.12) is proved.

(5.2) We turn next to (1.13). The number p is such that $1 < p < 2$. Let G be nondiscrete. Write G in the form $\mathbb{R}^a\times H$, and consider the related compact group $\mathbb{T}^a\times J$. Since G is nondiscrete, we have either $a > 0$ or J is infinite, and so $\mathbb{T}^a\times J$ is an infinite compact Abelian group. Its character group $\mathbb{Z}^a\times\mathsf{Z}$ is infinite and contains at least $2^{\aleph_0}$ infinite dissociate sets (see for example [7], Vol. II, (37.18)). Let Δ be a countably infinite dissociate subset of $\mathbb{Z}^a \times \mathsf{Z}$. Since Δ is a Sidon set (*loc. cit.* (37.15)), it is a Λ_2–set (*loc. cit.* (37.10)). As proved *loc. cit.* (37.9), we have

$$\sum_{\mathbf{n}\times(\chi\mathsf{A})\in\Delta} |\widehat{g}(\mathbf{n},\chi\mathsf{A})|^2 < \infty \tag{1}$$

for all $g \in \mathfrak{L}_p(\mathbb{T}^a \times J)$. Let ψ be a complex–valued function on $\mathbb{Z}^a \times \mathsf{Z}$ that vanishes outside of Δ, is in $l_2(\mathbb{Z}^a \times \mathsf{Z})$, and is in no space $l_\eta(\mathbb{Z}^a \times \mathsf{Z})$ for $\eta < 2$. For $(\mathbf{y},\omega) \in \mathbb{R}^a \times \mathsf{Y}$, define

$$w(\mathbf{y},\omega) = \psi(\mathbf{n},\chi\mathsf{A}) \qquad \text{if } n_j \le y_j < n_j + 1 \quad (j = 1,\ldots,a) \quad \text{and } \omega \in \chi\mathsf{A}. \tag{2}$$

If G contains no factor $\mathbb{R}^a$, there are obvious alterations to be made in the definition of ψ and w.

Now consider functions $g \in \mathfrak{L}_p(\mathbb{R}^a \times H)$ that vanish λ–almost everywhere outside of $[-\pi,\pi]^a \times J = F$. As above, we denote the set of these functions by $\mathfrak{L}_p(F)$. We may identify $\mathfrak{L}_p(F)$ with $\mathfrak{L}_p(\mathbb{T}^a \times J)$. Hence (1) holds for all $g \in \mathfrak{L}_p(F)$. From the definitions of ψ and w, we have

$$\sum_{(\mathbf{n},\chi\mathsf{A})\in\Delta} |w(\mathbf{n},\chi\mathsf{A})|\,|\widehat{g}(\mathbf{n},\chi\mathsf{A})| < \infty \tag{3}$$

for all $g \in \mathfrak{L}_p(F)$. As in the proof of Lemma (4.2), the closed graph theorem shows that there is a positive real number α such that

$$\sum_{(\mathbf{n},\chi\mathsf{A})\in\Delta} |w(\mathbf{n},\chi\mathsf{A})|\,|\widehat{g}(\mathbf{n},\chi\mathsf{A})| \le \alpha\|g\|_p \tag{4}$$

for all $g \in \mathfrak{L}_p(F)$. We will not repeat the details.

We will show that $\widehat{g}w \in \mathfrak{L}_1(\mathsf{X})$ for all $g \in \mathfrak{L}_{p,k}(G)$. All such functions g are linear combinations of translates of functions in $\mathfrak{L}_p(F)$. Hence we may suppose without loss of generality that $g \in \mathfrak{L}_p(F)$. We borrow the notation of (4.3), the result of (4.4), and the proof of (4.5), to compute as follows:

