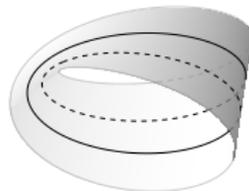
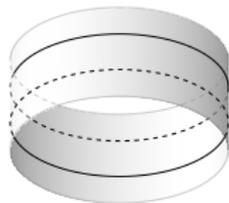


Understanding uncolored CFI-graphs

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Joint work with Jörg Flum (Freiburg) and Mingjun Liu (Regensburg)



In 1992, Cai, Fürer, and Immerman gave a construction of two graphs $X(G)$ and $\tilde{X}(G)$ from any given graph G . They are **not isomorphic** but look **locally the same everywhere**, like a strip vs. a Möbius strip.



Jin-Yi **C**ai



Martin **F**ürer



Neil **I**mmerman

The CFI-graphs were constructed to solve two then major open problems, both answered negatively using the CFI-graphs.

1. For any fixed $k \geq 1$, does the k -dimensional Weisfeiler-Leman algorithm (k -WL) decide the graph isomorphism problem?
2. Does fixed-point logic with counting capture polynomial time?

Theorem (Cai, Fürer, and Immerman, 1992)

For every $k \geq 1$, there exists a graph G whose vertices are assigned distinct colors such that the corresponding CFI-graphs $X(G)$ and $\tilde{X}(G)$ have the following properties.

1. $X(G)$ and $\tilde{X}(G)$ are not isomorphic.
2. $X(G)$ and $\tilde{X}(G)$ cannot be distinguished by k -WL.
3. Both $X(G)$ and $\tilde{X}(G)$ are 3-regular graphs with $O(k)$ vertices.

Moreover, there is a polynomial time algorithm accepting all $X(G)$'s and rejecting all $\tilde{X}(G)$'s.

Let $Y(G)$ and $\tilde{Y}(G)$ be obtained from $X(G)$ and $\tilde{X}(G)$ by removing all colors.

Theorem (C. , Flum, and Liu, 2025)

For every $k \geq 1$ there is a graph G such that:

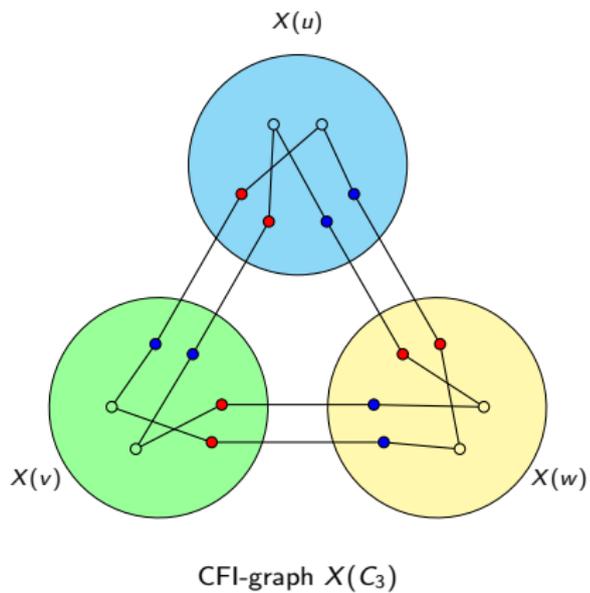
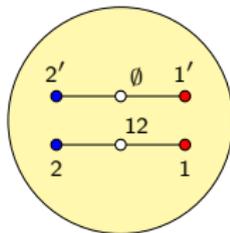
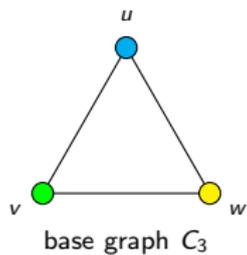
1. $Y(G)$ and $\tilde{Y}(G)$ are not isomorphic.
2. $Y(G)$ and $\tilde{Y}(G)$ cannot be distinguished by k -WL.
3. Both $Y(G)$ and $\tilde{Y}(G)$ are 3-regular graphs with $O(k)$ vertices.

