Model Theory

Moritz Müller

February 10, 2024

Contents

Preliminaries

This course builds on an introduction into mathematical logic. We recall some basic definitions and notations.

Sets

For a set A we write $A^2 = A \times A, A^3 = (A \times A) \times A, \dots$ and $P(A)$ for the power set of X. We often view a function f as sets ordered pairs and write $dom(f) = \{b \mid \exists a : (a,b) \in f\}$ for its domain, and $im(f) := \{a \mid \exists a : (a,b) \in f\}$ for its image. For $A \subseteq dom(f)$ we let $f{\restriction}A := f \cap (A \times im(f))$ denote the restriction of f to A. Note that a function g is a restriction of a function f if and only if $g \subseteq f$; then we call f an extension of g.

A set X is *finite* if there is a bijection from $\{0,\dots,n-1\}$ onto X for some $n \in \mathbb{N}$; otherwise it is infinite. E.g., $\varnothing$ is a bijection from $\{0,\dots,0-1\} = \varnothing$ onto $\varnothing$.

For a family of sets $(X_i)_{i\in I}$, the *cartesian product* $\prod_{i\in I} X_i$ is the set of functions f with domain I such that $f(i) \in X_i$ for all $i \in I$. It is non-empty if all X_i are nonempty (this statement is called the *axiom of choice*).

Structures

Let L be a *language*: a set *symbols*, namely function and relation symbols; every symbol has an *arity* (a natural number) associated to it. A *constant* is a function symbol of arity 0.

An L-*structure* $\mathfrak{A}$ is a pair $(A, (s^{\mathfrak{A}})_{s\in L})$ of a *universe* $A \neq \varnothing$ and *interpretations* $s^{\mathfrak{A}}$ of the symbols in $s \in L$: an r-ary relation symbol $R \in L$ is interpreted by a relation $R^{\mathfrak{A}} \subseteq A^r$, an r-ary function symbol $f \in L$ is interpreted by a function $f^{\mathfrak{A}} : A^r \to A$. For a constant $c \in L$ the function maps the unique element of A^0 to some value, and we identify $c^{\mathfrak{A}}$ with this value. We usually denote the universes of structures $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \dots$ by $A, B, C, \dots$.

Let $L' \subseteq L$ and $\mathfrak{A}$ be an L-structure. We call $\mathfrak{A}{\restriction}L' := (A, (s^{\mathfrak{A}})_{s\in L'})$ the L'-*reduct of* $\mathfrak{A}$, and $\mathfrak{A}$ an L-*expansion* of $\mathfrak{A}{\restriction}L'$.

An L-structure $\mathfrak{A}$ is a *substructure* of an L-structure $\mathfrak{B}$, and $\mathfrak{B}$ an *extension of* $\mathfrak{A}$, symbolically $\mathfrak{A} \subseteq \mathfrak{B}$, if $A \subseteq B$ and $R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap A^r$, $f^{\mathfrak{A}} = f^{\mathfrak{B}}{\restriction}A^r$ for all $r \in \mathbb{N}$, all r-ary relation and function symbols $R, f \in L$. For constants $c \in L$ this means $c^{\mathfrak{A}} = c^{\mathfrak{B}}$. Note that the universe A of $\mathfrak{A}$ is an L-*closed* subset of B, i.e., $A \supseteq im(f^{\mathfrak{B}}{\restriction}A^r)$ for every r-ary function symbol $f \in L$; in particular, $c^{\mathfrak{B}} \in A$ for all constants $c \in L$. Conversely, every L-closed subset $A \subseteq B$ is the universe of an L-structure.

Let $X \subseteq B$. If L contains a constant or if $X \neq \varnothing$, then the intersection of all L-closed $A \subseteq B$ with $X \subseteq A$ is the smallest L-closed superset of X. The substructure of $\mathfrak{B}$ with this universe is denoted $\langle X\rangle^{\mathfrak{B}}$ and said to be *generated (in* $\mathfrak{B}$*) by* X. If $X = \varnothing$ and L does not contain constants, the notation is undefined. An L-structure $\mathfrak{B}$ is *finitely generated* if $\mathfrak{B} = \langle X\rangle^{\mathfrak{B}}$ for some finite $X \subseteq B$.

Morphisms

Let L be a language and $\mathfrak{A}, \mathfrak{B}$ be L-structures. A function $\pi : A \to B$ is an *homomorphism from* $\mathfrak{A}$ *to* $\mathfrak{B}$, symbolically $\pi : \mathfrak{A} \to_h \mathfrak{B}$, if for all $r \in N$, all r-ary relation symbols $R \in L$, all r-ary functions symbols $f \in L$, and all $\bar{a} = (a_0, \ldots, a_{r-1}) \in A^r$:

$$\begin{aligned} &\bar{a} \in R^{\mathfrak{A}} \implies \pi(\bar{a}) := (\pi(a_0), \ldots, \pi(a_{r-1})) \in R^{\mathfrak{B}}, \\ &\pi(f^{\mathfrak{A}}(\bar{a}))) = f^{\mathfrak{B}}(\pi(a_0), \ldots, \pi(a_{r-1})). \end{aligned}$$

In particular, $\pi(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for all constants $c \in L$. It is *strong* if $\iff$ holds above, i.e., if $\pi(R^{\mathfrak{A}})(= \{\pi(\bar{a}) \mid \bar{a} \in R^{\mathfrak{A}}\}) = R^{\mathfrak{B}}$. If π is strong and injective, it is an *(algebraic) embedding of* $\mathfrak{A}$ *into* $\mathfrak{B}$, symbolically $\pi : \mathfrak{A} \to_a \mathfrak{B}$. If π is strong and bijective it is an *isomorphism of* $\mathfrak{A}$ *onto* $\mathfrak{B}$, symbolically $\pi : \mathfrak{A} \cong \mathfrak{B}$. If such π exists, then $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic*, symbolically $\mathfrak{A} \cong \mathfrak{B}$. An *automorphism of* $\mathfrak{A}$ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}$. Note $\pi : \mathfrak{A} \to_a \mathfrak{B}$ if and only if $\pi : \mathfrak{A} \cong \mathfrak{B}_0 \subseteq \mathfrak{B}$ for some $\mathfrak{B}_0$, and, $\mathfrak{A} \subseteq \mathfrak{B}$ if and only if the identity (on A) is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

Formulas

L-terms are obtained from variables $x_0, x_1, x_2, \ldots$ (often we use other symbols like $x, y, z, \ldots$) by composition: if $f \in L$ is an r-ary function symbol and $t_0, \ldots, t_{r-1}$ are L-terms, then so is $ft_0 \cdots t_{r-1}$; in particular every constant in L is an L-term. *L-atoms* have the form $t_0 = t_1$ or $Rt_0 \cdots t_{r-1}$ for L-terms t_i and an r-ary relation symbol $R \in L$. *L-formulas* are built from L-atoms by means of $\wedge, \neg, \forall x$. The symbols $\vee, \to, \leftrightarrow, \exists x$ are explained as suitable abbreviations. Let $\bar{x} = (x_0, \ldots, x_{k-1})$ be a tuple of variables, $k \in \mathbb{N}$. Writing a formula φ as $\varphi(\bar{x})$ indicates that the free variables of φ are among $x_0, \ldots, x_{k-1}$; similarly for terms. *Sentences* are formulas without free variables. In formulas, we write $\forall \bar{x}$ or $\forall x_0 \cdots x_{k-1}$ instead $\forall x_0 \cdots \forall x_{k-1}$ and similarly $\exists \bar{x}$. The *universal closure of* $\varphi(\bar{x})$ is the sentence $\forall \bar{x} \varphi(\bar{x})$.

Let $\mathfrak{A}$ be an L-structure. An *assignment in* $\mathfrak{A}$ is a map from the variables into A. The value of an L-term t under an assignment β in $\mathfrak{A}$ is denoted $t^{\mathfrak{A}}[\beta]$. $\mathfrak{A} \models \varphi[\beta]$ means that the L-formula φ is true in $\mathfrak{A}$ under β. For a set of formulas Φ, $\mathfrak{A} \models \Phi[\beta]$ means $\mathfrak{A} \models \varphi[\beta]$ for all $\varphi \in \Phi$. For an L-sentence φ and a term t without variables we omit β, so $t^{\mathfrak{A}} \in A$ and $\mathfrak{A} \models \varphi$ means that $\mathfrak{A}$ *satisfies* φ or φ is *true in* $\mathfrak{A}$.

If $\varphi = \varphi(x_0, \dots, x_{k-1})$ and $\beta(x_i) = a_i \in A$ we write $\mathfrak{A} \models \varphi[a_0, \dots, a_{k-1}]$ for this, and say $\bar{a} = (a_0, \dots, a_{k-1})$ *satisfies* $\varphi(\bar{x})$ in $\mathfrak{A}$. The formula $\varphi(\bar{x})$ *defines (in* $\mathfrak{A}$*)* the set

$$\varphi(\mathfrak{A}) := \{\bar{a} \in A^k \mid \mathfrak{A} \models \varphi[\bar{a}]\}.$$

The notation $t^{\mathfrak{A}}[\bar{a}]$ is similarly explained. If $\pi : \mathfrak{A} \to_h \mathfrak{B}$ then $\pi(t^{\mathfrak{A}}[\bar{a}]) = t^{\mathfrak{B}}[\pi(\bar{a})]$, and if $\pi : \mathfrak{A} \cong \mathfrak{B}$, then $\bar{a} \in \varphi(\mathfrak{A}) \iff \pi(\bar{a}) \in \varphi(\mathfrak{B})$.

Remark. Assume L contains a constant or $X \neq \emptyset$. The universe of $\langle X \rangle^{\mathfrak{A}}$ is the set of $t^{\mathfrak{A}}[\bar{a}]$ where t is an L-term and $\bar{a}$ a tuple from X.

Remark. If $\varphi(\bar{x})$ is quantifier-free and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\varphi(\mathfrak{A}) = \varphi(\mathfrak{B}) \cap A^k$. Hence, if $\mathfrak{B} \models \forall \bar{x} \varphi(\bar{x})$, then $\mathfrak{A} \models \forall \bar{x} \varphi(\bar{x})$. Further, if $\pi : \mathfrak{A} \to_a \mathfrak{B}$ and $\bar{a} \in \varphi(\mathfrak{A})$, then $\pi(\bar{a}) \in \varphi(\mathfrak{B})$.

Theories

Let L be a language and T be an L-*theory*, i.e., a set of L-sentences also called *axioms*. The *theory of* $\mathfrak{A}$ is

$$Th(\mathfrak{A}) := \{\varphi \mid \varphi \text{ is an } L\text{-sentence and } \mathfrak{A} \models \varphi\}.$$

We say $\mathfrak{A}$ *satisfies* T if $\mathfrak{A} \models T$, i.e., $T \subseteq Th(\mathfrak{A})$; if such $\mathfrak{A}$ exists, we call T is *satisfiable*. Two L-theories T, T' are *equivalent*, symbolically $T \equiv T'$, if they have the same models. The class of L-structures satisfying T is *axiomatized by* T. A class of L-structures is *axiomatizable* if it is axiomatized by some L-theory. A class of L-structures is *elementary* if it is axiomatized by some finite L-theory, equivalently, by $\{\varphi\}$ for some L-sentence φ.

Two L-structures $\mathfrak{A}, \mathfrak{B}$ are *elementary equivalent*, symbolically $\mathfrak{A} \equiv \mathfrak{B}$, if $Th(\mathfrak{A}) = Th(\mathfrak{B})$; e.g., $\cong$ implies $\equiv$. An L-theory T is *complete* if it is satisfiable and its models are pairwise elementarily equivalent. his happens if and only if for all L-sentences φ, either T proves φ or T proves $\neg\varphi$ (and not both). That T proves φ is written $T \vdash \varphi$ and characterized as follows – for this course this can be taken as a definition.

Completeness theorem *Let T be an L-theory and φ an L-formula. Then $T \vdash \varphi$ if and only if $\mathfrak{A} \models \varphi[\beta]$ for all L-structures $\mathfrak{A}$ with $\mathfrak{A} \models T$ and all assignments β in $\mathfrak{A}$.*

Remark. Let T be an L-theory.

1. Let $\varphi(\bar{x}, \bar{y})$ be an L-formula and $\bar{c}$ be a tuple of constants outside L. Then $T \vdash \varphi(\bar{c}, \bar{y})$ if and only if $T \vdash \forall \bar{x}\bar{y} \varphi(\bar{x}, \bar{y})$.
2. T is satisfiable if and only if it is *consistent*, i.e., $T \nvdash \neg x = x$.

The proof of the completeness theorem implies

Compactness theorem *Let T be an L-theory. Then T is satisfiable if and only if every finite subset of T is satisfiable.*

This could be called the *fundamental theorem of model theory*. We shall give a 'direct' proof, sidestepping the formal notion of proof.

Algebraic theories

1. The *language of (additively written) groups* is $L_{Gr} := \{+, -, 0\}$ for a binary function symbol $+$, a unary function symbol $-$ and a constant 0. We use infix notation and write $(t+s)$ for L_{Gr}-terms t, s instead $+ts$, The *theory of groups* contains (the universal closures of)

$$x + (y + z) = (x + y) + z, \quad x + 0 = x, \quad 0 + x = x, \quad x + (-x) = 0, \quad (-x) + x = 0.$$

2. The *theory of abelian groups* contains additionally $x + y = y + x$.
3. For $n > 0$ write nx for the term $x + \cdots + x$ (n times) where we omit parenthesis as is usual. We extend the notation to integers setting $0x := 0$ and $(-n)x := n(-x)$ (where $- \in L$ on the r.h.s.). The *theory of divisible abelian groups* is the theory of abelian groups plus $\exists y\ x = ny$ for every $n > 0$.
4. The *theory of ordered abelian groups* has language $L_{Gr} \cup \{<\}$ for a binary relation symbol $<$ and is the theory of abelian groups plus

$$\neg x < x, \quad (x < y \wedge y < z) \to x < z, \quad x < y \to x + z < y + z.$$

5. The *theory of divisible ordered abelian groups* is the union of the previous two theories.
6. The *language of rings* is $L_{Ring} := L_{Gr} \cup \{\cdot, 1\}$ for $\cdot$ a binary function symbol (with infix notation) and a constant 1. The *theory of (commutative unitary) rings* is the theory of abelian groups plus (omitting parentheses as usual)

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z, \quad x \cdot 1 = x, \quad x \cdot y = y \cdot x, \quad x \cdot (y + z) = x \cdot y + x \cdot z.$$

7. The *theory integral domains* is the theory of rings plus

$$\neg 0 = 1, \quad x \cdot y = 0 \to (x = 0 \vee y = 0)$$

8. The *theory of fields* is the theory of rings plus

$$\neg 0 = 1, \quad \neg x = 0 \to \exists y\ x \cdot y = 1.$$

9. The *theory of ordered fields* is the theory of fields plus

$$\begin{gathered} \neg x < x, \quad (x < y \wedge y < z) \to x < z \\ x < y \to x + z < y + z, \quad (x < y \wedge 0 < z) \to x \cdot z < y \cdot z. \end{gathered}$$

By a *group*, *abelian group*, etc. we mean a model of the corresponding theory.

Chapter 1

Ordinals and cardinals

This chapter is a crash course in set theory, developed naively, i.e., not axiomatically. We treat set theory as just another mathematical theory of a certain class of objects, and are not concerned with its philosophical role as a foundation of mathematics.

1.1 Orders

In this section we consider the language $L := \{<\}$ for a binary relation symbol $<$. An L-structure is a pair $\mathfrak{A} = (A, <^{\mathfrak{A}})$ for A a nonempty set and $<^{\mathfrak{A}} \subseteq A^2$. We use infix notation and write $x < y$ instead $< xy$. Given an $\{<\}$-structure $\mathfrak{A}$ and $a, b \in A$ (the universe of A), we similarly write $a <^{\mathfrak{A}} b$ instead $(a, b) \in <^{\mathfrak{A}}$, and $a \leqslant^{\mathfrak{A}} b$ means $a <^{\mathfrak{A}} b$ or $a = b$.

Definition 1.1. An $\{<\}$-structure $\mathfrak{A}$ is a *partial order* if it is *irreflexive* and *transitive*; this means, respectively, that it satisfies $\forall x \neg x < x$ and $\forall xyz((x < y \wedge y < z) \rightarrow x < z)$. A *linear order* additionally satisfies $\forall xy(x = y \vee x < y \vee y < x)$.

Let $\mathfrak{A}$ be a partial order and $B \subseteq A$. An element $b \in B$ is *minimal (in B)* if there does not exist $b' \in B$ with $b' <^{\mathfrak{A}} b$. $\mathfrak{A}$ is *well-founded* if every $\varnothing \neq B \subseteq A$ has minimal elements. A *well-order* is a well-founded linear order.

Examples 1.2.

1. $(A, <^{\mathfrak{A}})$ for $<^{\mathfrak{A}} := \varnothing$ is a partial order where all elements are both minimal and maximal.
2. Throughout this chapter we let $\mathfrak{N}, \mathfrak{Z}, \mathfrak{Q}, \mathfrak{R}$ denote the familiar linear orders with universes $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, respectively. Each is a substructure of the next. None has maximal elements. $\mathfrak{N}$ is a well-order. They are pairwise non-isomorphic. The first 3 are pairwise non-elementary equivalent. We shall see later that $\mathfrak{Q} \equiv \mathfrak{R}$ (Example 3.13).

Remark 1.3. Let $\mathfrak{A}$ be a partial order.

1. If $\varnothing \neq B \subseteq A$ then $\mathfrak{B} = (B, <^{\mathfrak{A}} \cap B^2) \subseteq \mathfrak{A}$ is a partial order. It is a linear order or a well-order if $\mathfrak{A}$ is.

2. The $\{<\}$-structure $\mathfrak{B} := (B, <^{\mathfrak{B}})$ with $B := A$ and $<^{\mathfrak{B}} := \leqslant^{\mathfrak{A}}$ is *reflexive* (satisfies $\forall x\, x < x$), transitive and *anti-symmetric* (satisfies $\forall xy((x < y \wedge y < x) \to x = y)$. Every such structure $\mathfrak{B}$ comes in this sense from a partial order $\mathfrak{A}$.

3. $\mathfrak{A}$ is well-founded if and only if in $\mathfrak{A}$ there is no infinite descending sequence

$$\ldots <^{\mathfrak{A}} a_2 <^{\mathfrak{A}} a_1 <^{\mathfrak{A}} a_0$$

   of elements $a_0, a_1, \ldots \in A$. Indeed: If $a_0, a_1, \ldots$ is such a sequence, then $B := \{a_0, a_1, \ldots\}$ has no minimal element. Conversely, assume $\varnothing \neq B \subseteq A$ has no minimal element. Choose $a_0 \in B$. Since a_0 is not minimal in B, there is $a_1 \in B$ with $a_1 <^{\mathfrak{A}} a_0$. Continue.

4. Finite partial orders are well-founded.

5. For a set X, $(P(X), \subsetneq)$ is a partial order. It is well-founded if and only if X is finite.

   Proof. If X is infinite, there are pairwise distinct $x_0, x_1 \ldots \in X$. Then $\{x_0, x_1, \ldots\} \supsetneq \{x_1, x_2, \ldots\} \supsetneq \{x_2, x_3, \ldots\} \supsetneq \ldots$ is decreasing in $(P(X), \subsetneq)$. $\square$

Exercise 1.4. Show that a well-order is infinite if and only if it contains an infinite *increasing* sequence.

Definition 1.5. Let $\mathfrak{A}, \mathfrak{B}$ be partial orders. The *ordered sum* $\mathfrak{A} + \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$ has universe $(\{0\} \times A) \cup (\{1\} \times B)$ and interprets $<$ setting $(i, c) <^{\mathfrak{A}+\mathfrak{B}} (i', c')$ if and only if $i < i'$, or, $i = i' = 0$ and $c <^{\mathfrak{A}} c'$, or, $i = i' = 1$ and $c <^{\mathfrak{B}} c'$.

The *(anti-lexicographic) product* $\mathfrak{A} \times \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$ has universe $A \times B$ and interprets $<$ setting $(a', b') <^{\mathfrak{A}\times\mathfrak{B}} (a, b)$ if and only if $b' <^{\mathfrak{B}} b$, or, $b' = b$ and $a' <^{\mathfrak{A}} a$.

Intuitively, $\mathfrak{A} + \mathfrak{B}$ is obtained by placing $\mathfrak{B}$ on top of $\mathfrak{A}$; and $\mathfrak{A} \times \mathfrak{B}$ is obtained by replacing every $b \in B$ by a copy of $\mathfrak{A}$.

Example 1.6. $\mathfrak{N} \times \mathfrak{Q} \not\cong \mathfrak{Q} \times \mathfrak{N}$. Indeed, the latter is *dense* (satisfies $\forall xy(x < y \to \exists z(x < z \wedge z < y)))$, the former is not.

Remark 1.7. Let $\mathfrak{A}, \mathfrak{B}$ be partial orders.

1. $\mathfrak{A} + \mathfrak{B}, \mathfrak{A} \times \mathfrak{B}$ are partial orders.

2. If $\mathfrak{A}, \mathfrak{B}$ are linear, then so are $\mathfrak{A} + \mathfrak{B}, \mathfrak{A} \times \mathfrak{B}$.

3. If $\mathfrak{A}, \mathfrak{B}$ are linear or well-founded, then so are $\mathfrak{A} + \mathfrak{B}, \mathfrak{A} \times \mathfrak{B}$.

   Proof. For $\mathfrak{A} + \mathfrak{B}$ this is easy to see. For $\mathfrak{A} \times \mathfrak{B}$, let $\varnothing \neq X \subseteq A \times B$. Since $\mathfrak{B}$ is well-founded, there is a minimal element b_0 of $\{b \in B \mid (a, b) \in X \text{ for some } a \in A\}$. Since $\mathfrak{A}$ is well-founded there is a minimal element a_0 of $\{a \in A \mid (a, b_0) \in X\}$. Then (a_0, b_0) is minimal in X. $\square$

Exercise 1.8 (Order arithmetic)**.** Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be partial orders. Show

$$(\mathfrak{A} + \mathfrak{B}) + \mathfrak{C} \cong \mathfrak{A} + (\mathfrak{B} + \mathfrak{C}),$$
$$(\mathfrak{A} \times \mathfrak{B}) \times \mathfrak{C} \cong \mathfrak{A} \times (\mathfrak{B} \times \mathfrak{C}),$$
$$\mathfrak{A} \times (\mathfrak{B} + \mathfrak{C}) \cong (\mathfrak{A} \times \mathfrak{B}) + (\mathfrak{A} \times \mathfrak{C}).$$

Exercise 1.9. For a partial order $\mathfrak{A}$ and $a \in A$, let $\mathfrak{A}_{<a}$ be the substructure with universe $A_{<a} := \{a' \in A \mid a' <^{\mathfrak{A}} a\}$; it is undefined if this set is empty, i.e., a is minimal; $\mathfrak{A}_{\leqslant a}$ is similarly explained. For partial orders $\mathfrak{A}, \mathfrak{B}$ and $(a, b) \in A \times B$, show

$$(\mathfrak{A} \times \mathfrak{B})_{<(a,b)} \cong (\mathfrak{A} \times \mathfrak{B}_{<b}) + \mathfrak{A}_{<a},$$

if neither a nor b are minimal (in A resp. B). What if one of a, b is minimal?

Definition 1.10. Let $\mathfrak{A}$ be a linear order with a minimal element $0 \in A$ and $\mathfrak{B}$ be another linear order. The $\{<\}$-structure $\mathfrak{A}^{\mathfrak{B}}$ has universe

$$A^{(B)} := \{f : B \to A \mid \mathit{supp}(f) \text{ is finite}\}$$

where $\mathit{supp}(f) := \{b \in B \mid f(b) \neq 0\}$. It interprets $<$ setting $f <^{\mathfrak{A}^{\mathfrak{B}}} g$ if and only if there is $b \in B$ such that $f(b) <^{\mathfrak{A}} g(b)$ and $f(b') = g(b')$ for all $b' \in B$ with $b <^{\mathfrak{B}} b'$.

Example 1.11. Let $2 := (\{0, 1\}, \{(0, 1)\})$. Then $2^{\mathfrak{N}}$ consists of infinite binary sequences that are eventually 0. To determine whether $f <^{2^{\mathfrak{N}}} g$ look for the maximal position where the functions differ: there should be a 1 in g and a 0 in f. E.g.

$$\begin{aligned} f &= 110110001001\mathbf{0}1101011000\cdots \\ g &= 010111101000\mathbf{1}1101011000\cdots \end{aligned}$$

Then, $2^{\mathfrak{N}} \cong \mathfrak{N}$ via the isomorphism $f \mapsto \sum_{i \in \mathbb{N}} f(i) \cdot 2^i$; note the sum is finite.

Exercise 1.12. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be linear orders. Show

$$\mathfrak{A}^{\mathfrak{B}+\mathfrak{C}} \cong \mathfrak{A}^{\mathfrak{B}} \times \mathfrak{A}^{\mathfrak{C}},$$
$$\mathfrak{A}^{\mathfrak{B}\times\mathfrak{C}} \cong (\mathfrak{A}^{\mathfrak{B}})^{\mathfrak{C}}.$$

Lemma 1.13. *If $\mathfrak{A}, \mathfrak{B}$ are well-orders, then so is $\mathfrak{A}^{\mathfrak{B}}$.*

Proof. It is straightforward to check that $\mathfrak{A}^{\mathfrak{B}}$ is a linear order. Let $\emptyset \neq X \subseteq A^{(B)}$. If the function constantly 0 is in X, it is minimal in X and we are done.

Otherwise, $\mathit{supp}(f) \neq \emptyset$ for all $f \in X$. For $f \in X$, let b^0_f be the maximal element of $\mathit{supp}(f)$ and let b_0 be minimal among these. Let a_0 be minimal in $\{f(b_0) \mid f \in X, b^0_f = b_0\}$. It is clear that $X_0 := \{f \in X \mid b^0_f = b_0, f(b_0) = a_0\}$ is an initial segment of X (i.e., its elements are smaller than all other elements in X). Hence it suffices to find a minimal element in X_0. If X_0 contains the function constantly 0 on $B \setminus \{b_0\}$, we are done.

Otherwise, $\mathit{supp}(f) \setminus \{b_0\} \neq \emptyset$ for all $f \in X_0$. For $f \in X_0$, let b^1_f be the maximal element of $\mathit{supp}(f) \setminus \{b_0\}$, and let b_1 be minimal among these. Then $b_1 <^{\mathfrak{B}} b_0$. Let a_1 be minimal in $\{f(b_1) \mid f \in X_0, b^1_f = b_1\}$. Then $X_1 := \{f \in X_0 \mid b^1_f = b_1, f(b_1) = a_1\}$ is an initial segment of X_0. If X_1 contains the function constantly 0 on $B \setminus \{b_0, b_1\}$, we are done.

Otherwise continue. This process eventually stops because $b_0, b_1, \ldots$ is decreasing. □

1.2 Ordinals

For a set X (of sets) we write $\bigcap X := \bigcap_{x\in X} x$ and $\bigcup X := \bigcup_{x\in X} x$. We have $\bigcup\varnothing = \varnothing$, and we consider $\bigcap\varnothing$ to be undefined.

Definition 1.14. A set X is *transitive* if $\bigcup X \subseteq X$. A set α is an *ordinal* if it is transitive, and, $\alpha = \varnothing$ or $(\alpha, \in_\alpha)$ is a well-order where

$$\in_\alpha := \{(x, y) \in \alpha \times \alpha \mid x \in y\}.$$

For an ordinal α we sometimes write $<$ instead of $\in$; as before, $x \leqslant y$ means $x \in y$ or $x = y$.

Remark 1.15. X is transitive if and only if $x \subseteq X$ for all $x \in X$, if and only if $y \in x \in X$ implies $y \in X$ for all x, y.

Exercise 1.16. α is an ordinal, then so is $\alpha^+ := \alpha \cup \{\alpha\}$. It has maximal element α.

Examples 1.17. The sets

$$\underline{0} := \varnothing,\ \underline{1} := \{\varnothing\},\ \underline{2} := \underline{1} \cup \{\underline{1}\} = \{\varnothing, \{\varnothing\}\}, \ldots$$

are ordinals. We shall usually omit the underlining, and, in particular, write 0 for $\varnothing$.

Remark 1.18. Let α, β be ordinals.