$$\begin{aligned}
\int_{C_\mathbf{n}\times(\chi\mathsf{A})} |w\widehat{g}|\,\mathrm{d}\theta =& |\psi(\mathbf{n},\chi\mathsf{A})| \int_{\chi\mathsf{A}}\int_{C_\mathbf{n}} |\widehat{g}(\mathbf{y},\omega)|\,\mathrm{d}\mathbf{y}\,\mathrm{d}\omega \\
=& |\psi(\mathbf{n},\chi\mathsf{A})| \int_{C_\mathbf{n}} |\widehat{g}(\mathbf{y},\chi\mathsf{A})|\,\mathrm{d}\mathbf{y} \\
=& |\psi(\mathbf{n},\chi\mathsf{A})| \int_{C_\mathbf{n}} |\sum_{\mathbf{k}} \frac{1}{\mathbf{k}!}\widehat{g}_\mathbf{k}(\mathbf{n},\chi\mathsf{A})(\mathbf{y}-\mathbf{n})^\mathbf{k}|\,\mathrm{d}\mathbf{y} \\
\le& \sum_{\mathbf{k}} |\psi(\mathbf{n},\chi\mathsf{A})| \frac{1}{\mathbf{k}!} |\widehat{g}_\mathbf{k}(\mathbf{n},\chi\mathsf{A})|.
\end{aligned} \tag{5}$$

Sum both ends of (5) over all $(\mathbf{n},\chi\mathsf{A})$ in Δ, note that $\|g_\mathbf{k}\|_p \le (2\pi)^{|\mathbf{k}|}\|g\|_p$, and use (4). We find that

$$\begin{aligned}
\int_{\mathsf{X}} |w\widehat{g}|\,\mathrm{d}\theta \le& \sum_{\mathbf{k}} \frac{1}{\mathbf{k}!} \sum_{(\mathbf{n},\chi\mathsf{A})\in\Delta} |\psi(\mathbf{n},\chi\mathsf{A})||\widehat{g}_\mathbf{k}(\mathbf{n},\chi\mathsf{A})| \\
\le& \sum_{\mathbf{k}} \frac{1}{\mathbf{k}!}\alpha(2\pi)^{|\mathbf{k}|}\|g\|_p \\
\le& \alpha\exp(2\pi a)\|g\|_p \\
<& \infty.
\end{aligned}$$

The definitions of ψ and w show that w is in no space $\mathfrak{S}_p(\mathsf{X})$ for $p < 2$, while at the same time it is in all of the spaces $\mathfrak{M}(\mathfrak{L}_{p,k}(G))$ for $1 < p \le 2$. This fact and Theorem (3.8) show that $\mathfrak{S}_p(\mathsf{X})$ is a proper subspace of $\mathfrak{M}(\mathfrak{L}_{p,k}(G))$ for all nondiscrete G, that is, (1.13) holds.

(5.3) *Proof of* (1.14). Let w be any function in $\mathfrak{M}(\mathfrak{L}_{1,k}(G))$, G being an arbitrary locally compact Abelian group. Let F be a compact subset of G. As in (4.2), we prove easily that there is a positive number α such that

$$\int_{\mathsf{X}} |w\widehat{g}|\,\mathrm{d}\theta \le \alpha\|g\|_1 \tag{1}$$

for all $g \in \mathfrak{L}_1(F)$. Let F now be a compact set whose interior contains the identity element of G. As shown in [7], Vol. II, Theorem (33.11), there is a net $\{h_\kappa\}_{\kappa\in K}$ of continuous functions on G with supports contained in F such that $\|h\|_1 = 1$, all Fourier transforms $\widehat{h}$ are real and nonnegative, and $\lim_\kappa \widehat{h}_\kappa(\chi) = 1$, uniformly on compact subsets of X. Thus, given a compact subset ϕ of X and a positive number $\eta < 1$, we find a member κ of the net K such that

$$\eta \int_\phi |w|\,\mathrm{d}\theta \le \int_\phi |w\widehat{h}_\kappa|\,\mathrm{d}\theta \le \int_{\mathsf{X}} |w\widehat{h}_\kappa|\,\mathrm{d}\theta \le \alpha. \tag{2}$$

The inequalities (2) prove that

$$\sup\{\int_\Phi |w|\,\mathrm{d}\theta : \Phi \text{ compact in } \mathsf{X}\} \le \alpha. \tag{3}$$

It is elementary to show from (3) that w differs from a function in $\mathfrak{L}_1(\mathsf{X})$ by a function that is locally θ–null.

