Maintaining Longest Common Extensions in Dyn-FO

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Dyn-FO

Dyn-FO [Patnaik and Immerman 1997]

- Dynamic algorithm where updates and queries are expressible in FO-logic

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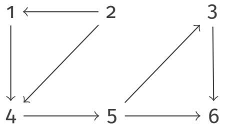
REACH in acyclical graphs [Patnaik and Immerman 1997, Theorem 4.2]

Query(a, b)	Is b reachable from a in G?



- Query$(1, 6)$ = true

Dyn-FO

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- Dynamic algorithm where updates and queries are expressible in FO-logic

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Data	
Ins(a, b)	Insert edge (a, b) into G. (assume G remains acyclical)
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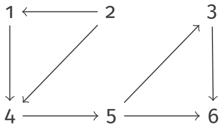
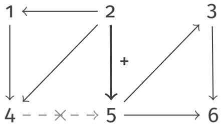
Query(a, b)	Is b reachable from a in G?



- Query$(1, 6) =$ true
- Del$(4, 5)$
- Ins$(2, 5)$



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Data	$R(x, y)$: node y is reachable from node x in G
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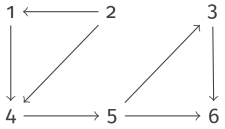
Del(a, b)	Delete edge (a, b) from G. $R'(x, y) \equiv$
Query(a, b)	Is b reachable from a in G? $R(a, b)$.



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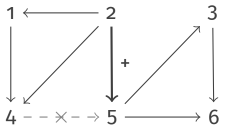

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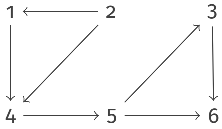

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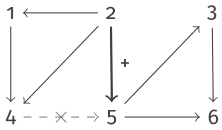

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Del(a,b)	Delete edge (a,b) from G.	
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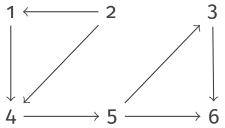

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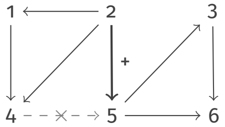

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Dyn-FO Continuation

Successor in $D \subseteq [1, n]$

Ins(a)	Insert a into D.
Del(a)	Remove a from D.
Query(a, b)	Is b the successor of a in D?

$D = \{3, 6, 7, 10, 11, 14, 16, 18, 20, 22, 25, 29, 31\}$

- Query$(23, 28) =$ false
- Query$(23, 25) =$ true
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  - We can probably do better.
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  - **Work-Sensitive Dyn-FO**

Work-Sensitive Dyn-FO

Follows from Immerman 1999, Corollary 5.10

Problem $\mathcal{P}$ is in Dyn-FO

$\Leftrightarrow$

There is a dynamic parallel constant-time algorithm for $\mathcal{P}$ on a common CRCW PRAM with polynomial work.

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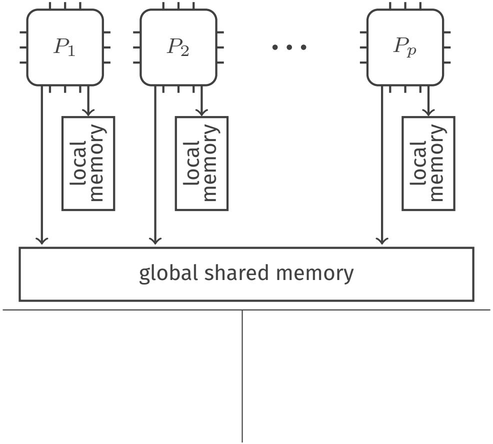
common CRCW PRAM

PRAM parallel machine with multiple processors, synchronized steps, shared memory



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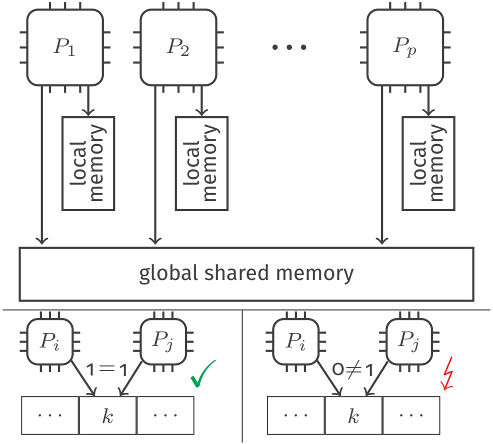
CRCW concurrent read and write allowed

common concurrent write only with same value



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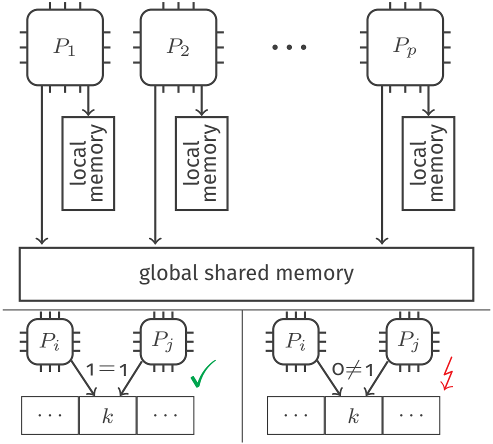

common CRCW PRAM

PRAM	parallel machine with multiple processors, synchronized steps, shared memory
CRCW	concurrent read and write allowed
common	concurrent write only with same value

number of processors $\times$ runtime $=$ work

▶ ***better* = less work**

Longest Common Extension (LCE)

Definition

Given: String $S[1, n]$, indices i, j

Longest Common Extension (LCE) $=$

$$\begin{array}{cccccccccccccccccccccc} & & & & & & & & & i=9 & & & & & & j=16 & & & & \\ & & & & & & & & & \downarrow & & & & & & \downarrow & & & & \\ S = & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{b} & \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{b} & \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{a} \end{array}$$

Longest Common Extension (LCE)

Definition

Given: String $S[1, n]$, indices i, j

Longest Common Extension (LCE) $= \begin{cases} \text{Longest Common Prefix (LCP) of } S[i, n] \text{ and } S[j, n] \\ \end{cases}$