1. If $\alpha \neq 0$, then $0 \in \alpha$ (it is the minimal element of α).
2. $\alpha \notin \alpha$ (by irreflexivity).
3. If $x \in \alpha$, then $x = \{y \in \alpha \mid y < x\}$ (since α is transitive).
4. If $x \in \alpha$, then x is an ordinal.

   Indeed: $z \in y \in x$ implies $z \in x$ (transitivity of $<=\in$), so x is transitive and $\in_x = \in_\alpha \cap (x \times x)$; thus, $x = 0$ or $(x, \in_x) \subseteq (\alpha, \in_\alpha)$, so $(x, \in_x)$ is a well-order (see Remark 1.7).
5. $\beta \subseteq \alpha$ if and only if $\beta = \alpha$ or $\beta \in \alpha$. We write $\alpha \leqslant \beta$ for $\alpha \subseteq \beta$ and $\alpha < \beta$ for $\alpha \in \beta$.

   Proof. $\Leftarrow$ is clear since α is transitive. $\Rightarrow$: assume $\beta \subsetneq \alpha$ and let x be minimal in $\alpha \setminus \beta$. Then $x = \{y \in \alpha \mid y < x\} \subseteq \beta$. We are left to show $\beta \subseteq \{y \in \alpha \mid y < x\}$. Let $y \in \beta$ and assume $y \nless x$. As $y \neq x$ (since $y \in \beta$) and $<$ is linear, we have $x < y$. Then $x \in y \in \beta$, so $x \in \beta$ (being transitive), a contradiction. □

Lemma 1.19. *Let X be a nonempty set of ordinals. Then $\bigcap X = \bigcap_{\alpha\in X} \alpha$ is the smallest element of X (i.e., $\bigcap X \in X$ and $\bigcap X \leqslant \alpha$ for all $\alpha \in X$).*

Proof. $\beta := \bigcap X$ is an ordinal because intersections of transitive sets are transitive, and substructures of well-orders are well-orders. Clearly, $\beta \subseteq \alpha$ for all $\alpha \in X$. We have to show $\beta \in X$: otherwise $\beta \in \alpha$ for all $\alpha \in X$ by Remark 1.18 (5), so $\beta \in \beta$ contradicting Remark 1.18 (2). □

Theorem 1.20. *Let α, β be ordinals. Then either $\alpha = \beta$, or $\beta \in \alpha$, or $\alpha \in \beta$.*

Proof. Clearly, no two of these conditions can hold: e.g. $\alpha \in \beta$ and $\beta \in \alpha$ implies $\beta \in \beta$ (since β is transitive), contradicting Remark 1.18 (2). To show at least one holds, apply the previous lemma on $X := \{\alpha, \beta\}$. Then $\alpha \cap \beta$ equals α or β. Hence $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$ and Remark 1.18 (5) implies the claim. □

Corollary 1.21. *Transitive sets of ordinals are ordinals.*

Proof. Linearly ordered by $\in$ by the theorem and well-founded by Lemma 1.19. □

Lemma 1.22. *Let X be a set of ordinals. Then $\bigcup X = \bigcup_{\alpha \in X} \alpha$ is an ordinal and for all ordinals $\gamma < \bigcup X$ there is $\alpha \in X$ such that $\gamma < \alpha$. We write $\sup_{\alpha \in X} \alpha := \bigcup X$.*

Proof. This follows from the corollary: $\bigcup X$ is transitive since it is a union of transitive sets, and its elements are ordinals by Remark 1.18 (4). □

Definition 1.23. An ordinal $\alpha \neq 0$ is a *successor* if $\alpha = \beta^+ = \beta \cup \{\beta\}$ for some ordinal β. Otherwise it is a *limit*.

Examples 1.24. All $\underline{n}$ for $n \in \mathbb{N} \setminus \{0\}$ are successors. $\omega := \{\underline{n} \mid n \in \mathbb{N}\} = \bigcup \{\underline{n} \mid n \in \mathbb{N}\}$ is a limit ordinal. We have $\mathfrak{N} \cong (\omega, \in_\omega)$ via $n \mapsto \underline{n}$.

Exercise 1.25. Show that an ordinal $\lambda \neq 0$ is a limit if and only if $\lambda = \bigcup \lambda$. Show ω is the set of all finite ordinals.

Lemma 1.26. *Let α, β be ordinals and $f : (\alpha, <) \to (\beta, <)$ be an embedding. Then $\gamma \leqslant f(\gamma)$ for all $\gamma \in \alpha$, so $\alpha \leqslant \beta$. If f is bijective, then $\alpha = \beta$ and f is the identity. In particular, $(\alpha, <)$ is* rigid *in that its only automorphism is the identity.*

Proof. Assume there exists $\gamma \in \alpha$ with $f(\gamma) < \gamma$. Let γ_0 be minimal such. Since f is an embedding, $f(f(\gamma_0)) < f(\gamma_0)$ which contradicts minimality.

This implies $\alpha \leqslant \beta$. If f is bijective, then both f and f^{-1} are isomorphisms. Hence f^{-1} is an embedding, so $\beta \leqslant \alpha$ and thus $\alpha = \beta$. Further, $\gamma \leqslant f^{-1}(\gamma)$, so $f(\gamma) \leqslant f(f^{-1}(\gamma)) = \gamma$. □

Theorem 1.27 (Order types)**.** *Every well-order $\mathfrak{A}$ is isomorphic to $(\alpha, \in_\alpha)$ for some ordinal α, called its* order type*. The ordinal and the isomorphism are unique.*

Proof. Uniqueness: assume π, π' are isomorphisms from $\mathfrak{A}$ onto $(\alpha, \in_\alpha), (\alpha', \in_{\alpha'})$. Then $\pi' \circ \pi^{-1} : (\alpha, \in_\alpha) \cong (\alpha', \in_{\alpha'})$. By the lemma, $\alpha = \alpha'$ and $\pi' \circ \pi^{-1}$ is the identity, i.e., $\pi = \pi'$.

To see existence, recall the notations $\mathfrak{A}_{\leqslant a}, \mathfrak{A}_{<a}$ from Exercise 1.9. Call $a \in A$ *good* if there is $\pi_a : \mathfrak{A}_{\leqslant a} \cong (\alpha_a, \in)$ for some ordinal α_a; not good means *bad*. We claim all $a \in A$ are good. Otherwise there is a minimal bad $b \in A$. Clearly, b is not the minimal element of $\mathfrak{A}$. Note $a' <^{\mathfrak{A}} a <^{\mathfrak{A}} b$ implies that $A_{\leqslant a'}$ is an initial segment of $A_{<a}$. As an isomorphism, π_a maps $A_{\leqslant a'}$ onto an initial segment of α_a. This is an ordinal. By uniqueness, π_a agrees with $\pi_{a'}$ on $A_{\leqslant a'}$. It follows that $a \mapsto \pi_a(a)$ is an isomorphism from $\mathfrak{A}_{<b}$ onto $(\beta, \in_\beta)$ where $\beta := \sup_{a <^{\mathfrak{A}} b} \alpha_a$. Extend to an isomorphism from $\mathfrak{A}_{\leqslant b}$ onto β^+ by mapping b to β. Hence b is good, a contradiction.

If all $a \in A$ are good, as before $a \mapsto \pi_a(a)$ shows $\mathfrak{A} \cong (\alpha, \in_\alpha)$ for $\alpha := \sup_{a \in A} \alpha_a$. □

1.3 Ordinal arithmetic

Definition 1.28. Let α, β be non-empty ordinals.

1. $\alpha + \beta$ is the order type of $(\alpha, \in_\alpha) + (\beta, \in_\beta)$; further, we define $0 + \alpha = \alpha + 0 = \alpha$.
2. $\alpha \cdot \beta$ is the order type of $(\alpha, \in_\alpha) \times (\beta, \in_\beta)$; further, we define $0 \cdot \alpha = \alpha \cdot 0 = 0$.
3. α^β is the order type of $(\alpha, \in_\alpha)^{(\beta, \in_\beta)}$; further, we define $\alpha^0 = 0^0 = 1$ and $0^\alpha = 0$.

We omit parentheses as usual: e.g. $\alpha \cdot \beta^\gamma + \delta$ stands for $((\alpha \cdot (\beta^\gamma)) + \delta$.

Remark 1.29 (Ordinal addition)**.** Let α, β, γ be ordinals.

1. $+$ is associative (Exercise 1.8), and $\alpha + 1 = \alpha^+$.
2. Successor recursion: $\alpha + \beta^+ = (\alpha + \beta)^+$ (since $\alpha + (\beta + 1) = (\alpha + \beta) + 1$).
3. $\alpha < \beta$ if and only if $\alpha + \delta = \beta$ for some ordinal $\delta > 0$.

   Indeed, for $\Rightarrow$ take δ such that $(\delta, \in_\delta) \cong (\beta \setminus \alpha, <)$ and check $(\alpha, \in_\alpha) + (\delta, \in_\delta) \cong (\beta, \in_\beta)$. Further, δ is unique by left cancellation below.
4. Right monotonicity: If $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.

   Indeed, given $\beta < \gamma$, choose $\delta > 0$ with $\beta + \delta = \gamma$; then $\alpha + \gamma = (\alpha + \beta) + \delta > \alpha + \beta$.
5. Left cancellation: if $\alpha + \beta = \alpha + \gamma$, then $\beta = \gamma$.

   Indeed, $\beta \neq \gamma$ implies $\beta < \gamma$ or $\gamma < \beta$ by Theorem 1.20; then apply the previous.

Remark 1.30 (Ordinal multiplication)**.** Let α, β, γ be ordinals.

1. 1 is both left- and right-neutral, and $\cdot$ is associative (Exercise 1.8).
2. Left distributivity: $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$ (Exercise 1.8).
3. Successor recursion: $\alpha \cdot (\beta^+) = \alpha \cdot \beta + \alpha$.
4. Right monotonicity: if $\alpha \neq 0$ and $\beta < \gamma$, then $\alpha \cdot \beta < \alpha \cdot \gamma$.

   Indeed write $\gamma = \beta + \delta$ for some $\delta > 0$ and note $\alpha \cdot \delta > 0$ since both $\alpha, \delta \neq 0$. By left distributivity and right monotonicity of $+$: $\alpha \cdot \gamma = \alpha \cdot \beta + \alpha \cdot \delta > \alpha \cdot \beta + 0 = \alpha \cdot \beta$.
5. Left cancellation: if $\alpha \neq 0$ and $\alpha \cdot \beta = \alpha \cdot \gamma$, then $\beta = \gamma$.

   Indeed, if $\beta \neq \gamma$, then $\beta < \gamma$ or $\gamma < \beta$ by Theorem 1.20; then apply the previous.

Exercise 1.31. Let $\alpha \geqslant \omega$. Commutativity fails: $1 + \alpha \neq \alpha + 1$ and $2 \cdot \omega \neq \omega \cdot 2$. Right cancellation fails: $0 + \alpha = 1 + \alpha$. Right distributivity fails: $(\alpha + 1) \cdot 2 \neq \alpha \cdot 2 + 2$.

Proposition 1.32 (Euclidian division)**.** *Let α, β be ordinals and $\alpha \neq 0$. Then there exists a unique pair (ρ, μ) of ordinals such that $\rho < \alpha$ and $\beta = \alpha \cdot \mu + \rho$.*

Proof. Uniqueness: assume $\alpha\cdot\mu+\rho = \alpha\cdot\mu'+\rho'$. By left cancellation, it suffices to show $\mu = \mu'$. Otherwise, say $\mu < \mu'$, so $\mu^+ \leqslant \mu'$. Then we get a contradiction using right monotonicity: $\alpha\cdot\mu+\rho < \alpha\cdot\mu+\alpha = \alpha\cdot\mu^+ \leqslant \alpha\cdot\mu' \leqslant \alpha\cdot\mu'+\rho'$.

Existence: for $\beta = 0$, the claim is trivial, so assume $\beta > 0$. Note $\gamma \mapsto (0,\gamma)$ is an embedding from $(\beta,\in_\beta)$ into $\mathfrak{A} := (\alpha,\in_\alpha)\times(\beta,\in_\beta)$. Hence $\beta \leqslant \alpha\cdot\beta$ by Lemma 1.26. If =, set $(\rho,\mu) := (0,\beta)$. Otherwise $\beta \in \alpha\cdot\beta$. Let $f : (\alpha\cdot\beta,\in_{\alpha\cdot\beta}) \cong \mathfrak{A}$ and set $(\rho,\mu) := f(\beta)$. Not both ρ,μ are 0. We distinguish cases and use Exercise 1.9:

1. $\rho \neq 0$ and $\mu \neq 0$. Then $\mathfrak{A}_{<(\rho,\mu)} \cong (\alpha,\in_\alpha)\times(\beta,\in_\beta)_{<\mu} + (\alpha,\in_\alpha)_{<\rho}$.
2. $\rho = 0$ and $\mu \neq 0$. Then $\mathfrak{A}_{<(\rho,\mu)} \cong (\alpha,\in_\alpha)\times(\beta,\in_\beta)_{<\mu}$.
3. $\rho \neq 0$ and $\mu = 0$. Then $\mathfrak{A}_{<(\rho,\mu)} \cong (\alpha,\in_\alpha)_{<\rho}$.

The l.h.s. has order type β and the r.h.s. has order type $\alpha\cdot\mu+\rho$ in all cases. By Lemma 1.26, $\beta = \alpha\cdot\mu+\rho$. □

Remark 1.33 (Ordinal exponentiation)**.** Let α,β,γ be ordinals.

1. $\alpha^{\beta+\gamma} = \alpha^\beta\cdot\alpha^\gamma$ and $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$ (Exercise 1.12).
2. Successor recursion: $\alpha^{\beta^+} = \alpha^\beta\cdot\alpha$ (since $\alpha^{\beta+1} = \alpha^\beta\cdot\alpha^1$ and $\alpha^1 = \alpha$).
3. Right monotonicity: if $\alpha > 1$ and $\beta < \gamma$, then $\alpha^\beta < \alpha^\gamma$.

   Indeed, write $\beta+\delta = \gamma$ for some $\delta > 0$, so $\alpha^\gamma = \alpha^\beta\cdot\alpha^\delta$. Now, $\alpha^\delta > 1$ since the set $\alpha^{(\delta)}$ has more than one element. Right monotonicity of $\cdot$ gives $\alpha^\beta\cdot\alpha^\delta > \alpha^\beta\cdot 1 = \alpha^\beta$.
4. Left cancellation: if $\alpha > 1$ and $\alpha^\beta = \alpha^\gamma$, then $\beta = \gamma$.

Proposition 1.34 (Limit recursion)**.** *Let α,λ be ordinals and λ a limit.*

1. $\alpha+\lambda = \sup_{\beta<\lambda}(\alpha+\beta)$.
2. $\alpha\cdot\lambda = \sup_{\beta<\lambda}(\alpha\cdot\beta)$.
3. $\alpha^\lambda = \sup_{\beta<\lambda}(\alpha^\beta)$.

Proof. $\geqslant$ is clear in each case. We prove $\leqslant$ for each case.

1: It suffices to find for $\gamma < \alpha+\lambda$ some $\beta < \lambda$ such that $\gamma < \alpha+\beta$. We can assume $\gamma \geqslant \alpha$ and write $\gamma = \alpha+\delta$ for some δ. Then $\delta < \lambda$ (otherwise $\delta \geqslant \lambda$ by Theorem 1.20, so $\gamma \geqslant \alpha+\lambda$), and we set $\beta := \delta^+$. Note $\delta^+ < \lambda$ since $\delta^+ \leqslant \lambda$ and λ is a limit.

2: It suffices to find for $\gamma < \alpha\cdot\lambda$ some $\beta < \lambda$ such that $\gamma < \alpha\cdot\beta$. Euclidian division gives (μ,ρ) with $\gamma = \alpha\cdot\mu+\rho$ and $\rho < \alpha$. Then $\mu < \lambda$. We set $\beta := \mu^+ < \lambda$ and argue by right monotonicity of +: $\gamma < \alpha\cdot\mu+\alpha = \alpha\cdot(\mu^+)$.

3: It suffices to find for f in $(\alpha,\in_\alpha)^{(\lambda,\in_\lambda)} =: \mathfrak{A}$ some $\beta < \lambda$ such that $\mathfrak{A}_{\leqslant f}$ embeds into $(\alpha,\in_\alpha)^{(\beta,\in_\beta)}$ – then the order type of $\mathfrak{A}_{\leqslant f}$ is $\leqslant \alpha^\beta$ by Lemma 1.26. We can assume $supp(f) \neq \varnothing$ and set $\beta := (\max supp(f))^+$. Note every $g \leqslant^{\mathfrak{A}} f$ is a function $g : \lambda \to \alpha$ that is constantly 0 on arguments $\geqslant \beta$. The desired embedding just restricts g to β. □

Exercise 1.35 (Continuity)**.** This proposition states that the operations are in a natural sense continuous in their second argument. Explain why.

1.4 Cardinals

We intend to compare the sizes of arbitrary sets X, Y. The intuition is that the size of X is at most the size of Y if there is an injection from X into Y, equivalently there is a surjection from Y onto X. That X, Y have the same size should mean that they are bijective (there is a bijection from X onto Y). The following is vital for this idea.

Theorem 1.36 (Schröder-Bernstein). *Let X, Y be sets and assume there are injections f from X into Y and g from Y into X. Then there is a bijection from X onto Y.*

Proof. For x in X define the *preimage sequence*

$$g^{-1}(x),\ f^{-1}(g^{-1}(x)),\ g^{-1}(f^{-1}(g^{-1}(x))),\ldots$$

as long as it is defined. E.g., this is the empty sequence if x is not in the image of g. Then

$$h(x) := \begin{cases} g^{-1}(x) & \text{if the preimage sequence of } x \text{ has odd length,} \\ f(x) & \text{if the preimage sequence of } x \text{ has even or infinite length.} \end{cases}$$

defines a bijection from X onto Y. Injectivity is easy to see. We verify surjectivity: given $y \in Y$, set $x := g(y)$. Then the preimage sequence of x is not empty. If its length is odd, then $h(x) = g^{-1}(x) = y$. If it is even or infinite, then it has length at least 2, so $x' := f^{-1}(y)$ exists and the length of its preimage sequence is also even or infinite; hence $h(x') = f(x') = y$. □

The following shows that our desired notion of 'size' is non-trivial for infinite sets.

Proposition 1.37 (Cantor). *Let X be a set. There is no injection from $P(X)$ into X.*

Proof. Otherwise there is a surjection f from X onto $P(X)$. Let $D := \{x \in X \mid x \notin f(x)\}$ and choose $d \in X$ such that $f(d) = D$. Then $d \in D$ if and only if $d \notin f(d) = D$, contradiction. □

Definition 1.38. An ordinal κ is a *cardinal* if there is no injection from κ into a smaller ordinal.

Remark 1.39.

1. All $\underline{n}$ for $n \in \mathbb{N}$ and ω are cardinals.
2. If X is a set of cardinals, then $\sup_{\kappa \in X} \kappa = \bigcup X$ is a cardinal.

   Proof. Assume f is an injection from $\lambda := \sup_{\kappa \in X} \kappa$ into $\alpha < \lambda$. Then $\alpha < \kappa$ for some $\kappa \in X$, so the restriction of f to κ is an injection into $\alpha < \kappa$. But κ is a cardinal. □

The following shows that arbitrarily large cardinals exist.

Definition 1.40. For a set X let $W(X)$ be the set of all well-orders $\mathfrak{A}$ with $A \subseteq X$. *Hartog's aleph* $\mathcal{H}(X)$ *of* X is the set of order types of the well-orders in $W(X)$.

Theorem 1.41 (Hartog)**.** *For every set X, $\mathcal{H}(X)$ is an ordinal, namely the smallest such that there is no injection of $\mathcal{H}(X)$ into X. In particular, $\mathcal{H}(X)$ is a cardinal.*

Proof. Assume $\alpha < \beta \in \mathcal{H}(X)$, say $f : (\beta, \in_\beta) \cong \mathfrak{A} \in W(X)$. Then the restriction $f \upharpoonright \alpha$ (has image in $W(X)$ and) shows $\alpha \in \mathcal{H}(X)$. Thus, $\mathcal{H}(X)$ is a transitive set of ordinals, so an ordinal itself by Corollary 1.21.

If there is an injection f from $\mathcal{H}(X)$ into X, then $f : \mathcal{H}(X) \cong \mathfrak{A} \in W(X)$ where $\mathfrak{A}$ has as universe $im(f)$ and as order the one imported by f (i.e., $\{(x, y) \mid f^{-1}(x) \in f^{-1}(y)\}$). Hence $\mathcal{H}(X) \in \mathcal{H}(X)$, contradicting Remark 1.18 (2). □

Corollary 1.42. *For every ordinal α, $\mathcal{H}(\alpha)$ is the smallest cardinal $> \alpha$.*

We now enumerate all infinite cardinals:

Definition 1.43. For every ordinal α define $\aleph_\alpha$ as follows.

1. $\aleph_0 := \omega$,
2. $\aleph_{\beta+1} := \mathcal{H}(\aleph_\beta)$,
3. $\aleph_\lambda := \sup_{\beta<\lambda} \aleph_\beta$ if λ is a limit.

Exercise 1.44. $\aleph_\alpha$ is a cardinal $\geqslant \alpha$ for every ordinal α, and for every cardinal $\kappa \geqslant \omega$ there exists an ordinal α such that $\aleph_\alpha = \kappa$.

Exercise 1.45. Show there exist arbitrarily large cardinals κ with $\aleph_\kappa = \kappa$.

Hint: Given α, take the supremum of $\aleph_\alpha, \aleph_{\aleph_\alpha}, \aleph_{\aleph_{\aleph_\alpha}}, \ldots$.

1.5 Cardinal arithmetic

Definition 1.46. A set X is *well-orderable* if there exists a well-order with universe X. In this case, the *cardinality* $|X|$ *of* X is the least order-type of a well-order with universe X. If $|X| = \aleph_0$, X is *countable*; if $|X| \leqslant \aleph_0$, X is *at most countable*; if $|X| > \aleph_0$, X is *uncountable*.

Remark 1.47.

1. $|X|$ is a cardinal for every well-orderable set X
2. A set X is well-orderable if and only if X is bijective to some ordinal, if and only if there is an injection of X into some ordinal.

   Indeed, if $f : X \to \alpha$ is an injection, then $\{(x, y) \in X^2 \mid f^{-1}(x) \in f^{-1}(y)\}$ is a well-order on X.
3. Being finite means being bijective to $\{0, \ldots, n-1\}$ for some $n \in \mathbb{N}$; hence, finite sets are well-orderable.
4. $\mathbb{N} \times \mathbb{N}$ is well-orderable because it is bijective to $\mathbb{N}$, hence to ω.

5. The set of finite binary strings is well-orderable since it can be injected into $\mathbb{N}$ (hence into ω): put a 1 in front and view the result as the binary expansion of a natural.

Exercise 1.48. Let X, Y be well-orderable. There is an injection from X into Y if and only if $|X| \leqslant |Y|$. There is a bijection from X onto Y if and only if $|X| = |Y|$.

Thus, the notion of cardinality realizes our initial idea how to measure the size of sets – but only for well-orderable ones. This begs the question which sets are well-orderable. The so-called axiom of choice, or equivalently, Zorn's lemma imply Zermelo's theorem stating that *all* sets are well-orderable. Zermelo's theorem is actually equivalent to Zorn's lemma. These remarks can be made precise only within an axiomatic development of set theory as a foundation of mathematics which is outside the scope of this course.

Our development sofar did not use Zorn's lemma but we adopt it from now on (as is usual nowadays). We recall its statement and use it to prove Zermelo's theorem.

Definition 1.49. A partial order $\mathfrak{A}$ is *inductive* if every chain X in $\mathfrak{A}$ has an upper bound: an element $a \in A$ such that $x \leqslant^{\mathfrak{A}} a$ for every $x \in X$. Here, a *chain in* $\mathfrak{A}$ is a linearly ordered subset, i.e., a nonempty subset $X \subseteq A$ such $\langle X \rangle^{\mathfrak{A}} \subseteq \mathfrak{A}$ is a linear order.

Examples 1.50. Finite partial orders are trivially inductive. $(P(X), \subseteq)$ is inductive for every set X. The linear orders $\mathfrak{N}, \mathfrak{Z}, \mathfrak{Q}, \mathfrak{R}$ are not inductive. The set of consistent theories in a given language, the set of linearly independent subsets of a given vectorspace, the set of proper ideals of a given unitary ring, all partially ordered by $\subseteq$, are inductive.

Zorn's lemma *Inductive partial orders have maximal elements.*

Theorem 1.51 (Zermelo)**.** *Every set is well-orderable.*

Proof. Let X be a set and consider the set F of all injections $f : \alpha \to X$ where α is an ordinal (note $\alpha < \mathcal{H}(X)$). Consider the partial order $(F, \subseteq)$. For a chain C note $\bigcup C$ is an injection whose domain is the union of the domains of functions in C. This is an ordinal by Lemma 1.22. Hence, $\bigcup C \in F$ is an upper bound of C. Thus, $(F, \subseteq)$ is inductive. By Zorn's lemma it contains a maximal element f.

We are left to show that f is surjective. Otherwise choose $x \in X \setminus im(f)$ and let $\alpha := dom(f)$. Then extend f mapping α to x. This defines an injection with domain α^+, contradicting the maximality of f. $\square$

We define arithmetic operations on cardinals. They should not be confused with their ordinal variants although we use the same notation. Again we omit parentheses as usual.

Definition 1.52. Let κ, λ be cardinals.

1. $\kappa + \lambda := |(\{0\} \times \kappa) \cup (\{1\} \times \lambda)|$.
2. $\kappa \cdot \lambda := |\kappa \times \lambda|$.
3. $\kappa^\lambda := |\{f \mid f : \lambda \to \kappa\}|$.

Remark 1.53. Let κ, λ, μ be cardinals.

1. $+, \cdot$ are associative and commutative with neutral elements 0, respectively 1.
2. Distributive law: $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$.
3. $\kappa^{\lambda+\mu} = \kappa^{\lambda} \cdot \kappa^{\mu}$, $\kappa^{\lambda\cdot\mu} = (\kappa^{\lambda})^{\mu}$, $(\kappa \cdot \lambda)^{\mu} = \kappa^{\mu} \cdot \lambda^{\mu}$.

   Proof. These are easy to see. E.g., $\kappa^{\lambda\cdot\mu}$ is bijective to the set of functions $f : \lambda \times \mu \to \kappa$. Such f and $\alpha \in \mu$ determines $F(\alpha) : \lambda \to \kappa$ given by $\beta \mapsto f(\beta, \alpha)$. Thus f determines a function $\alpha \mapsto F(\alpha)$ from μ into the set of functions from λ to κ. The set of such functions is bijective to $(\kappa^{\lambda})^{\mu}$. □

4. $\kappa < 2^{\kappa}$. Indeed: By Cantor it suffices to show that $P(\kappa)$ is bijective to 2^{κ}: map $X \subseteq \kappa$ to its characteristic function "if $\alpha \in X$ then 1, else 0".

Exercise 1.54. For sets X, Y let X^Y denote the set of functions from Y into X. Show $|Y| < |X^Y|$ if $|X| > 1$. Show $|P(\mathbb{N})| = |\{0,1\}^{\mathbb{N}}| = |\mathbb{R}| = |\mathbb{N}^{\mathbb{N}}|$.

Theorem 1.55 (Hessenberg)**.** *$\kappa \cdot \kappa = \kappa$ for all infinite cardinals κ.*

Proof. It suffices to show $\leqslant$. Assume not, and let α be minimal such that $\aleph_\alpha < \aleph_\alpha \cdot \aleph_\alpha$. Define $\prec$ on $\aleph_\alpha \times \aleph_\alpha$ setting $(\beta, \gamma) \prec (\beta', \gamma')$ if and only if one of the following holds:

– $\max\{\beta, \gamma\} < \max\{\beta', \gamma'\}$, or,

– $\max\{\beta, \gamma\} = \max\{\beta', \gamma'\}$ and $\beta < \beta'$, or,

– $\max\{\beta, \gamma\} = \max\{\beta', \gamma'\}$ and $\beta = \beta$ and $\gamma < \gamma$.

It is easy to check that this defines a well-order. Let γ be its order type and $f : (\gamma, \in_\gamma) \cong (\aleph_\alpha \times \aleph_\alpha, \prec)$. It suffices to show $\gamma \leqslant \aleph_\alpha$.