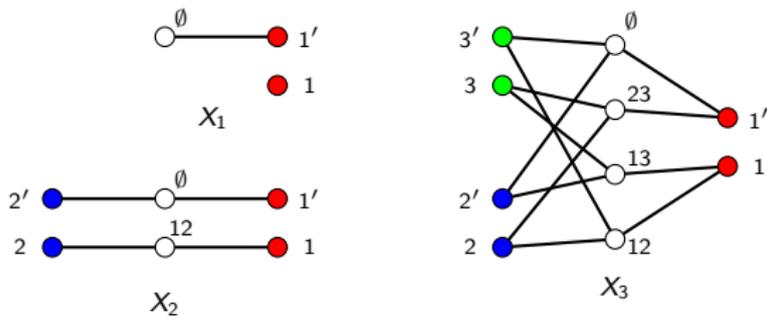
Moreover, there is a *polynomial time algorithm* accepting all $Y(G)$'s and rejecting all $\tilde{Y}(G)$'s.

Why we wanted to remove the colors?

1. For some applications, we might be more interested in **uncolored graphs**:
 - Can some k -WL decide graph isomorphism on uncolored 3-regular graphs?
 - The question whether fixed-point logic with counting captures polynomial time only makes sense on graphs with **bounded number of colors**.
2. Clearly we can remove colors by adding gadgets [CFI 1992; Otto 1997; Dawar and Richerby, 2007], but some important parameters of the resulting graphs deteriorate, e.g., **size** and **degree**.
3. Recently, Roberson (also [Fuhlbrück et al., 2021]) gave an uncolored variant of the CFI-graphs with many of the same properties of the original CFI-graphs. But the degree of Roberson's variant is very large, and the corresponding polynomial time graph isomorphism testing is unknown.

1. For every $d \geq 1$ we construct a CFI-gadget X_d .
2. Given a base graph G , we replace every vertex $v \in V(G)$ with a copy $X(v)$ of X_d where $d = \deg(v)$. For every edge $uv \in E(G)$ we add two edges between $X(u)$ and $X(v)$. This gives us the first graph $X(G)$.
3. To obtain $\tilde{X}(G)$ we choose an edge $uv \in E(G)$ and twist two edges between $X(u)$ and $X(v)$ in $X(G)$, like turning a strip to a Möbius strip.





1. The **middle** vertices with color white are not adjacent to any vertex outside X_d .
2. Vertices colored **red**, **green**, **blues**, etc. will be adjacent to vertices in other CFI-gadgets.

(i, i') is a pair of same-colored vertices which play a role for the i -th adjacent edge of a given vertex.

Let G be a graph, i.e., the **base** graph. The graph $X(G)$ is defined as follows.

1. Replace every $u \in V(G)$ by a copy $X(u)$ of X_d with $d = \deg(u)$. Assume $uv \in E(G)$ is the i -th edge adjacent to u , we rename

$$i \mapsto a(u, v) \quad \text{and} \quad i' \mapsto b(u, v).$$

2. For every edge $uv \in E(G)$ we add two edges

$$\{a(u, v), a(v, u)\} \quad \text{and} \quad \{b(u, v), b(v, u)\}$$

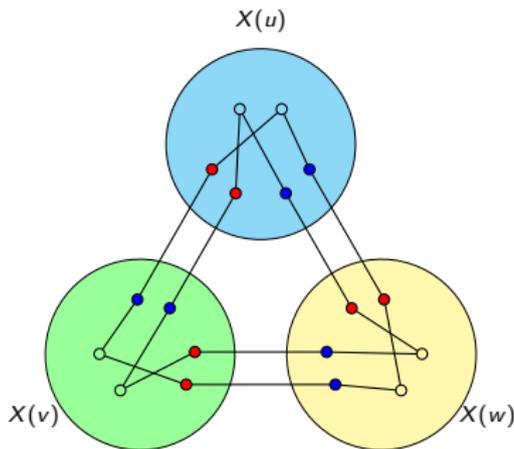
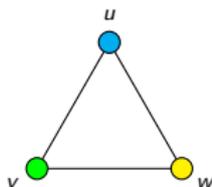
between $X(u)$ and $X(v)$.