$i = 9$ ↓ $\quad j = 16$ ↓

$S =$ a b a b c a a b | b a b c a | a b | b a b c a |

$\mathrm{lcp}_S(3, 9)$ $\quad$ $\mathrm{lcp}_S(3, 9)$

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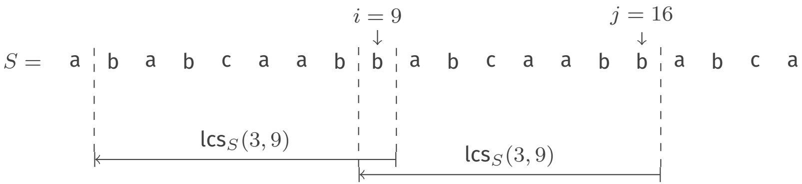

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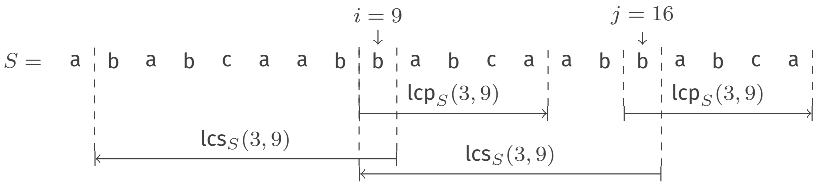

Observation

- LCP and LCS are symmetrical
  - From now only LCP

Dynamic LCP

init($\{a, b, c\}$) $S = \varepsilon$

Problem

init(Σ)

- Initialization $S = \varepsilon$
- Alphabet Σ

Dynamic LCP

$\mathsf{init}(\{a, b, c\})$ $S = \varepsilon$

$\mathsf{ins}(1, \mathsf{a})$ $S = \boxed{\mathbf{a}}$

$\mathsf{ins}(1, \mathsf{c})$ $S = \boxed{\mathbf{c}}$ a

⋮

$\mathsf{ins}(4, \mathsf{b})$ $S =$ a b a $\boxed{\mathbf{b}}$ c a a b b a b c a a b b a b c a

Problem

$\mathsf{init}(\Sigma)$

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$\mathsf{ins}(i, \sigma)$

- Insert σ at i

Dynamic LCP

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lcp(9, 16) = 5	$S =$ a b a b c a a b [**b** a b c a] a b [**b** a b c a]

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lcp(i, j)

- Compute lcp$_S(i, j)$

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lcp$(9, 16) = 5$	$S =$ a b a b c a a b [**b** a b c a] a b [**b** a b c a]
del(3)	$S =$ a b ‖ b c a a b b a b c a a b b a b c a
lcp$(1, 13) = 3$	$S =$ [**a** b b] c a a b b a b c a [**a** b b] a b c a

Problem

init(Σ)
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ins(i, σ)
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del(i)
- Delete position i

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LCP Algorithms

Computing LCP from scratch

sequential naïve in $\mathcal{O}(n)$ time

parallel constant time with $\mathcal{O}(n)$ work (slightly more complicated)

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- Given S, build data structure $\mathcal{D}$
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- Shun 2014: Parallel, Time $\mathcal{O}(\log^2 n)$, Work $\mathcal{O}(n)$

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Dynamic Algorithms

- Maintain data structure $\mathcal{D}$ for changing S
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  - Related problem, Sequential
  - Time $\mathcal{O}(\log n)$ with high probability

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▶ Work-Sensitive Dyn-FO

For any $\varepsilon > 0$,

$\mathcal{O}(n^\varepsilon)$ work for updates and queries

in parallel constant time on common CRCW PRAM.

Overview

▶ Work-Sensitive Dyn-FO

For any $\varepsilon > 0$, parallel constant time with $\mathcal{O}(n^\varepsilon)$ work for updates and queries on common CRCW PRAM.

Ingredients

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For any $\varepsilon > 0$, parallel constant time with $\mathcal{O}(n^{\varepsilon})$ work for updates and queries on common CRCW PRAM.

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String Synchronizing Sets

- tool from string algorithms
- answering queries easy
- constructing difficult
- maintaining even worse

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Kempa and Kociumaka 2022

- dynamic string algorithm
- maintains suffix-array
- can maintain string synchronizing sets (with some modifications)
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- constant-time updates compute non-constant-time algorithm
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+ technical details and auxiliary algorithms, skipped here

String Synchronizing Sets

String Synchronizing Set B^τ (e.g. Kociumaka et al. 2024, Section 4)

Set of occurrences (starting positions) of length-τ substrings within a string S with:

$S =$ **a** b **a** b c a a b b **a** b **c** a a b b a b c b

B^4 : 1 2 2 3

1: abab

2: abca

3: caab

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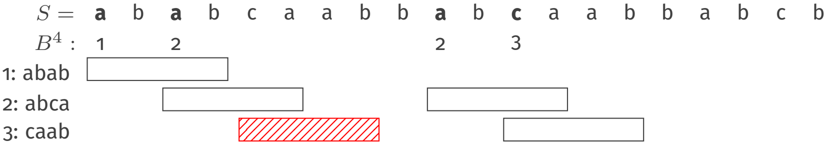

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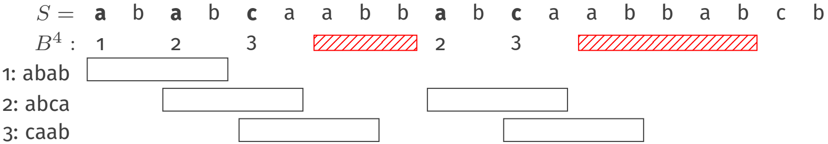

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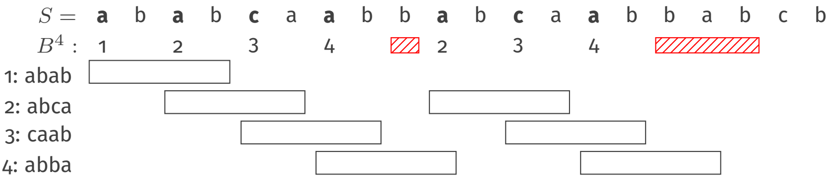