Otherwise $\aleph_\alpha \in \gamma$. Let $f(\aleph_\alpha) = (\beta_0, \gamma_0)$ and $\delta_0 := \max\{\beta_0, \gamma_0\}^+$. Since cardinals are limit ordinals, $\delta_0 < \aleph_\alpha$. Then $f : \aleph_\alpha \to \delta_0 \times \delta_0$ because pairs (β, γ) with $\max\{\beta, \gamma\} \geqslant \delta_0$ are $\succ (\beta_0, \gamma_0)$. Hence $\aleph_\alpha \leqslant |\delta_0 \times \delta_0|$. But $|\delta_0| < \aleph_\alpha$, so $|\delta_0| = |\delta_0| \cdot |\delta_0| = |\delta_0 \times \delta_0|$ – contradiction. □

Corollary 1.56. *Let $\kappa \geqslant \omega$ and $\lambda > 0$ be cardinals. Then $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$.*

Proof. Let $\mu := \max\{\kappa, \lambda\}$. Then $\mu \leqslant \kappa + \lambda \leqslant \mu + \mu = 2 \cdot \mu \leqslant \mu \cdot \mu = \mu$ by Hessenberg. Similarly, $\mu \leqslant \kappa \cdot \lambda \leqslant \mu \cdot \mu = \mu$. □

Exercise 1.57.

1. Let X, Y be nonempty sets, at least one infinite. Show $|X \cup Y| = |X \times Y| = \max\{|X|, |Y|\}$.
2. Let $(X_i)_{i \in I}$ be a family of sets. Show $|\bigcup_{i \in I} X_i| \leqslant \max\{\sup_{i \in I} |X_i|, |I|\}$.
3. (König) Assume $(\kappa_i)_{i \in I}$ is a family of cardinals such that $|X_i| < \kappa_i$. Show

$$|\textstyle\bigcup_{i \in I} X_i| < |\prod_{i \in I} \kappa_i|.$$

1.6 Cofinality and cardinal exponentiation

We saw in the previous section that cardinal addition and multiplication are trivial. We introduce the key concept to understand exponentiation:

Definition 1.58. Let $\mathfrak{A}$ be a linear order. The *cofinality* $\mathrm{cf}(\mathfrak{A})$ *of* $\mathfrak{A}$ is the smallest ordinal α such that there exists an unbounded function $f : \alpha \to A$, i.e., for all $a \in A$ there is $\beta \in \alpha$ such that $a \leqslant^{\mathfrak{A}} f(\beta)$. We write $\mathrm{cf}(\alpha) \coloneqq \mathrm{cf}((\alpha, \in_\alpha))$ for ordinals α.

Remark 1.59.

1. $\mathrm{cf}(\mathfrak{A}) \leqslant |A|$ for every linear order $\mathfrak{A}$.
2. $\mathrm{cf}(\alpha^+) = 1$ for every ordinal α: map 0 to the maximal element α.
3. $\mathrm{cf}(\mathfrak{A})$ is a cardinal for every linear order $\mathfrak{A}$ because if $f : \alpha \to A$ is unbounded and $g : |\alpha| \to \alpha$ is a bijection, then $f \circ g$ is unbounded.

Exercise 1.60. $\mathrm{cf}(\aleph_\lambda) = \mathrm{cf}(\lambda)$ for every limit ordinal λ.

Intuitively, $\mathrm{cf}(\mathfrak{A})$ is the minimal number of steps required to 'climb up' $\mathfrak{A}$:

Lemma 1.61. *Let $\mathfrak{A}$ be a linear order. Then $\mathrm{cf}(\mathfrak{A})$ is the smallest ordinal α such that there is an unbounded embedding f from $(\alpha, \in_\alpha)$ into $\mathfrak{A}$.*

Proof. It suffices to find $f : (\alpha, \in_\alpha) \to_a \mathfrak{A}$ for some $\alpha \leqslant \mathrm{cf}(\mathfrak{A}) =: \kappa$. Let $g : \kappa \to A$ be unbounded. Set $X \coloneqq \{\beta \in \kappa \mid g(\gamma) <^{\mathfrak{A}} g(\beta) \text{ for all } \gamma < \beta\}$. Then the restriction of g to X is unbounded: given $a \in A$ there is $\beta \in \kappa$ such that $a \leqslant^{\mathfrak{A}} g(\beta)$ and a minimal such β is in X.

Let α be the order type of $(X, <)$ and choose $h : (X, <) \cong (\alpha, \in_\alpha)$. Then $f \coloneqq g \circ h^{-1} : \alpha \to A$ is unbounded. Since $h^{-1} : (\alpha, \in_\alpha) \to_a (\kappa, \in_\kappa)$, Lemma 1.26 implies $\alpha \leqslant \kappa$. □

Corollary 1.62. $\mathrm{cf}(\mathfrak{A}) = \mathrm{cf}(\mathrm{cf}(\mathfrak{A}))$ *for every linear order* $\mathfrak{A}$.

Proof. Let $\kappa \coloneqq \mathrm{cf}(\mathfrak{A})$ and $\lambda \coloneqq \mathrm{cf}(\mathrm{cf}(\mathfrak{A}))$. By Remark 1.59 (1), $\kappa \geqslant \lambda$. Conversely, the previous lemma gives unbounded embeddings $f : (\lambda, \in_\lambda) \to (\kappa, \in_\kappa)$ and $g : (\kappa, \in_\kappa) \to \mathfrak{A}$. Then $g \circ f$ is an unbounded embedding of $(\lambda, \in_\lambda)$ into $\mathfrak{A}$, so $\lambda \leqslant \kappa$. □

Definition 1.63. A cardinal κ is *regular* if $\mathrm{cf}(\kappa) = \kappa$, and otherwise *singular*.

Proposition 1.64. $\aleph_{\alpha^+}$ *is regular for every ordinal* α.

Proof. Assume $\kappa \coloneqq \mathrm{cf}(\aleph_{\alpha^+}) < \aleph_{\alpha^+}$, so $\kappa \leqslant \aleph_\alpha$. Let $f : \kappa \to \aleph_{\alpha^+}$ be unbounded. Then $\bigcup_{\beta<\kappa} f(\beta) = \aleph_{\alpha^+} > \aleph_\alpha$. But by Exercise 1.57 (2), $|\bigcup_{\beta<\kappa} f(\beta)| \leqslant \max\{\sup_{\beta<\kappa} |f(\beta)|, \kappa\} \leqslant \max\{\aleph_\alpha, \kappa\} = \aleph_\alpha$, a contradiction. □

Remark 1.65. The usual axioms of set theory (assuming their consistency) do not prove that there exist *weakly inaccessible* cardinals: regular $\aleph_\lambda$ for limit λ.

Proposition 1.66. $\mathrm{cf}(2^\kappa) > \kappa$ *for every cardinal* κ.

Proof. Let $\lambda := \mathrm{cf}(2^\kappa) \leqslant \kappa$ and assume $f : \lambda \to 2^\kappa$ is unbounded. Then $\bigcup_{\beta\in\lambda} f(\beta) = 2^\kappa$ and $|f(\beta)| < 2^\kappa$ for every $\beta < \lambda$. Using Exercise 1.57 (3) and Corollary 1.56, $|\bigcup_{\beta\in\lambda} f(\beta)| < |\prod_{\beta\in\lambda} 2^\kappa| = 2^{\kappa\cdot\lambda} \leqslant 2^\kappa$, a contradiction. □

Remark 1.67. $2^{\aleph_0} \neq \aleph_\omega$. Indeed: $\mathrm{cf}(2^{\aleph_0}) > \aleph_0 = \mathrm{cf}(\omega) = \mathrm{cf}(\aleph_\omega)$ (see Exercise 1.60).

Theorem 1.68 (Cardinal exponentiation)**.** *Let κ, λ be infinite cardinals.*

1. *If $\kappa \leqslant \lambda$, then $\kappa^\lambda = 2^\lambda$.*
2. *If $\mathrm{cf}(\kappa) \leqslant \lambda \leqslant \kappa$, then $\kappa < \kappa^\lambda \leqslant 2^\kappa$.*
3. *If $\lambda < \mathrm{cf}(\kappa)$, then $\kappa \leqslant \kappa^\lambda \leqslant 2^{<\kappa} := \sup_{\alpha<\kappa} 2^{|\alpha|}$.*

Proof. 1: $2^\lambda \leqslant \kappa^\lambda \leqslant (2^\kappa)^\lambda \leqslant 2^{\kappa\cdot\lambda} = 2^\lambda$.

2: $\kappa^\lambda \leqslant (2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^\kappa$. For the lower bound, let $f : \lambda \to \kappa$ be unbounded. Then $\kappa = \bigcup_{\alpha<\lambda} f(\alpha)$. Note $|f(\alpha)| \leqslant f(\alpha) < \kappa$. Hence, by Exercise 1.57 (3):

$$\kappa = |\textstyle\bigcup_{\alpha<\lambda} f(\alpha)| < |\prod_{\alpha<\lambda} \kappa| = \kappa^\lambda.$$

3: If $\kappa = \aleph_0$, this is clear. Assume $\kappa = \aleph_{\alpha^+}$ for some α. Note $\kappa \leqslant 2^{\aleph_\alpha}$ and $\lambda \leqslant \aleph_\alpha$. Hence

$$\kappa \leqslant \kappa^\lambda \leqslant (2^{\aleph_\alpha})^\lambda = 2^{\aleph_\alpha\cdot\lambda} = 2^{\aleph_\alpha} = 2^{<\kappa}.$$

Now assume $\kappa = \aleph_\alpha$ for a limit α, so $\kappa = \sup_{\beta<\alpha} \aleph_\beta$. Every $f : \lambda \to \kappa$ is bounded, so has image in $\aleph_\beta$ for some $\beta < \alpha$. Using Exercise 1.57 (2)

$$\kappa^\lambda = |\textstyle\bigcup_{\beta<\alpha}\{f \mid f : \lambda \to \aleph_\beta\}| \leqslant \max\{|\alpha|, \sup_{\beta<\alpha} \aleph_\beta^\lambda\} \leqslant \max\{|\alpha|, \sup_{\beta<\alpha} 2^{\aleph_\beta\cdot\lambda}\}.$$

Note $\lambda \leqslant \aleph_\beta$ for some $\beta < \alpha$. Further, $|\alpha| \leqslant \alpha = \bigcup_{\beta<\alpha} \beta \subseteq \bigcup_{\beta<\alpha} 2^{\aleph_\beta} = 2^{<\kappa}$. Thus

$$\kappa^\lambda \leqslant \max\{|\alpha|, \sup_{\beta<\alpha} 2^{\aleph_\beta}\} = 2^{<\kappa}.$$ □

This theorem does not give complete information. How large is the gap between κ and 2^κ? Of particular interest is the value of $2^{\aleph_0} = |\mathbb{R}|$ (see Exercise 1.54). Cantor conjectured the *continuum hypothesis CH*: $2^{\aleph_0} = \aleph_1$. The *generalized continuum hypothesis GCH* states $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α. Deep results state that these hypotheses are independent from the usual axioms of set theory (assuming their consistency).

Exercise 1.69. Assume GCH. Given ordinals α, β determine γ such that $\aleph_\alpha^{\aleph_\beta} = \aleph_\gamma$, distinguishing cases as in the previous theorem.

Chapter 2

Boolean algebras and ultraproducts

2.1 Boolean algebras

Let $L_{BA} := \{\cup, \sim, 1\}$ where $\cup$ is a binary function symbol, $\sim$ a unary function symbol, and 1 a constant. We use infix notation and write $(t \cup t)'$ instead $\cup tt'$ for L_{BA}-terms t, t' and usually omit outer parentheses in terms.. We use the abbreviations

$$(t \cap t) :=\sim (\sim t \cup \sim t'), \quad 0 :=\sim 1.$$

Definition 2.1. *Boolean algebra* is a L_{BA}-structure $\mathfrak{B}$ that models the *theory of Boolean algebras.* This is the set of universal closures of the equations:

1. Commutativity: $x \cup y = y \cup x$.
2. Associativity: $(x \cup y) \cup z = x \cup (y \cup z)$.
3. Distributivity: $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$.
4. Absorption: $(x \cup y) \cap y = y$.
5. Complement: $\sim\sim x = x,\ x \cup \sim x = 1$

Obviously, substructures of Boolean algebras are Boolean algebras. Boolean algebras fall on every mathematician's table recognized as such or not:

Examples 2.2.

1. If $\mathfrak{B} \vDash 1 = 0$, then $B = \{1^{\mathfrak{B}}\}$ and $\mathfrak{B}$ is *trivial.* Indeed, using some rules verified in Lemma 2.4 below, we have $b = b \cup^{\mathfrak{B}} 0^{\mathfrak{B}} = b \cup^{\mathfrak{B}} 1^{\mathfrak{B}} = 1^{\mathfrak{B}}$ for every $b \in B$.
2. For a set X the *power set algebra* $\mathfrak{P}(X)$ *of* X has universe $P(X)$ and interpretations given by: $1^{\mathfrak{P}(X)} := X, Y \cup^{\mathfrak{P}(X)} Z := Y \cup Z$ (the union of sets), and $\sim^{\mathfrak{P}(X)} Y = X \setminus Y$ for all $Y, Z \in P(X)$. If $X = \varnothing$, then $\mathfrak{P}(X)$ is trivial.
3. Let L be a language and $\mathfrak{A}$ an L-structure. The *Boolean algebra* $\mathfrak{D}(\mathfrak{A})$ *of definable sets* is the substructure of $\mathfrak{P}(A)$ consisting of the sets $\varphi(\mathfrak{A})$ where $\varphi(x)$ is some L-formula.

4. Let L be a language and T an L theory. Declare formulas φ and ψ equivalent if if $T \vdash (\varphi \leftrightarrow \psi)$. The *Lindenbaum algebra* $\mathfrak{L}(T)$ *of* T has as universe the set of equivalence classes φ/T for L-formulas φ. It interprets 1 by $(\forall x\ x = x)/T$, and $\cup, \sim$ by $(\varphi/T, \psi/T) \mapsto (\varphi \vee \psi)/T$ and $\varphi/T \mapsto \neg\varphi/T$.

   For $n \in \mathbb{N}$, the *n-th Lindenbaum algebra* $\mathfrak{L}_n(T)$ *of* T is the substructure of $\mathfrak{L}(T)$ whose universe consists of φ/T for $\varphi = \varphi(x_0, \dots, x_{n-1})$.

5. For a linear order $\mathfrak{A} = (A, <^{\mathfrak{A}})$ with a smallest element. A *half-open interval* is a set $[a, b) := \{c \in A \mid a \leqslant^{\mathfrak{A}} c <^{A} b\}$ or $[a, \infty) := \{c \in A \mid a \leqslant^{\mathfrak{A}} c\}$. The *interval algebra* $\mathfrak{I}(\mathfrak{A})$ *of* $\mathfrak{A}$ is the substructure of $\mathfrak{P}(A)$ whose universe consists of the sets that are finite unions of disjoint half-open intervals.

6. Let X be a set and $\tau \subseteq P(X)$ a topology on X. The *Boolean algebra of clopen sets* $\mathfrak{C}(X, \tau)$ *of* (X, τ) is the substructure of $\mathfrak{P}(X)$ whose universe consists of the clopen subsets Y of X (i.e., $Y, X \smallsetminus Y \in \tau$).

Exercise 2.3. If $\mathfrak{A} \models T$, then there is a homomorphism from $\mathfrak{L}_1(T)$ into $\mathfrak{D}(\mathfrak{A})$. It is an isomorphism, if T is complete.

Lemma 2.4. *The theory of Boolean algebras proves*

1. *DeMorgan:* $\sim (x \cup y) = \sim x \cap \sim y,\ \sim (x \cap y) = \sim x \cup \sim y$.

2. *the* duals *of the equations in Definition 2.1: swap* $\cap/\cup$ *and* $1/0$.

3. *Idempotencies:* $x \cup x = x,\ x \cap x = x$.

4. *Neutralities:* $x \cup 0 = x,\ x \cap 1 = x,\ x \cup 1 = 1,\ x \cap 0 = 0$.

Proof. Let $\mathfrak{B}$ be a Boolean algebra. We omit superscripts, and write e.g. $a \cup b$ instead $a \cup^{\mathfrak{B}} b$. Let $a, b, c \in B$.

1: Using complement: $\sim (a \cup b) = \sim (\sim\sim a \cup \sim\sim b) = \sim a \cap \sim b$ and $\sim (a \cap b) = \sim\sim (\sim a \cup \sim b) = (\sim a \cup \sim b)$.

2: Commutativity: $\sim a \cup \sim b = \sim b \cup \sim a$. Apply $\sim$ to both sides: $\sim (\sim a \cup \sim b) = \sim (\sim b \cup \sim a)$. This equals dual commutativity: $a \cap b = b \cap a$.

Using complement and DeMorgan: $(a \cap b) \cap c = \sim (\sim a \cup \sim b) \cap \sim\sim c = \sim ((\sim a \cup \sim b) \cup \sim c)$. By associativity: $= \sim (\sim a \cup (\sim b \cup \sim c)) = \sim\sim a \cap \sim (\sim b \cup \sim c) = a \cap (b \cap c)$.

Dual distributivity is similar.

Absorption is $\sim (\sim (a \cup b) \cup \sim b) = b$. Apply $\sim$ to both sides and use complement: $\sim (a \cup b) \cup \sim b = \sim b$. Replace a by $\sim a$ and b by $\sim b$: $\sim (\sim a \cup \sim b) \cup \sim\sim b = \sim\sim b$. By complement: $\sim (\sim a \cup \sim b) \cup b = b$. This equals dual absorption: $(a \cap b) \cup b = b$.

Apply $\sim$ to complement: $\sim (a \cup \sim a) = 0$. Replace a by $\sim a$: $a \cap \sim a = \sim (\sim a \cup \sim\sim a) = 0$

3: Plug $(a \cup a)$ for a, and a for b in dual absorption: $((a \cup a) \cap a) \cup a = a$. By absorption: $(a \cup a) \cap a) = a$. Hence $a \cup a = a$. Using this and complement: $a \cap a = \sim (\sim a \cup \sim a) = \sim\sim a = a$.
4: $a \cup 0 = a \cup (a \cap \sim a)$ by 3, $= (\sim a \cap a) \cup a$ by commutativity of $\cup$ and $\cap$, $= a$ by dual absorption. Similarly, $a \cap 1 = a \cap (a \cup \sim a) = (\sim a \cup a) \cap a = a$. Using idempotency: $a \cup 1 = a \cup (a \cup \sim a) = (a \cup a) \cup \sim a = a \cup \sim a = 1$. Similarly, $a \cap 0 = 0$. $\square$

Exercise 2.5 (Boolean rings)**.** This exercise shows that Boolean algebras are conceptually equivalent to Boolean rings in algebra: a ring $\mathfrak{A} = (A, +^{\mathfrak{A}}, -^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, 0^{\mathfrak{A}}, 1^{\mathfrak{A}})$ is *Boolean* if $a \cdot^{\mathfrak{A}} a = a$ for all $a \in A$. We omit superscripts.

1. Let X be a set. In the power set algebra $\mathfrak{P}(X)$ define $0 := \varnothing, 1 := X$, $Y \cdot Z := Y \cap Z$ and $Y + Z := (Y \smallsetminus Z) \cup (Y \smallsetminus Z)$ (the *symmetric difference* of Y and Z) for $Y, Z \in P(X)$. Show that $(P(X), +, -, \cdot, 0, 1)$ is a Boolean ring.

   For finite X show it is isomorphic to the ring product $\mathfrak{Z}_2 \times \cdots \times \mathfrak{Z}_2$ ($|X|$ times) where $\mathfrak{Z}_2$ is the Boolean ring of integers modulo 2.
2. Let $\mathfrak{A}$ be a Boolean ring. Show that commutativity "is automatic" and $a = -a$ for all $a \in A$. Define $a \cup b := a + b + a \cdot b$ and $\sim a := 1 + a$. Show $(A, \cup, \sim)$ is a Boolean algebra with $a \cap b = a \cdot b$.
3. Conversely, let $\mathfrak{B}$ be a Boolean algebra. Define $a + b := (a \cup b) \cap \sim (a \cap b)$, $-a := a$ and $a \cdot b := a \cap b$. Show $(B, +, -, \cdot, 0, 1)$ is a Boolean ring.

2.2 Classification of finite Boolean algebras

Lemma 2.6. *Finitely generated Boolean algebras are finite.*

Proof. Let $\mathfrak{B} = \langle A \rangle^{\mathfrak{B}}$ where $A = \{a_0, \dots, a_{n-1}\}$ and $n \in \mathbb{N}$. For $n = 0$, $B = \{0^{\mathfrak{B}}, 1^{\mathfrak{B}}\}$ is finite. Inductively assume $n > 0$ and $\mathfrak{B}_0 := \langle A \smallsetminus \{a_0\} \rangle^{\mathfrak{B}} = \langle A \smallsetminus \{a_0\} \rangle^{\mathfrak{B}_0}$ is finite. It suffices to show that every $c \in B$ is *good*: there are $b, b' \in B_0$ (where B_0 is the universe of $\mathfrak{B}_0$) such that

$$c = (b \cap a_0) \cup (b' \cap \sim a_0).$$

All elements $c \in B_0$ are good because $c = c \cap 1 = c \cap (a_0 \cup \sim a_0) = (c \cap a_0) \cup (c \cap \sim a_0)$. Also $a_0 = (1 \cap a_0) \cup (0 \cap a)$ is good. We are left to show that the set of good elements is L_{BA}-closed. Clearly, $1 \in B_0$ is good, and closure under $\cup$ is easy. We show closure under $\sim$.

Let c be good, say $c = (b \cap a_0) \cup (b' \cap a_0)$ for $b, b' \in B_0$, then

$$\begin{aligned} \sim c &= \sim (b \cap a_0) \cap \sim (b' \cap \sim a_0) = (\sim b \cup \sim a_0) \cap (\sim b' \cup a_0) \\ &= (\sim b \cap (\sim b' \cup a_0)) \cup (\sim a_0 \cap (\sim b' \cup a_0)) \\ &= (\sim b \cap \sim b') \cup (\sim b \cap a_0) \cup (\sim a_0 \cap \sim b') \cup (\sim a_0 \cap a_0) \end{aligned}$$

Thus $\sim c$ is good, as the union of good $(\sim b \cap \sim b') \in B_0$ and $(\sim b \cap a_0) \cup (\sim a_0 \cap \sim b')$. □

Remark 2.7. For a Boolean algebra $\mathfrak{B}$ and $a, b \in B$ define $a \leqslant b$ if and only if $a \cap b = a$, and $a < b$ if $a \leqslant b$ and $a \neq b$. Let $a, b \in B$.

1. $\leqslant$ is reflexive, transitive and anti-symmetric.

   Indeed: $a \leqslant a$ since $a \cap a = a$; if $a \leqslant b$ and $b \leqslant c$, then $a \cap c = (a \cap b) \cap c = a \cap (b \cap c) = a \cap b = a$; if $a \leqslant b$ and $b \leqslant a$, then $a = a \cap b = b \cap a = b$.

2. $(B,<)$ is a partial order with (unique) minimal element 0 and maximal element 1.
3. $a \leqslant b$ if and only if $a \cap \sim b = 0$, if and only if $\sim a \cup b = 1$.

   Proof. If $a \leqslant b$, then $a \cap \sim b = (a \cap b) \cap \sim b = a \cap (b \cap \sim b) = 0$.

   If $a \cap \sim b = 0$, then $\sim a \cup b = \sim (a \cap \sim b) = \sim 0 = 1$.

   If $\sim a \cup b = 1$, then $a = a \cap (\sim a \cup b) = (a \cap \sim a) \cup (a \cap b) = a \cap b$, so $a \leqslant b$. □

4. $a \leqslant b$ if and only if $\sim b \leqslant \sim a$.

   Indeed, the l.h.s. is equivalent to $a \cap \sim b = 0$, and the r.h.s. to $\sim b \cap \sim\sim a = 0$.
5. $a \leqslant a \cup b$ since $a \cap (a \cup b) = a$ (absorption).

Examples 2.8.

1. In a power set algebra $\mathfrak{P}(X)$ we have $\leqslant = \subseteq$.
2. In a Lindenbaum algebra $\mathfrak{L}(T)$ we have $\varphi/T \leqslant \psi/T$ if and only if $(\varphi \wedge \psi)/T = \varphi/T$, if and only if $T \vdash ((\varphi \wedge \psi) \leftrightarrow \varphi)$, if and only if $T \vdash (\varphi \to \psi)$.

Definition 2.9. An *atom* of a non-trivial Boolean algebra $\mathfrak{B}$ is a minimal element of the partial order $(B \setminus \{0\}, <)$. If there are no atoms, $\mathfrak{B}$ is *atomless*. $\mathfrak{B}$ is *atomic* if for every $b \in B \setminus \{0\}$ there is an atom a such that $a \leqslant b$.

Lemma 2.10. *Let $\mathfrak{B}$ be a Boolean algebra. The following are equivalent for $a \in B \setminus \{0\}$:*

1. *$a \in B$ is an atom.*
2. *If $a = b \cup c$ for some $b, c \in B$, then $a = b$ or $a = c$.*
3. *$a \leqslant b$ or $a \leqslant \sim b$ for every $b \in B$.*

Proof. $(a) \Rightarrow (b)$: If $a = b \cup c$ then by absorption $b \leqslant a$ and $c \leqslant a$. By minimality, $b = 0$ or $b = a$, and, $c = 0$ or $c = a$. But not both $c = b = 0$ since $a \neq 0$.

$(b) \Rightarrow (c)$: let $b \in B$; then $a = a \cap 1 = a \cap (b \cup \sim b) = (a \cap b) \cup (a \cap \sim b)$. By (b), $a = a \cap b$ or $a = a \cap \sim b$, i.e., $a \leqslant b$ or $a \leqslant \sim b$.

$(c) \Rightarrow (a)$: Let $0 \neq b \leqslant a$. We have to show $a = b$, i.e., $a \leqslant b$. Otherwise by (c), $a \leqslant \sim b$, so $b \leqslant \sim b$, so $b = b \cap \sim b = 0$ a contradiction. □

Examples 2.11.

1. Let X be a non-empty set. $\mathfrak{P}(X)$ is atomic, the atoms are the singletons $\{x\}, x \in X$.
2. The interval algebra $\mathfrak{I}(\mathfrak{Q}_{\geqslant 0})$ is atomless where $\mathfrak{Q}_{\geqslant 0}$ is the usual order on non-negative rationals.
3. $\mathfrak{I}(\mathfrak{N} + \mathfrak{Q})$ is neither atomic nor atomless. Its atoms are $[(0,n),(0,n+1)) = \{(0,n)\}$ for $n \in \mathbb{N}$. E.g. $[(1,0),(1,1))$ has no atom below.
4. Every finite non-trivial Boolean algebra $\mathfrak{B}$ is atomic.

Proof. Let $b \in B \setminus \{0\}$. If b is minimal (in $(B \setminus \{0\}, <)$), it is an atom and we are done. Otherwise choose $0 \neq b_1 < b$; if b_1 is minimal we are done. Otherwise choose $0 \neq b_2 < b_1 \ldots$ this process has to stop because B is finite. □

Later, in Corollary 3.16, we shall see that $\mathfrak{I}(\mathfrak{Q})$ is isomorphic to the following:

Exercise 2.12. Call $X \subseteq \mathbb{N}$ *periodic* if there is $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$:

$$\chi_X(m) = \chi_X(n+m).$$

Here, $\chi_X : \mathbb{N} \to \{0,1\}$ denotes the characteristic function of X. Show that the set of periodic sets is the universe of a countable atomless subalgebra of $\mathfrak{P}(\mathbb{N})$.

Theorem 2.13 (Classification of finite Boolean algebras)**.** *Every finite Boolean algebra is isomorphic to a power set algebra.*

Proof. We show $\pi : \mathfrak{B} \cong \mathfrak{P}(A)$ where A is the set of atoms of $\mathfrak{B}$ and π maps $b \in B$ to

$$\pi(b) := \{a \in A \mid a \leqslant b\}.$$

In particular, if $\mathfrak{B}$ is trivial, then $A = \varnothing$ and $\pi : \mathfrak{B} \cong \mathfrak{P}(\varnothing)$. Assume $\mathfrak{B}$ is not trivial.

Note $1^{\mathfrak{P}(A)} = A = \pi(b) \cup \pi({\sim}\, b)$ by Lemma 2.10. The union is disjoint: $a \leqslant b$ and $a \leqslant {\sim}\, b$ imply $a = a \cap a = (a \cap b) \cap (a \cap {\sim}\, b) = a \cap (b \cap {\sim}\, b) = a \cap 0 = 0$. Hence $\pi({\sim}\, b) = A \setminus \pi(b)$.