The coloring of $X(G)$

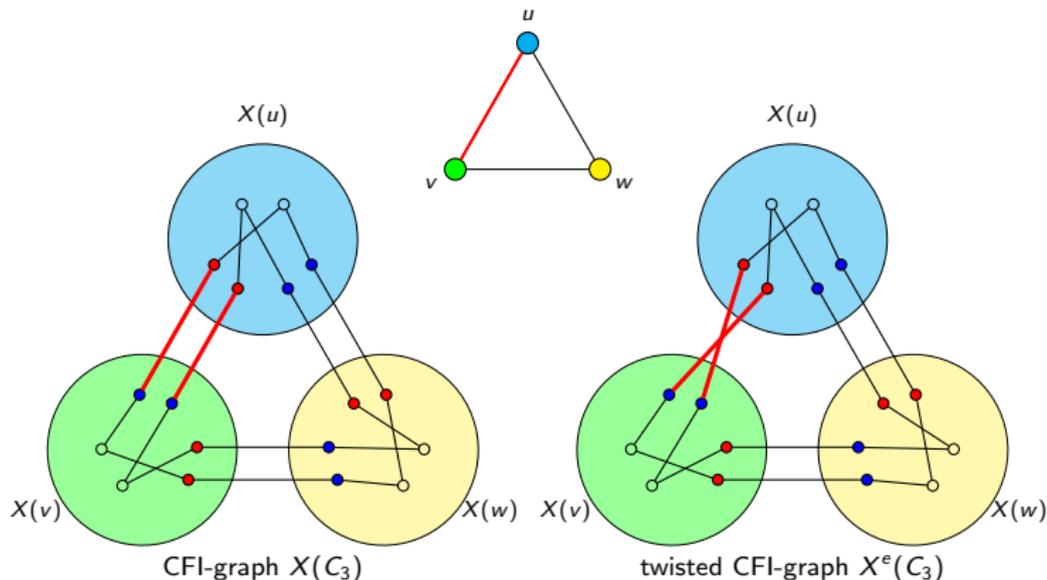
1. We assume every $v \in V(G)$ has a **distinct color**, then all vertices in $X(v)$ inherit the color of v .
2. Assume in $X(u)$ we have renamed

$$i \mapsto a(u, v) \quad \text{and} \quad i' \mapsto b(u, v).$$

Then the color of $a(u, v)$ and $b(u, v)$ consists of the color of u and the color of i and i' .



Let $e = uv \in E(G)$. Then $X^e(G)$ is obtained from $X(G)$ by **twisting e** , i.e., changing $\{a(u, v), a(v, u)\}$ and $\{b(u, v), b(v, u)\}$ to $\{a(u, v), b(v, u)\}$ and $\{b(u, v), a(v, u)\}$.



Lemma

Let G be a connected graph and $e, e' \in E(G)$. Then $X^e(G)$ and $X^{e'}(G)$ are isomorphic.