§6. Examples and Remarks

(6.1) *Explanation.* Applying the inequalities of Hausdorff–Young and Hölder, we see at once that $\mathfrak{L}_p(\mathsf{X}) \subset \mathfrak{M}(\mathfrak{C}_k(G))$ for $1 \le p \le 2$. It is easy to see that the sum $\mathfrak{L}_1(\mathsf{X}) + \mathfrak{L}_2(\mathsf{X}) = \{\mathsf{g}_1 + \mathsf{g}_2 : \mathsf{g}_\mathsf{j} \in \mathfrak{L}_\mathsf{j}(\mathsf{X})$ for $j = 1, 2\}$ is the smallest linear space containing all of the spaces $\mathfrak{L}_p(\mathsf{X})$ $(1 \le p \le 2)$. One might conjecture that $\mathfrak{M}(\mathfrak{C}_k(G)) = \mathfrak{L}_1(\mathsf{X}) + \mathfrak{L}_2(\mathsf{X})$. We will show that this is *not* the case unless G is compact or discrete.

(6.2) **Lemma.** *Let $(X, \mathcal{M}, \mu)$ be an arbitrary measure space. Let p be a real number greater than 1. Let h be a bounded, complex–valued, $\mathcal{M}$–measurable function on X such that $h = f + g$ with $f \in \mathfrak{L}_1(X, \mu)$ and $g \in \mathfrak{L}_p(X, \mu)$. Then both of the functions f and h are also in $\mathfrak{L}_p(X, \mu)$.*

Proof. Write

$$h_0 = \max\{\mathrm{Re}\, h, 0\}, \quad h_1 = \max\{\mathrm{Im}\, h, 0\},$$
$$h_2 = -\mathrm{Re}\, h + h_0, \qquad h_3 = -\mathrm{Im}\, h + h_1.$$

Let f_0, f_1, f_2, f_3, and g_0, g_1, g_2, g_3 be the analogous decompositions of f and g respectively. Plainly the functions h_l, f_l, and g_l are nonnegative, the h_l's are bounded, and also the identities

$$\begin{aligned} h_0 - h_2 &= \mathrm{Re}\, h = (f_0 + g_0) - (f_2 + g_2) \\ h_1 - h_3 &= \mathrm{Im}\, h = (f_1 + g_1) - (f_3 + g_3) \end{aligned} \tag{1}$$

obtain. We also have

$$\min\{w_0, w_2\} = \min\{w_1, w_3\} = 0$$

for w any of our functions h, f, and g. From this fact and (1) we infer that

$$h_l \le f_l + g_l \qquad (l = 0, 1, 2, 3). \tag{2}$$

Let ψ_l be the function equal to $h_l(f_l + g_l)^{-1}$ where $f_l + g_l > 0$ and zero where $f_l + g_l = 0$. It is plain that

$$h_l = \psi_l f_l + \psi_l g_l \tag{3}$$

and that $\psi_l f_l \in \mathfrak{L}_1(X, \mu)$ and $\psi_l g_l \in \mathfrak{L}_p(X, \mu)$. Since $\psi_l g_l$ is nonnegative, $\psi_l f_l$ is bounded. Being bounded and in $\mathfrak{L}_1(X, \mu)$, $\psi_l f_l$ is in $\mathfrak{L}_p(X, \mu)$. From (3) we infer that h_l is in $\mathfrak{L}_p(X, \mu)$, and since

$$h = \sum_{l=0}^{3} i^l h_l,$$

it follows that h is in $\mathfrak{L}_p(X, \mu)$. □

(6.3) We will construct bounded functions in $\bigcap_{1<p\le 2} \mathfrak{S}_p(\mathsf{X})$ that are not in $\mathfrak{L}_1(\mathsf{X}) + \mathfrak{L}_2(\mathsf{X})$, for all character groups X that are neither compact nor discrete. Suppose that $\mathsf{X} = \mathbb{R}^a \times \mathsf{Y}$ with $a > 0$. In this case, we define w on $\mathbb{R}^a \times \mathsf{Y}$ by:

$$w(u_1, \ldots, u_a, \chi) = \begin{cases} 1 & \text{if for some } n \in \mathbb{N}, \text{ we have } n \le u_1 \le n + n^{-1}, \\ & 0 \le u_j \le 1 \quad \text{for } j = 2, \ldots, a, \text{ and } \chi \in \mathsf{A}; \\ 0 & \text{for all other points } (u_1, \ldots, u_a, \chi) \text{ in } \mathbb{R}^a \times \mathsf{Y}. \end{cases}$$

It is plain that

$$\int_{B_\iota} |w|\,\mathrm{d}\theta = \begin{cases} \frac{1}{n} & \text{for the blocks } B_\iota \text{ on which } w \neq 0, \\ 0 & \text{for all other blocks } B_\iota. \end{cases}$$

