Shapley Value $\rightleftarrows$ Model Counting

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AIMoTh'26, Passau

Shapley Value

- The Shapley value quantifies the fair contribution of a player to a wealth function shared by a set of players in a cooperative game
- It was introduced by Lloyd Shapley in 1951

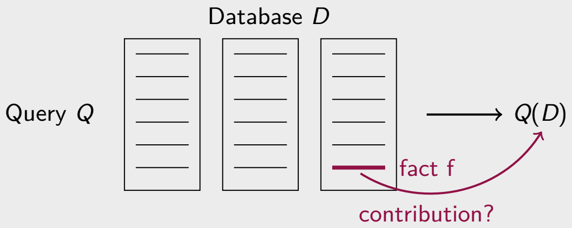
Some applications of the Shapley Value:

- Measuring the importance of features in machine learning
- Measuring the centrality and power of genes
- Finding key players in social networks
- Explaining query results

Explaining Query Results Using Shapley Value

Livshits et al. quantify the contribution of database facts to query results using the Shapley value

Problem Setting



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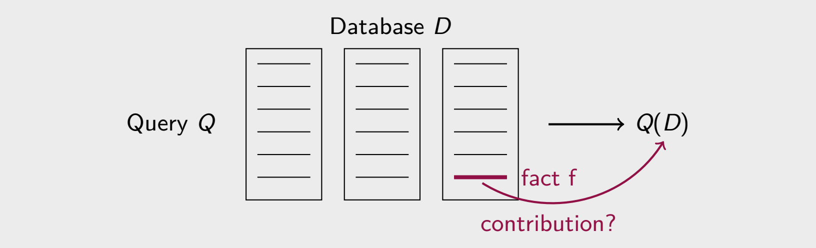

High-level recipe of computing the Shapley value of a database fact f:

- Interpret database facts as players and sub-databases as coalitions
- Evaluate Q on sub-databases $D' \subseteq D \setminus \{f\}$ and $D' \cup \{f\}$ and compare the results

Our work is about the complexity of this computation

Explaining Query Results Using Shapley Value - Prior Results

Theorem [Livshits et al., 21]

Let Q be a self-join-free Boolean query.

- If Q is hierarchical, Shapley value computation is in PTIME.
- Otherwise, Shapley value computation is #P-hard.

- Authors ask: Is there a simple proof based on model counting?

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There is a polynomial-time reduction from Shapley value computation to probabilistic query evaluation.

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Our results give answers to both questions

Overview of Our Work

- We use the Shapley value to quantify the contribution of

  Boolean variables to the models of Boolean functions

- We give polynomial-time reductions between

  computing the Shapley value of Boolean variables

  and

  counting the models of Boolean functions

- We show applications of our result to the computation of Shapley values of database facts in query evaluation

Shapley Value of Boolean Variables

Shapley Value of Boolean Variables - Example

$$F = X \wedge (Y \vee \neg Z)$$

What is the contribution of Y to the models of F?

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permutations of variables
XYZ
XZY
YXZ
YZX
ZYX
ZXY

Shapley Value of Boolean Variables - Example

$$F = X \wedge (Y \vee \neg Z)$$

What is the contribution of Y to the models of F?

permutations of variables	Impact of setting Y to 1 to the Boolean value of F		
XYZ	$F[XY] - F[X]$	$=$	$1 - 1 = 0$
XZY			
YXZ			
YZX			
ZYX			
ZXY			

$F[\boldsymbol{X}]$: Boolean value of F, when only the variables in $\boldsymbol{X}$ are set to 1

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YXZ	$F[Y] - F[\emptyset]$	$=$	$0 - 0 = 0$
YZX	$F[Y] - F[\emptyset]$	$=$	$0 - 0 = 0$
ZYX	$F[ZY] - F[Z]$	$=$	$0 - 0 = 0$
ZXY	$F[ZXY] - F[ZX]$	$=$	$1 - 0 = 1$

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sum $= 2$

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What is the contribution of Y to the models of F?