String Synchronizing Sets

String Synchronizing Set B^τ (e.g. Kociumaka et al. 2024, Section 4)

Set of occurrences (starting positions) of length-τ substrings within a string S with:

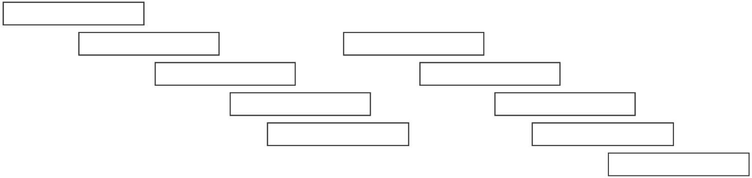
Consistency: Contains either none or all occurrences of a substring ✓

Density: There is an occurrence at least every $\frac{1}{2}\tau$ positions.* ✓ *except periodic (repeating) parts

Sparseness: The total length of all occurrences is $\mathcal{O}(n)$. ¯_(ツ)_/¯

$S =$	**a**	b	**a**	b	**c**	a	**a**	**b**	b	**a**	b	**c**	a	**a**	**b**	b	**a**	b	c	b
B^4 :	1		2		3		4	5		2		3		4	5		6			

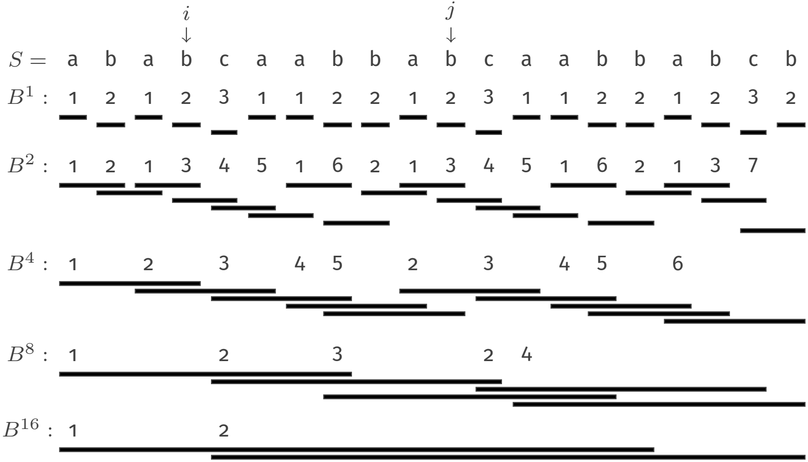
1: abab
2: abca
3: caab
4: abba
5: bbab
6: abcb



Answering Queries with String Synchronizing Sets

$S =$	a	b	a	b	c	a	a	b	b	a	b	c	a	a	b	b	a	b	c	b
B^1 :	1	2	1	2	3	1	1	2	2	1	2	3	1	1	2	2	1	2	3	2
B^2 :	1	2	1	3	4	5	1	6	2	1	3	4	5	1	6	2	1	3	7	
B^4 :	1		2		3		4	5		2		3		4	5		6			
B^8 :	1				2			3				2	4							
B^{16} :	1				2															

Answering Queries with String Synchronizing Sets



Compute lcp(i, j)

- Search on $m' = \text{lcp}(i, j)$
  - Binary-Search-like
  - $\mathcal{O}(n^{\varepsilon})$ equality queries

"$S[i, i+m) = S[j, j+m)$?"

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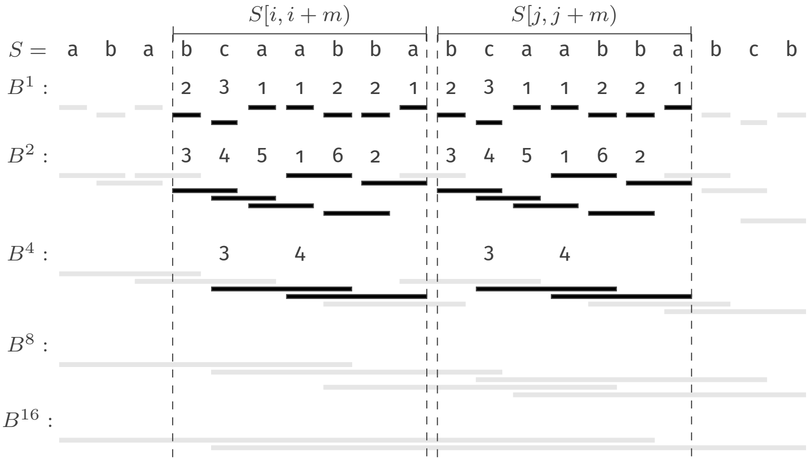

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Equality Query

$$S[i, i+m) = S[j, j+m)$$
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both have same occurrences

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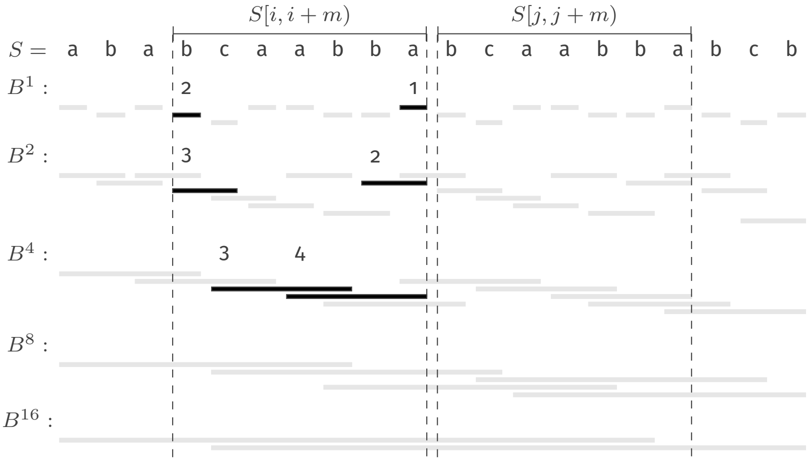

