A Bisimulation-Invariance-Based Approach to the Separation of Polynomial Complexity Classes

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joint work with Florian Bruse, TU Munich

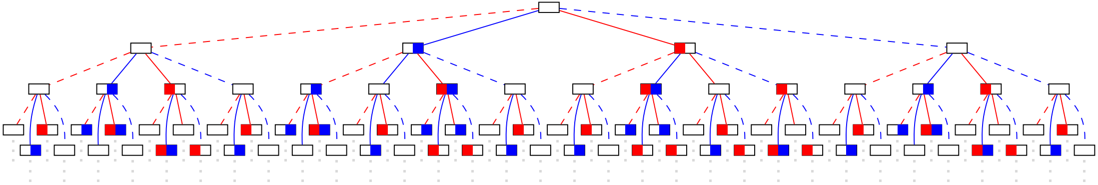
This Talk is All About Languages of Coloured Trees

…and everything is parameterised by $d = 1, 2, \ldots$

we consider unranked, unordered trees with

- multi-coloured nodes (with d available colours)
- uniquely coloured edges (with solid/dashed variant of one of d colours)

Ex.: $d = 2$



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well-understood notion of regular tree language, here (based on Janin-Walukiewicz): definability in modal μ-calulus $\mathcal{L}_\mu$

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$$L = L(\varphi)$$

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$$L \cap R \;=\; L(\varphi) \cap R$$

Ex.: careful! non-regular languages can be regular relative to non-regular ones

- L = "*there is* $n \geq 1$ *s.t. all nodes on level* n *are coloured red*"
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N.B.: proof of relative non-regularity by pumping more difficult: pumping must take tree from $L \cap R$ to $\overline{L} \cap R$

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$\text{1N}\textsc{on}\text{U}\textsc{niv}\text{NFA}_d$ = "there is some $n \geq 0$ such that for every colour i and every path from the root to node v:

there are n edges of solid colour i after the last dashed edge of colour i

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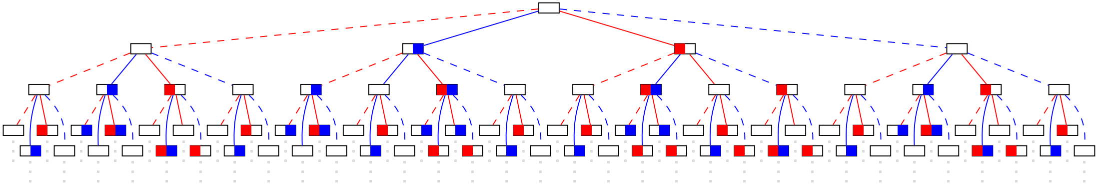

Asynchronous d-Bisimulations and the Languages Power$_d$

consider largest relations $\approx_{ij}$ such that for all $i, j \in [d]$ and nodes u, v: $u \approx_{ij} v \Longrightarrow \ldots$

- u has colour i iff v has colour j
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intuition: if $u = (u_0, \ldots, u_{d-1})$, $v = (v_0, \ldots, v_{d-1})$ then $u \approx_{ij} v$ iff $u_i \sim v_j$

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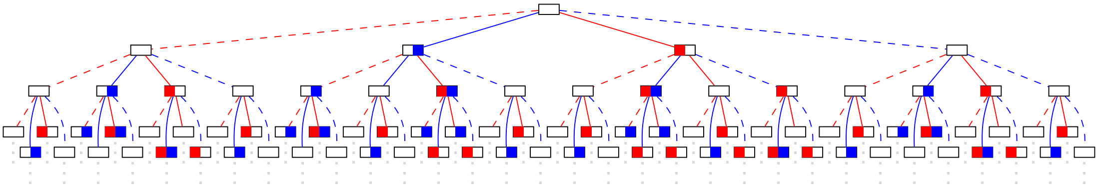
Definition 3

Power_d = "the relations $(\approx_{ij})_{i,j \in [d]}$ satisfy the following properties:

- *a solid or dashed edge of colour i preserves the node colouring w.r.t. all colours $j \neq i$*
- there is a dashed edge of colour i to $v \implies v \approx_{ii} v_I$
- $v_I \approx_{ij} v_I$ for all $i, j \in [d]$

where v_I is the tree's root"

Example



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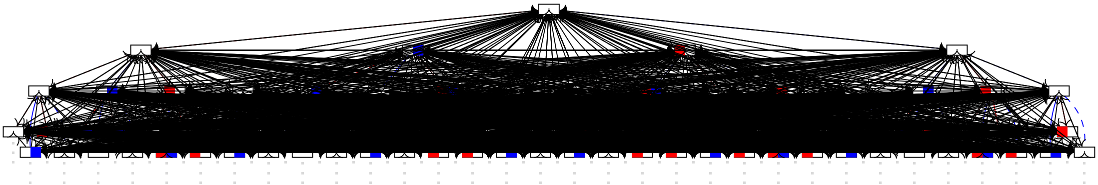

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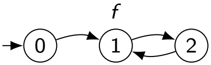
Question

Is $1\textsc{NonUnivNFA}_d$ non-regular relative to $\textsc{Power}_d$ for every $d \geq 1$?