For $b, b' \in B$ we have $\pi(b \cup b') = \pi(b) \cup \pi(b)$: $\supseteq$ is clear; conversely, let $a \leqslant b \cup b'$ be an atom; then $a = a \cap (b \cup b') = (a \cap b) \cup (a \cap b')$. By Lemma 2.10, $a = (a \cap b)$ or $a = (a \cap b')$.

π is injective: assume $b \neq b'$, say $b \nleqslant b'$. Then $b \cap {\sim}\, b' \neq 0$. Since $\mathfrak{B}$ is atomic (Examples 2.11 (4)), there is an atom $a \leqslant b \cap {\sim}\, b'$. Then $a \in \pi(b)$ and $a \in \pi({\sim}\, b') = A \setminus \pi(b')$.

To see π is surjective, first note that $a \cap a' = 0$ for distinct atoms a, a': say, $a \nleqslant a'$, i.e., $a \cap a' \neq a$. Since $a \cap a' \leqslant a$ and a is minimal in $B \setminus \{0\}$, we have $a \cap a' = 0$.

Clearly, $\varnothing = \pi(0)$. Let $\varnothing \neq X \in P(A)$, say $X = \{a_0, \ldots, a_n\}$ for some $n \in \mathbb{N}$. Set $b := a_0 \cup \cdots \cup a_n$. Clearly, $a_i \leqslant b$, so $X \subseteq \pi(b)$. Conversely, if $a \in \pi(b)$, then $a = a \cap b = (a \cap a_0) \cup \cdots \cup (a \cap a_n)$, so $a \cap a_i \neq 0$ for some $i \leqslant n$; then $a = a_i$, i.e., $a \in X$. □

Remark 2.14. Recall Exercise 2.5. The above theorem implies that every finite Boolean ring is isomorphic to a finite product $\mathfrak{Z}_2 \times \cdots \times \mathfrak{Z}_2$. The following exercise asks for a direct algebraic proof of this fact. This constitutes a second proof of the above theorem.

Exercise 2.15 (Classification of finite Boolean rings)**.** Let $\mathfrak{A} = (A, +, -, \cdot, 0, 1)$ be a Boolean ring. For $B \subseteq A$ and $a \in A$, write $Ba := \{b \cdot a \mid b \in B\}$.

1. Write $\mathfrak{A}' := \mathfrak{A} \upharpoonright \{+, -, \cdot, 0\}$, i.e., forget the unit. Then $\langle Aa \rangle^{\mathfrak{A}'}$ becomes a Boolean ring $\mathfrak{A}a$ setting $1^{\mathfrak{A}a} := a$. Then $\mathfrak{A}a \times \mathfrak{A}(1+a) \cong \mathfrak{A}$.

2. The universe of $\langle B \rangle^{\mathfrak{A}}$ consists of finite sums of finite products of elements of B: we agree that the empty product equals 1. Further, $\langle Ba \rangle^{\mathfrak{A}a} = \mathfrak{A}a$.

3. If $\mathfrak{A}$ is finite, then $\mathfrak{A}$ is isomorphic to a finite ring product $\mathfrak{Z}_2 \times \cdots \times \mathfrak{Z}_2$.

   Hint: use induction on $n :=$ the minimal size of some $B \subseteq A$ generating $\mathfrak{A}$.

We derive a form of completeness of the equations in Definition 2.1: they imply all equations (whose universal closure is) true in all power set algebras. In fact,

Corollary 2.16. *The theory of Boolean algebras proves every universal L_{BA}-sentence that is true in all finite power set algebras.*

Proof. Assume the theory does not prove $\forall \bar{x}\varphi(\bar{x})$ where $\varphi(\bar{x})$ is quantifier-free. Then there is a Boolean algebra $\mathfrak{B}$ and a tuple $\bar{b}$ in B such that $\mathfrak{B} \models \neg\varphi[\bar{b}]$. Since $\neg\varphi(\bar{x})$ is quantifier-free, we have $\langle \bar{b}\rangle^{\mathfrak{B}} \models \neg\varphi[\bar{b}]$. By Lemma 2.6, $\langle \bar{b}\rangle^{\mathfrak{B}}$ is finite. Hence, it is isomorphic to a power set algebra, so $\forall \bar{x}\varphi(\bar{x})$ is false in it. □

For equalities we shall prove in Section 2.4:

Corollary 2.17. *Let $t(\bar{x}), t'(\bar{x})$ be L_{BA}-terms. If $\forall \bar{x}\ t(\bar{x}) = t'(\bar{x})$ is true in some non-trivial Boolean algebra, then it is true in all Boolean algebras.*

2.3 Stone representation theorem

Theorem 2.13 is not true in general for infinite Boolean algebras. For example, a countable Boolean algebra cannot be isomorphic to a power set algebra because power set algebras are either finite or uncountable. In this section we show that infinite Boolean algebras can be embedded in power set algebras.

Definition 2.18. Let $\mathfrak{B}$ be a Boolean algebra. A set $A \subseteq B$ has the *finite intersection property (fip)* if $A \neq \emptyset$ and $a_0 \cap \cdots \cap a_k \neq 0$ for all $k \in \mathbb{N}$ and $a_0, \ldots, a_k \in A$.

A set $F \subseteq B$ is a *filter (in $\mathfrak{B}$)* if

1. $0 \notin F \neq \emptyset$.
2. For all $a \in F, b \in B$: if $a \leqslant b$, then $b \in F$.
3. For all $a, b \in F$: $a \cap b \in F$.

An *ultrafilter (in $\mathfrak{B}$)* is a maximal filter (i.e., no proper superset is a filter).

Lemma 2.19. *Let $\mathfrak{B}$ be a Boolean algebra.*

1. *Let $A \subseteq B$ have the fip. Then A* generates *the filter*

$$F_A := \{ b \in B \mid \text{ there is } k \in \mathbb{N} \text{ and } a_0, \ldots, a_k \in A \text{ such that } a_0 \cap \cdots \cap a_k \leqslant b \ \}$$

   This is the smallest filter that contains A (i.e., if $F \supseteq A$ is a filter, then $F_A \subseteq F$)

2. *Let F be a filter. The following are equivalent:*

   (a) F is an ultrafilter.

   (b) For all $b \in B$ either $b \in F$ or $\sim b \in F$.

   (c) For all $a, b \in B$, if $a \cup b \in F$, then $a \in F$ or $b \in F$.

Proof. The first statement being easy, we prove the second.

$(a) \Rightarrow (b)$: Assume $b, \sim b \notin F$. To show F is not maximal, it suffices to show that $F \cup \{b\}$ or $F \cup \{b\}$ has the fip. Otherwise there are $a_0 \cap \cdots \cap a_{k-1} \cap b = 0$ and $a'_0 \cap \cdots \cap a'_{\ell-1} \cap \sim b = 0$ for certain $a_i, a'_j \in F$. For $c := a_0 \cap \cdots \cap a_{k-1} \cap a'_0 \cap \cdots \cap a'_{\ell-1}$ we have $c \cap b = 0$ and $c \cap \sim b = 0$. Then $c = c \cap (b \cup \sim b) = 0 \cup 0 = 0$, contradicting fip.

$(b) \Rightarrow (a)$: is trivial, and so is $(c) \Rightarrow (b)$: $b \cup \sim b = 1 \in F$ implies $b \in F$ or $\sim b \in F$; clearly not both $b, \sim b \in F$ (else $0 = b \cap \sim b \in F$).

$(b) \Rightarrow (c)$: if $a, b \notin F$, then $\sim a, \sim b \in F$ by (b), so $\sim a \cap \sim b = \sim (a \cup b) \in F$, so $(a \cup b) \notin F$. $\square$

Lemma 2.20. *If $\mathfrak{B}$ is a Boolean algebra and $A \subseteq B$ has the fip, then there exists an ultrafilter F with $A \subseteq F$.*

Proof. Consider the set $\mathcal{F}$ of filters F with $F \supseteq A$. It is not empty because $F_A \in \mathcal{F}$. The partial order $(\mathcal{F}, \subseteq)$ is inductive: if $\mathcal{C} \subseteq \mathcal{F}$ is a chain, then $\bigcup \mathcal{C} \in \mathcal{F}$ is an upper bound. Now apply Zorn's lemma. $\square$

Exercise 2.21. If $\mathfrak{B}$ is a Boolean algebra and $a \in B \setminus \{0\}$, then $F_{\{a\}} = \{b \in B \mid a \leqslant b\}$ is an ultrafilter if and only if a is an atom.

This motivates the following definition:

Definition 2.22. An ultrafilter F in a Boolean algebra $\mathfrak{B}$ is *principal* if $F = F_{\{a\}}$ for some atom $a \in B$. Otherwise, F is *free.*

Proposition 2.23. *If $\mathfrak{B}$ is infinite, then there exists a free ultrafilter in $\mathfrak{B}$.*

Proof. If $\mathfrak{B}$ is atomless, every ultrafilter is free and ultrafilters exists by Lemma 2.20. Otherwise, the set $\{\sim a \mid a \text{ atom}\}$ is nonempty and by Lemma 2.20 is suffices to show it has the fip. Assume not. Then $\sim a_0 \cap \cdots \cap \sim a_k = 0$ for certain atoms a_i, so $a_0 \cup \cdots \cup a_k = 1$. For every $b \in B$ we have $b = b \cap 1 = (b \cap a_0) \cup \cdots \cup (b \cap a_k)$. Since $b \cap a_i \leqslant a_i$ we have $b \cap a_i \in \{0, a_i\}$. Thus $|B| \leqslant 2^{k+1}$ is finite. $\square$

Theorem 2.24 (Stone representation)**.** *Every Boolean algebra is embeddable into a power set algebra.*

Proof. Let U be the set of ultrafilters in $\mathfrak{B}$. We claim $\pi : \mathfrak{B} \to_a \mathfrak{P}(U)$ where π maps $b \in B$ to $\{p \in U \mid b \in p\}$.

Clearly, $\pi(1^{\mathfrak{B}}) = U = 1^{\mathfrak{P}(U)}$. Further, $\pi(b) = U \setminus \pi(\sim b)$ because for $p \in U$: $p \in \pi(b)$ if and only if $b \in p$, if and only if $\sim b \notin p$, if and only if $p \notin \pi(\sim b)$.

Let $a, b \in B$. Then $\pi(a \cup b) = \pi(a) \cup \pi(b)$: $p \in \pi(a \cup b)$ if and only if $a \cup b \in p$, if and only if $a \in p$ or $b \in p$, if only if $p \in \pi(a)$ or $p \in \pi(b)$.

π is injective: if $a \neq b$, say $a \nleqslant b$, then $a \cap \sim b \neq 0$, so $\{a, \sim b\}$ has the fip. By Lemma 2.20 there is $p \in U$ with $a, \sim b \in p$. Then $p \in \pi(a)$ and $p \in \pi(\sim b) = U \setminus \pi(b)$, so $\pi(a) \neq \pi(b)$. $\square$

Exercise 2.25 (Stone topology)**.** We describe the image of the embedding $\pi : \mathfrak{B} \to_a \mathfrak{P}(U)$ of the above proof. Show $im(\pi)$ is the basis of a topology τ on U, the *Stone topology.* It is Hausdorff and compact. Further, $\pi : \mathfrak{B} \cong \mathfrak{C}(U, \tau)$, the algebra of clopen sets.

Exercise 2.26. The following are equivalent for a Boolean algebra $\mathfrak{B}$:

1. $\mathfrak{B}$ is finite.
2. All ultrafilters in $\mathfrak{B}$ are principal.
3. There are only finitely many ultrafilters in $\mathfrak{B}$.

Exercise 2.27. A nontrivial Boolean algebra $\mathfrak{B}$ that is not atomic has a least $2^{\aleph_0}$ many ultrafilters.

Hint: Assume there is no atom below $b \in B \setminus \{0\}$. Since b is not an atom, $b = b_0 \cup b_1$ for certain $b_0, b_1 > 0$ with $b_0 \cap b_1 = 0$. Similarly write $b_0 = b_{00} \cup b_{01}$ and $b_1 = b_{10} \cup b_{11}$, etc. This defines a binary tree below b and its branches are contained in pairwise distinct ultrafilters.

2.4 Reduced products and Horn formulas

Let L be a language and $(\mathfrak{A}_i)_{i\in I}$ a family of L-structures for a nonempty set I.

Definition 2.28 (Products)**.** The *product* $\mathfrak{A} := \prod_{i\in I} \mathfrak{A}_i$ *of* $(\mathfrak{A}_i)_{i\in I}$ has universe $A := \prod_{i\in I} A_i$ (where A_i is the universe of $\mathfrak{A}_i$), that is, the set of functions a with $dom(a) = I$ and $a(i) \in A_i$ for all $i \in I$. We occasionally write $a \in A$ as

$$a = \langle a(i) \mid i \in I \rangle.$$

For $r \in \mathbb{N}$ and $\bar{a} = (a_0, \ldots, a_{r-1}) \in A^r$ and $i \in I$ write $\bar{a}(i) := (a_0(i), \ldots, a_{r-1}(i))$. For r-ary relation and function symbols $R, f \in L$ the interpretations are given by

$$\begin{aligned} &\bar{a} \in R^{\mathfrak{A}} \ :\Longleftrightarrow\ \bar{a}(i) \in R^{\mathfrak{A}_i} \text{ for all } i \in I,\\ &f^{\mathfrak{A}}(\bar{a}) := \langle f^{\mathfrak{A}_i}(\bar{a}(i)) \mid i \in I \rangle. \end{aligned}$$

If $\mathfrak{A}_i = \mathfrak{B}$ for all $i \in I$ we write $\mathfrak{B}^I := \prod_{i\in I} \mathfrak{A}_i$.

Proposition 2.29. *Let φ be $\forall \bar{x}\ t(\bar{x}) = s(\bar{x})$ for L-terms $t(\bar{x}), s(\bar{x})$. If $\mathfrak{A}_i \vDash \varphi$ for all $i \in I$, then $\mathfrak{A} := \prod_{i\in I} \mathfrak{A}_i \vDash \varphi$.*

Proof. A straightforward induction shows for every L-term $t(\bar{x})$ and $\bar{a}$ from A:

$$t^{\mathfrak{A}}[\bar{a}] = \langle t^{\mathfrak{A}_i}[\bar{a}(i)] \mid i \in I \rangle.$$

Assume $\mathfrak{A} \nvDash \varphi$ and choose $\bar{a}$ such that $t^{\mathfrak{A}}[\bar{a}] \neq s^{\mathfrak{A}}[\bar{a}]$. Then there exists $i \in I$ such that $t^{\mathfrak{A}_i}[\bar{a}(i)] \neq s^{\mathfrak{A}_i}[\bar{a}(i)]$, i.e., $\mathfrak{A}_i \nvDash t = s[\bar{a}(i)]$, so $\mathfrak{A}_i \nvDash \varphi$. □

Examples 2.30.

1. If all $\mathfrak{A}_i$ are Boolean algebras, abelian groups or rings, then so is $\prod_{i\in I} \mathfrak{A}_i$.
2. For the linear order $\mathfrak{Q}$, $\mathfrak{Q}^{\mathbb{N}}$ is not a linear order: neither $\langle 0, 0, 0, \ldots \rangle <^{\mathfrak{Q}^{\mathbb{N}}} \langle 1, 0, 0, \ldots \rangle$ nor vice-versa.

Proof of Corollary 2.17. Let $\mathfrak{B}$ be a nontrivial Boolean algebra satisfying $\varphi := \forall \bar{x}\ t(\bar{x}) = t'(\bar{x})$. By Corollary 2.16 it suffices to show that $\mathfrak{P}(I) \vDash \varphi$ for every set I.

Let $\mathfrak{B}_0 \subseteq \mathfrak{B}$ have universe $\{0^{\mathfrak{B}}, 1^{\mathfrak{B}}\}$. Since φ is universal, $\mathfrak{B}_0 \vDash \varphi$. If $I = \varnothing$, then $\mathfrak{P}(I)$ has one element, so satisfies all equalities. If $I \neq \varnothing$, then $\mathfrak{B}_0^I$ is a Boolean algebra that satisfies φ by Proposition 2.29. But $\mathfrak{P}(I) \cong \mathfrak{B}_0^I$ via $X \mapsto \langle 1^{\mathfrak{B}} \mid i \in X\rangle \cup \langle 0^{\mathfrak{B}} \mid i \notin X\rangle$. □

Remark 2.31. By a filter, ultrafilter, ... *on* I we mean one in $\mathfrak{P}(I)$.

1. The principal ultrafilters on I are $F_{\{i\}} = \{X \subseteq I \mid i \in X\}$ (Examples 2.11). There exist free ultrafilters on I if and only if I is infinite (Exercise 2.26).
2. Let I be infinite. The *Frechet filter F_{cf} (on I)* is the set of all co-finite subsets of I, i.e., those $X \subseteq I$ with $I \setminus X$ finite. For an ultrafilter F on I the following are equivalent:

   (a) F is free.

   (b) F does not contain finite sets.

   (c) $F_{cf} \subseteq F$.

   Proof. $(b) \Rightarrow (a)$: if F is principal, say $F = F_{\{i\}}$, then $\{i\} \in F$.

   $(a) \Rightarrow (b)$: If F contains a finite set $\{i_0, \ldots, i_k\} = \{i_0\} \cup \cdots \cup \{i_k\}$ for some $k \in \mathbb{N}$, then it contains $\{i_j\}$ for some $j \leqslant k$ (cf. Lemma 2.19 (2c)), so $F = F_{\{i_j\}}$ is principal.

   $(b) \Leftrightarrow (c)$: F contains no finite set if and only if F contains all complements of finite sets (cf. Lemma 2.19 (2b)). □

Lemma 2.32. *Let F be a filter on I. Define a binary relation $\sim_F$ on $\prod_{i\in I} A_i$ by*

$$a \sim_F b \iff \{a = b\} := \{i \in I \mid a(i) = b(i)\} \in F.$$

Then for all $r \in \mathbb{N}$, all r-ary relation symbols $R \in L$, all r-ary function symbols $f \in L$ and all $\bar{a} = (a_0, \ldots, a_{r-1}), \bar{b} = (b_0, \ldots, b_{r-1}) \in A^r$ with $a_0 \sim_F b_1, \ldots, a_{r-1} \sim_F b_{r-1}$:

1. *$\sim_F$ is an equivalence relation.*
2. *$f^{\mathfrak{A}}(\bar{a}) \sim_F f^{\mathfrak{A}}(\bar{b})$*
3. *$\{i \in I \mid (\bar{a}(i)) \in R^{\mathfrak{A}_i}\} \in F \iff \{i \in I \mid (\bar{b}(i)) \in R^{\mathfrak{A}_i}\} \in F.$*

Proof. 1: $a \sim_F a$ because $\{a = a\} = I \in F$. If $a \sim_F b$, then $b \sim_F a$ because $\{a = b\} = \{b = a\}$. If $a \sim_F b$ and $b \sim_F c$, then $\{a = b\} \cap \{b = c\} \in F$, so $\{a = b\} \cap \{b = c\} \subseteq \{a = c\} \in F$.

2: $X := \bigcap_{j<r}\{a_j = b_j\} \in F$ and $X \subseteq \{i \in I \mid f^{\mathfrak{A}_i}(\bar{a}(i)) = f^{\mathfrak{A}_i}(\bar{b}(i))\} \in F$.

3: Let Y, Z denote the sets on the left and right respectively. If $Y \in F$, then $X \cap Y \in F$, so $X \cap Y \subseteq Z \in F$. The converse is analogous. □

This lemma enables the

Definition 2.33 (Reduced products)**.** Let F be a filter on I. The *reduced product* $\mathfrak{A} := \prod_F \mathfrak{A}_i$ has as universe A the set of equivalence classes a^F for $a \in \prod_{i \in I} A_i$. For $r \in \mathbb{N}$ and $\bar{a} = (a_0, \dots, a_{r-1}) \in (\prod_F \mathfrak{A}_i)^r$ write $\bar{a}^F := (a_0^F, \dots, a_{r-1}^F) \in A^r$.

For r-ary relation and function symbols $R, f \in L$, the interpretations are given by:

$$\begin{aligned} \bar{a}^F \in R^{\mathfrak{A}} \; &:\Longleftrightarrow \; \{i \in I \mid \bar{a}(i) \in R^{\mathfrak{A}_i}\} \in F \\ f^{\mathfrak{A}}(\bar{a}^F) &:= \langle f^{\mathfrak{A}_i}(\bar{a}(i)) \mid i \in I \rangle^F, \end{aligned}$$

If $\mathfrak{A}_i = \mathfrak{B}$ for all $i \in I$, we write $\mathfrak{B}^I_F := \prod_F \mathfrak{A}_i$.

Exercise 2.34. Prove:

1. Let $\varnothing \neq I_0 \subseteq I$ and $F := F_{I_0} = \{X \subseteq I \mid I_0 \subseteq X\}$. Then $\prod_F \mathfrak{A}_i \cong \prod_{i \in I_0} \mathfrak{A}_i$. Hence, $\prod_{\{I\}} \mathfrak{A}_i \cong \prod_{i \in I} \mathfrak{A}_i$ and $\prod_{F_{\{i\}}} \mathfrak{A}_i \cong \mathfrak{A}_i$ for every $i \in I$.
2. If $F \subseteq F'$ are filters, then there is an homomorphism from $\prod_F \mathfrak{A}_i$ into $\prod_{F'} \mathfrak{A}_i$.
3. Infer an analogue of Proposition 2.29.

Example 2.35. Let F be the Frechet filter on $\mathbb{N}$. Consider $\mathfrak{R}^{\mathbb{N}}_F$ for $\mathfrak{R}$ the field of reals. Then $\langle r_0, r_1, r_2, \dots \rangle^F = \langle s_0, s_1, s_2, \dots \rangle^F$ if and only if $r_n = s_n$ for all sufficiently large $n \in \mathbb{N}$. Further, $\langle 1, 0, 1, 0, \dots \rangle^F$ and $\langle 0, 1, 0, 1, \dots \rangle^F$ are elements $\neq 0^{\mathfrak{R}^{\mathbb{N}}_F}$ with product $0^{\mathfrak{R}^{\mathbb{N}}_F}$.

We generalize Proposition 2.29.

Definition 2.36. A *Horn formula* is a CNF (a conjunction of disjunctions of literals (atomic formulas or a negations thereof)) whose disjunctions contain at most one literal that is not a negation. A formula is *universal Horn* if it is obtained from a Horn formula by universal quantification.

Proposition 2.37. *Let F be a filter on I. Let φ be a universal Horn sentence and assume $\mathfrak{A}_i \vDash \varphi$ for all $i \in I$. Then $\prod_F \mathfrak{A}_i \vDash \varphi$.*

Proof. Write $\mathfrak{A} := \prod_F \mathfrak{A}_i$. For every L-term $t(\bar{x})$ and $\bar{a}$ from $\prod_{i \in I} A_i$:

$$t^{\mathfrak{A}}[\bar{a}^F] = \langle t^{\mathfrak{A}_i}[\bar{a}(i)] \mid i \in I \rangle^F.$$

We prove this by induction on t. If $t = x_j$, then both sides equal a_j^F. If $t = f t_0 \cdots t_{r-1}$ and the claim holds for the $t_j(\bar{x})$, then

$$\begin{aligned} t^{\mathfrak{A}}[\bar{a}^F] &= f^{\mathfrak{A}}(t_0^{\mathfrak{A}}[\bar{a}^F], \dots, t_{r-1}^{\mathfrak{A}}[\bar{a}^F]) = f^{\mathfrak{A}}\big(\langle t_0^{\mathfrak{A}_i}[\bar{a}(i)] \mid i \in I \rangle^F, \dots, \langle t_{r-1}^{\mathfrak{A}_i}[\bar{a}(i)] \mid i \in I \rangle^F\big) \\ &= \big\langle f^{\mathfrak{A}_i}(t_0^{\mathfrak{A}_i}[\bar{a}(i)], \dots, t_{r-1}^{\mathfrak{A}_i}[\bar{a}(i)]) \mid i \in I \big\rangle^F = \big\langle t^{\mathfrak{A}_i}[a_0(i), \dots, a_{k-1}(i)] \mid i \in I \big\rangle^F. \end{aligned}$$

We next claim for every atomic $\varphi(\bar{x})$ and $\bar{a}$ from $\prod_{i \in I} A_i$:

$$\mathfrak{A} \vDash \varphi[\bar{a}^F] \iff \big\{ i \in I \mid \mathfrak{A}_i \vDash \varphi[\bar{a}(i)] \big\} \in F.$$

Assume $\varphi = Rt_0\cdots t_{r-1}$ (the case $t_0 = t_1$ is similar). Then

$$\begin{aligned}\mathfrak{A} \vDash \varphi[\bar{a}^F] &\iff \big(\langle t_0^{\mathfrak{A}_i}[\bar{a}(i)] \mid i \in I\rangle^F, \dots, \langle t_{r-1}^{\mathfrak{A}_i}[\bar{a}(i)] \mid i \in I\rangle^F\big) \in R^{\mathfrak{A}} \\ &\iff \big\{i \in I \mid (t_0^{\mathfrak{A}_i}[\bar{a}(i)], \dots, t_{r-1}^{\mathfrak{A}_i}[\bar{a}(i)]) \in R^{\mathfrak{A}_i}\big\} \in F \\ &\iff \big\{i \in I \mid \mathfrak{A}_i \vDash \varphi[\bar{a}(i)]\big\} \in F.\end{aligned}$$

Let $\varphi(\bar{x}) = \forall \bar{x} \bigwedge_j \bigvee_k \lambda_{jk}(\bar{x})$ be universal Horn, the λ_{jk} being literals. Assume $\mathfrak{A} \nvDash \varphi$ and $\mathfrak{A}_i \vDash \varphi$ for all $i \in I$. Choose $\bar{a}^F$ and j such that $\mathfrak{A} \nvDash \bigvee_k \lambda_{jk}[\bar{a}^F]$. Assume the disjunction contains an atom χ; the case that all literals are negations is similar. Then the disjunction is logically equivalent to $((\psi_0 \wedge \dots \wedge \psi_{\ell-1}) \to \chi)$ for certain atoms $\psi_j(\bar{x}), j < \ell$. Hence $\mathfrak{A} \vDash \psi_j[\bar{a}^F]$, for $j < \ell$, and $\mathfrak{A} \nvDash \chi[\bar{a}^F]$. By the claim, $X_j := \{i \in I \mid \mathfrak{A}_i \vDash \psi_j[\bar{a}(i)]\} \in F$ and $Y := \{i \in I \mid \mathfrak{A}_i \vDash \chi[\bar{a}(i)]\} \notin F$. Then $X := \bigcap_{j<\ell} X_j \in F$. For every $i \in X$ we have $\mathfrak{A}_i \vDash \psi_j[\bar{a}(i)]$ for all $j < \ell$ and hence $\mathfrak{A}_i \vDash \chi[\bar{a}(i)]$ because $\mathfrak{A}_i \vDash \varphi$. Hence $X \subseteq Y$, so $Y \in F$, a contradiction. □

Examples 2.38. Reduced products of partial orders are partial orders. Same for Boolean algebras, rings and abelian groups.

2.5 Ultraproducts

Let L be a language and $(\mathfrak{A}_i)_{i\in I}$ a family of L-structures for a nonempty set I.

Definition 2.39 (Ultraproducts)**.** An *ultraproduct of* $(\mathfrak{A}_i)_{i\in I}$ is a reduced product $\prod_F \mathfrak{A}_i$ for some ultrafilter F on I. It is an *ultrapower of* $\mathfrak{B}$ if $\mathfrak{B} = \mathfrak{A}_i$ for all $i \in I$; written $\mathfrak{B}^I_F$.

Theorem 2.40 (Łos)**.** *Let F be an ultrafilter on I and φ be an L-sentence. Then*

$$\prod_F \mathfrak{A}_i \vDash \varphi \iff \{i \in I \mid \mathfrak{A}_i \vDash \varphi\} \in F.$$

Proof. Let $\mathfrak{A} := \prod_F \mathfrak{A}_i$. We show that for all L-formulas $\varphi(\bar{x})$and all $\bar{a}^F$ from A:

$$\mathfrak{A} \vDash \varphi[\bar{a}^F] \iff \big\{i \in I \mid \mathfrak{A}_i \vDash \varphi[\bar{a}(i)]\big\} \in F.$$

Call formulas *good* if they satisfy this claim. We saw in the proof of Proposition 2.37 that atomic formulas are good.