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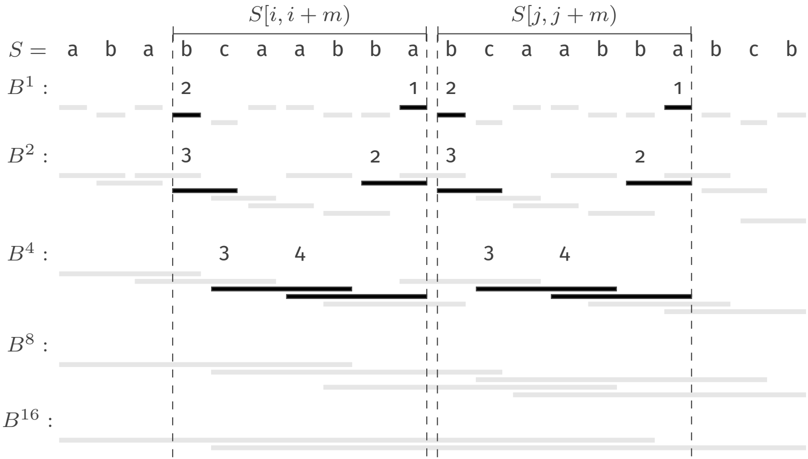
Answering Queries with String Synchronizing Sets



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$$S[i, i+m) = S[j, j+m)$$
$$\Leftrightarrow$$
both have same occurrences

1. Cover $S[i, i+m)$ with few ($\mathcal{O}(\log n)$) occurrences.
2. Check if $S[j, j+m)$ has same covering.

Maintaining String Synchronizing Sets

Sequential Dynamic Algorithm by Kempa and Kociumaka 2022

- maintains the suffix array of a dynamic string
- sequential, $\mathcal{O}(\log^4 n)$ time for queries, $\mathcal{O}(\log^{3+o(1)} n)$ time for updates.
- **maintains dynamic String Synchronizing Sets**

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- keeps the String Synchronizing Sets and what is used to construct them in memory

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- updates recalculate parts of this information
  - updates may only affect a small part
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  - uses B^τ to compute $B^{2\tau}$

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Difficulties for parallel constant-time

- updates have to traverse logarithmic number of levels, but still run in constant time
  - muddling

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Difficulties for parallel constant-time

- updates have to traverse logarithmic number of levels, but still run in constant time
  - muddling
- need data structures to store sets in constant time and maintain consistent names for occurrences
  - technical, auxiliary algorithms (skipped here)

Achieving Parallel Constant Time

Ideas from the Muddling Lemma [Datta et al. 2019]

- Spread the computation of a time $T(n)$ algorithm across $T(n)$ updates.

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Time

Level

Set	Record: Delay
B^1	0
B^2	1
B^4	2
B^8	3
B^{16}	4

Delayed Processing

- Sets B^τ are delayed

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Delayed Processing

- Sets B^{τ} are delayed
- Changes from an update are not all computed immediately.

Time →

Level ↓

	Record:	[ins$(1, \text{a})$,
Set	Delay	↘
B^1	0	a
B^2	1	ε
B^4	2	ε
B^8	3	ε
B^{16}	4	ε

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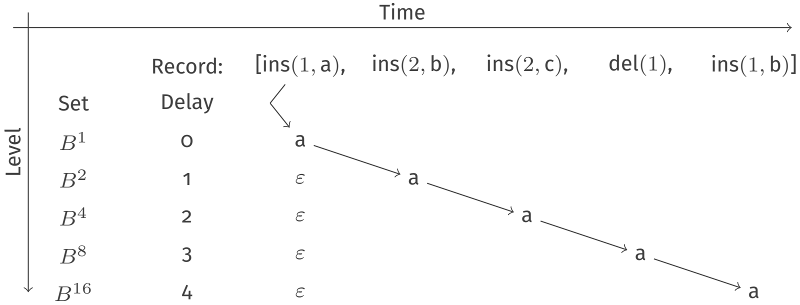

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Time →, Level ↓

Set	Delay	Record: [ins(1, a),	ins(2, b),	ins(2, c),	del(1),	ins(1, b)]
B^1	0	a	ab	acb	ab	bcb
B^2	1	ε	a	ab	acb	ab
B^4	2	ε	ε	a	ab	acb
B^8	3	ε	ε	ε	a	ab
B^{16}	4	ε	ε	ε	ε	a

Delayed Processing

- Sets B^τ are delayed
- Changes from an update are not all computed immediately.
- Spreads computation across following updates.
- Computes changes from several updates in parallel.

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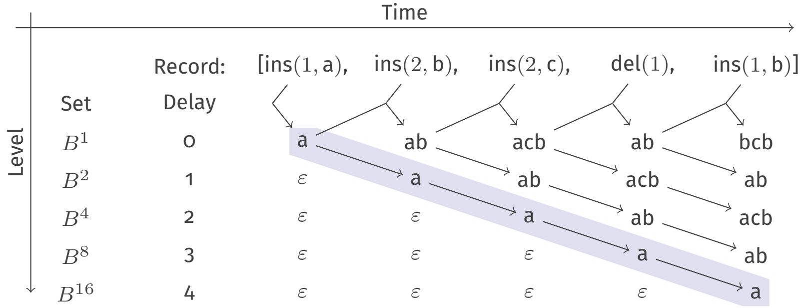

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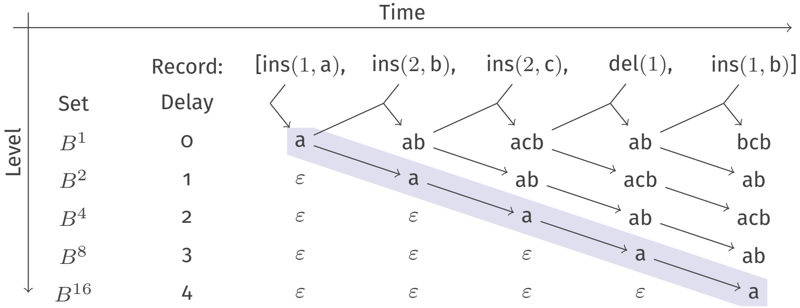
- Computes changes from several updates in parallel.