One Last Thing: Asynchronous Powers of Graphs

Ex.: the 2nd power of 

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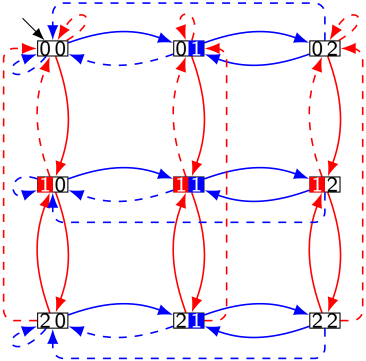

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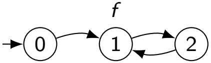
Ex.: the 2nd power of 

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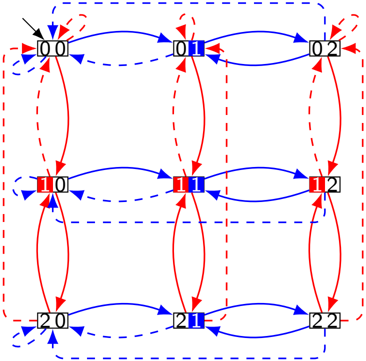

easily extends to d-powers

- solid edge with colour i = move in i-th component
- dashed edge with colour i = reset to root node in i component

Separability of P and NP

Theorem 4

P ≠ NP iff $\text{1N}\textsc{on}\text{U}\textsc{niv}\text{NFA}_d$ *is non-regular relative to* $\text{P}\textsc{ower}_d$ *for every* $d \geq 1$

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$P \neq NP$ iff $1\textsc{NonUnivNFA}_d$ is non-regular relative to $\textsc{Power}_d$ for every $d \geq 1$

key ingredients:

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Lemma 6: Non-Universality for 1-letter NFA is in $\mathrm{NP}/_{\sim}$

Proof of Thm. 4

"$\Leftarrow$" suppose $\mathrm{P} = \mathrm{NP}$, then also $\mathrm{P}/_{\sim} = \mathrm{NP}/_{\sim}$

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by Lem. 1 and NP-completeness of 1-letter NFA-non-universality: $\mathrm{P} = \mathrm{NP}$ $\square$

Conclusion

main contribution: bisimulation-invariant complexity theory allows machinery from formal tree language theory to be imported

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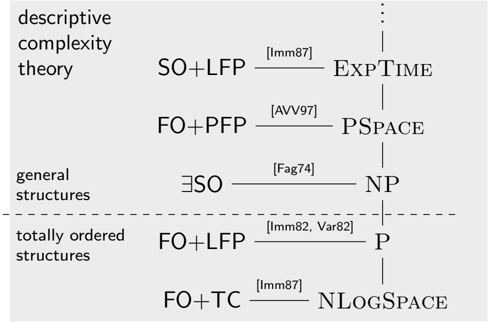

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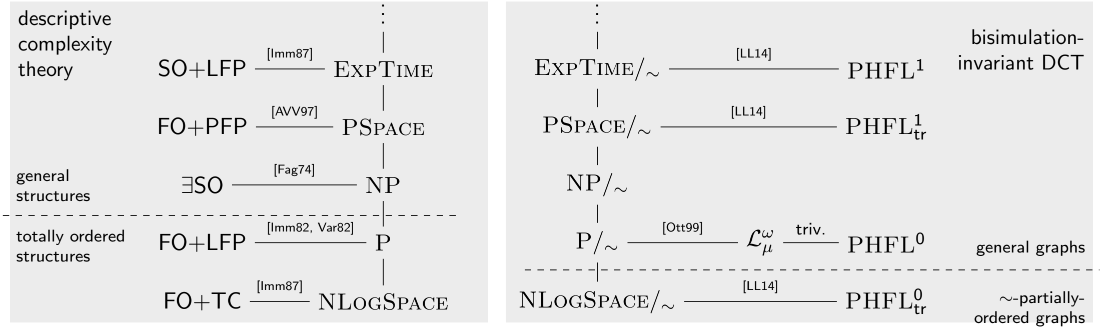

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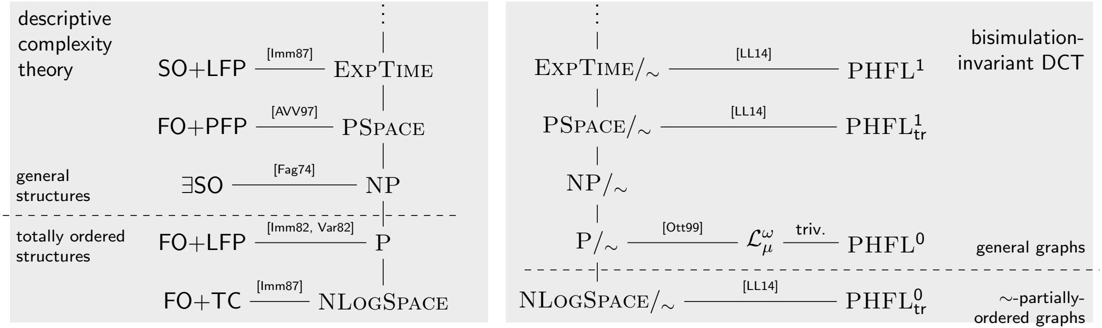

analogous theorem for $\mathrm{P} \neq \mathrm{PSPACE}$, built on 2-letter non-universality problem for NFA

in general: characterisation of bisimulation-invariant $\mathcal{C}$-complete problem can be used to separate P from $\mathcal{C}$

The End