Thus we have

$$\sum_{\iota \in I} \Big[\int_{B_\iota} |w|\,\mathrm{d}\theta \Big]^p = \sum_{n=1}^{\infty} n^{-p} < \infty,$$

that is, w is in all of the spaces $\mathfrak{S}_p(\mathsf{X})$ with $p > 1$. It is also clear that $\int_{\mathsf{X}} |w|^2\,\mathrm{d}\theta = \sum_{n=1}^{\infty} n^{-1} = \infty$. By Lemma (6.2), w does not have the form $f + g$ for $f \in \mathfrak{L}_1(\mathsf{X})$ and $f \in \mathfrak{L}_2(\mathsf{X})$.

Next, suppose that G has the form H, where the compact open subgroup J of H is infinite. This implies that the quotient group Y/A, which is isomorphic with the character group of J, is also infinite. Each block B_ι is a coset $\chi\mathsf{A}$. Choose a countably infinite family $\{\chi_n\mathsf{A}\}_{\mathsf{n}=1}^{\infty}$ of distinct cosets. In each coset $\chi_n\mathsf{A}$, select a measurable subset Γ_n such that $\theta(\Gamma_n) = n^{-1}$. Define the function w as $\sum_{n=1}^{\infty} \xi_{\Gamma_n}$. A simple computation as above shows that $w \in \mathfrak{S}_p(\mathsf{X})$ for all $q > 1$ while $w \notin \mathfrak{L}_1(\mathsf{X}) + \mathfrak{L}_2(\mathsf{X})$.

(6.4) There is a considerable latitude in the choice of the blocks B_ι. First, as noted in (1.3), J can be *any* compact open subgroup of H. (There may be exactly one such subgroup, or there may, as in the case for example of the p–adic numbers, be an infinite number of such subgroups.) The annihilator A of J in Y plainly varies with the choice of J. Second, the sets $C_{\mathbf{n}}$ in $\mathbb{R}^a$ need not be translates of the unit hypercube. In fact, they need not be cubes at all. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_a)$ be any sequence of positive real numbers, and let $D_{\mathbf{n}} = \{\mathbf{y} \in \mathbb{R}^a : n_j\alpha_j \leq y_j \leq (n_j + 1)\alpha_j \text{ for } j = 1, 2, \ldots, \alpha\}$. Our entire proof goes through unaltered if we replace $C_{\mathbf{n}}$ by $D_{\mathbf{n}}$ throughout. Only some details need to be changed.

(6.5) *Remark.* Theorem (1.12) admits a small extension, as follows. Let G be arbitrary, and let A be any open subset of G such that $0 < \lambda(A) < \infty$. It is easy to show, using partitions of unity, that every function in $\mathfrak{C}_k(G)$ is a finite linear combination of translates of continuous functions vanishing outside of A. From this it follows that $\mathfrak{M}(\mathfrak{C}(A)) = \mathfrak{S}_2(\mathsf{X})$.

(6.6) *Another Proof of Gronwall's Theorem.* In 1921, Gronwall published the following theorem. Let ω be any nonnegative real–valued function on $[0, \infty[$ such that $\lim_{t\to\infty} \omega(t) = \infty$. There exists a function $f \in \mathfrak{C}(\mathbb{T})$ such that

$$\sum_{n=1}^{\infty} |\widehat{f}(n)|^2 \omega(|\widehat{f}(n)|^{-1}) = \infty.$$

For the history of the theorem, see the discussion in [6], §3. Gronwall's theorem was extended in [6], §3 to all nondiscrete locally compact Abelian groups G, as follows.

Let ω be a Lebesgue measurable function mapping $]0, \infty[$ into some half line $[\alpha, \infty[$ with the property that $\lim_{t\to\infty} \omega(t) = \infty$. There is a function $\phi \in \mathfrak{C}_k(G)$ such that

$$\int_{\mathsf{X}} |\widehat{\phi}|^2 \omega(|\widehat{\phi}|^{-1})\,\mathrm{d}\theta = \infty. \tag{1}$$

The proof in [6] of this theorem is explicit but it is also complicated. Since the theorem follows easily from Theorem (1.12), it seems worthwhile to sketch the proof here. We follow Paley [11], p. 122, who pointed out that Gronwall's original theorem follows from the Orlicz–Paley–Sidon theorem (1.1).