$\Rightarrow$ Time available to compute updates grows with number of levels

Achieving Parallel Constant Time

Ideas from the Muddling Lemma [Datta et al. 2019]

- Spread the computation of a time $T(n)$ algorithm across $T(n)$ updates.
- Answer queries with outdated information + a record of recent changes.



Delayed Processing

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- Changes from an update are not all computed immediately.
- Spreads computation across following updates.
- Computes changes from several updates in parallel.

⇒ Time available to compute updates grows with number of levels

Dealing with outdated information

Problem

- Only the topmost String Synchronizing Set B^1 represents the current string.
- The other sets are all delayed.

Set	Delay	Delayed String																					
B^1	0	a	b		b	c	a	a	b	b	a	b	c	a	a	b	b	c	a	[a]	b		a
B^2	1	a	b		b	c	a	a	b	b	a	b	c	a	a	b	b	c	a		b	□	a
B^4	2	a	b		b	c	a	a	b	b	a	b	c	a	a	b	b	[c]	a		b	c	a
B^8	3	a	b	□	b	c	a	a	b	b	a	b	c	a	a	b	b		a		b	c	a
B^{16}	4	a	b	a	[b]	c	a	a	b	b	a	b	c	a	a	b	b		a		b	c	a

Record:

Update	Age
ins$(18, \text{a})$	0
del(19)	1
ins$(16, \text{c})$	2
del(3)	3
ins$(4, \text{b})$	4

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Modified Queries

1. Cut out parts that were recently changed.

Set	Delay	Delayed String																				
B^1	0	a	b		b	c	a	a	b	b	a	b	c	a	a	b	b	c	a	a	b		a
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Modified Queries

1. Cut out parts that were recently changed.
   - Remaining parts are fully up-to-date
2. Answer sub-queries on remaining parts
3. Compare cut-out positions naïvely

Set	Delay	Delayed String																					
B^1	0	a	b		b	c	a	a	b	b	a	b	c	a	a	b	b	c	a	a	b		a
B^2	1	a	b		b	c	a	a	b	b	a	b	c	a	a	b	b	c	a		b		a
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Summary and Applications

Summary

There is a parallel constant-time algorithm for dynamic LCP on a common CRCW PRAM with $\mathcal{O}(n^{\varepsilon})$ work, for any $\varepsilon > 0$.

- Combines techniques from Dyn-FO (Muddling) and String-Algorithms (String Synchronizing Sets)
- Only for integer alphabets $\Sigma = [1, n]$ (common assumption for string algorithms)

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Squares

$S[i, j]$ is a square if $S[i, j] = uu$

Input	String $S[1, n]$
Set(i, σ)	Set $S' = S[1, i-1]\sigma S[i+1, n]$
Query$()$	What is the longest square in S?

Based on Amir et al. 2019, Theorem 26:

Parallel constant-time algorithm with work $\mathcal{O}(n^\varepsilon)$, for any $\varepsilon > 0$.

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Dyck Languages D_k

D_k: well-matched parentheses with k types

Input	String $S[1, n]$
Set(i, σ)	Set $S' = S[1, i-1]\sigma S[i+1, n]$
Query$()$	Is $S \in D_k$?

Due to Schmidt et al. 2021, Lemma 6.9:

Parallel constant-time algorithms with work $\mathcal{O}(n^\varepsilon)$, for D_k, $k \in \mathbb{N}$ and any $\varepsilon > 0$.

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Answering Queries with String Synchronizing Sets (1)

$\Rightarrow$ Compute lcp(i, j) via search on m similar to a binary search, using equality queries "$S[i, i+m) = S[j, j+m)$?"

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1. Cover $S[i, i+m)$ with few ($\mathcal{O}(\log n)$) occurrences from $B^1, B^2, B^4, \ldots$
2. If $S[i, i+m) = S[j, j+m)$, then $S[j, j+m)$ has the same covering (due to *Consistency*)

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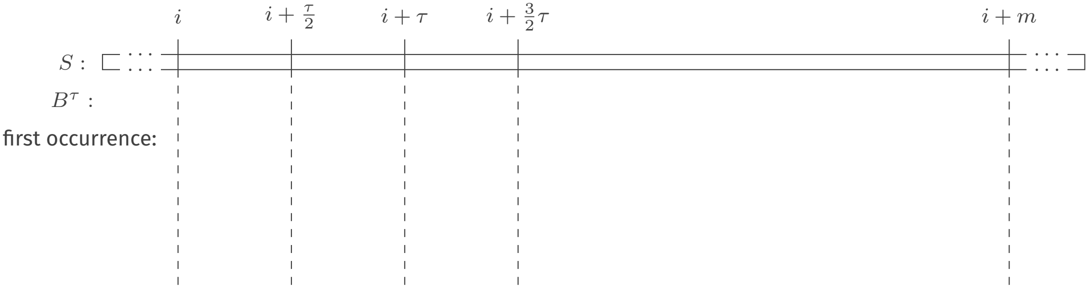

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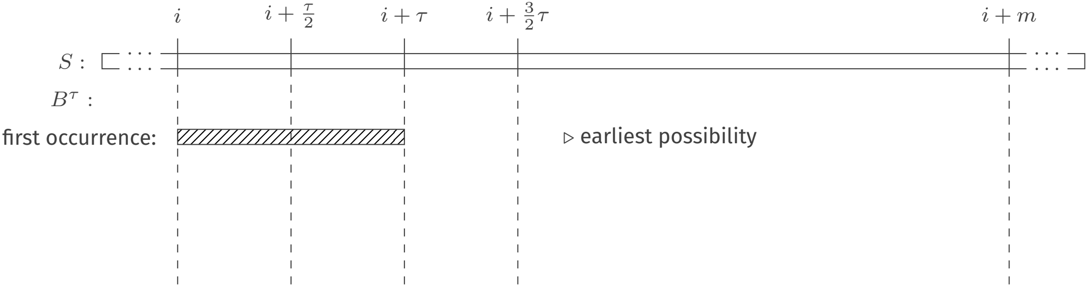