$$\begin{array}{lllll}
\text{permutations} & \text{Impact of setting } Y \text{ to } 1 & & & \\
\text{of variables} & \text{to the Boolean value of } F & & & \\
\hline
XYZ & F[XY] - F[X] & = & 1 - 1 = 0 & \\
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\end{array} \left.\vphantom{\begin{array}{l}1\\1\\1\\1\\1\\1\end{array}}\right\} \text{sum} = 2$$

$F[\boldsymbol{X}]$: Boolean value of F, when only the variables in $\boldsymbol{X}$ are set to 1

Shapley value of Y in F: $Shap(Y, F) = \frac{\text{sum of contributions of } F}{\text{number of permutations}} = \frac{2}{6}$

Shapley Value of Boolean Variables

- Let F be a Boolean function with variables $X_1, \ldots, X_n$
- S_n: Set of all permutations of $X_1, \ldots, X_n$
- $\Pi^{<i}$: set of variables that appear before X_i in the permutation Π

Shapley value of variable X_i in F

$$Shap(X_i, F) = \frac{1}{n!} \sum_{\Pi \in S_n} (F[\Pi^{<i} \cup \{X_i\}] - F[\Pi^{<i}])$$

Problems and Main Results

Computational Problems

Let $\mathcal{C}$ be a class of Boolean functions

Shap($\mathcal{C}$) (Shapley Computation)

Given:	$F \in \mathcal{C}$ over variable set $\boldsymbol{X}$
Compute:	$Shap(X, F)$, $\forall X \in \boldsymbol{X}$

$\#\mathcal{C}$ (Model Counting)

Given:	$F \in \mathcal{C}$ over variable set $\boldsymbol{X}$
Compute:	$\sum_{\boldsymbol{Y} \subseteq \boldsymbol{X}} F[\boldsymbol{Y}]$ (number of models of F)

$\#_*\mathcal{C}$ (k-Model Counting)

Given:	$F \in \mathcal{C}$ over variable set $\boldsymbol{X}$
Compute:	$\sum_{\substack{\boldsymbol{Y} \subseteq \boldsymbol{X} \\ \lvert\boldsymbol{Y}\rvert = k}} F[\boldsymbol{Y}]$, $\forall k \leq \lvert\boldsymbol{X}\rvert$ (number of models of F of size k)

OR-Substitutions

$F \xrightarrow{\text{OR}} G$ (function F OR-substitutes into function G) if:

- G results from F by substituting each variable by a (possibly empty) disjunction of fresh variables.

Example

$$X \wedge (Y \vee \neg Z) \xrightarrow{\text{OR}} X^1 \wedge (\bot \vee \neg(Z^1 \vee Z^2 \vee Z^3))$$

$$\widetilde{\mathcal{C}} \stackrel{\text{def}}{=} \{G \mid \exists F \in \mathcal{C} \text{ with } F \xrightarrow{\text{OR}} G\}$$

Our Main Result

Theorem

For any class $\mathcal{C}$ of Boolean functions, the following polynomial-time reductions hold:

$$\begin{aligned} \mathit{Shap}(\mathcal{C}) &\leq^P \#_*\widetilde{\mathcal{C}} \\ \#_*\mathcal{C} &\leq^P \#\widetilde{\mathcal{C}} \\ \#\mathcal{C} &\leq^P \mathit{Shap}(\widetilde{\mathcal{C}}) \end{aligned}$$

Corollary

If $\widetilde{\mathcal{C}} = \mathcal{C}$, then $\mathit{Shap}(\mathcal{C}) \equiv^P \#_*\mathcal{C} \equiv^P \#\mathcal{C}$

Implications of Our Main Result

For the following classes $\mathcal{C}$, it holds $\widetilde{\mathcal{C}} = \mathcal{C}$ and $\#\mathcal{C}$ is in PTIME:

- Positive β-acyclic CNF-formulas
- Deterministic and decomposable Boolean circuits

For each such class $\mathcal{C}$, our result implies:

- *Shap*($\mathcal{C}$) is in PTIME.