First define a function Ω on $]0, \infty[$ by

$$\Omega(x) = x\omega(x^{-1}).$$

Replacing ω if necessary by a smaller function, we may arrange matters so that Ω is continuous, strictly increasing, and also has the property that $\lim_{x\to 0} \Omega(x) = 0$. We omit the details. We define $\Omega(0)$ as 0. Let Ψ be the inverse function to Ω. It is easy to see that

$$\lim_{y\to 0} \Psi(y)y^{-1} = 0. \tag{2}$$

A simple geometric argument shows that

$$\beta\gamma \leq \beta\Psi(\beta) + \gamma\Omega(\gamma) \tag{3}$$

for all nonnegative real numbers β and γ. On account of (2), there is a sequence $(\beta_n)_{n=1}^{\infty}$ of positive real numbers such that

$$\sum_{n=1}^{\infty} \beta_n^2 = \infty \tag{4}$$

and

$$\sum_{n=1}^{\infty} \beta_n \Psi(\beta_n) < \infty. \tag{5}$$

Since the group G is nondiscrete, there is an infinite sequence $(B_n)_{n=1}^{\infty}$ of pairwise disjoint blocks in the character group X. Define a function v on X by

$$v = \sum_{n=1}^{\infty} \beta_n \xi_{B_n}.$$

By (4), v does not belong to the space $\mathfrak{S}_2(\mathsf{X})$. By Theorem (1.12), there exists a function ϕ in $\mathfrak{C}_k(G)$ such that

$$\int_{\mathsf{X}} v|\widehat{\phi}| \,\mathrm{d}\theta = \infty. \tag{6}$$

Using (6) and (3), we write

$$\infty = \int_{\mathsf{X}} v|\widehat{\phi}| \,\mathrm{d}\theta \leq \int_{\mathsf{X}} v\Psi(v) \,\mathrm{d}\theta + \int_{\mathsf{X}} |\widehat{\phi}|\Omega(|\widehat{\phi}|) \,\mathrm{d}\theta = \sum_n \beta_n \Psi(\beta_n) + \int_{\mathsf{X}} |\widehat{\phi}|\Omega(|\widehat{\phi}|) \,\mathrm{d}\theta.$$

Referring to (5) and the definition of Ω from ω, we see that (1) holds for the above choice of ϕ.

§7. An Application to Hyperbolic Differential Equations

(7.1) *Explanation and Notation.* Let $\mathcal{D}'(\mathbb{R}^a)$ denote the space of all distributions on $\mathbb{R}^a$ and $\mathcal{S}'(\mathbb{R}^a)$ the space of all temperate distributions on $\mathbb{R}^a$. Let P be a polynomial in the space $\mathbb{C}[X_1, \ldots, X_a]$ of complex polynomials in a indeterminates. Let $P(D_1, \ldots, D_a)$ be the corresponding linear differential operator with constant coefficients on $\mathbb{R}^a$ (as usual, D_k is $-i\frac{\partial}{\partial x_k}$). Let P_0 be the principal part of P, and $\mathbf{b}$ a nonzero element of $\mathbb{R}^a$. Let $H_{\mathbf{b}}$ denote the set $\{\mathbf{y} \in \mathbb{R}^a : \langle \mathbf{y}, \mathbf{b}\rangle \geq 0\}$. We shall need the following theorem of L. Gårding (see for example [4], p. 407).

(7.2) **Theorem**. *Let $P(D_1, \ldots, D_a)$ be any differential operator as in* (7.1). *The following are equivalent:*

(i) *there is an elementary solution E of $P(D_1, \ldots, D_a)$ for which* $\mathrm{supp}(E) \cap H_{\mathbf{b}} = \{\mathbf{0}\}$;

(ii) *we have $P_0(\mathbf{B}) \neq 0$ and there is a real number t_0 such that $P(\mathbf{y}+it\mathbf{b}) \neq 0$ for all $t > t_0$ and all $\mathbf{y} \in \mathbb{R}^a$.*

(7.3) As is well known (see for example [4], 5.18.5, p. 407) the differential operator $P(D_1, \ldots, D_a)$ is called *hyperbolic* if it has an elementary solution E whose support is contained in a proper cone with vertex at $\mathbf{0}$. In [13], Ritter has given a sufficient condition that such an E should be a Radon measure. With the use of Theorem (1.12) above, the result of [13] can be sharpened.