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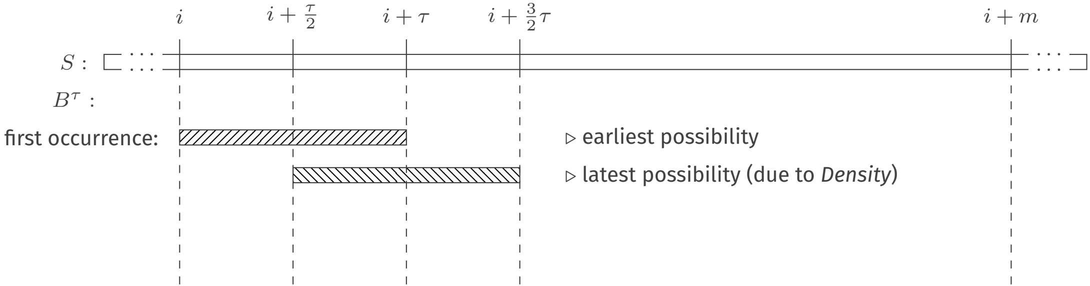

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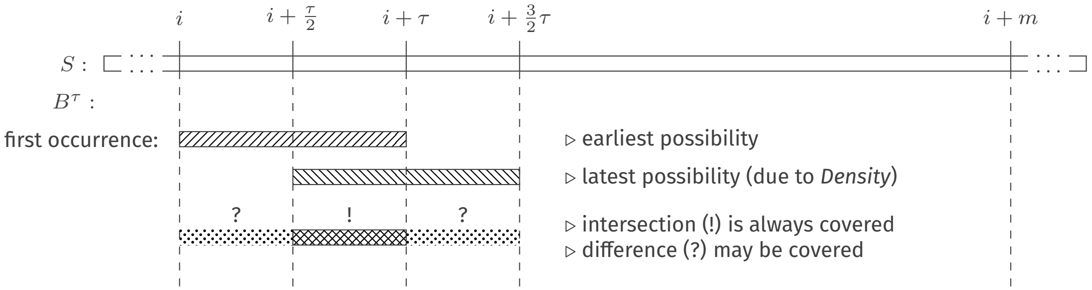

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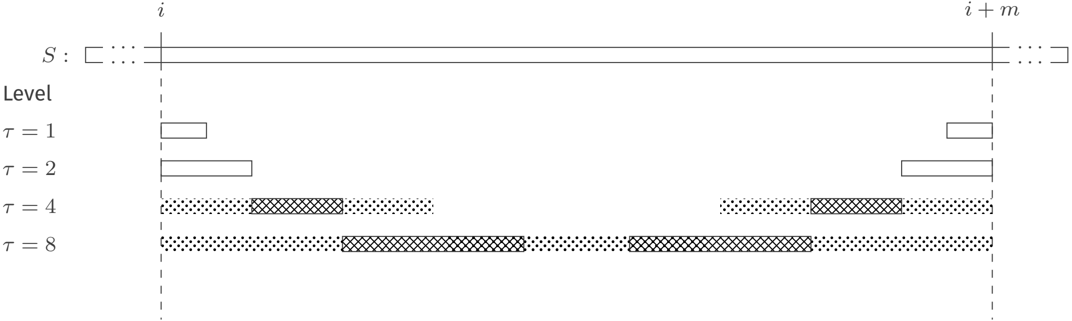

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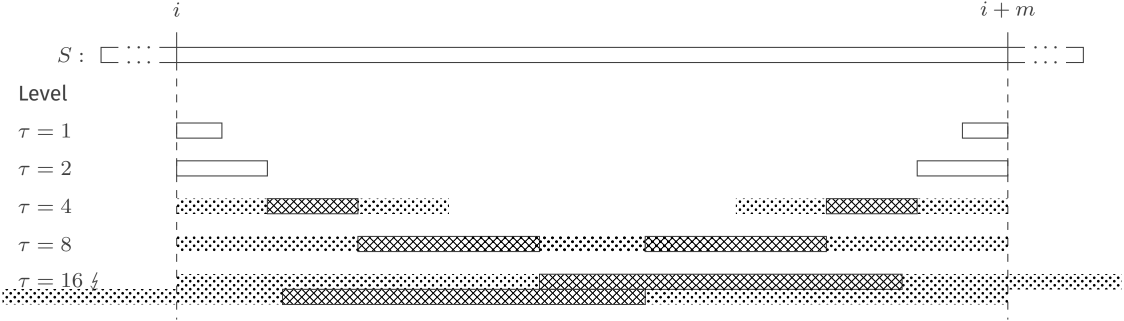

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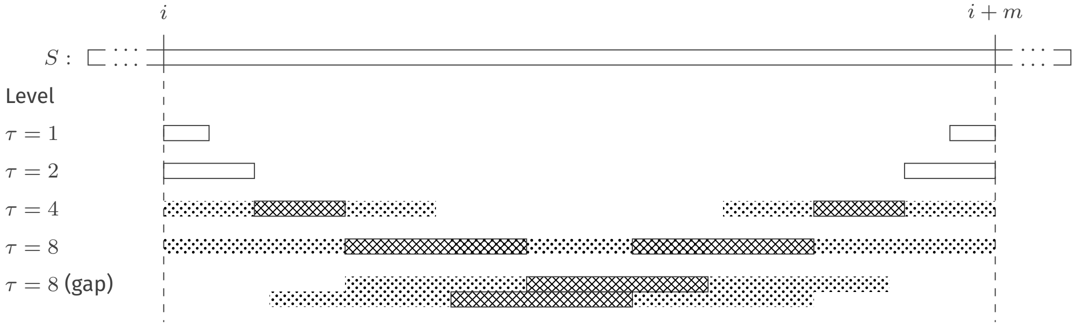