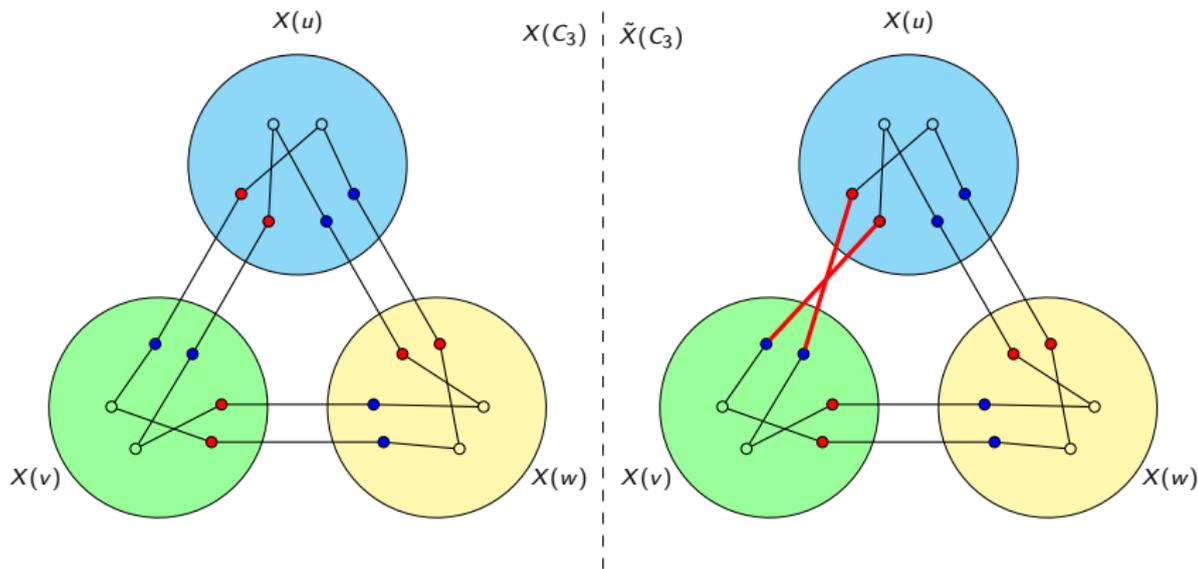
Definition

For a **connected** graph G we let

$$\tilde{X}(G) := X^e(G).$$

for an arbitrary $e \in E(G)$.

$$X(G) \not\cong \tilde{X}(G)$$



Lemma (Cai, Fürer, and Immerman, 1992)

$X(G)$ and $\tilde{X}(G)$ are not isomorphic.

What role does the coloring play in CFI's work?

Recall:

Theorem (Cai, Fürer, and Immerman, 1992)

For every $k \geq 2$ there is a pair of CFI-graphs $X(G)$ and $\tilde{X}(G)$ such that:

- 1. $X(G)$ and $\tilde{X}(G)$ are not isomorphic.*
- 2. $X(G)$ and $\tilde{X}(G)$ cannot be distinguished by k -WL.*
- 3. Both $X(G)$ and $\tilde{X}(G)$ are 3-regular graphs with $O(k)$ vertices.*

Moreover, there is a polynomial time algorithm accepting all $X(G)$'s and rejecting all $\tilde{X}(G)$'s.

CFI's argument for $X(G) \not\cong \tilde{X}(G)$ uses the coloring of $X(G)$ and $\tilde{X}(G)$ in some crucial steps. And their polynomial time algorithm also exploits this coloring.

Theorem (Implicit in Cai, Fürer, and Immerman, 1992)

For every $k \geq 2$ there is a pair of *uncolored graphs* G_k and H_k such that:

1. G_k and H_k are not isomorphic.
2. G_k and H_k cannot be distinguished by k -WL.
3. Both G_k and H_k are graphs with $O(k^2)$ vertices.

Moreover, all pairs of G_k and H_k can be distinguished in polynomial time.

This implies:

To determine an uncolored graph $G = (V, E)$ up to isomorphisms, we need k -WL with dimension at least $\Omega(\sqrt{|V|})$.

Ideally we would like to have the optimal $\Omega(|V|)$.

There are other issues, e.g., the degree of G_k and H_k are no more 3.

We obtain $Y(G)$ and $\tilde{Y}(G)$ from $X(G)$ and $\tilde{X}(G)$ by forgetting their colors.