(7.4) **Theorem**. *Let $P(D_1, \ldots, D_a)$ be hyperbolic, and let E be an elementary solution whose support is contained in a proper cone with vertex at* $\mathbf{0}$ *and for which* $\mathrm{supp}(E) \cap H_{\mathbf{b}} = \{\mathbf{0}\}$. *Let t_0 be as in* (7.2). *For $t > t_0$, the following are equivalent:*

(i) *the function $\mathbf{y} \to [P(\mathbf{y} + it\mathbf{b})]^{-1}$ belongs to $\mathfrak{S}_2(\mathbb{R}^a)$;*

(ii) *the elementary solution E is a Radon measure with the property that for all $\phi \in \mathfrak{C}_k(\mathbb{R}^a)$, we have*

(a) $E * \phi(\mathbf{y}) = \exp(-t\langle \mathbf{y}, \mathbf{b}\rangle)\widehat{f}(\mathbf{y})$
for some $f \in \mathfrak{L}_1(\mathbb{R}^a)$ and all $\mathbf{y} \in \mathbb{R}^a$.

Proof. Let F be the distribution

$$\exp(t\langle \cdot, \mathbf{b}\rangle)E.$$

We have

$$\widehat{F}(\mathbf{y}) = [P(\mathbf{y} + it\mathbf{b})]^{-1}. \tag{1}$$

From [4], (5.18.10), p. 406, we infer that $\widehat{F}$ is bounded. Therefore $\widehat{F}$ and F are in $\mathcal{S}'(\mathbb{R}^a)$.

Suppose that (i) holds. By Theorem (1.12), we have

$$\widehat{\phi}[P(\cdot + it\mathbf{b})]^{-1} \in \mathfrak{L}_1(\mathbb{R}^a) \tag{2}$$

for all $\phi \in \mathfrak{C}_k(\mathbb{R}^a)$. From (1), (2), and [4], (5.15.28), p. 388, we find

$$\widehat{F * \phi} = \widehat{F}\widehat{\phi} \in \mathfrak{L}_1(\mathbb{R}^a)$$

and so

$$F * \phi \in \widehat{\mathfrak{L}}_1(\mathbb{R}^a) \subset \mathfrak{C}_0(\mathbb{R}^a). \tag{3}$$

A classical theorem of L. Schwartz (see [15], p. 192) implies that F is a Radon measure. Hence E is also a Radon measure. On the other hand, if $\phi \in \mathfrak{C}_k(\mathbb{R}^a)$, we have

$$\begin{aligned} E * \phi(\mathbf{x}) &= \int_{\mathbb{R}^a} \phi(\mathbf{x} - \mathbf{s})\, \mathrm{d}E(\mathbf{s}) \\ &= \int_{\mathbb{R}^a} \phi(\mathbf{x} - \mathbf{s}) \exp(-t\langle \mathbf{s}, \mathbf{b}\rangle)\, \mathrm{d}F(\mathbf{s}) \\ &= \exp(-t\langle \mathbf{x}, \mathbf{b}\rangle) \int_{\mathbb{R}^a} \phi(\mathbf{x} - \mathbf{s}) \exp(t\langle \mathbf{x} - \mathbf{s}, \mathbf{b}\rangle)\, \mathrm{d}F(\mathbf{s}) \\ &= \exp(-t\langle \mathbf{x}, \mathbf{b}\rangle)\big[F * (\exp(t\langle \cdot, \mathbf{b}\rangle)\phi)\big](\mathbf{x}). \end{aligned}$$

From (3) we infer that $E * \phi$ has the form

$$\exp(-\tau\langle \cdot, \mathbf{b}\rangle)\widehat{f}$$

for some $f \in \mathfrak{L}_1(\mathbb{R}^a)$.

Suppose conversely that (ii) holds. Property (a) and the argument just used show that

$$F * \phi \in \widehat{\mathfrak{L}}_1(\mathbb{R}^a) \qquad \text{for all } \phi \in \mathfrak{C}_k(\mathbb{R}^a).$$

Thus we have

$$\widehat{\phi}[P(\cdot + it\mathbf{b})]^{-1} = \widehat{F}\widehat{\phi} = \widehat{F * \phi} \in \mathfrak{L}_1(\mathbb{R}^a).$$

By (1.12), the function $[P(\cdot + it\mathbf{b})]^{-1}$ belongs to $\mathfrak{S}_2(\mathbb{R}^a)$. □

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