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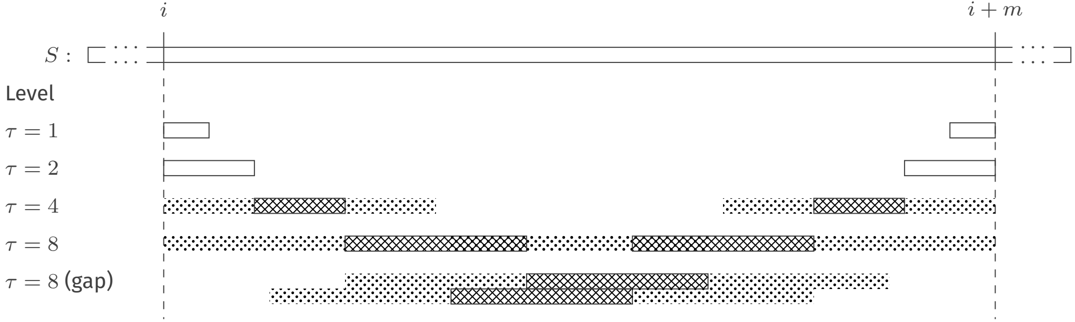

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- Positions in $S[i, i+m)$ are always covered
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$\Rightarrow S[i, j) = S[j, m)$ iff both produce the same covering

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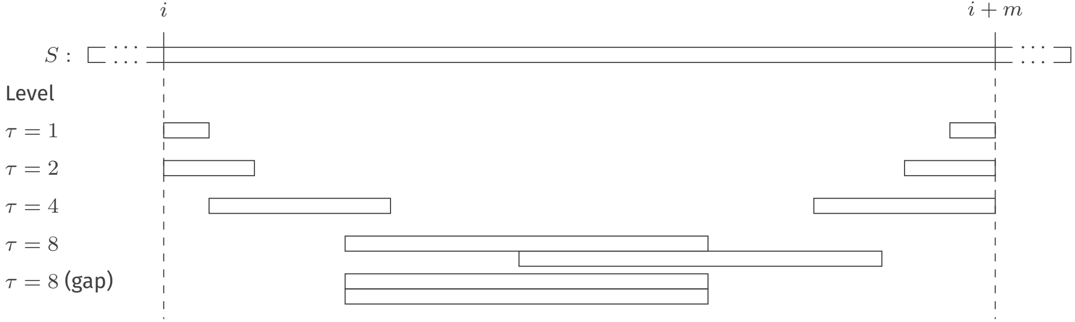

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Decompose S into factors $S = u_1 u_2 \dots u_m$ with similar properties to String Synchronizing Sets

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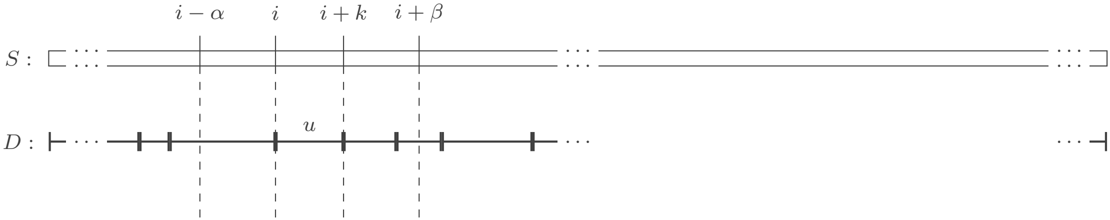

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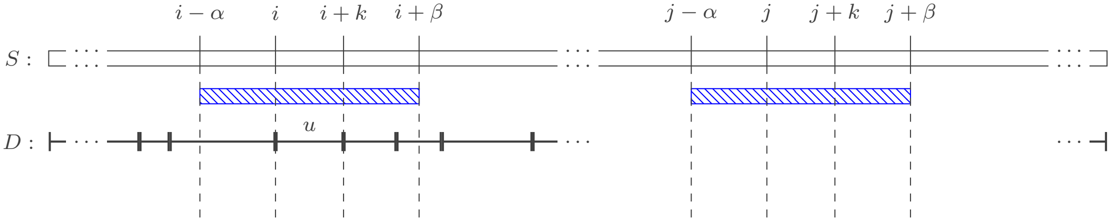

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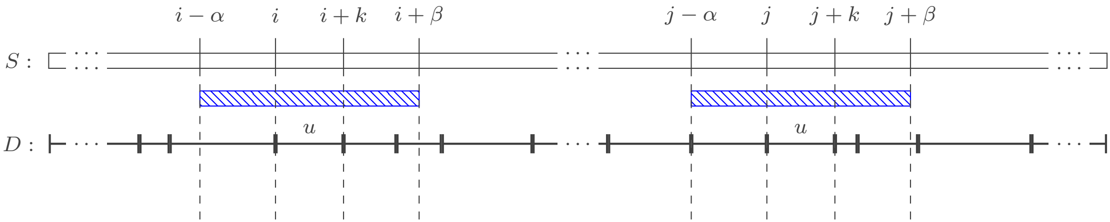

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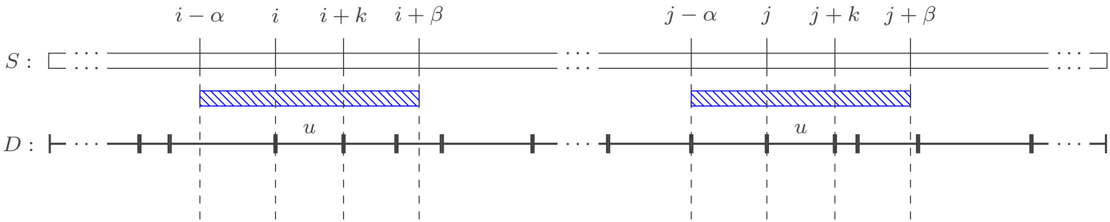
Density: Max length of factors limited*