Theorem (C. , Flum, and Liu, 2025)

For every $k \geq 2$ there is a graph G such that:

1. $Y(G)$ and $\tilde{Y}(G)$ are not isomorphic.
2. $Y(G)$ and $\tilde{Y}(G)$ cannot be distinguished by k -WL.
3. Both $Y(G)$ and $\tilde{Y}(G)$ are 3-regular graphs with $O(k)$ vertices.

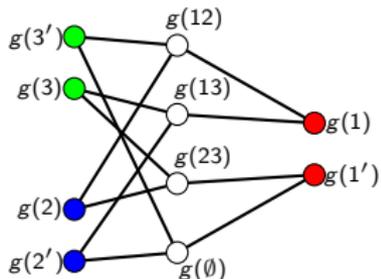
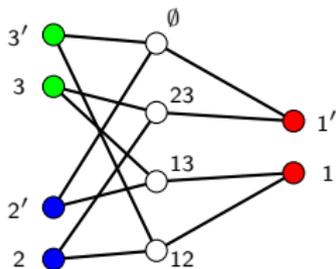
Moreover, $Y(G)$ and $\tilde{Y}(G)$ can be distinguished in polynomial time.

Why $Y(G) \not\cong \tilde{Y}(G)$ is nontrivial?

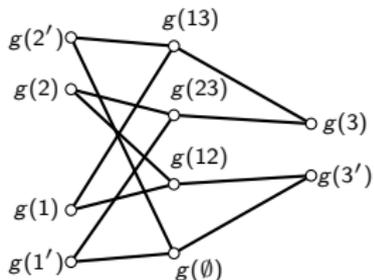
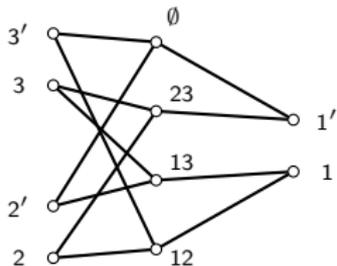
In CFI's proof of $X(G) \not\cong \tilde{X}(G)$, we assume that there is an isomorphism $g : X(G) \rightarrow \tilde{X}(G)$. Then the coloring guarantees:

1. g maps every $X(v)$ to $X(v)$, i.e., it preserves gadgets.
2. g maps every $\{i, i'\}$ to $\{i, i'\}$ inside $X(v)$, i.e., it preserves the internal structure of $X(v)$.

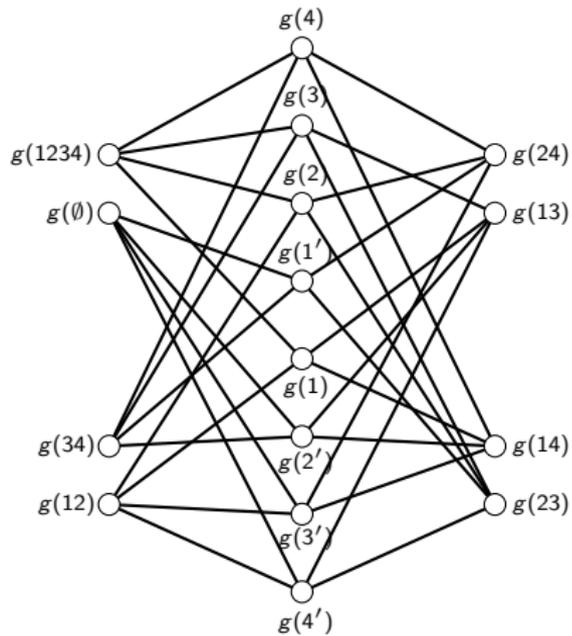
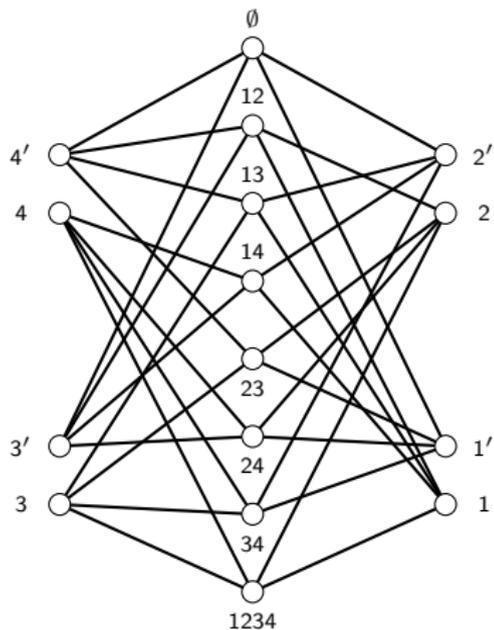
The automorphisms of CFI-gadgets