If $\varphi(\bar{x})$ is good, so is $\neg\varphi(\bar{x})$: note, since F is an ultrafilter,

$$\big\{i \in I \mid \mathfrak{A}_i \vDash \varphi[\bar{a}(i)]\big\} \notin F \iff \big\{i \in I \mid \mathfrak{A}_i \nvDash \varphi[\bar{a}(i)]\big\} \in F.$$

It is easy to see that goof formulas are closed under conjunctions. We show they are closed under universal quantification. Assume $\varphi(\bar{x}, y)$ is good. We show $\forall y \varphi(\bar{x}, y)$ is good:

$$\begin{aligned}\mathfrak{A} \vDash \forall y \varphi[\bar{a}^F] &\iff \mathfrak{A} \vDash \varphi[\bar{a}^F, a^F] \text{ for all } a \in \textstyle\prod_{i\in I} A_i \\ &\iff X_a := \big\{i \in I \mid \mathfrak{A}_i \vDash \varphi[\bar{a}(i), a(i)]\big\} \in F \text{ for all } a \in \textstyle\prod_{i\in I} A_i \\ &\iff X := \big\{i \in I \mid \mathfrak{A}_i \vDash \forall y \varphi[\bar{a}(i)]\big\} \in F.\end{aligned}$$

For the last equivalence argue as follows. If $X \in F$, then $X_a \in F$ because $X \subseteq X_a$. If $X \notin F$, then $I \setminus X \in F$ being an ultrafilter. Choose for each $i \notin X$ some $a(i)$ such that $\mathfrak{A}_i \nvDash \psi[\bar{a}(i), a(i)]$; for $i \in X$ let $a(i) \in A_i$ be arbitrary. For this a we have $I \setminus X_a \supseteq I \setminus X \in F$, so $I \setminus X_a \in F$ and $X_a \notin F$. □

Corollary 2.41 (Compactness theorem)**.** *Let T be an L-theory such that every finite subset of T is satisfiable. Then T is satisfiable.*

Proof. Let I be the set of finite subsets of T. For $i \in I$ choose an L-structure $\mathfrak{A}_i$ with $\mathfrak{A}_i \vDash i$. For $i \in I$ let $X(i) := \{j \in I \mid i \subseteq j\}$. These sets have the fip: if $i_0, \dots, i_k \in I$, then

$$X(i_0) \cap \cdots \cap X(i_k) = X(i_0 \cup \cdots \cup i_k) \neq \varnothing.$$

Let F be an ultrafilter on I containing all $X(i), i \in I$ (Lemma 2.20). Let $\varphi \in T$. Then

$$\{i \in I \mid \mathfrak{A}_i \vDash \varphi\} \supseteq \{i \in I \mid \varphi \in i\} = X(\{\varphi\}) \in F.$$

By Łos, $\prod_F \mathfrak{A}_i \vDash \varphi$. Thus, $\prod_F \mathfrak{A}_i \vDash T$. □

2.5.1 Periodic and torsion-free abelian groups

Let $\mathfrak{G}$ be an abelian group. For $n \in \mathbb{N}$ and $g \in G$ let $ng := g +^{\mathfrak{G}} \cdots +^{\mathfrak{G}} g$ (n times) and let nx be the term $x + \cdots + x$ (n times); for $n = 0$ we undertand $ng = 0^{\mathfrak{G}}$ and $nx = 0$.

$\mathfrak{G}$ is *periodic* if all elements have finite order, i.e., for all $g \in G$ there is $n > 0$ such that $ng = 0^{\mathfrak{G}}$. $\mathfrak{G}$ is *torsion-free* if $ng \neq 0^{\mathfrak{G}}$ for all $n > 0$ and $g \in G \setminus \{0^{\mathfrak{G}}\}$.

Proposition 2.42.

1. *The class of periodic abelian groups is not axiomatizable.*
2. *The class of torsion-free abelian groups is axiomatizable but not elementary.*

Proof. We give a proof without using the compactness theorem.

1: Assume for contradiction that T is an axiomatization. Let F be a free ultrafilter on the set of primes. Let $\mathfrak{Z}_p$ be the additive group of integers modulo p. Then $\mathfrak{Z}_p \vDash T$. By Łos, $\mathfrak{A} := \prod_F \mathfrak{Z}_p \vDash T$. But this is false, $\mathfrak{A}$ is in fact torsion-free: assume $\mathfrak{A} \vDash nx = 0[a^F]$ and $a^F \neq 0^{\mathfrak{A}}$ and $n > 0$; by Łos, $X := \{p \mid p \text{ divides } n{\cdot}a(p)\} \in F$ and $Y := \{p \mid a(p) \neq 0\} \in F$; hence, $X \cap Y \in F$ is infinite (Remark 2.31 (2)), but every $p \in X \cap Y$ divides n, a contradiction.

2: For an axiomatization just add, for every $n > 0$, the sentence $\forall x(\neg x = 0 \to \neg nx = 0)$ to the theory of abelian groups. For contradiction, assume $\{\varphi\}$ is an axiomatization. Then $\mathfrak{A} \vDash \varphi$, so $\{p \mid \mathfrak{Z}_p \vDash \varphi\} \in F$ by Łos. But this set is empty, a contradiction. □

2.5.2 Ideals versus filters in products of fields

Let I be a nonempty set and for each $i \in I$ let $\mathfrak{A}_i$ be a field. By Examples 2.30 (3), $\prod_{i \in I} \mathfrak{A}_i$ is a ring. We show that ideals in this ring naturally correspond to filters on I.

Proposition 2.43. *There exists a bijection $J \mapsto F_J$ from the set of proper ideals J of $\prod_{i\in I}\mathfrak{A}_i$ onto the set of filters on I such that*

$$\prod_{i\in I}\mathfrak{A}_i/J = \prod_{F_J}\mathfrak{A}_i.$$

Furthermore, J is a maximal ideal if and only if F_J is an ultrafilter.

Proof. For $a \in \prod_{i\in I} A_i$ let $0_a := \{i \in I \mid a(i) = 0^{\mathfrak{A}_i}\}$. For a proper ideal J of $\prod_{i\in I}\mathfrak{A}_i$ we define

$$F_J := \{0_a \mid a \in J\}.$$

We claim $a \in J \iff 0_a \in F_J$ for all $a \in \prod_{i\in I} A_i$. Indeed, if $0_a \in F_J$, then $0_a = 0_b$ for some $b \in J$, so $a = c \cdot b \in J$ for $c := \langle 0 \mid i \in 0_b\rangle \cup \langle a(i)/b(i) \mid i \notin 0_b\rangle$.

The exercise below shows F_J is a filter and similarly $F \mapsto J_F := \{a \mid 0_a \in F\}$ maps filters F to ideals J_F. Then $F_{J_F} = F$ is trivial, and $J_{F_J} = J$ follows from the claim:

$$a \in J_{F_J} \iff 0_a \in F_J \iff a \in J.$$

It follows that both maps are bijective. To see $\prod_{i\in I}\mathfrak{A}_i/J = \prod_{F_J}\mathfrak{A}_i$ we check using the claim:

$$a = b \mod J \iff b - a \in J \iff 0_{b-a} \in F_J \iff \{i \in I \mid a(i) = b(i)\} \in F_J \iff a \sim_{F_J} b.$$

The exercise below shows that F_J is an ultrafilter if and only if $\prod_{F_J}\mathfrak{A}_i$ is a field. But $\prod_{i\in I}\mathfrak{A}_i/J$ is a field if and only if J is a maximal ideal. □

Exercise 2.44. In the notation of the above proof, show: F_J is a filter on I for every proper ideal J, and J_F is an ideal for every filter F on I. Further show for every filter F on I that $\prod_F\mathfrak{A}_i$ is a field if and only if F is an ultrafilter.

Chapter 3

Back and Forth

3.1 Partial isomorphisms

Let L be a language, and $\mathfrak{A}, \mathfrak{B}$ be L-structures. Recall we identify functions with their graphs, i.e., view them as sets of ordered pairs.

Definition 3.1. A *partial isomorphism p from $\mathfrak{A}$ to $\mathfrak{B}$* is an injection with $dom(p) \subseteq A$ and $im(p) \subseteq B$ such that for all $r \in \mathbb{N}$, all r-ary relations symbols $R \in L$, all r-ary function symbols $f \in L$ and all $\bar{a} = (a_0, \ldots, a_{r-1}) \in dom(p)^r, a \in dom(p)$:

$$\begin{aligned} \bar{a} \in R^{\mathfrak{A}} &\iff p(\bar{a}) := (p(a_0), \ldots, p(a_{r-1})) \in R^{\mathfrak{B}}, \\ f^{\mathfrak{A}}(\bar{a}) = a &\iff f^{\mathfrak{B}}(p(\bar{a})) = p(a). \end{aligned}$$

$\mathfrak{A}$ and $\mathfrak{B}$ are *partially isomorphic*, symbolically $\mathfrak{A} \cong_p \mathfrak{B}$, if $I : \mathfrak{A} \cong_p \mathfrak{B}$ for some nonempty set I of partial isomorphisms. This means:

1. (Forth) for every $p \in I$ and $a \in A$ there is $q \in I$ such that $p \subseteq q$ and $a \in dom(q)$,
2. (Back) for every $p \in I$ and $b \in B$ there is $q \in I$ such that $p \subseteq q$ and $b \in im(q)$.

Remark 3.2. If $dom(p)$ is the universe of a substructure $\mathfrak{A}_0 \subseteq \mathfrak{B}$, then $im(p)$ is the universe of a substructure $\mathfrak{B}_0 \subseteq \mathfrak{B}$ and $p : \mathfrak{A}_0 \cong \mathfrak{B}_0$. Indeed: for $\bar{a} \in A^r$ with $f^{\mathfrak{A}}(\bar{a}) \in dom(p)$ the condition for f above is equivalent to $p(f^{\mathfrak{A}}(\bar{a})) = f^{\mathfrak{B}}(p(\bar{a}))$.

The following partial isomorphisms will play a central role.

Definition 3.3. The *skeleton of* $\mathfrak{A}$ is the class $Sk(\mathfrak{A})$ of L-structures that are isomorphic to finitely generated substructures of $\mathfrak{A}$. $I_{Sk}(\mathfrak{A}, \mathfrak{B})$ is the set of isomorphisms from a finitely generated substructure of $\mathfrak{A}$ onto a finitely generated substructure of $\mathfrak{B}$.

Remark 3.4 (Ehrenfeucht games)**.** Consider the following two-player game between Spoiler and Duplicator. The Spoiler chooses an element in one of the structures, the Duplicator responds by choosing an element in the other structure. After k rounds they determined tuples $\bar{a} = (a_0, \dots, a_{k-1}) \in A^k$ and $\bar{b} = (b_0, \dots, b_{k-1}) \in B^k$. The Spoiler wins once

$$\bar{a} \mapsto \bar{b} := \{(a_i, b_i) \mid i < k\}$$

is not a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Otherwise, Duplicator wins (the infinite play).

Then $I : \mathfrak{A} \cong_p \mathfrak{B}$ can be seen as a winning strategy for Duplicator. Section 3.2 studies this game truncated to a fixed number k of rounds.

Definition 3.5. L-formula is *term-reduced* if all its atomic subformulas have the form

$$R\bar{x}, \;\; x = y, \;\; f\bar{x} = y$$

for variables $\bar{x}, x, y$ and $R, f \in L$ relation, resp. function symbols.

Exercise 3.6. Every L-formula is logically equivalent to a term-reduced L-formula.

Exercise 3.7. $\bar{a} \mapsto \bar{b}$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ if and only if $\bar{a}$ in $\mathfrak{A}$ satisfies the same term-reduced atoms as $\bar{b}$ does in $\mathfrak{B}$.

The following two theorems give two meanings according to which partially isomorphic structures are 'similar'.

Theorem 3.8. *Assume $\mathfrak{A}, \mathfrak{B}$ are at most countable. Then*

$$\mathfrak{A} \cong_p \mathfrak{B} \iff \mathfrak{A} \cong \mathfrak{B}.$$

Moreover, if $p \in I : \mathfrak{A} \cong_p \mathfrak{B}$ then there is $\pi : \mathfrak{A} \cong \mathfrak{B}$ with $p \subseteq \pi$.

Proof. If $\pi : \mathfrak{A} \cong \mathfrak{B}$, then $\{\pi\} : \mathfrak{A} \cong_p \mathfrak{B}$. Conversely, assume $I : \mathfrak{A} \cong_p \mathfrak{B}$ and let $p \in I$. Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$ (possibly finite). We define a chain $p_0 \subseteq p_1 \subseteq \cdots$ of partial isomorphisms in I. We set $p_0 := p$. Having defined p_n distinguish cases: if n is even, choose $p_n \subseteq p_{n+1} \in I$ with $a_{\lfloor n/2 \rfloor} \in dom(p_{n+1})$ according to (Forth); if n is odd, choose $p_n \subseteq p_{n+1} \in I$ with $b_{\lfloor n/2 \rfloor} \in im(p_{n+1})$ according to (Back). Then $\pi := \bigcup_{n \in \mathbb{N}} p_n$ is the desired isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. □

Theorem 3.9. *If $\mathfrak{A} \cong_p \mathfrak{B}$, then $\mathfrak{A} \equiv \mathfrak{B}$.*

Proof. Let $I : \mathfrak{A} \cong_p \mathfrak{B}$. By Exercise 3.6 is suffices to show for every term-reduced L-formula $\varphi(\bar{x})$ that for all $p \in I$ and all tuples $\bar{a}$ from $dom(p)$:

$$\mathfrak{A} \models \varphi[\bar{a}] \iff \mathfrak{B} \models \varphi[p(\bar{a})].$$

Call an L-formula *good* if it is term-reduced and satisfies this claim. Term-reduced atoms are good by Exercise 3.7. Clearly, good formulas are closed under $\neg, \wedge$. We show that $\varphi := \exists y \psi(\bar{x}, y)$ is good, if $\psi(\bar{x}, y)$ is. Let $\bar{a}$ be a tuple from $dom(p)$ and $p \in I$. $\Rightarrow$: if $\mathfrak{A} \models \varphi[\bar{a}]$, choose $a \in A$ such that $\mathfrak{A} \models \psi[\bar{a}, a]$; by (Forth) choose $p \subseteq q \in I$ with $a \in dom(q)$; since ψ is good, $\mathfrak{B} \models \psi[q(\bar{a}), q(a)]$, so $\mathfrak{B} \models \varphi[p(\bar{a})]$. $\Leftarrow$ is similar using (Back). □

We shall later prove a more fine-grained result later (Theorem 3.27). The converse of Theorem 3.9 fails – Example 3.36 is a natural counterexample.

3.1.1 Back and forth in dense orders

Definition 3.10. The *theory of dense linear orders without endpoints* has axioms:

$$\forall x\ \neg x < x,\ \forall xyz((x < y \wedge y < z) \to x < z),$$
$$\forall xy(x < y \to \exists z(x < z \wedge z < y)),\ \forall x \exists y\ x < y, \forall y \exists y\ y < x.$$

Lemma 3.11. *Let $\mathfrak{A}, \mathfrak{B}$ be dense linear orders without endpoints. Then $I_{Sk}(\mathfrak{A}, \mathfrak{B}) : \mathfrak{A} \cong_p \mathfrak{B}$.*

Proof. Note $\emptyset \in I \neq \emptyset$. Let $p \in I$ and let $a_0, \ldots, a_{k-1}$ list $dom(p)$ for some $k \in \mathbb{N}$; let $b_i := p(a_i)$ for $i < k$. To verify (Back) ((Forth) is similar), let $b \in B$ be given. There exists $a \in A$ that $<^{\mathfrak{A}}$-compares to the a_i exactly as b $<^{\mathfrak{B}}$-compares to the b_i: e.g., if $b_i <^{\mathfrak{B}} b <^{\mathfrak{B}} b_{i+1}$ for some $i < k-1$, then we can choose $a \in A$ such that $a_i <^{\mathfrak{A}} a <^{\mathfrak{A}} a_{i+1}$ because $\mathfrak{A}$ is dense. Then $p \cup \{(a, b)\} \in I$. □

By Theorems 3.8 and 3.9:

Corollary 3.12.

1. *Every countable dense linear order without endpoints is isomorphic to $\mathfrak{Q}$.*
2. *The theory of dense linear orders without endpoints is complete.*

Example 3.13. $\mathfrak{Q} + \mathfrak{Q} \cong \mathfrak{Q} \times \mathfrak{Q} \cong \mathfrak{Q} \times \mathfrak{N} \cong \mathfrak{Q} \cong_p \mathfrak{R}$, in particular, $\mathfrak{Q} \equiv \mathfrak{R}$.

3.1.2 Back and forth in atomless Boolean algebras

Definition 3.14. The theory of *(nontrivial) atomless Boolean algebras* is the theory of Boolean algebras plus the axioms

$$\neg 0 = 1,\ \forall x(\neg x = 0 \to \exists y(\neg y = x \wedge \neg y = 0 \wedge y \cap x = y)).$$

Lemma 3.15. *Let $\mathfrak{A}, \mathfrak{B}$ be atomless Boolean algebras. Then $I_{Sk}(\mathfrak{A}, \mathfrak{B}) : \mathfrak{A} \cong_p \mathfrak{B}$.*

Proof. Note $I \neq \emptyset$ because $0, 1$ (we omit superscripts) generate isomorphic subalgebras of $\mathfrak{A}$ and $\mathfrak{B}$. We verify (Forth) ((Back) is similar). Let $p \in I$ and $a \in A$. Then $p : \mathfrak{A}_0 \cong \mathfrak{B}_0$ where $\mathfrak{A}_0 \subseteq \mathfrak{A}$ is the substructure with universe $dom(p)$ and $\mathfrak{B}_0 \subseteq \mathfrak{B}$ is the substructure with universe $im(p)$. Let $a_0, \ldots, a_{k-1}$ list the atoms of $\mathfrak{A}_0$. Then $p(a_0), \ldots, p(a_{k-1})$ lists the atoms of $\mathfrak{B}_0$. Let $a'_j := a \cap a_j$ and set $J := \{j < k \mid a'_j \neq 0\}$. For $j \in J$ choose $0 < b'_j < b_j$ in $\mathfrak{B}$. Then $\{a'_j, a_j \cap \sim a'_j \mid j \in J\} \cup \{a_j \mid j \notin J\}$ are the atoms of the subalgebra $\mathfrak{A}_1 \subseteq \mathfrak{A}$ they generate, and $\{b'_j, b_j \cap \sim b'_j \mid j \in J\} \cup \{b_j \mid j \notin J\}$ are the atoms of the subalgebra $\mathfrak{B}_1 \subseteq \mathfrak{B}$ they generate. The natural bijection between these to sets extends to an isomorphism $q : \mathfrak{A}_1 \cong \mathfrak{B}_1$. Then $p \subseteq q$ and $a \in dom(q)$. □

Corollary 3.16.

1. *Every countable atomless Boolean algebra is isomorphic to $\mathfrak{I}(\mathfrak{Q}_{\geqslant 0})$.*
2. *The theory of atomless Boolean algebras is complete.*

3.1.3 Back and forth in algebraically closed fields

In the language $L_{Ring} = \{+, -, \cdot, 0, 1\}$ of rings and fields write $x^n := x \cdot \ldots \cdot x$ (n times).

Definition 3.17. The *theory ACF of algebraically closed fields* is the theory of fields together with for every $d > 0$ the sentence

$$\forall y_0 \cdots y_{d-1} \exists x \; x^d + y_{d-1} \cdot x^{d-1} + \cdots + y_1 \cdot x + y_0 = 0$$

For a prime p, the *theory ACF_p of algebraically closed fields of characteristic p* additionally has axiom $\chi_p := 1 + \cdots + 1 = 0$ (p times 1).

The *theory ACF_0 of algebraically closed fields of characteristic* 0 additionally has axioms $\neg\chi_p$ for all primes p.

We recall some facts from algebra. Let $\mathfrak{A}$ be a field and $B \subseteq A$. Then $\langle B \rangle^{\mathfrak{A}}$ is a ring. The smallest subfield of $\mathfrak{A}$ containing B is $\langle\!\langle B \rangle\!\rangle^{\mathfrak{A}}$, it has universe $t^{\mathfrak{A}_B} / s^{\mathfrak{A}_B}$ for $L_{Ring}(B)$-terms t, s. A subfield of $\mathfrak{A}$ is *finitely generated* if it equals $\langle\!\langle B \rangle\!\rangle^{\mathfrak{A}}$ for some finite $B \subseteq A$.

Let $\mathfrak{A}_0 \subseteq \mathfrak{A}$ be a subfield and $a \in A$. Write $\mathfrak{A}_0(a)$ for $\langle\!\langle A_0 \cup \{a\} \rangle\!\rangle^{\mathfrak{A}}$. The element $a \in A$ is *transcendental over* $\mathfrak{A}_0$ if $P(a) \neq 0$ for all $P \in \mathfrak{A}_0[X]$, the polynomial ring over $\mathfrak{A}_0$. Then $\mathfrak{A}_0(a) \cong \mathfrak{A}_0(X)$, the quotient field of $\mathfrak{A}_0[X]$. The isomorphism fixes $\mathfrak{A}_0$ and maps a to X.

If $a \in A$ is not transcendental over $\mathfrak{A}_0$, it is *algebraic over* $\mathfrak{A}_0$. The unique monic polynomial p of minimal degree with $P(a) = 0$ is the *minimal polynomial of a over* $\mathfrak{A}_0$. Then $\mathfrak{A}_0(a) = \langle A_0 \cup \{a\} \rangle^{\mathfrak{A}} \cong \mathfrak{A}_0[X]/(P)$; here, (P) denotes the ideal of $\mathfrak{A}_0[X]$ generated by P. The isomorphism fixes $\mathfrak{A}_0$ and maps a to $X \mod (P)$.

Lemma 3.18. *Let $\mathfrak{A}, \mathfrak{B}$ be algebraically closed fields of the same characteristic. Assume $\mathfrak{A}$ is "large" in the sense that for every finitely generated subfield $\mathfrak{A}_0$ of $\mathfrak{A}$ there exists $a \in A$ which is transcendental over $\mathfrak{A}_0$. Assume also $\mathfrak{B}$ is "large" in this sense.*

Then $I : \mathfrak{A} \cong_p \mathfrak{B}$ for I the set of partial isomorphisms p from $\mathfrak{A}$ to $\mathfrak{B}$ such that $dom(p)$ is the universe of a finitely generated subfield of $\mathfrak{A}$.

Proof. Since $\mathfrak{A}, \mathfrak{B}$ have the same characteristic, $I \neq \emptyset$. Indeed, $\langle\!\langle \emptyset \rangle\!\rangle^{\mathfrak{A}}, \langle\!\langle \emptyset \rangle\!\rangle^{\mathfrak{B}}$ are both isomorphic to the field of rationals if the characteristic is 0, and to the field of integers modulo p if the characteristic is $p > 0$.

We verify (Forth) ((Back) is similar). Let $p \in I$ and $a \in A$. Let $\mathfrak{A}_0$ be the subfield of $\mathfrak{A}$ with universe $dom(p)$ and let $\mathfrak{B}_0$ be the subfield of $\mathfrak{B}$ with universe $im(p)$.

Assume first that a is algebraic over $\mathfrak{A}_0$, say with minimal polynomial P. Let $Q \in \mathfrak{B}_0[X]$ be obtained from P by replacing all coefficients a by $p(a)$. Since $p : \mathfrak{A}_0 \cong \mathfrak{B}_0$, also $\mathfrak{A}_0(a) \cong \mathfrak{A}_0[X]/(P) \cong \mathfrak{B}_0[X]/(Q)$. The latter is $\cong \mathfrak{B}_0(b)$ for $b \in B$ such that $Q(b) = 0$. Such b exists because $\mathfrak{B}$ is algebraically closed. Composing gives an isomorphism $q : \mathfrak{A}_0(a) \cong \mathfrak{B}_0(b)$ that extends p and maps a to b.

Now assume a is transcendental over $\mathfrak{A}_0$. Then $\mathfrak{A}_0(a) \cong \mathfrak{A}_0(X)$. Since $\mathfrak{B}$ is "large" there exists $b \in B$ transcendental over $\mathfrak{B}_0$. Then $\mathfrak{B}_0(b) \cong \mathfrak{B}_0(X)$. But $p : \mathfrak{A}_0 \cong \mathfrak{B}_0$ implies $\mathfrak{A}_0(X) \cong \mathfrak{B}_0(X)$. Composing gives an isomorphism $q : \mathfrak{A}_0(a) \cong \mathfrak{B}_0(b)$ that extends p and maps a to b. $\square$

Exercise 3.19. Let $\mathfrak{A}, \mathfrak{B}$ be fields and I as in Lemma 3.18. Then:

$$I : \mathfrak{A} \cong \mathfrak{B} \iff I_{Sk}(\mathfrak{A}, \mathfrak{B}) : \mathfrak{A} \cong_p \mathfrak{B}.$$

In Section 5.1 we shall prove:

Lemma 3.20. *For every algebraically closed field $\mathfrak{A}$ there exists $\mathfrak{A}^* \equiv \mathfrak{A}$ that is "large" in the sense of the previous lemma.*

Corollary 3.21. *Let p be a prime or 0. The theory ACF_p is complete.*

Proof. We have to show that any two models $\mathfrak{A}, \mathfrak{B}$ of ACF_p are elementarily equivalent. Then $\mathfrak{A} \equiv \mathfrak{A}^* \cong_p \mathfrak{B}^* \equiv \mathfrak{B}$ for suitable $\mathfrak{A}^*, \mathfrak{B}^*$. By Theorem 3.9, $\mathfrak{A}^* \equiv \mathfrak{B}^*$. □

Exercise 3.22. Let $\mathfrak{C}$ be the field of complex numbers and φ be an L_{Ring}-sentence. Use the compactness theorem to show that $\mathfrak{C} \models \varphi$ if and only if φ is true in all algebraically closed fields of sufficiently large characteristic.

3.2 Ehrenfeucht-Fraïssé theory

Let L be a *finite* language, and $\mathfrak{A}, \mathfrak{B}$ be L-structures.

Definition 3.23. Let $k \in \mathbb{N}$. $\mathfrak{A}$ and $\mathfrak{B}$ are *k-isomorphic*, symbolically $\mathfrak{A} \cong_k \mathfrak{B}$, if there is a sequence $(I_j)_{j \leqslant k}$ of sets I_j of partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ such that $(I_j)_{j \leqslant k} : \mathfrak{A} \cong_k \mathfrak{B}$, i.e., $I_k \neq \varnothing$ and

1. (Forth) For all $0 < j \leqslant k$, $p \in I_j$, $a \in A$ there is $q \in I_{j-1}$ such that $p \subseteq q$ and $a \in dom(q)$.
2. (Back) For all $0 < j \leqslant k$, $p \in I_j$, $a \in A$ there is $q \in I_{j-1}$ such that $p \subseteq q$ and $b \in im(q)$.

$\mathfrak{A}, \bar{a} \cong_k \mathfrak{B}, \bar{b}$ means that there exists such $(I_j)_{j \leqslant k}$ and some $p \in I_{k-1}$ with $p(\bar{a}) = \bar{b}$.

Lemma 3.24. *Let $k > 0$ and $\bar{a}, \bar{b}$ be tuples from A, B. Then $\mathfrak{A}, \bar{a} \cong_k \mathfrak{B}, \bar{b}$ if and only if*

for all $a \in A$ there is $b \in B$: $\mathfrak{A}, \bar{a}a \cong_{k-1} \mathfrak{B}, \bar{b}b$, and,

for all $b \in B$ there is $a \in A$: $\mathfrak{A}, \bar{a}a \cong_{k-1} \mathfrak{B}, \bar{b}b$.

Proof. $\Rightarrow$: clear. $\Leftarrow$: for every $a \in A$ choose $b \in B$ and $(I^a_j)_{j \leqslant k-1}$ witnessing $\mathfrak{A}, \bar{a}a \cong_{k-1} \mathfrak{B}, \bar{b}b$, and for all $b \in B$ choose $a \in A$ and $(I^b_j)_{j \leqslant k-1}$ witnessing $\mathfrak{A}, \bar{a}a \cong_{k-1} \mathfrak{B}, \bar{b}b$. For $j \leqslant k-1$ define $I_j := \bigcup_{a \in A} I^a_j \cup \bigcup_{b \in B} I^b_j$ and $I_k := \{\bar{a} \mapsto \bar{b}\}$. Then $(I_j)_{j \leqslant k} : \mathfrak{A} \cong_k \mathfrak{B}$. □

Definition 3.25. The *quantifier rank* $qr(\varphi)$ of an L-formula φ is defined setting $qr(\varphi) = 0$ for atomic φ and recursively

$$qr(\neg\varphi) = qr(\varphi), \quad qr((\varphi \wedge \psi)) = \max\{qr(\varphi), qr(\psi)\}, \quad qr(\forall x \varphi) = 1 + qr(\varphi).$$

$\mathfrak{A}$ and $\mathfrak{B}$ are *k-equivalent*, symbolically $\mathfrak{A} \equiv_k \mathfrak{B}$, if and only if $\mathfrak{A}, \mathfrak{B}$ satisfy the same term-reduced L-sentences of quantifier rank at most k.