Application:
Explaining Query Results

Query Lineage

- The lineage of a Boolean query over a database is a propositional DNF formula over variables associated with database facts
- The lineage contains one clause for each tuple of joining facts

Example

Query $Q = R(A, B), S(B, C)$

Database D

R	A	B	
	a_1	b_1	X_1
	a_2	b_1	X_2
	a_1	b_2	X_3

S	B	C	
	b_1	c_1	Y_1
	b_2	c_1	Y_2

lineage variables

Lineage of Q over D: $F_{Q,D} = (X_1 \wedge Y_1) \vee (X_2 \wedge Y_1) \vee (X_3 \wedge Y_2)$

$$\mathcal{C}_Q = \{F_{Q,D} \mid D \text{ is a database}\}$$

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The Shapley value of a lineage variable quantifies the contribution of the corresponding fact to the query result

OR-Substitution of Lineage Expressions

- The stretching of a query is obtained by adding a fresh variable to each atom

Example

Query	$Q =$	$R(A, B),$	$S(B, C)$
Stretching	$\widetilde{Q} =$	$R(Z_1, A, B),$	$S(Z_2, B, C)$

OR-Substitution of Lineage Expressions

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Query	$Q =$	$R(A, B)$,	$S(B, C)$
Stretching	$\widetilde{Q} =$	$R(Z_1, A, B)$,	$S(Z_2, B, C)$

- OR-substitution of lineage corresponds to stretching of queries

$$\begin{tikzcd} Q \arrow[r, "\text{stretch}"] \arrow[d, "D"'] & \widetilde{Q} \arrow[d, "\widetilde{D}"] \\ F_{Q,D} \arrow[r, "\text{OR-substitute}"'] & \widetilde{F_{Q,D}} = F_{\widetilde{Q},\widetilde{D}} \end{tikzcd}$$

$$\Longrightarrow \mathcal{C}_{\widetilde{Q}} = \widetilde{\mathcal{C}}_Q$$

Back to the Open Questions in Prior Work

Shapley Values of Database Facts 1/2

Theorem [Livshits et al., 21]

Let Q be a self-join-free Boolean query.

- If Q is hierarchical, $\mathit{Shap}(\mathcal{C}_Q)$ is in PTIME
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- Otherwise, it is #P-hard.

- Authors ask: Is there a simple proof based on model counting?
- Our answer: Yes, our results allow for a simple proof!
- Main ingredients of the proof:
  - $\mathcal{C}_{\widetilde{Q}} = \widetilde{\mathcal{C}}_Q$
  - $\widetilde{Q}$ is hierarchical for any hierarchical query Q
  - $\#\mathcal{C}_Q$ is in PTIME for hierarchical queries [Olteanu & Huang, 2008]
  - $\#\mathcal{C}_Q$ is #P-hard for non-hierarchical queries [Provan & Ball, 1983]

Shapley Values of Database Facts 2/2

Theorem [Deutch et. al., 22]

For any Boolean query Q:

$$Shap(\mathcal{C}_Q) \leq^P \mathrm{PQE}(Q)$$

PQE(Q): Evaluation of Q over probabilistic databases

- Authors ask: $Shap(\mathcal{C}_Q) \equiv^P \mathrm{PQE}(Q)$?

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- Our answer: $Shap(\mathcal{C}_Q) \equiv^P \#\mathcal{C}_{\widetilde{Q}}$

Shapley Computation in Practice

We implemented systems for computing the Shapley value

[SIGMOD'24]

- Shapley (Banzhaf) computation for select-project-join-union queries
  - Exact algorithm
  - Anytime deterministic approximation algorithm
  - Algorithm for ranking and top-k
- Our algorithms significantly outperform algorithms from prior work on real-world datasets
- Lower bound: Shapley-based ranking is tractable only for hierarchical queries

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[VLDB'25]

- First practical approach for queries with aggregates

Proof Techniques for the Main Result

Expressing Shapley Value Using k-Model Counting

Lemma [Livshits et al., 21]

The Shapley value of a variable X of a Boolean function F is:

$$Shap(X, F) = \sum_{k=0}^{n-1} c_k \big(\#_k F[X := 1] - \#_k F[X := 0]\big)$$

where $c_k = \frac{k!(n-k-1)!}{n!}$.