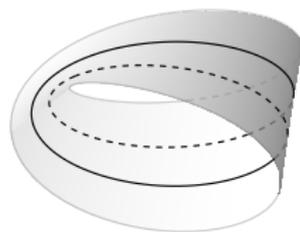
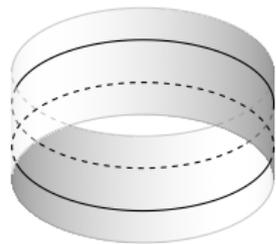
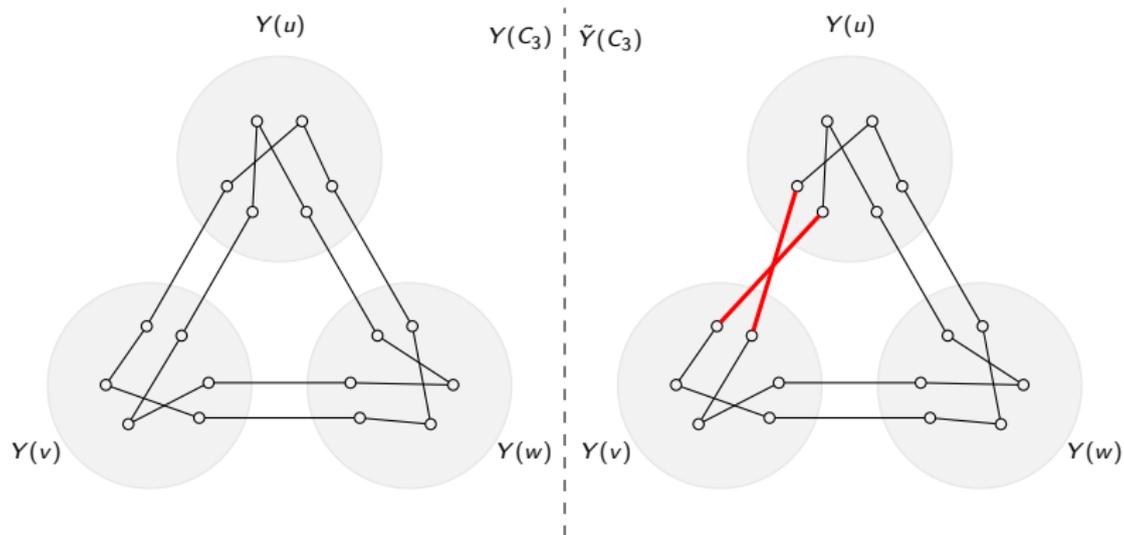
But after losing the colors, we might have



The automorphisms of uncolored CFI-gadgets



Why $Y(G) \not\cong \tilde{Y}(G)$ is nontrivial?



How we show $Y(G) \not\cong \tilde{Y}(G)$

1. We study all the automorphisms of $X(d)$ which behave well if $d \geq 5$.
2. If G has a vertex of degree at least 3, then any hypothetical isomorphism from $Y(G)$ to $\tilde{Y}(G)$ must preserve gadgets.
3. Any hypothetical gadget-preserving isomorphism from $Y(G)$ to $\tilde{Y}(G)$ can be “decomposed” into an automorphism of G and an isomorphism from $X(G)$ to $\tilde{X}(G)$ which does not exist by CFI.