Remark 3.26.

1. $\mathfrak{A} \equiv_0 \mathfrak{B}$ means that $\mathfrak{A}, \mathfrak{B}$ interpret all 0-ary relation symbols in the same way. This is equivalent to $\varnothing$ being a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, and to $\mathfrak{A} \cong_0 \mathfrak{B}$.
2. $\mathfrak{A} \equiv \mathfrak{B}$ if and only if $\mathfrak{A} \equiv_k \mathfrak{B}$ for all $k \in \mathbb{N}$ (Exercise 3.6).
3. $\mathfrak{A} \cong_p \mathfrak{B}$ implies $\mathfrak{A} \cong_k \mathfrak{B}$ for all $k \in \mathbb{N}$. The converse fails: see Example 3.36.

Theorem 3.27 (Ehrenfeucht-Fraïssé)**.** *For all $k \in \mathbb{N}$, $\mathfrak{A} \cong_k \mathfrak{B}$ if and only if $\mathfrak{A} \equiv_k \mathfrak{B}$.*

The following two lemmas give stronger versions of both directions.

Lemma 3.28. *Let $k, \ell \in \mathbb{N}$ and $\bar{a} \in A^\ell, \bar{b} \in B^\ell$ and assume $\mathfrak{A}, \bar{a} \cong_k \mathfrak{B}, \bar{b}$. Then for all term-reduced L-formulas $\varphi(\bar{x})$ of quantifier rank at most k:*

$$\mathfrak{A} \vDash \varphi[\bar{a}] \iff \mathfrak{B} \vDash \varphi[\bar{b}].$$

Proof. We proceed by induction on k. For $k = 0$, $\mathfrak{A}, \bar{a} \cong_k \mathfrak{B}, \bar{b}$ implies that there exists a partial isomorphism mapping $\bar{a}$ to $\bar{b}$. The set of formulas satisfying our claim is clearly closed under $\wedge, \neg$ and it contains all term-reduced atoms by Definition 3.1. Hence, all quantifier free term-reduced formulas satisfy our claim.

Let $k > 0$. The formulas satisfying our claim are closed under $\wedge, \neg$. Let $\varphi(\bar{x}) = \exists x \psi(\bar{x}, x)$ have quantifier rank k. Assuming $\mathfrak{A} \vDash \varphi[\bar{a}]$, we show $\mathfrak{B} \vDash \varphi[\bar{b}]$ (the other direction is analogous). Choose $a \in A$ such that $\mathfrak{A} \vDash \psi[\bar{a}, a]$. By Lemma 3.24, there is $b \in B$ such that $\mathfrak{A}, \bar{a}a \cong_{k-1} \mathfrak{B}, \bar{b}b$. Since $qr(\psi) \leqslant k-1$, by induction $\mathfrak{B} \vDash \psi[\bar{b}, b]$, so $\mathfrak{B} \vDash \varphi[\bar{b}]$. □

It is for the following lemma that we need the assumption that L is finite.

Lemma 3.29. *For all $k, \ell \in \mathbb{N}$ with $k + \ell > 0$, and all $\bar{a} \in A^\ell$ there is a term-reduced L-formula $\tau^k_{\mathfrak{A},\bar{a}}(x_0, \ldots, x_{\ell-1})$ of quantifier rank k such that for all L-structures $\mathfrak{B}$ and $\bar{b} \in B^\ell$:*

$$\mathfrak{A}, \bar{a} \cong_k \mathfrak{B}, \bar{b} \iff \mathfrak{B} \vDash \tau^k_{\mathfrak{A},\bar{a}}[\bar{b}].$$

Moreover, for all $k, \ell \in \mathbb{N}$ the set $\{\tau^k_{\mathfrak{A},\bar{a}} \mid \mathfrak{A}$ is an L-structure and $\bar{a} \in A^\ell\}$ is finite.

Proof. We use induction on k. For $k = 0$, we can assume $\ell > 0$. Define $\tau^0_{\mathfrak{A},\bar{a}}$ to be the conjunction of all term-reduced literals in the variables $\bar{x} = (x_0, \ldots, x_{\ell-1})$ that are satisfied by $\bar{a}$ in $\mathfrak{A}$. By Exercise 3.7, $\bar{b}$ satisfies $\tau^0_{\mathfrak{A},\bar{a}}(\bar{x})$ in $\mathfrak{B}$ if and only if $\bar{a} \mapsto \bar{b}$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, i.e., $\mathfrak{A}, \bar{a} \cong_0 \mathfrak{B}, \bar{b}$.

Let $k > 0$. Given $\bar{a} \in A^\ell$ we define

$$\tau^k_{\mathfrak{A},\bar{a}} \quad := \quad \bigwedge_{a \in A} \exists x_\ell \, \tau^{k-1}_{\mathfrak{A},\bar{a}a}(\bar{x}, x_\ell) \wedge \forall x_\ell \bigvee_{a \in A} \tau^{k-1}_{\mathfrak{A},\bar{a}a}(\bar{x}, x_\ell).$$

Note the conjunction and disjunction are finite by the moreover-part for $k - 1$. The moreover-part for k follows. By induction, the formulas $\tau^{k-1}_{\mathfrak{A},\bar{a}a}$ have quantifier rank $k-1$, so $\tau^k_{\mathfrak{A},\bar{a}}$ has quantifier rank k. It follows from Lemma 3.24 that $\tau^k_{\mathfrak{A},\bar{a}}$ satisfies our claim. □

Proof of Theorem 3.27. $\Rightarrow$ follows from Lemma 3.28. $\Leftarrow$: for $k = 0$, see Remark 3.26 (1). For $k > 0$, $\tau^k_{\mathfrak{A}}$ (empty tuple $\bar{a}$ not written) is a sentence of quantifier rank k and is true in $\mathfrak{A}$ (Lemma 3.29 for $\mathfrak{B} := \mathfrak{A}$). Hence $\mathfrak{B} \vDash \tau^k_{\mathfrak{A}}$, so $\mathfrak{A} \cong_k \mathfrak{B}$ by Lemma 3.29. □

Corollary 3.30. *Let $k, \ell \in \mathbb{N}$ with $k + \ell > 0$. Every term-reduced L-formula $\varphi(x_0, \dots, x_{\ell-1})$ of quantifier rank at most k is logically equivalent to*

$$\bigvee\{\tau^k_{\mathfrak{A},\bar{a}}(\bar{x}) \mid \mathfrak{A} \text{ is an } L\text{-structure and } \bar{a} \in \varphi(\mathfrak{A})\}.$$

In particular, for every tuple of variables $\bar{x}$ there are, up to logical equivalence, only finitely many term-reduced L-formulas of quantifier rank at most k with free variables among $\bar{x}$.

Proof. Let $\psi(\bar{x})$ denote the displayed formula. Let $\mathfrak{B} \vDash \varphi[\bar{b}]$. Then $\tau^k_{\mathfrak{B},\bar{b}}$ is a disjunct in ψ satisfied by $\bar{b}$ in $\mathfrak{B}$, so $\mathfrak{B} \vDash \psi[\bar{b}]$. Conversely, if $\mathfrak{B} \vDash \psi[\bar{b}]$, then $\mathfrak{B} \vDash \tau^k_{\mathfrak{A},\bar{a}}$ for some $\mathfrak{A}, \bar{a}$ with $\mathfrak{A} \vDash \varphi[\bar{a}]$. Then $\mathfrak{A}, \bar{a} \cong_k \mathfrak{B}, \bar{b}$ by Lemma 3.29, so $\mathfrak{B} \vDash \varphi[\bar{b}]$ by Lemma 3.28. □

Exercise 3.31. Assume $\mathfrak{A}$ is finite. Then $\mathfrak{B} \vDash \tau^{|A|+1}_{\mathfrak{A}}$ if and only if $\mathfrak{B} \cong \mathfrak{A}$.

Exercise 3.32. Let $\mathcal{C}$ be a class of L-structures. Then $\mathcal{C}$ is elementary if and only if there is $k \in \mathbb{N}$ such that $\mathcal{C}$ is $\cong_k$-*closed*: $\mathfrak{A} \cong_k \mathfrak{B} \in \mathcal{C}$ implies $\mathfrak{A} \in \mathcal{C}$.

3.2.1 Back and forth in discrete orders

Definition 3.33. The *theory of discrete linear orders without endpoints* has axioms ($x \leqslant y$ abbreviates $(x < y \vee x = y)$):

$$\begin{gathered}\forall x\ \neg x < x,\ \ \forall xyz((x < y \wedge y < z) \to x < z),\\ \forall x \exists y(x < y \wedge \forall z(x < z \to y \leqslant z)),\ \ \forall x \exists y(y < x \wedge \forall z(z < x \to z \leqslant y)).\end{gathered}$$

Notation: for a linear order $\mathfrak{A}$, $a, a' \in A$ and $j \in \mathbb{N}$ define

$$\begin{aligned} d^{\mathfrak{A}}(a, a') &:= |\{c \in A \mid a \leqslant^{\mathfrak{A}} c <^{\mathfrak{A}} a' \text{ or } a' \leqslant^{\mathfrak{A}} c <^{\mathfrak{A}} a\}|,\\ d^{\mathfrak{A}}_j(a, a') &:= \max\{d^{\mathfrak{A}}(a, a'), 2^j\}.\end{aligned}$$

Lemma 3.34. *Let $\mathfrak{A}, \mathfrak{B}$ be discrete linear orders without endpoints and $k \in \mathbb{N}$. Then $(I_j)_{j \leqslant k} : \mathfrak{A} \cong_k \mathfrak{B}$ for I_j the set of partial isomorphisms p from $\mathfrak{A}$ to $\mathfrak{B}$ such that $dom(p)$ is finite and $d^{\mathfrak{A}}_j(a, a') = d^{\mathfrak{B}}_j(p(a), p(a'))$ for all $a, a' \in dom(p)$.*

Proof. Note $\varnothing \in I_k \neq \varnothing$. We show (Forth) ((Back) is similar). Let $j < k$ and $p \in I_{j+1}$. Let $a_0 <^{\mathfrak{A}} \dots <^{\mathfrak{A}} a_{\ell-1}$ list $dom(p)$, and write $b_i := p(a_i)$. Let $a \in A \smallsetminus dom(p)$. Assume $a_i <^{\mathfrak{A}} a <^{\mathfrak{A}} a_{i+1}$; the cases that $a <^{\mathfrak{A}} a_0$ or $a_{\ell-1} <^{\mathfrak{A}} a$ are similar.

Case: $d := d^{\mathfrak{A}}_j(a_i, a) < 2^j$. Choose $b \in B$ with $b_i <^{\mathfrak{B}} b <^{\mathfrak{B}} b_{i+1}$ such that $d^{\mathfrak{B}}(b_i, b) = d$: it exists because $d^{\mathfrak{B}}(b_i, b_{i+1}) \geqslant d^{\mathfrak{B}}_{j+1}(b_i, b_{i+1}) = d^{\mathfrak{A}}_{j+1}(a_i, a_{i+1}) \geqslant d^{\mathfrak{A}}_j(a_i, a_{i+1}) = d$.

We claim that $d^{\mathfrak{A}}_j(a, a_{i+1}) = d^{\mathfrak{B}}_j(b, b_{i+1})$. If $d^{\mathfrak{A}}_{j+1}(a_i, a_{i+1}) = d^{\mathfrak{B}}_{j+1}(b_i, b_{i+1}) = 2^{j+1}$, then $d^{\mathfrak{A}}_j(a, a_{i+1}) = d^{\mathfrak{B}}_j(b, b_{i+1}) = 2^j$. If $d^{\mathfrak{A}}_{j+1}(a_i, a_{i+1}) = d^{\mathfrak{B}}_{j+1}(b_i, b_{i+1}) < 2^{j+1}$, then $d^{\mathfrak{A}}(a_i, a_{i+1}) = d^{\mathfrak{B}}(b_i, b_{i+1})$, so even $d^{\mathfrak{A}}(a, a_{i+1}) = d^{\mathfrak{B}}(b, b_{i+1})$.

Case $d_j^{\mathfrak{A}}(a, a_{i+1}) < 2^j$ is analogous.

Case $d_j^{\mathfrak{A}}(a_i, a) = 2^j$ and $d_j^{\mathfrak{A}}(a, a_{i+1}) = 2^j$. It suffices to find $b \in B$ with $b_i <^{\mathfrak{B}} b <^{\mathfrak{B}} b_{i+1}$ such that $d_j^{\mathfrak{B}}(b_i, b) = d_j^{\mathfrak{B}}(b, b_{i+1}) = 2^j$. But $d^{\mathfrak{A}}(a_i, a_{i+1}) \geqslant 2 \cdot 2^j$ so $d_{j+1}^{\mathfrak{A}}(a_i, a_{i+1}) = 2^{j+1} = d_{j+1}^{\mathfrak{B}}(b_i, b_{i+1})$, so $d^{\mathfrak{B}}(b_i, b_{i+1}) \geqslant 2^{j+1}$. Hence, there exists $b_i <^{\mathfrak{B}} b <^{\mathfrak{B}} b_{i+1}$ with $d^{\mathfrak{B}}(b_i, b) = 2^j$. Then $d_j^{\mathfrak{B}}(b_i, b) = 2^j$. Since $d^{\mathfrak{B}}(b, b_{i+1}) \geqslant 2^j$, also $d_j^{\mathfrak{B}}(b, b_{i+1}) = 2^j$. □

Corollary 3.35. *The theory of discrete linear orders without endpoints is complete.*

Proof. Let $\mathfrak{A}, \mathfrak{B}$ be discrete linear orders without endpoints. By the above lemma, $\mathfrak{A} \cong_k \mathfrak{B}$ for all $k \in \mathbb{N}$. By Theorem 3.27, $\mathfrak{A} \equiv_k \mathfrak{B}$ for all $k \in \mathbb{N}$, so $\mathfrak{A} \equiv \mathfrak{B}$ by Remark 3.26. □

Example 3.36. Let $\mathfrak{Z}$ be the linear order on $\mathbb{Z}$. Then $\mathfrak{Z} \equiv \mathfrak{Z} + \mathfrak{Z}$ by the above corollary and $\mathfrak{Z} \cong_k \mathfrak{Z} + \mathfrak{Z}$ for all $k \in \mathbb{N}$ via Lemma 3.34 and $\mathfrak{Z} \not\cong_p \mathfrak{Z} + \mathfrak{Z}$ by Theorem 3.8.

3.3 Fraïssé limits

Let L be an most countable language, and let $\mathcal{K}$ be a class of finitely generated L-structures that is closed under isomorphism. Note that the structures in $\mathcal{K}$ are at most countable.

Definition 3.37. An L-structure $\mathfrak{A}$ is *ultrahomogenous* if every isomorphism between finitely generated substructures of $\mathfrak{A}$ extends to an automorphism of $\mathfrak{A}$.

Remark 3.38. Let $\mathfrak{A}, \mathfrak{B}$ be L-structures, both with skeleton $\mathcal{K}$.

1. If $\mathfrak{A}$ is countable and $I_{Sk}(\mathfrak{A}, \mathfrak{A}) : \mathfrak{A} \cong_p \mathfrak{A}$, then $\mathfrak{A}$ is ultrahomogeneous (Theorem 3.8).
2. $I_{Sk}(\mathfrak{A}, \mathfrak{B})$ has (Forth) if and only if $\mathfrak{B}$ is $\mathcal{K}$-*saturated*: if $f : \mathfrak{K}_0 \to_a \mathfrak{B}$ and $\mathfrak{K}_0 \subseteq \mathfrak{K}_1 \in \mathcal{K}$, then there exists $g : \mathfrak{K}_1 \to_a \mathfrak{B}$ with $f \subseteq g$.

   $\Leftarrow$: let $f \in I_{Sk}(\mathfrak{A}, \mathfrak{B})$ and $a \in A$. Say, f embeds $\mathfrak{K}_0 \subseteq \mathfrak{A}$ into $\mathfrak{B}$. Let $\mathfrak{K}_1 := \langle K_0 \cup \{a\} \rangle^{\mathfrak{A}} \subseteq \mathfrak{A}$. Then $\mathfrak{K}_0, \mathfrak{K}_1 \in \mathcal{K}$. Choose $f \subseteq g : \mathfrak{K}_1 \to_a \mathfrak{B}$. Then $g \in I_{Sk}(\mathfrak{A}, \mathfrak{B})$ and $a \in dom(g)$.

   $\Rightarrow$: let $f : \mathfrak{K}_0 \to_a \mathfrak{B}$ and $\mathfrak{K}_0 \subseteq \mathfrak{K}_1 \in \mathcal{K}$. We can assume $\mathfrak{K}_1 \subseteq \mathfrak{A}$. Then $f \in I_{Sk}(\mathfrak{A}, \mathfrak{B})$. By (Forth) there is $f \subseteq g \in I_{Sk}(\mathfrak{A}, \mathfrak{B})$ defined on a finite set of generators of $\mathfrak{K}_1$.
3. If $\mathfrak{A}, \mathfrak{B}$ are countable and $\mathcal{K}$-saturated, then $\mathfrak{A} \cong \mathfrak{B}$.
4. If $\mathfrak{A}$ is ultrahomogeneous, then $\mathfrak{A}$ is $\mathcal{K}$-saturated.

   Indeed, let $f : \mathfrak{K}_0 \to_a \mathfrak{A}$ and $\mathfrak{K}_0 \subseteq \mathfrak{K}_1 \in \mathcal{K}$. Choose $f_1 : \mathfrak{K}_1 \to_a \mathfrak{A}$. Then $f_1 \circ f^{-1} \in I_{Sk}(\mathfrak{A}, \mathfrak{A})$, so extends to an automorphism g of $\mathfrak{A}$. Then $g^{-1} \circ f_1 : \mathfrak{K}_1 \to_a \mathfrak{A}$ extends f: for $a \in K_0$ we have $g(f(a)) = f_1(a)$ by choice of g, so $f(a) = g^{-1}(f_1(a))$.
5. If $\mathfrak{A}, \mathfrak{B}$ are countable and ultrahomogenous, then $\mathfrak{A} \cong \mathfrak{B}$.

Definition 3.39. $\mathcal{K}$ is a *Fraïssé class* if it has the following properties.

1. $\mathcal{K}$ contains at most countably many L-structures up to isomorphism.
2. *Heredity:* $Sk(\mathfrak{K}) \subseteq \mathcal{K}$ for all $\mathfrak{K} \in \mathcal{K}$.

3. *Joint Embedding:* for all $\mathfrak{K}_0, \mathfrak{K}_1 \in \mathcal{K}$ there exist $\mathfrak{K} \in \mathcal{K}$ with $\mathfrak{K}_0, \mathfrak{K}_1 \in Sk(\mathfrak{K})$.
4. *Amalgamation:* for all $\mathfrak{K}, \mathfrak{K}_0, \mathfrak{K}_1 \in \mathcal{K}$ and all embeddings $f_0 : \mathfrak{K} \to_a \mathfrak{K}_0, f_1 : \mathfrak{K} \to_a \mathfrak{K}_1$ there exist $\mathfrak{K}^* \in \mathcal{K}$ and embeddings $g_0 : \mathfrak{K}_0 \to_a \mathfrak{K}^*, g_1 : \mathfrak{K}_1 \to_a \mathfrak{K}^*$ with $g_0 \circ f_0 = g_1 \circ f_1$.

$$\begin{tikzcd} & \mathfrak{K}_0 \arrow[dr, "g_0"] & \\ \mathfrak{K} \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & & \mathfrak{K}^* \\ & \mathfrak{K}_1 \arrow[ur, "g_1"'] & \end{tikzcd}$$

Exercise 3.40. Let E be a binary relation symbol. The *theory of graphs* is the $\{E\}$-theory $\{\forall x \neg Exx, \forall xy(Exy \leftrightarrow Eyx)\}$. Show that the class of finite graphs is a Fraïssé class.

Exercise 3.41. If $\mathfrak{A}$ is ultrahomogeneous, then $Sk(\mathfrak{A})$ is a Fraïssé class.

Theorem 3.42 (Fraïssé). *If $\mathcal{K}$ is a Fraïssé class, then there exists a countable ultrahomogeneous L-structure with skeleton $\mathcal{K}$. It is unique up to isomorphism and called the* Fraïssé limit *of $\mathcal{K}$.*

Examples 3.43. Finite linear orders, and finite Boolean algebras are Fraïssé classes. Their Fraïssé limits are the rational order $\mathfrak{Q}$ and the atomless Boolean algebra $\mathfrak{I}(\mathfrak{Q}_{\geqslant 0})$.

Proof. $\mathfrak{Q}$ is ultrahomogeneous by Remark 3.38 (1) and Lemma 3.11. By Exercise 3.41, $Sk(\mathfrak{Q})$ is a Fraïssé class. This is easily seen to be the class of finite linear orders.

The proof for Boolean algebras is analogous using Lemma 3.15. □

The proof of Fraïssé's remarkable theorem is based on the following method of model construction which we study in Section 4.4.

Definition 3.44. Let $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots$ be L-structures. The *union* $\bigcup_n \mathfrak{A}_n$ is the L-structure with universe $\bigcup_n A_n$ that interprets every symbol $s \in L$ by $\bigcup_n s^{\mathfrak{A}_n}$.

It is straightforward to check that $\bigcup_n \mathfrak{A}_n$ is well-defined and $\mathfrak{A}_n \subseteq \bigcup_n \mathfrak{A}_n$ for all $n \in \mathbb{N}$ (cf. Remark 4.37).

Proof of Theorem 3.42. Uniqueness holds by Remark 3.38 (5). To prove existence, let $\mathfrak{K}_0, \mathfrak{K}_1, \ldots$ list $\mathcal{K}$ up to isomorphism; we assume $K_i \subseteq \mathbb{N}$. Let $f_0, f_1, \ldots$ list all finite partial bijections from $\mathbb{N}$ to $\mathbb{N}$. Given an L-structure $\mathfrak{B}$ with $B \subseteq \mathbb{N}$ and $i, j \in \mathbb{N}$ we abuse notation and write $f_i : \mathfrak{K}_j \to \mathfrak{B}$ if $dom(f_i)$ generates $\mathfrak{K}_j$ and f_i extends to an embedding $\mathfrak{K}_j \to \mathfrak{B}$; note there is at most one such extension. Fix some surjection $\beta : \mathbb{N} \to \mathbb{N}^4$ such that each quadruple has infinitely many pre-images.

The desired structure is $\mathfrak{A} := \bigcup_n \mathfrak{A}_n$ for a certain chain $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots$ of structures from $\mathcal{K}$ constructed as follows. We choose all $\mathfrak{A}_n$ to have universe $A_n \subseteq \mathbb{N}$. Let $\mathfrak{A}_0 \in \mathcal{K}$ be arbitrary (with $A_0 \subseteq \mathbb{N}$). Assume $\mathfrak{A}_{2n}$ is defined.

For $\mathfrak{A}_{2n+1}$ choose, by Joint Embedding, a structure in $\mathcal{K}$ such that there are embeddings from $\mathfrak{A}_{2n}$ and $\mathfrak{K}_n$ into $\mathfrak{A}_{2n+1}$; we can assume $\mathfrak{A}_{2n} \subseteq \mathfrak{A}_{2n+1}$.

To find $\mathfrak{A}_{2n+2}$ consider the tuple $(\mathfrak{K}_{n_0}, \mathfrak{K}_{n_1}, f_{n_2}, f_{n_3})$ where $\beta(n) = (n_0, n_1, n_2, n_3)$. Check whether $f_{n_2} : \mathfrak{K}_{n_0} \to_a \mathfrak{A}_{2n+1}, f_{n_3} : \mathfrak{K}_{n_0} \to_a \mathfrak{K}_{n_1}$. If this is false, set $\mathfrak{A}_{2n+2} := \mathfrak{A}_{2n+1}$. If it is true, use Amalgamation to choose $\mathfrak{A}_{2n+2} \in \mathcal{K}$ and $g_0 : \mathfrak{A}_{2n+1} \to_a \mathfrak{A}_{2n+2}$ and $g_1 : \mathfrak{K}_{n_2} \to_a \mathfrak{A}_{2n+2}$ with $g_0 \circ f_{n_2} = g_1 \circ f_{n_3}$; we can assume g_0 is the identity, i.e., $\mathfrak{A}_{2n+1} \subseteq \mathfrak{A}_{2n+2}$.

We claim $\mathit{Sk}(\mathfrak{A}) = \mathcal{K}$. $\subseteq$: a finitely generated substructure of $\mathfrak{A}$ is one of $\mathfrak{A}_n$ for suitable $n \in \mathbb{N}$, so in $\mathit{Sk}(\mathfrak{A}_n) \subseteq \mathcal{K}$ (Heredity). $\supseteq$: $\mathfrak{K}_n \to_a \mathfrak{A}_{2n+1} \subseteq \mathfrak{A}$ by construction.

By Remark 3.38 (1), (2) we are left to show that $\mathfrak{A}$ is $\mathcal{K}$-saturated. So let $f : \mathfrak{B} \to_a \mathfrak{A}$ and $\mathfrak{B} \subseteq \mathfrak{C} \in \mathcal{K}$. Choose $m \in \mathbb{N}$ such that $f : \mathfrak{B} \to_a \mathfrak{A}_m$. Choose (n_0, n_1, n_2, n_3) such that $\mathfrak{B}, \mathfrak{C}$ are isomorphic to $\mathfrak{K}_{n_0}, \mathfrak{K}_{n_1}$, say, via π_0, π_1, and $f_{n_2} = f \circ \pi_0^{-1}$ and $f_{n_3} = \pi_1 \circ \pi_0^{-1}$.

$$\begin{array}{ccccccc} & & & & \mathfrak{C} & \overset{\pi_1}{\to} & \mathfrak{K}_{n_1} \\ & & & & \cup & & \uparrow f_{n_3} \\ \mathfrak{A}_{2n+1} & \supseteq & \mathfrak{A}_m & \overset{f}{\leftarrow} & \mathfrak{B} & \overset{\pi_0}{\to} & \mathfrak{K}_{n_0} \end{array}$$

Choose $n \geqslant m$ such that $\beta(n) = (n_0, n_1, n_2, n_3)$. Then $f_{n_2} : \mathfrak{K}_{n_0} \to_a \mathfrak{A}_{2n+1}$ and $f_{n_3} : \mathfrak{K}_{n_0} \to_a \mathfrak{K}_{n_1}$. By construction $\mathfrak{A}_{2n+1} \subseteq \mathfrak{A}_{2n+2}$ and $g_1 : \mathfrak{K}_{n_1} \to_a \mathfrak{A}_{2n+2}$ with $f_{n_2} = g_1 \circ f_{n_3}$. Then $g_1 \circ \pi_1 : \mathfrak{C} \to_a \mathfrak{A}_{2n+2} \subseteq \mathfrak{A}$ extends f. Indeed, for all $b \in B$

$$f(b) = f_{n_2}(\pi_0(b)) = g_1(f_{n_3}(\pi_0(b))) = g_1(\pi_1(b)). \qquad \square$$

3.3.1 The random graph

Definition 3.45. The *random graph* is the Fraïssé limit of the class of finite graphs.

Theorem 3.46. *The theory of the random graph is equivalent to the theory of graphs plus for all $n > 0$ and all $X \subseteq \{0, \ldots, n-1\}$ the* extension axiom $\epsilon_{n,X}$*: the universal closure of*

$$\textstyle\bigwedge_{i<j<n} \neg x_i = x_j \;\to\; \exists z\big(\bigwedge_{i<n} \neg z = x_i \wedge \bigwedge_{i \in X} Ezx_i \wedge \bigwedge_{i \notin X} \neg Ezx_i\big).$$

Every countable model of this theory is isomorphic to the random graph.

Proof. Let $\mathfrak{R}$ be the random graph and $\mathcal{K} := \mathit{Sk}(\mathfrak{R})$ the class of finite graphs. By Remark 3.38 (4), $\mathfrak{R}$ is $\mathcal{K}$-saturated. This implies that $\mathfrak{R}$ satisfies the extension axioms.