Vandermonde Matrices

Definition

Let $x_1, \ldots, x_n$ be scalars. The $n \times n$ Vandermonde matrix generated by these numbers is:

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ & & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$

Lemma [Golub et al., 96]

Let $x_1, \ldots, x_n$ pairwise distinct scalars. Then, the $n \times n$ Vandermonde matrix generated by these numbers is invertible.

From Shapley Computation to Model Counting 1/2

Lemma

For any class $\mathcal{C}$ of Boolean functions, it holds:

$$\#\mathcal{C} \;\leq^{P} \mathit{Shap}(\widetilde{\mathcal{C}})$$

Given

$F \in \mathcal{C}$

Oracle for $\mathit{Shap}(\widetilde{\mathcal{C}})$

Compute

$\sum_{\boldsymbol{Y} \subseteq \boldsymbol{X}} F[\boldsymbol{Y}]$ (number of models of F)

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Oracle for $\mathit{Shap}(\widetilde{\mathcal{C}})$

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Proof steps:

- Let $F \in \mathcal{C}$ with variables $X_1, \ldots, X_n$
- For $i \in \{1, \ldots, n\}$, let $F^{(\ell,i)} = F[X_p := \bigvee_{j \in \{1,\ldots,\ell\}} X_p^j]_{p \neq i}$
- We have $F^{(\ell,i)} \in \widetilde{\mathcal{C}}$
- Compute $\mathit{Shap}(F^{(\ell,i)}, X_i)$ using oracle for $\mathit{Shap}(\widetilde{\mathcal{C}})$

From Shapley Computation to Model Counting 2/2

- Show:

$$Shap(F^{(\ell,i)}, X_i) = \sum_{k=0}^{n-1} (2^\ell - 1)^k \underbrace{c_k(\#_k F[X_i := 1] - \#_k F[X_i := 0])}_{\Gamma_k F}$$

From Shapley Computation to Model Counting 2/2

- Show:

$$Shap(F^{(\ell,i)}, X_i) = \sum_{k=0}^{n-1} (2^\ell - 1)^k \underbrace{c_k(\#_k F[X_i := 1] - \#_k F[X_i := 0])}_{\Gamma_k F}$$

- Create one equation for each $\ell \in \{1, \ldots, n\}$:

$$\underbrace{\begin{pmatrix} Shap(F^{(1,i)}, X_i) \\ \vdots \\ Shap(F^{(n,i)}, X_i) \end{pmatrix}}_{\text{known}} = \underbrace{\begin{pmatrix} (2^1 - 1)^1 & \cdots & (2^1 - 1)^{n-1} \\ \vdots & \vdots & \vdots \\ (2^{n+1} - 1)^1 & \cdots & (2^{n+1} - 1)^{n-1} \end{pmatrix}}_{\text{Vandermonde}} \underbrace{\begin{pmatrix} \Gamma_0 F \\ \vdots \\ \Gamma_n F \end{pmatrix}}_{\text{unknown}}$$

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- Compute all differences $\#_k F[X_i := 1] - \#_k F[X_i := 0])$

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- Show

$$\sum_{i=1}^{n} (\#_k F[X_i := 1] - \#_k F[X_i := 0]) = (k+1)\#_{k+1} F - (n-k)\#_k F \quad (*)$$

From Shapley Computation to Model Counting 2/2

- Show:
$$Shap(F^{(\ell,i)}, X_i) = \sum_{k=0}^{n-1} (2^\ell - 1)^k \underbrace{c_k(\#_k F[X_i := 1] - \#_k F[X_i := 0])}_{\Gamma_k F}$$
- Create one equation for each $\ell \in \{1, \ldots, n\}$:
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- Compute all differences $\#_k F[X_i := 1] - \#_k F[X_i := 0])$
- Show
$$\sum_{i=1}^{n} (\#_k F[X_i := 1] - \#_k F[X_i := 0]) = (k+1)\#_{k+1} F - (n-k)\#_k F \qquad (*)$$
- Compute $\#_0 F$ and then inductively $\#_k F$ for $k \in \{1, \ldots, n\}$ using (*)