Theorem (Cai, Fürer, and Immerman, 1992)

Let G be a graph without *separator* of size $\leq k$. Then k -WL cannot distinguish $X(G)$ and $\tilde{X}(G)$.

Built on [Dawar and Richerby, 2007] and [Roberson, 2022]:

Theorem (C. , Flum, and Liu, 2025)

k -WL can distinguish $Y(G)$ and $\tilde{Y}(G)$ if and only if the *tree-width* of G is at most k .

How tree-width is related to CFI-graphs?

1. Dawar and Richerby showed that if $\text{tw}(G) \geq k$, then k -WL cannot distinguish $X(G)$ and $\tilde{X}(G)$. This immediately implies that k -WL cannot distinguish $Y(G)$ and $\tilde{Y}(G)$.
2. Using a **linear-algebraic** machinery developed by Roberson we show that a topological minor of G has different **homomorphism counts** in $Y(G)$ and $\tilde{Y}(G)$. Then $Y(G)$ and $\tilde{Y}(G)$ can be distinguished by k -WL for $k = \text{tw}(G) + 1$ by [Dvorák 2010; Dell, Grohe, and Ratten, 2018]
3. We give a new proof of the result of Dawar and Richerby using Roberson's linear-algebraic machinery too.

For two graphs F and G ,

$$\text{hom}(F, G) := |\{f \mid f \text{ a homomorphism from } F \text{ to } G\}|.$$

Theorem (Dvorák 2010; Dell, Grohe, and Ratten, 2018)

Two graphs G and H can be distinguished by k -WL, if and only if there is an F with $\text{tw}(F) \leq k$ with

$$\text{hom}(F, G) \neq \text{hom}(F, H).$$

Theorem (C. , Flum, and Liu, 2025)

k -WL can distinguish $Y(G)$ and $\check{Y}(G)$ if and only if $\text{tw}(G) \leq k$. Equivalently

- There is an F with $\text{tw}(F) = \text{tw}(G)$ and $\text{hom}(F, Y(G)) \neq \text{hom}(F, \check{Y}(G))$.
- $\text{hom}(F, Y(G)) = \text{hom}(F, \check{Y}(G))$ holds for all F with $\text{tw}(F) < \text{tw}(G)$.

Our linear-algebraic proof à la Roberson

1. Both $\text{hom}(F, Y(G))$ and $\text{hom}(F, \tilde{Y}(G))$ can be understood as the number of solutions of some appropriate **systems of linear equations over \mathbb{F}_2** .
2. The systems for $\text{hom}(F, Y(G))$ are **homogeneous**, and the systems for $\text{hom}(F, \tilde{Y}(G))$ are some corresponding **nonhomogeneous** system.
Recall that a homogeneous system always has the **zero solution**. And if a nonhomogeneous system has **a solution**, then the number of its solutions equals the number of solutions of the corresponding homogeneous solution.
3. By taking F as some appropriate **topological minor** of G , in particular $\text{tw}(F) = \text{tw}(G)$, we can show that the nonhomogeneous systems for $\text{hom}(F, \tilde{Y}(G))$ are **not solvable**.
4. If $\text{tw}(F) < \text{tw}(G)$, we can show that the nonhomogeneous systems for $\text{hom}(F, \tilde{Y}(G))$ are always **solvable**.

1. The CFI-graphs prove that each k -WL is an incomplete algorithm for GI.
2. The original CFI construction is colored graphs. We show that the coloring can be removed and resulting graphs basically serve the same purpose as the original CFI-graphs.

In particular, k -WL is strictly weaker than $k + 1$ -WL on uncolored graphs.

3. There are many other applications of CFI-graphs, e.g., in finite model theory, in proof complexity, and in circuit complexity. The uncolored CFI-graphs might yield cleaner and better bounds.

Thank You!