Conversely, let $\mathfrak{A}$ be a graph that satisfies the extension axioms. We claim $\mathfrak{R} \equiv \mathfrak{A}$. By Theorem 3.9 it suffices to show $I_{Sk}(\mathfrak{R}, \mathfrak{A}) : \mathfrak{R} \cong_p \mathfrak{A}$. (Forth) follows from $\mathfrak{A}$ satisfying the extension axioms, and (Back) from $\mathfrak{R}$ being $\mathcal{K}$-saturated and Remark 3.38 (2).

The 2nd statement follows from $\mathfrak{R} \cong_p \mathfrak{A}$ and Theorem 3.8. $\square$

Theorem 3.47 (0-1 law)**.** *For an $\{E\}$-sentence φ and $n > 0$ let $\Pr_n[\varphi]$ be the probability that a graph chosen uniformly at random among those with universe $\{0, \ldots, n-1\}$ satisfies φ. Then*

$$\lim_n \Pr_n[\varphi] \in \{0, 1\}.$$

Moreover, the random graph satisfies exactly those φ with $\lim_n \Pr_n[\varphi] = 1$.

Proof. We claim $\lim_n \Pr_n[\epsilon_{m,X}] = 1$. Let $n > m$ and $a_0, \ldots, a_{m-1} < n$ be pairwise distinct and let G be a random graph on $\{0, \ldots, n-1\}$. For $a < n$ distinct from the a_i let E_a be the event that a has an edge (in G) to $a_i, i \in X$, and not to $a_j, j \notin X$. Then $\Pr[E_a] = 2^{-m}$. The events E_a are independent. Thus no E_a occurs with probability $(1 - 2^{-m})^{n-m}$. That this happens for some $a_0, \ldots, a_{n-1}$ has probability $\leqslant \binom{n}{m}(1 - 2^{-m})^{n-m}$. This bounds $\Pr_n[\neg\epsilon_{m,X}]$ and tends to 0 (m fixed and $n \to \infty$).

If the random graph satisfies φ, then its theory implies it. By compactness, there are finitely many extension axioms E such that any graph satisfying them, also satisfies φ. As $\lim_n \Pr_n[\bigwedge E] = 1$, also $\lim_n \Pr[\varphi] = 1$. If the random graph satisfies $\neg\varphi$, then $\lim_n \Pr_n[\varphi] = 1 - \lim_n \Pr_n[\neg\varphi] = 0$. Both statements follow. $\square$

Chapter 4

Diagrams

Let L be a language and $\mathfrak{A}, \mathfrak{B}$ be L-structures. For $A_0 \subseteq A$ we let $L(A_0)$ be the language $L \cup \{c_a \mid a \in A_0\}$ for pairwise distinct constants $c_a \notin L$. We let $\mathfrak{A}_{A_0}$ denote the $L(A_0)$-expansion of $\mathfrak{A}$ interpreting c_a by a.

4.1 Algebraic diagrams

Recall from the preliminaries that $\mathfrak{A}$ is embeddable into $\mathfrak{B}$, symbolically $\mathfrak{A} \to_a \mathfrak{B}$, if there is an embedding $\pi : \mathfrak{A} \to_a \mathfrak{B}$, i.e., an isomorphism from $\mathfrak{A}$ onto some substructure of $\mathfrak{B}$.

Definition 4.1. A formula φ is *universal (existential)* if it is logically equivalent to $\forall \bar{x} \psi$ ($\exists \bar{x} \psi$) for some quantifier free ψ.

Definition 4.2. The *algebraic diagram* $D_a(\mathfrak{A})$ *of* $\mathfrak{A}$ is the set of $L(A)$-literals true in $\mathfrak{A}_A$.

Roughly, the following states that the models of $D_a(\mathfrak{A})$ are the extensions of $\mathfrak{A}$.

Lemma 4.3. *For an $L(A)$-structure $\mathfrak{C}$, the following are equivalent.*

1. $\mathfrak{C} \vDash D_a(\mathfrak{A})$.
2. $a \mapsto c_a^{\mathfrak{C}}$ *is an embedding from* $\mathfrak{A}$ *into* $\mathfrak{C} \restriction L$.
3. *There exist* $\mathfrak{B} \supseteq \mathfrak{A}$ *and an isomorphism* $\pi : \mathfrak{B} \cong \mathfrak{C} \restriction L$ *that extends* $a \mapsto c_a^{\mathfrak{C}}$.

Proof. $1 \Rightarrow 2$: The map is injective: if $a \neq a'$, then $\neg c_a = c_{a'} \in D_a(\mathfrak{A})$, so $c_a^{\mathfrak{C}} \neq c_{a'}^{\mathfrak{C}}$. If e.g. $f \in L$ is a binary function symbol and $a := f^{\mathfrak{A}}(a_0, a_1)$, we have to show $f^{\mathfrak{C}}(c_{a_0}^{\mathfrak{C}}, c_{a_1}^{\mathfrak{C}}) = c_a^{\mathfrak{C}}$; this follows from $f c_{a_0} c_{a_1} = c_a \in D_a(\mathfrak{A})$.

$2 \Rightarrow 3$: Choose a set B' disjoint from $A \cup C$ and a bijection π from $B := B' \cup A$ onto C that extends $a \mapsto c_a^{\mathfrak{C}}$. Define an $L(A)$-structure $\mathfrak{B}$ with universe B such that $\pi : \mathfrak{B} \cong \mathfrak{C} \restriction L$: e.g. define $f^{\mathfrak{B}}(b, b') := \pi^{-1}(f^{\mathfrak{C}}(\pi(b), \pi(b')))$. Since $\pi(a) = c_a^{\mathfrak{C}}$ we have $\pi : \mathfrak{B}_A \cong \mathfrak{C}$, so $\mathfrak{B}_A \vDash D_a(\mathfrak{A})$. By $1 \Rightarrow 2$, the identity $a \mapsto c_a^{\mathfrak{B}_A} = a$ is an embedding from $\mathfrak{A}$ into $\mathfrak{B}$, i.e., $\mathfrak{A} \subseteq \mathfrak{B}$.

$3 \Rightarrow 1$ follows from $\mathfrak{B}_A \vDash D_a(\mathfrak{A})$ and $\pi : \mathfrak{B}_A \cong \mathfrak{C}$. □

The following is a general tool to construct extensions with certain desired properties.

Lemma 4.4. *Let $L' \supseteq L$ be a language and T' an L'-theory. The following are equivalent.*

1. *$\mathfrak{A}$ satisfies every universal L-sentence φ such that $T' \vdash \varphi$.*
2. *$T' \cup D_a(\mathfrak{A})$ is consistent.*
3. *There is $\mathfrak{B}' \vDash T'$ such that $\mathfrak{A} \subseteq \mathfrak{B}' \upharpoonright L$.*
4. *Every finitely generated substructure of $\mathfrak{A}$ is embeddable into the L-reduct of some model of T'.*

Proof. $1 \Rightarrow 2$: Assume $T' \cup D_a(\mathfrak{A})$ is inconsistent. By compactness there are $\varphi_0, \ldots, \varphi_{\ell-1} \in D_a(\mathfrak{A})$ for some $\ell \in \mathbb{N}$ such that $T' \vdash \neg \bigwedge_{i<\ell} \varphi_i$. Write this $L(A)$-sentence as $\psi(\bar{c})$ where $\psi(\bar{x})$ is a quantifier free L-formula and $\bar{c}$ are constants outside L. We can assume they are outside L'. Then $T' \vdash \forall x \psi(\bar{x})$. Since $\mathfrak{A} \nvDash \forall \bar{x} \psi(\bar{x})$, (1) fails.

$2 \Rightarrow 3$: given $\mathfrak{C}' \vDash T' \cup D_a(\mathfrak{A})$, set $\mathfrak{C} := \mathfrak{C}' \upharpoonright L(A)$ and choose $\mathfrak{B} \supseteq \mathfrak{A}$ and $\pi : \mathfrak{B} \cong \mathfrak{C} \upharpoonright L$ by Lemma 4.3. Let $\mathfrak{B}'$ be the L'-expansion of $\mathfrak{B}$ with $\pi : \mathfrak{B}' \cong \mathfrak{C}' \upharpoonright L'$.

$3 \Rightarrow 4$ is trivial. $4 \Rightarrow 1$: Let $T' \vdash \forall \bar{x} \psi(\bar{x})$ where ψ is a quantifier free L-formula. Assume $\mathfrak{A} \vDash \neg\psi[\bar{a}]$ for some $\bar{a}$. Then $\mathfrak{A}_0 := \langle \bar{a} \rangle^{\mathfrak{A}} \vDash \neg\psi[\bar{a}]$ since $\neg\psi$ is quantifier free. By (4), $\mathfrak{A}_0$ is embeddable into some $\mathfrak{C} \vDash T'$, say $\pi : \mathfrak{A}_0 \cong \mathfrak{C}_0 \subseteq \mathfrak{C} \upharpoonright L$. Then $\mathfrak{C}_0 \vDash \neg\psi[\pi(\bar{a})]$, so $\mathfrak{C} \vDash \neg\psi[\pi(\bar{a})]$, so $\mathfrak{C} \nvDash \forall \bar{x} \psi(\bar{x})$. Since $\mathfrak{C} \vDash T'$ this contradicts $T' \vdash \forall \bar{x} \psi(\bar{x})$. □

Exercise 4.5. In the above lemma, assume $L' \setminus L$ contains only relation symbols and T' is universal. Then the statements are equivalent to:

5. $\mathfrak{A}$ has an L'-expansion that models T'.

Exercise 4.6. Every nontrivial Boolean algebra is embeddable into an atomless one.

4.1.1 Łos-Tarski

Lemma 4.7. *Let T_0, T be L-theories. Assume for all $\mathfrak{A}, \mathfrak{B} \vDash T_0$:*

$$\mathfrak{A} \subseteq \mathfrak{B} \vDash T \implies \mathfrak{A} \vDash T.$$

Then there exists a universal L-theory U such that $T_0 \cup T$ is equivalent to $T_0 \cup U$.

Proof. Let U be the set of universal L-sentences φ such that $T_0 \cup T \vdash \varphi$. It suffices to show that every model $\mathfrak{A}$ of $T_0 \cup U$ is a model of T. By $1 \Rightarrow 3$ of Lemma 4.4, $\mathfrak{A}$ embeds into a model $\mathfrak{B}$ of $T_0 \cup T$, say $\mathfrak{A} \cong \mathfrak{B}_0 \subseteq \mathfrak{B}$. Then $\mathfrak{B}_0 \vDash T$ by assumption, so $\mathfrak{A} \vDash T$. □

Theorem 4.8 (Łos-Tarski for theories)**.** *Let T be an L-theory. Then T is equivalent to a universal theory if and only if for all L-structures $\mathfrak{A}, \mathfrak{B}$:*

$$\mathfrak{A} \subseteq \mathfrak{B} \vDash T \implies \mathfrak{A} \vDash T$$

Proof. The forward direction is clear. For the converse apply Lemma 4.7 with $T_0 := \varnothing$. □

Theorem 4.9 (Łos-Tarski for formulas)**.** *Let T_0 be an L-theory and $\varphi(\bar{x})$ be an L-formula. The following are equivalent.*

1. *For all $\mathfrak{A}, \mathfrak{B} \vDash T_0$ and all tuples $\bar{a}$ from A of suitable length:*

$$\mathfrak{A} \subseteq \mathfrak{B} \vDash \varphi[\bar{a}] \implies \mathfrak{A} \vDash \varphi[\bar{a}].$$

2. *There exists a universal L-formula $\psi(\bar{x})$ such that $T_0 \vdash (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.*

Proof. $2 \Rightarrow 1$: if $\mathfrak{B} \vDash \varphi[\bar{a}]$, then $\mathfrak{B} \vDash \psi[\bar{a}]$ since $\mathfrak{B} \vDash T_0$, then $\mathfrak{A} \vDash \psi[\bar{a}]$ since ψ is universal, then $\mathfrak{A} \vDash \varphi[\bar{a}]$ since $\mathfrak{A} \vDash T_0$.

$1 \Rightarrow 2$: choose new constants $\bar{c}$. It suffices to show $T_0 \vdash (\varphi(\bar{c}) \leftrightarrow \psi)$ for some universal $L \cup \{\bar{c}\}$-sentence ψ. Our assumption implies the assumption of Lemma 4.7 for $T := \{\varphi(\bar{c})\}$ (and $L \cup \{\bar{c}\}$ in place of L). Choose a universal $L \cup \{\bar{c}\}$-theory U accordingly. Then $T_0 \cup U \cup \{\neg\varphi(\bar{c})\}$ is inconsistent, so by compactness $T_0 \cup U_0 \cup \{\neg\varphi(\bar{c})\}$ is inconsistent for some finite $U_0 \subseteq U$. Then $T_0 \vdash (\psi \rightarrow \varphi(\bar{c}))$ for $\psi := \bigwedge U_0$. And $\leftarrow$ is clear because $T_0 \vdash (\varphi(\bar{c}) \rightarrow \chi)$ for all $\chi \in U$. □

Exercise 4.10. Formulate and prove a variant of the above for existential $\psi(\bar{x})$.

4.1.2 Orderable and divisible abelian groups

We need the following result from algebra. Let $\mathfrak{Z}$ denote the additive group of integers, and $\mathfrak{Z}_n$ the additive group of integers modulo n.

Theorem 4.11. *Every finitely generated abelian group is isomorphic to*

$$\mathfrak{Z}^k \times \mathfrak{Z}_{p_0^{k_0}} \times \cdots \times \mathfrak{Z}_{p_{r-1}^{k_{r-1}}}$$

for some $r, k, k_0, \ldots, k_{r-1} \in \mathbb{N}$ and primes $p_0, \ldots, p_{r-1}$.

The *Prüfer p-group* $\mathfrak{Z}_{p^\infty}$ for a prime p is the subgroup of the multiplicative group on $\mathbb{C} \setminus \{0\}$ with universe $\mathbb{Z}_{p^\infty} := \{c \in \mathbb{C} \mid c^{p^k} = 1 \text{ for some } k \in \mathbb{N}\}$.

Lemma 4.12. *$\mathfrak{Z}_{p^\infty}$ is a divisible abelian group and $\mathfrak{Z}_{p^k} \rightarrow_a \mathfrak{Z}_{p^\infty}$ for all $k \in \mathbb{N}$.*

Proof. For $k \in \mathbb{N}$ let $\mathfrak{G}_k$ be the cyclic subgroup of $\mathfrak{Z}_{p^\infty}$ of p^kth roots of unity. Then $\mathfrak{Z}_{p^k} \cong \mathfrak{G}_k$, so $\mathfrak{Z}_{p^k} \rightarrow_a \mathfrak{Z}_{p^\infty}$. Further, $\mathbb{Z}_{p^\infty} = \bigcup_k G_k$. Let $a \in \mathbb{Z}_{p^\infty}$ and choose k such that $a \in G_k$. It suffices to find for every prime q some $b \in \mathbb{Z}_{p^\infty}$ such that $b^q = a$.

Case $q = p$. Let b generate G_{k+1} and let $c := b^p$. Then c has order p^k and generates G_k. Hence $a = c^\ell$ for some ℓ. Then $a = (b^\ell)^p$.

Case $q \neq p$. Choose $z_0, z_1 \in \mathbb{Z}$ such that $z_0 p^k + z_1 q = 1$. Then $a = a^{z_0 p^k} a^{z_1 q} = (a^{z_1})^q$. □

Theorem 4.13. *Every abelian group is embeddable into a divisible abelian group.*

Proof. Let $\mathfrak{A}$ be an abelian group. By Lemma 4.4 it suffices to embed every finitely generated substructure of $\mathfrak{A}$ in a divisible abelian group. These have the form in Theorem 4.11. But every factor $\mathfrak{Z}$ embeds into the rationals, and every factor $\mathfrak{Z}_{p_i^{k_i}}$ into $\mathfrak{Z}_{p^\infty}$. □

This illustrates a natural use of Lemma 4.4. Here is a direct construction:

Exercise 4.14 (Divisible hull)**.** Let $\mathfrak{A}$ be an abelian group. Consider pairs (a, n) where $a \in A, n > 0$. Declare (a, n) equivalent to (a', n') if $n'a = na'$, and let a/n be the equivalence class of (a, n). Define $a/n + a'/n' := (n'a + na)/nn'$. This defines the *divisible hull of* $\mathfrak{A}$. Verify it is a divisible abelian group and $a \mapsto a/1$ embeds $\mathfrak{A}$ into it.

Definition 4.15. An abelian group is *orderable* if it has an $L_{Gr} \cup \{<\}$ expansion that satisfies theory of ordered abelian groups.

Theorem 4.16. *An abelian group is orderable if and only if it is torsion-free.*

Proof. Forward is easy: say $a < 0$ (omitting superscripts), then $a < 0, 2a < a, 3a < 2a, \ldots$, so $na \neq 0$ for all $n \in \mathbb{N} \smallsetminus \{0\}$. Conversely, let $\mathfrak{A}$ be a torsion-free abelian group. Note we ask for an expansion interpreting a relation symbol $<$ that satisfies a universal theory. By Exercise 4.5 it suffices to find a suitable expansion for every finitely generated substructure of $\mathfrak{A}$. These have the form in Theorem 4.11 with $k = 0$ (being torsion-free). Thus it suffices to show $\mathfrak{Z}^k$ is orderable. This is easy, in fact, if $\mathfrak{A}, \mathfrak{A}'$ are orderable abelian groups, say with orders $<, <'$, then so is $\mathfrak{A} \times \mathfrak{A}'$ - take the lexicographic order on $A \times A'$: $(a_0, a_0') <_{lex} (a_1, a_1')$ if $a_0 < a_1$, or, $a_0 = a_1$ and $a_0' <' a_1'$. $\square$

4.2 Model completeness

Definition 4.17. $\mathfrak{A}$ is an *elementary substructure of* $\mathfrak{B}$ and $\mathfrak{B}$ an *elementary extension of* $\mathfrak{A}$, symbolically $\mathfrak{A} \preccurlyeq \mathfrak{B}$, if $\mathfrak{A} \subseteq \mathfrak{B}$ and for every L-formula $\varphi(\bar{x})$ and every $\bar{a}$ from A:

$$\mathfrak{A} \models \varphi[\bar{a}] \iff \mathfrak{B} \models \varphi[\bar{a}].$$

Remark 4.18.

1. Equivalently, one can replace $\iff$ by $\Rightarrow$ or $\Leftarrow$ above: the missing direction follows using $\neg\varphi$ instead φ.
2. $\mathfrak{A} \preccurlyeq \mathfrak{B}$ if and only if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A}_A \equiv \mathfrak{B}_A$.

Exercise 4.19. Let $\mathfrak{N}_{>0} \subseteq \mathfrak{N} \subseteq \mathfrak{Q} \subseteq \mathfrak{R}$ be the usual linear orders with universes $\{n \in \mathbb{N} \mid n > 0\}, \mathbb{N}, \mathbb{Q}, \mathbb{R}$. Which $\subseteq$ are $\preccurlyeq$? *Hint:* recall Lemma 3.11.

Lemma 4.20 (Tarski's test)**.** *For $A_0 \subseteq B$ the following are equivalent.*

1. *A_0 is the universe of an elementary substructure of $\mathfrak{B}$.*
2. *For every L-formula $\varphi(\bar{x}, y)$ and every tuple $\bar{a}$ from A_0 of suitable length: if $\mathfrak{B} \models \varphi[\bar{a}, b]$ for some $b \in B$, then $\mathfrak{B} \models \varphi[\bar{a}, a]$ for some $a \in A_0$.*

Proof. $1 \Rightarrow 2$: Let $\mathfrak{A}_0 \preccurlyeq \mathfrak{B}$ have universe A_0. If $\mathfrak{B} \models \varphi[\bar{a}, b]$ for some $b \in B$, then $\mathfrak{B} \models \exists y \varphi[\bar{a}]$, then $\mathfrak{A}_0 \models \exists y \varphi[\bar{a}]$, then $\mathfrak{A}_0 \models \varphi[\bar{a}, a]$ for some $a \in A_0$.

$2 \Rightarrow 1$: (2) for the formulas $y = y, f\bar{x} = y$ show A_0 is nonempty and L-closed, so universe of a substructure $\mathfrak{A}_0 \subseteq \mathfrak{B}$. Call a $\varphi(\bar{x})$ *good* if $\mathfrak{B} \models \varphi[\bar{a}] \iff \mathfrak{A}_0 \models \varphi[\bar{a}]$ for all $\bar{a}$ from A_0. We claim all formulas a good. Atomic formulas are good since $\mathfrak{A}_0 \subseteq \mathfrak{B}$, and good formulas are closed under $\neg, \wedge$. We are left to show closure under $\exists y$. So assume $\varphi(\bar{x}, y)$ is good and let $\bar{a}$ be a tuple from A_0. By (2), $\mathfrak{B} \models \exists y \varphi[\bar{a}]$ if and only if $\mathfrak{B} \models \varphi[\bar{a}, a]$ for some $a \in A_0$. Since φ is good, this is equivalent to $\mathfrak{A}_0 \models \varphi[\bar{a}, a]$, so $\mathfrak{A}_0 \models \exists y \varphi[\bar{a}]$. □

Definition 4.21. An L-theory T is *model complete* if for all $\mathfrak{A}, \mathfrak{B} \models T$:

$$\mathfrak{A} \subseteq \mathfrak{B} \iff \mathfrak{A} \preccurlyeq \mathfrak{B}.$$

An L-formula $\varphi(\bar{x})$ is *T-provably equivalent* to an L-formula $\psi(\bar{x})$ if $T \vdash (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.

Exercise 4.22. The theory of discrete linear orders without endpoints is not model complete. The theory of dense linear orders without endpoints is model complete. Find abelian groups $\mathfrak{A} \subseteq \mathfrak{B}$ such that $\mathfrak{A}$ is not existentially closed in $\mathfrak{B}$. Same for fields.

Exercise 4.23. T is model complete if and only if $T \cup D_a(\mathfrak{A})$ is complete for every $\mathfrak{A} \models T$.

Theorem 4.24. *Let T be an L-theory. The following are equivalent.*

1. *T is model complete.*
2. (Robinson's test) *For all $\mathfrak{A}, \mathfrak{B} \models T$, if $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A}$ is* existentially closed in $\mathfrak{B}$*: for every existential L-formula $\varphi(\bar{x})$ and all tuples $\bar{a}$ from A of suitable length:*

$$\mathfrak{B} \models \varphi[\bar{a}] \implies \mathfrak{A} \models \varphi[\bar{a}].$$

3. *Every existential L-formula is T-provably equivalent to a universal one.*
4. *Every L-formula is T-provably equivalent to a universal one.*

Proof. $1 \Rightarrow 2$ is clear, and $2 \Leftrightarrow 3$ is Theorem 4.9.

$3 \Rightarrow 4$: an L-formula $\varphi(\bar{x})$ is logically equivalent to one of the form

$$\forall \bar{x}_0 \exists \bar{y}_0 \cdots \forall \bar{x}_{k-1} \exists \bar{y}_{k-1} \ \psi(\bar{x}, \bar{x}_0, \bar{y}_0, \ldots, \bar{x}_{k-1}, \bar{y}_{k-1})$$

for ψ quantifier free. Using (3) replace $\exists \bar{y}_{k-1} \psi$ by a universal formula χ. Then replace $\forall x_{k-1} \chi$ by an existential formula χ' (apply (3) to the existential formula $\neg \forall x_{k-1} \chi$). This gives an expression of the above form with k decreased by 1. Proceed.

$4 \Rightarrow 1$: Let $\mathfrak{A} \subseteq \mathfrak{B}$ be models of T. By Remark 4.18 (1), it suffices to show

$$\mathfrak{B} \models \varphi[\bar{a}] \Longrightarrow \mathfrak{A} \models \varphi[\bar{a}]$$

for all L-formulas $\varphi(\bar{x})$ and tuples $\bar{a}$ from A. This clearly follows from (4). □

Remark 4.25. Assume $\mathfrak{A} \subseteq \mathfrak{B}$. Then $\mathfrak{A}$ is existentially closed in $\mathfrak{B}$ if and only if every primitive $L(A)$-sentence true in $\mathfrak{B}$ is true in $\mathfrak{A}$; being *primitive* means to be of the form $\exists \bar{x} \psi$ where ψ is a conjunction of literals. Indeed, every existential formula is logically equivalent to a disjunction of primitive formulas.

4.2.1 Existentially closed subfields

Recall, $\mathfrak{A}[\bar{x}]$ denotes the polynomial ring over a field $\mathfrak{A}$ with variables $\bar{x}$.

Proposition 4.26. *Let $\mathfrak{A} \subseteq \mathfrak{B}$ be fields. Then $\mathfrak{A}$ is existentially closed in $\mathfrak{B}$ if and only if for all $k \in \mathbb{N}$ and all variable tuples $\bar{x}$ and all $P_0(\bar{x}), \ldots, P_{k-1}(\bar{x}), Q(\bar{x}) \in \mathfrak{A}[\bar{x}]$:*

$$P_0(\bar{x}) = 0, \ \ldots, \ P_{k-1}(\bar{x}) = 0, \ Q(\bar{x}) \neq 0$$

has a solution in $\mathfrak{B}$, then it has a solution in A.

Proof. By Remark 4.25 it suffices to show that the truth of a primitive $L_{Ring}(A)$-sentence is equivalent to the solvability of a suitable system, and vice-versa.

Every polynomial $P(\bar{x}) \in \mathfrak{A}[\bar{x}]$ is *equivalent* to some $L_{Ring}(A)$-term $t_P(\bar{x})$ in the sense that the functions $\bar{a} \mapsto P(\bar{a})$ and $\bar{a} \mapsto t_P^{\mathfrak{A}_A}[\bar{a}]$ are equal. Conversely, every such term $t(\bar{x})$ is equivalent to a polynomial $P_t(\bar{x}) \in \mathfrak{A}[\bar{x}]$.

Hence, a system as displayed has a solution in $\mathfrak{B}$ (resp. $\mathfrak{A}$) if and only if $\mathfrak{B}_A$ (resp. $\mathfrak{A}_A$) satisfies the $L_{Ring}(A)$-sentence

$$\exists \bar{x}(t_{P_0}(\bar{x}) = 0 \wedge \ldots \wedge t_{P_{k-1}}(\bar{x}) = 0 \wedge \neg t_Q(\bar{x}) = 0).$$

Conversely, note that the theory fields proves $(t = s \leftrightarrow t + (-s) = 0)$, $(\neg t = 0 \wedge \neg s = 0 \leftrightarrow \neg t \cdot s = 0)$, for all L_{Ring}-terms t, s, and hence also for all $L_{Ring}(A)$-terms t, s. Hence, the theory proves every primitive $L_{Ring}(A)$-sentence φ equivalent to one of the form

$$\exists \bar{x}(t_0(\bar{x}) = 0 \wedge \ldots \wedge t_{k-1}(\bar{x}) = 0 \wedge \neg t_k(\bar{x}) = 0).$$

This sentence is true in $\mathfrak{B}_A$ (resp. $\mathfrak{A}_A$), if and only if $\mathfrak{B}$ (resp. $\mathfrak{A}$) contains a solution of

$$P_{t'_0}(\bar{x}) = 0, \ldots, \ P_{t'_{k-1}}(\bar{x}) = 0, \ P_{t'_k}(\bar{x}) \neq 0. \qquad \square$$

4.3 Elementary diagrams

Definition 4.27. $\pi : A \to B$ is an *elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$*, symbolically $\pi : \mathfrak{A} \to_e \mathfrak{B}$, if $\pi : \mathfrak{A} \cong \mathfrak{B}' \preccurlyeq \mathfrak{B}$ for some $\mathfrak{B}'$. If there is such π, then $\mathfrak{A}$ is *elementarily embedabble into* $\mathfrak{B}$, symbolically $\mathfrak{A} \to_e \mathfrak{B}$.

The *elementary diagram of* $\mathfrak{A}$ is $D_e(\mathfrak{A}) := Th(\mathfrak{A}_A)$.

The following are analogous to Lemmas 4.3 and 4.4.

