# Arity Hierarchies for Quantifiers closed under Partial Polymorphisms

Anuj Dawar[1], Lauri Hella[2], **Benedikt Pago**[1]

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[1]University of Cambridge

[2]Tampere University

# Generalized quantifiers

*Generalized quantifiers*, also called *Lindström quantifiers* (1966) are to first-order logic what **oracles** are to Turing machines.

# Generalized quantifiers

**Definition**

Let $\tau = \{R_1, \ldots, R_m\}$ and $\mathcal{K}$ be a class of $\tau$-structures. Then $\boldsymbol{Q}_{\mathcal{K}}$ is the *generalized quantifier* defining $\mathcal{K}$. Its *arity* is the maximum arity of any relation in $\tau$.

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**Usage:** Let $\Psi = (\psi_{R_1}(\bar{y}), \ldots, \psi_{R_m}(\bar{y}))$ be a tuple of $\sigma$-formulas, one for each $R_i \in \tau$.

Let $\mathbf{A}$ be a $\sigma$-structure. Then $\mathbf{A} \models Q_{\mathcal{K}}\bar{y}.\Psi$ if and only if $\Psi(\mathbf{A}) \in \mathcal{K}$ .

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**Example (Counting quantifiers)**

Express the **counting quantifier** $\exists^{=t} x$ as a (unary) generalized quantifier $Q_{\mathcal{K}}$:
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$$\exists^{=t}x.\varphi(x) \equiv Q_{\mathcal{K}}x.(\varphi(x)).$$

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**Classic example:** Let $\mathcal{Q}_r$ be the class of all quantifiers of *arity* $\leq r$.

**Theorem (Hella, 1992)**

*For every* $r \in \mathbb{N}$,

$$\mathcal{L}(\mathcal{Q}_r) \lneq \mathcal{L}(\mathcal{Q}_{r+1}).$$

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- [Dawar, Hella, CSL'24]: Quantifiers closed under partial polymorphisms.

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Identities satisfied by polymorphism $p$ of the solution domain determine the complexity:

- **Near-unanimity:** $p(x,x,\ldots,x,y) = p(x,x,\ldots,y,x) = p(x,y,\ldots,x,x) = p(y,x,\ldots,x,x) = x$
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We consider *partial* near-unanimity/Maltsev polymorphisms:
Only defined when the arguments match the pattern of the identities.

# Logics with generalized quantifiers closed under partial polymorphisms

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- [Dawar, Hella '24] introduced games for these logics and proved an **arity hierarchy** for $\mathcal{L}(\mathcal{Q}^{\mathcal{N}_\ell})$ in terms of $\ell$.

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- **In this work:** Interplay between $\ell$ and $r$?

# Arity hierarchies for near-unanimity

**Theorem (CSL'26)**

*For every $r \geq 3$,*

$$\mathcal{L}(\mathcal{Q}_r^{\mathcal{N}_r}) \equiv \mathcal{L}(\mathcal{Q}_{r-1}).$$

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*They do!*

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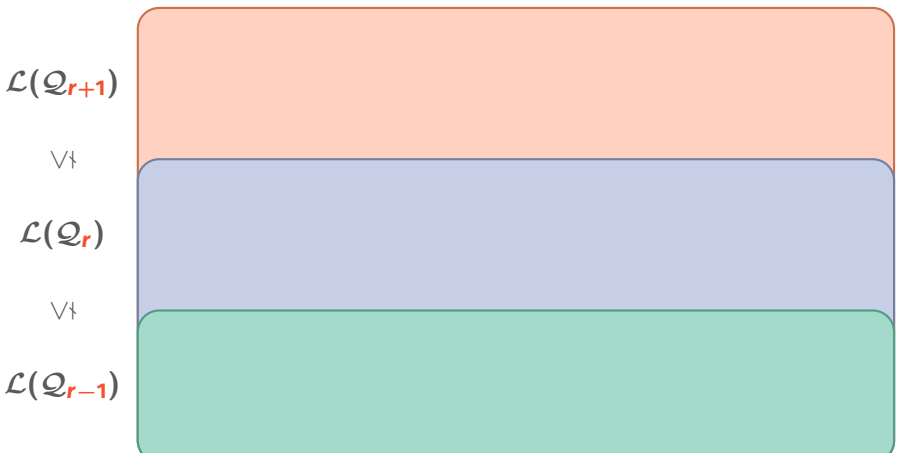

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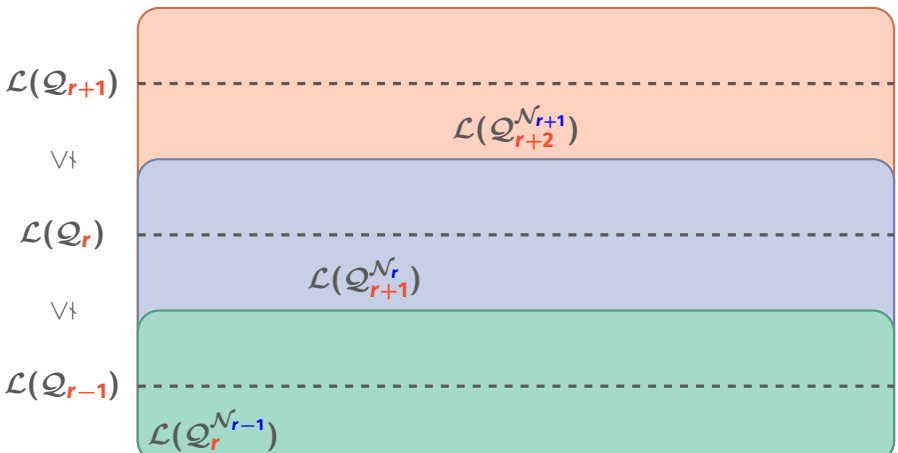

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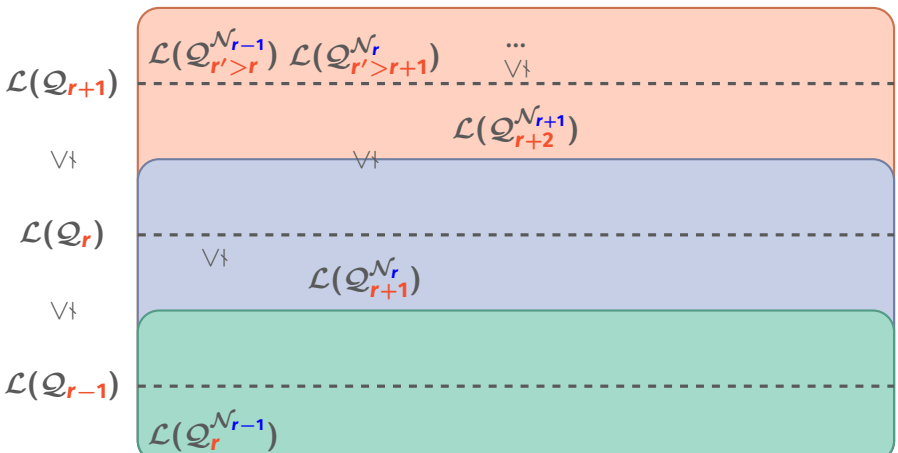

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**Theorem (CSL'26)**

*The k-variable fragments $\mathcal{L}^k$ are separated:*

$$\mathcal{L}^k(\mathcal{Q}_k^{\mathcal{M}}) \leq \mathcal{L}^k(\mathcal{Q}_k).$$