Sparseness: Short factors can only exist surrounded by long factors

*except periodic (repeating) parts



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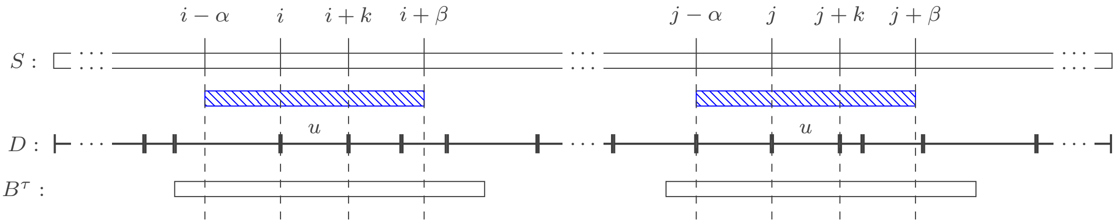
$\Rightarrow$ Turn each factor in D into an occurrence in B^τ



Computing Consistent Decompositions

Iterative Construction

1. Start with singleton factors

Example

$S =$ a b a b c a a b b a b c a a b b a b c b

$D^1 =$ 

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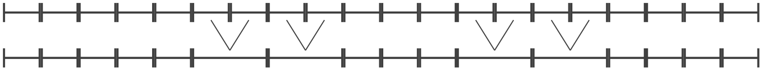

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Consistent Merging

- Partition unique factors into $\triangleleft, \triangleright, \bot$.

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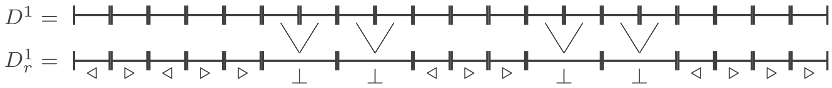

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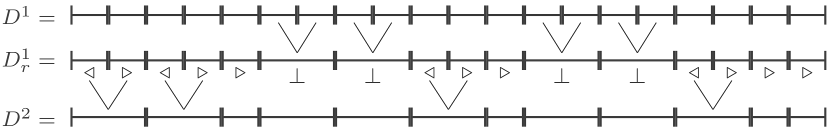

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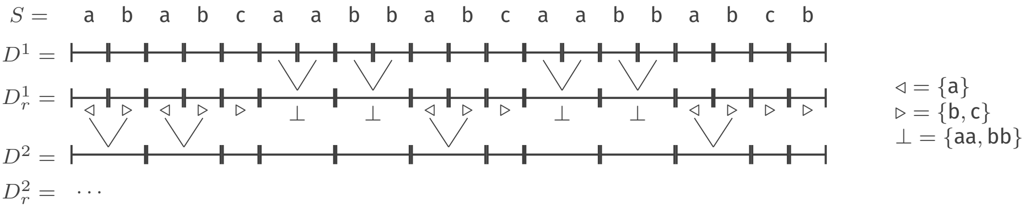
- Choose Partition…
  - At random (e.g. Lipták, Masillo, and Navarro 2024)
  - High number of merges (e.g. Kociumaka et al. 2024)

Example



Maintaining String Synchronizing Sets

Sequential Dynamic Algorithm by Kempa and Kociumaka 2022

Keep decompositions and synchronizing sets in memory. On updates, recalculate affected parts.

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stronger: Sparseness holds within small segments

weaker: More occurrences by $\log^*$-factor

String Synchronizing Sets Hierarchy

$S =$ a b a b c a a b b a b c a a b b a b c b

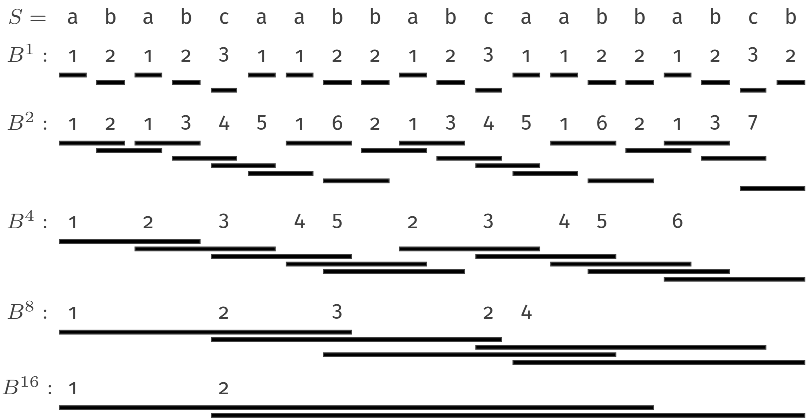
B^4 : 1 2 3 4 5 2 3 4 5 6

String Synchronizing Sets Hierarchy

Hierarchy

$B^1, B^2, B^4, B^8, B^{16}$

B^τ for all
$\tau = 2^i, 0 \leq i \leq \log n$



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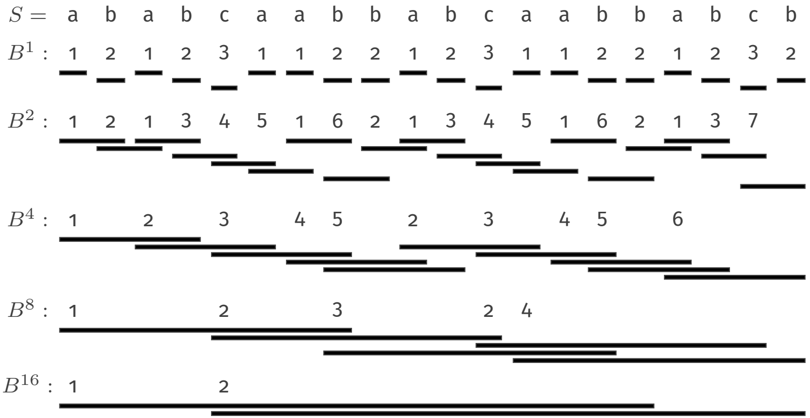

Observation 1

$B^1 = [1, n]$
$B^2 = [1, n-1]$
due to *Density*