Lemma 4.28. *For an $L(A)$-structure $\mathfrak{C}$, the following are equivalent.*

1. $\mathfrak{C} \models D_e(\mathfrak{A})$.
2. *$a \mapsto c_a^{\mathfrak{C}}$ is an elementary embedding from $\mathfrak{A}$ into $\mathfrak{C} \restriction L$.*
3. *There exist $\mathfrak{B} \succcurlyeq \mathfrak{A}$ and an isomorphism $\pi : \mathfrak{B} \cong \mathfrak{C} \restriction L$ that extends $a \mapsto c_a^{\mathfrak{C}}$.*

Proof. $1 \Rightarrow 2$: by Lemma 4.3, $a \mapsto c_a^{\mathfrak{C}}$ witnesses $\mathfrak{A} \cong \mathfrak{C}_0 \subseteq \mathfrak{C}{\restriction}L$ for some $\mathfrak{C}_0$, hence also $\mathfrak{A}_A \cong \mathfrak{C}'_0 := \langle C_0 \rangle^{\mathfrak{C}}$. But $\mathfrak{C}'_0 \equiv \mathfrak{C}$ as models of the complete theory $D_e(\mathfrak{A})$. Up to renaming constants, $\mathfrak{C}'_0$ is $(\mathfrak{C}_0)_{C_0}$ and $\mathfrak{C}$ is $(\mathfrak{C}{\restriction}L)_{C_0}$. Hence $\mathfrak{C}_0 \preccurlyeq \mathfrak{C}{\restriction}L$ by Remark 4.18 (2).

$2 \Rightarrow 3$ and $3 \Rightarrow 1$ are analogous to Lemma 4.3. □

Remark 4.29. $\pi : \mathfrak{A} \to_e \mathfrak{B}$ if and only if for all L-formulas $\varphi(\bar{x})$ and all tuples $\bar{a}$ from A of suitable length: $\mathfrak{A} \vDash \varphi[\bar{a}] \iff \mathfrak{B} \vDash \varphi[\pi(\bar{a})]$.

Proof. Let $\mathfrak{C}$ be the $L(A)$-expansion of $\mathfrak{B}$ that interprets c_a by $\pi(a)$. Then the r.h.s. is equivalent to (1) in the previous lemma, and the l.h.s. to (2). □

Lemma 4.30. *Let $L' \supseteq L$ be a language and T' an L'-theory. The following are equivalent.*

1. *$\mathfrak{A}$ satisfies every L-sentence φ such that $T' \vdash \varphi$.*
2. *$T' \cup Th(\mathfrak{A})$ is consistent.*
3. *$T' \cup D_e(\mathfrak{A})$ is consistent.*
4. *There is $\mathfrak{B}' \vDash T'$ with $\mathfrak{A} \preccurlyeq \mathfrak{B}'{\restriction}L$.*

Proof. $1 \Rightarrow 2$: by compactness, if $T' \cup Th(\mathfrak{A})$ is inconsistent, then T' proves $\neg\varphi$ for φ a finite conjunction of sentences in $Th(\mathfrak{A})$. Then $\varphi \in Th(\mathfrak{A})$, so $\mathfrak{A} \nvDash \neg\varphi$.

$2 \Rightarrow 3$: by compactness, if $T' \cup D_e(\mathfrak{A})$ is inconsistent, then T' proves $\neg\varphi$ for $\varphi \in D_e(\mathfrak{A})$. Write $\varphi = \psi(\bar{c})$ for constants $\bar{c}$ outside L and $\psi(\bar{x})$ and L-formula. We assume $L' \cap L(A) = \varnothing$. Then $T' \vdash \forall\bar{x}\neg\psi(\bar{x})$ but $\exists\bar{x}\psi(\bar{x}) \in Th(\mathfrak{A})$. Hence $T' \cup Th(\mathfrak{A})$ is inconsistent.

$3 \Rightarrow 4$: let $\mathfrak{C} \vDash T' \cup D_e(\mathfrak{A})$. By Lemma 4.28, there are $\mathfrak{B} \succcurlyeq \mathfrak{A}$ and $\pi : \mathfrak{B} \cong \mathfrak{C}{\restriction}L$. Define an L'-expansion $\mathfrak{B}'$ of $\mathfrak{B}$ such that $\pi : \mathfrak{B}' \cong \mathfrak{C}{\restriction}L'$.

$4 \Rightarrow 1$ is trivial. □

Exercise 4.31. $\mathfrak{A} \equiv \mathfrak{B}$ if and only if $\mathfrak{A} \to_e \mathfrak{C} \succcurlyeq \mathfrak{B}$ for some $\mathfrak{C}$.

Exercise 4.32 (Ultrapower embedding)**.** Let F be an ultrafilter on $I \neq \varnothing$ and consider the ultrapower $\mathfrak{A}^I_F$. The *diagonal map* d maps $a \in A$ to $d(a) := f_a^F$ where $f_a : I \to A$ is the function constantly equal to a. Then d is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{A}^I_F$.

Exercise 4.33 (Definable ultrapower)**.** A *Skolem-function* for $\varphi(\bar{x}, y)$ (in $\mathfrak{A}$), say $\bar{x} = (x_0, \ldots x_{n-1})$, is a function $f : A^n \to A$ such that for all $\bar{a} \in A^n$, if $\mathfrak{A} \vDash \exists y\varphi[\bar{a}]$, then $\mathfrak{A} \vDash \varphi[\bar{a}, f(\bar{a})]$. Assume $\mathfrak{A}$ has *definable Skolem-functions*: every $\varphi(\bar{x}, y)$ has a Skolem-function which is definable in $\mathfrak{A}_A$.

Consider an ultrapower $\mathfrak{A}^I_F$ for $I := A$ and a free ultrafilter F on I. Show that the set of functions from A to A that are definable in $\mathfrak{A}_A$ is the universe of an elementary substructure of $\mathfrak{A}^I_F$. It is called a *definable ultrapower of* $\mathfrak{A}$ *modulo* F.

Lemma 4.34. *Assume $\mathfrak{B}$ is infinite and $A_0 \subseteq B$. Then there exists $\mathfrak{A} \preccurlyeq \mathfrak{B}$ with $A_0 \subseteq A$ and $|A| = \max\{|A_0|, |L|, \aleph_0\}$.*

Proof. We define sets $A_0 \subseteq A_1 \subseteq \cdots$ of cardinality $\leqslant \kappa := \max\{|A_0|, |L|, \aleph_0\}$. Assume A_k is defined. For every $n \in \mathbb{N}$ and $\bar{a} \in A_k^n$ and every L-formula $\varphi(\bar{x}, y)$ with $\mathfrak{B} \vDash \exists y \varphi[\bar{a}]$ choose $b \in B$ such that $\mathfrak{B} \vDash \varphi[\bar{a}, b]$. Define A_{k+1} by adding all such chosen b to A_k.

There are $\aleph_0 \leqslant \kappa$ many $n \in \mathbb{N}$, $\max\{|L|, \aleph_0\} \leqslant \kappa$ many φ, and $|A_k|^n \leqslant \kappa$ many $\bar{a}$, hence $\leqslant \kappa^3 = \kappa$ many b are added; thus $|A_k| \leqslant |A_k| + \kappa = \kappa$. Then $A := \bigcup_k A_k$ has cardinality $\leqslant \aleph_0 \cdot \kappa = \kappa$. We verify Tarski's test for A and $\mathfrak{B}$: assume $\mathfrak{B} \vDash \varphi[\bar{a}, b]$ for $b \in B$ and $\bar{a} \in A^n$; then $\bar{a} \in A_k^n$ for some $k \in \mathbb{N}$ and A_{k+1} contains some a with $\mathfrak{B} \vDash \varphi[\bar{a}, a]$. □

Theorem 4.35 (Löwenheim-Skolem-Tarski)**.** *Assume $\mathfrak{A}$ is infinite and $\kappa \geqslant |L|$ is an infinite cardinal.*

1. *If $\kappa \leqslant |A|$, then $\mathfrak{A}$ has an elementary substructure of cardinality κ.*
2. *If $\kappa \geqslant |A|$, then $\mathfrak{A}$ has an elementary extension of cardinality κ.*

Proof. 1: choose $A_0 \subseteq A$ of cardinality κ and apply the previous lemma.

2: By compactness, $T' := D_e(\mathfrak{A}) \cup \{\neg c_\alpha = c_\beta \mid \alpha, \beta < \kappa, \alpha \neq \beta\}$ is consistent; the c_α's are pairwise distinct new constants. Let $\mathfrak{B} \vDash T'$. The language of this model has cardinality $|L| + |A| + \kappa = \kappa$, and $\kappa \leqslant |B|$. By (1), there is $\mathfrak{B}' \preccurlyeq \mathfrak{B}$ of cardinality κ. By Lemma 4.28, $\mathfrak{B}' \upharpoonright L$ is isomorphic to an elementary extension of $\mathfrak{A}$. □

4.4 Directed systems

Definition 4.36. Assume $\mathfrak{I} = (I, <^{\mathfrak{I}})$ is a *directed* partial order, i.e., it satisfies

$$\forall xy \exists z (x \leqslant z \wedge y \leqslant z).$$

Then $(\mathfrak{A}_i)_{i \in I} = (\mathfrak{A}_i)_I$ is an *directed system* if $\mathfrak{A}_i \subseteq \mathfrak{A}_j$ for all $i, j \in I$ with $i <^{\mathfrak{I}} j$; it is *elementary* if $\mathfrak{A}_i \preccurlyeq \mathfrak{A}_j$ for all $i, j \in I$ with $i <^{\mathfrak{I}} j$. A *chain* or *ω-chain* is a directed system for $\mathfrak{I}$ a linear order, resp. the linear order on $\mathbb{N}$.

The *union* $\bigcup_I \mathfrak{A}_i =: \mathfrak{A}$ *of* $(\mathfrak{A}_i)_I$ is the L-structure with universe $A := \bigcup_{i \in I} A_i$ (where A_i is the universe of $\mathfrak{A}_i$) that interprets every symbol $s \in L$ by $s^{\mathfrak{A}} := \bigcup_{i \in I} s^{\mathfrak{A}_i}$ (recall we view functions as sets of ordered pairs)

Remark 4.37. The following imply that $\mathfrak{A}$ is well-defined. Let $R \in L$ be an r-ary relation symbol, and $f \in L$ an r-ary function symbol, and $\bar{a} = (a_0, \dots, a_{r-1}) \in A^r$

1. $\bar{a} \in A_i^r$ for some $i \in I$: for $j < r$ choose $i_j \in I$ such that $a_j \in A_{i_j}$; since $\mathfrak{I}$ is directed there is $i \in I$ such that $i_0, \dots, i_{r-1} \leqslant^{\mathfrak{I}} i$; then $\bar{a} \in A_i^r$.
2. if $\bar{a} \in A_i^r \cap A_j^r$, then $f^{\mathfrak{A}_i}(\bar{a}) = f^{\mathfrak{A}_j}(\bar{a})$ and $\bar{a} \in R^{\mathfrak{A}_i} \Leftrightarrow \bar{a} \in R^{\mathfrak{A}_j}$.

   Indeed: let $k \in I$ such that $i, j \leqslant^{\mathfrak{I}} k$; since $\mathfrak{A}_i, \mathfrak{A}_j \subseteq \mathfrak{A}_k$ we have $f^{\mathfrak{A}_i}(\bar{a}) = f^{\mathfrak{A}_k}(\bar{a}) = f^{\mathfrak{A}_j}(\bar{a})$, and $\bar{a} \in R^{\mathfrak{A}_i} \Leftrightarrow \bar{a} \in R^{\mathfrak{A}_k} \Leftrightarrow \bar{a} \in R^{\mathfrak{A}_j}$.
3. $f^{\mathfrak{A}} \upharpoonright A_i^r = f^{\mathfrak{A}_i}$ and $R^{\mathfrak{A}} \cap A_i^r = R^{\mathfrak{A}_i}$, that is, $\mathfrak{A}_i \subseteq \mathfrak{A}$.

Remark 4.38. If A is countable, say $A = \{a_0, a_1, \ldots\}$, then $\mathfrak{A}$ is the union of the ω-chain $(\langle a_0, \ldots, a_n\rangle^{\mathfrak{A}})_{n\in\mathbb{N}}$. In general, let I be the set of finite nonempty subsets of A, and let $i <^{\mathfrak{I}} j$ mean $i \subsetneq j$. Then $\mathfrak{A}$ is the union of the directed system $(\langle i\rangle^{\mathfrak{A}})_{i\in I}$.

Lemma 4.39 (Tarski)**.** *Let $(\mathfrak{A}_i)_I$ be an elementary directed system and $\mathfrak{A} := \bigcup_I \mathfrak{A}_i$. Then $\mathfrak{A}_i \preccurlyeq \mathfrak{A}$ for all $i \in I$.*

Proof. We show that for all L-formulas $\varphi(\bar{x})$ and all $i \in I$ and all $\bar{a}$ from A_i:

$$\mathfrak{A} \vDash \varphi[\bar{a}] \iff \mathfrak{A}_i \vDash \varphi[\bar{a}].$$

Call a formula *good* if it satisfies this claim. The set of good formulas is closed under $\neg, \wedge$ and contains atoms by Remark 4.37. We are left to show it is closed under $\forall y$. Let $\psi(\bar{x}, y)$ be good, $\varphi = \forall y \psi$, $i \in I$ and let $\bar{a}$ be from A_i.

If $\mathfrak{A} \vDash \varphi[\bar{a}]$, then $\mathfrak{A} \vDash \psi[\bar{a}, a]$ for all $a \in A$, so $\mathfrak{A}_i \vDash \psi[\bar{a}, a]$ for all $a \in A_i$ since ψ is good, so $\mathfrak{A}_i \vDash \varphi[\bar{a}]$. Conversely, assume $\mathfrak{A}_i \vDash \varphi[\bar{a}]$ and let $a \in A$. We have to show $\mathfrak{A} \vDash \psi[\bar{a}, a]$. Choose $j \in I$ with $a \in A_j$, and, by directedness, choose $k \in I$ with $i, j \leqslant^{\mathfrak{I}} k$. The $\bar{a}, a$ are from A_k. As $\mathfrak{A}_i \preccurlyeq \mathfrak{A}_k$ we have $\mathfrak{A}_k \vDash \varphi[\bar{a}]$, so $\mathfrak{A}_k \vDash \psi[\bar{a}, a]$. As ψ is good, $\mathfrak{A} \vDash \psi[\bar{a}, a]$. □

4.4.1 Chang - Łoś -Suszko

Definition 4.40. An $\forall\exists$-formula is an L-formula logically equivalent to one of the form $\forall \bar{x} \exists \bar{y} \psi$ with ψ quantifier free. An $\forall\exists$-*theory* is an L-theory containing only $\forall\exists$-sentences.

For an L-theory T, let $T_{\forall\exists}$ be the set of $\forall\exists$-sentences proved by T.

Lemma 4.41. *$\mathfrak{A} \vDash T_{\forall\exists}$ if and only if $\mathfrak{A}$ has an extension $\mathfrak{B} \vDash T$ such that $\mathfrak{A}$ is existentially closed in $\mathfrak{B}$.*

Proof. Assume the r.h.s. and $T \vdash \forall \bar{x} \exists \bar{y} \psi(\bar{x}, \bar{y})$ for ψ quantifier free. Let $\bar{a}$ be a tuple from A. We have to find $\bar{a}'$ from A such that $\mathfrak{A} \vDash \psi[\bar{a}, \bar{a}']$. This follows from existential closure: since $\mathfrak{B} \vDash T$, there is $\bar{b}$ such that $\mathfrak{B} \vDash \psi[\bar{a}, \bar{b}]$.

Conversely, let T' be T together with the universal $L(A)$-sentences true in $\mathfrak{A}_A$. We claim T' is consistent. Otherwise, by compactness, $T \vdash \neg\psi$ for some universal $L(A)$-sentence ψ true in $\mathfrak{A}_A$. Write $\psi = \psi'(\bar{c})$ for some L-formula $\psi'(\bar{x})$ and $\bar{c}$ outside L. Then $\forall \bar{x} \neg \psi'(\bar{x})$ is proved by T and false in $\mathfrak{A}$. But $\forall \bar{x} \neg \psi'(\bar{x})$ is an $\forall\exists$-sentence, contradicting $\mathfrak{A} \vDash T_{\forall\exists}$.

Let $\mathfrak{B}' \vDash T'$. By Lemma 4.3, there are $\mathfrak{B} \supseteq \mathfrak{A}$ and $\pi : \mathfrak{B} \cong \mathfrak{B}' \restriction L$ with $\pi(a) = c_a^{\mathfrak{B}'}$ for all $a \in A$. Thus $\mathfrak{B}_A \cong \mathfrak{B}'$ models T', and hence T. Further, $\mathfrak{B}_A$ satisfies the universal $L(A)$-sentences true in $\mathfrak{A}_A$, equivalently, $\mathfrak{A}$ is existentially closed in $\mathfrak{B}$. □

Like the Łos-Tarski theorem does for universal theories, the following characterizes $\forall\exists$-theories semantically.

Theorem 4.42 (Chang, Łoś, Suszko)**.** *Let T be an L-theory. The following are equivalent.*

1. *$T \equiv T_{\forall\exists}$.*

2. *If* $(\mathfrak{A}_i)_I$ *is a directed system of models of* T*, then* $\bigcup_I \mathfrak{A}_i \vDash T$.
3. *If* $(\mathfrak{A}_n)_n$ *is an* ω*-chain of models of* T*, then* $\bigcup_n \mathfrak{A}_n \vDash T$.

Proof. $1 \Rightarrow 2$: we claim $\bigcup_I \mathfrak{A}_i \vDash \varphi$ for every $\forall\exists$-sentence φ that is true in all $\mathfrak{A}_i$. Say, φ is $\forall x_0 \cdots x_{k-1} \exists y_0 \cdots y_{\ell-1} \psi(\bar{x}, \bar{y})$ for quantifier free ψ. Given $\bar{a} \in A^k$, we have to find $\bar{b} \in A^\ell$ such that $\mathfrak{A} \vDash \psi[\bar{a}, \bar{b}]$. Choose $i \in I$ such that $\bar{a} \in A_i^k$. As $\mathfrak{A}_i \vDash \varphi$ there is $\bar{b} \in A_i^\ell$ such that $\mathfrak{A}_i \vDash \psi[\bar{a}, \bar{b}]$. Then $\mathfrak{A} \vDash \psi[\bar{a}, \bar{b}]$ since $\mathfrak{A}_i \subseteq \mathfrak{A}$ and ψ is quantifier free.

$2 \Rightarrow 3$ is trivial. $3 \Rightarrow 1$: given $\mathfrak{A}_0 \vDash T_{\forall\exists}$, we have to show $\mathfrak{A}_0 \vDash T$. We first construct a *sandwich*: $\mathfrak{B}_0 \vDash T$ and $\mathfrak{A}_1 \succcurlyeq \mathfrak{A}_0$ such that

$$\mathfrak{A}_0 \subseteq \mathfrak{B}_0 \subseteq \mathfrak{A}_1.$$

Choose $\mathfrak{A}_0 \subseteq \mathfrak{B}_0$ according to the previous lemma. By existential closure, $(\mathfrak{B}_0)_{A_0}$ satisfies all universal $L(A_0)$-sentences that are true in $(\mathfrak{A}_0)_{A_0}$, or, equivalently, proved by $D_e(\mathfrak{A}_0)$. By Lemma 4.4, $D_a((\mathfrak{B}_0)_{A_0}) \cup D_e(\mathfrak{A}_0)$ has a model. By Lemma 4.3, we can assume its $L(A_0)$-reduct $\mathfrak{A}'$ extends $(\mathfrak{B}_0)_{A_0}$. By Lemma 4.28, $a \mapsto c_a^{\mathfrak{A}'}$ is an elementary embedding from $\mathfrak{A}_0$ into $\mathfrak{A}' \restriction L$. But this map is the identity ($c_a^{\mathfrak{A}'} = c_a^{(\mathfrak{B}_0)_{A_0}} = a$), so $\mathfrak{A}_0 \preccurlyeq \mathfrak{A}' \restriction L =: \mathfrak{A}_1$.

This completes the construction of the sandwich. As $\mathfrak{A}_1 \vDash T_{\forall\exists}$ we get a sandwich $\mathfrak{B}_1, \mathfrak{A}_2$ with $\mathfrak{A}_1$ in the role of $\mathfrak{A}_0$, and so on:

$$\begin{array}{ccccccccccccccc} \mathfrak{A}_0 & \subseteq & \mathfrak{B}_0 & \subseteq & \mathfrak{A}_1 & \subseteq & \mathfrak{B}_1 & \subseteq & \mathfrak{A}_2 & \subseteq & \mathfrak{B}_2 & \subseteq & \mathfrak{A}_3 & \subseteq & \cdots \\ \mathfrak{A}_0 & & \preccurlyeq & & \mathfrak{A}_1 & & \preccurlyeq & & \mathfrak{A}_2 & & \preccurlyeq & & \mathfrak{A}_3 & \preccurlyeq & \cdots \end{array}$$

where all $\mathfrak{B}_n$ satisfy T. Then $\bigcup_n \mathfrak{A}_n = \bigcup_n \mathfrak{B}_n$. By (3), this structure satisfies T. By Tarski's Lemma 4.39, $\mathfrak{A}_0 \preccurlyeq \bigcup_n \mathfrak{A}_n$, so $\mathfrak{A}_0 \vDash T$ as was to be shown. □

Exercise 4.43. The class of partial orders with minimal elements is not $\forall\exists$-axiomatizable.

Corollary 4.44. *If* T *is model complete, then* $T \equiv T_{\forall\exists}$

Proof. By model completeness, chains of models of T are elementary. By Tarski's lemma, Theorem 4.42 (3) follows. □

4.4.2 Ax-Grothendieck on polynomial maps

Let $\mathfrak{I} = (I, <^{\mathfrak{I}})$ where I is the set of positive naturals and $i <^{\mathfrak{I}} j$ if i is a proper divisor of j. Then $\mathfrak{I} = (I, <^{\mathfrak{I}})$ is a directed partial order.

Let p be a prime. From algebra we know that for every $i \in I$ there is, up to $\cong$, exactly one field $\mathfrak{F}_{p^i}$ of size p^i; moreover, $\mathfrak{F}_{p^i} \subseteq \mathfrak{F}_{p^j}$ if $i <^{\mathfrak{I}} j$. In other words, $(\mathfrak{F}_{p^i})_I$ is a directed system. It is also known from algebra that $\tilde{\mathfrak{F}}_p := \bigcup_I \mathfrak{F}_{p^i}$ is algebraically closed.

Theorem 4.45. *The field of complex numbers satisfies every* $\forall\exists$*-sentence that is true in all finite fields.*

Proof. Let φ be such a sentence. By Exercise 3.22 it suffices to show $ACF_p \vdash \varphi$ for all primes p. Since ACF_p is complete, it suffices to show $\tilde{\mathfrak{F}}_p \vDash \varphi$. Since φ is $\forall\exists$ it suffices to show $\mathfrak{F}_{p^i} \vDash \varphi$ for all $i > 0$. This follows from the assumption. □

Let $\mathfrak{A}$ be a field and $k, \ell > 0$ be naturals. A function $f : A^k \to A^\ell$ is *polynomial* if there are polynomials $P_0(\bar{x}), \ldots, P_{\ell-1}(\bar{x})$ (with coefficients in A) such that for all $\bar{a} \in A^k$:

$$f(\bar{a}) = (P_0(\bar{a}), \ldots, P_{\ell-1}(\bar{a})).$$

Corollary 4.46 (Ax-Grothendieck). *Let $k > 0$ be natural. Every injective polynomial function from $\mathbb{C}^k$ into $\mathbb{C}^k$ is surjective.*

Proof. Assume $f : \mathbb{C}^k \to \mathbb{C}^k$ is an injective polynomial function that is not surjective. Say, f is given by the polynomials $P_0(\bar{x}), \ldots, P_{k-1}(\bar{x})$. Recall P_i is equivalent to an $L(\mathbb{C})$-term $t_{P_i}(\bar{x})$ (Section 4.2.1). Let $t_i(\bar{x}, \bar{y}_i)$ be obtained by replacing the constants $c_a, a \in \mathbb{C}$, in $t_{P_i}(\bar{x})$ by variables $\bar{y}_i$. Let $\bar{y}$ collect all $\bar{y}_i$. For a tuple of variables $\bar{z} = (z_0, \ldots, z_{k-1})$ let $\bar{t}(\bar{x}, \bar{y}) = \bar{z}$ abbreviate $\bigwedge_{i<k} t_i(\bar{x}, \bar{y}_i) = z_i$; further, let $\bar{x} \neq \bar{x}'$ abbreviate $\bigvee_{i<k} \neg x_i = x'_i$. Set

$$\varphi(\bar{y}) \;:=\; \exists \bar{x}\bar{x}'\bar{z}\big(\bar{x} \neq \bar{x}' \wedge \bar{t}(\bar{x}, \bar{y}) = \bar{z} \wedge \bar{t}(\bar{x}', \bar{y}) = \bar{z}\big) \;\vee\; \forall \bar{z} \exists \bar{x}\; \bar{t}(\bar{x}, \bar{y}) = \bar{z}.$$

Then the coefficients of the P_i's do not satisfy $\varphi(\bar{y})$ in $\mathfrak{C}$. Hence, $\mathfrak{C} \not\models \forall \bar{y} \varphi(\bar{y})$. But this is an $\forall\exists$-sentence that is obviously true in all finite fields – contradicting Theorem 4.45. □

4.5 Model companions

Let T be an L-theory. Let $T_\forall$ be the set of universal L-sentences proved by T.

Remark 4.47. Let T^* be an L-theory. Then $T^* \vdash T_\forall$ if and only if every model of T^* embeds into a model of T. In particular, $T^*_\forall = T_\forall$ if and only if models of T^* embed into models of T and vice-versa.

Proof. $\Rightarrow$: assume $T^* \vdash T_\forall$ and let $\mathfrak{A}^* \models T^*$. Then $\mathfrak{A}^* \models T_\forall$. By Lemma 4.4, $\mathfrak{A}^* \to_a \mathfrak{B} \models T$ for some $\mathfrak{B}$. $\Leftarrow$: let $\mathfrak{A}^* \models T^*$. We have to show $\mathfrak{A}^* \models T_\forall$. This is clear by the r.h.s.: $\mathfrak{A}^* \to_a \mathfrak{B} \models T$ for some $\mathfrak{B}$. □

Definition 4.48. Let T be an L-theory. A *model companion of T* is a model complete L-theory T^* with $T^*_\forall = T_\forall$. A model $\mathfrak{A}$ of T is *existentially closed in T* if $\mathfrak{A}$ is existentially closed in all extensions that model T; the class of such models is denoted $\mathcal{E}(T)$.

Example 4.49. The theory of dense linear orders without endpoints is the model companion of the theory of linear orders.

Indeed: it is model complete by Exercise 4.22; every linear oder $\mathfrak{A}$ embeds into a dense linear order without endpoints, e.g., $\mathfrak{Q} \times \mathfrak{A}$ for $\mathfrak{Q}$ the order of the rationals.

We shall see many natural examples of model companions in Section 6.3.1.

Lemma 4.50.

1. *If T is a $\forall\exists$-theory and T^* is a model companion of T, then $T^* \vdash T$.*
2. *T has at most one model companion up to equivalence.*

Proof. 1: Let $\mathfrak{A}_0 \models T^*$. Choose $\mathfrak{B}_0 \models T$ with $\mathfrak{A}_0 \subseteq \mathfrak{B}_0$. Choose $\mathfrak{A}_1 \models T^*$ with $\mathfrak{B}_0 \subseteq \mathfrak{A}_1$, etc.. Since T^* is model complete, $\mathfrak{A}_0 \preccurlyeq \mathfrak{A}_1 \preccurlyeq \cdots$, so $\bigcup_n \mathfrak{A}_n \models T^*$. But $\bigcup_n \mathfrak{A}_n = \bigcup_n \mathfrak{B}_n \models T$ since T is $\forall\exists$. As $\mathfrak{A}_0 \preccurlyeq \bigcup_n \mathfrak{A}_n$ by Tarski's lemma, $\mathfrak{A}_0 \models T$.

2: Let T^*, T^+ be model companions of T. Then $T^*_\forall = T_\forall = T^+_\forall$, so T^* is a model companion of T^+. By Corollary 4.44, $T^+ \equiv T^+_{\forall\exists}$. By (1), $T^* \vdash T^+$. □

Theorem 4.51. *Assume T is a $\forall\exists$-theory. An L-theory T^* is a model companion of T if and only if T^* axiomatizes $\mathcal{E}(T)$.*

Proof. $\Rightarrow$: We first show that every $\mathfrak{A} \models T^*$ is in $\mathcal{E}(T)$. By the previous lemma, $\mathfrak{A} \models T$. Let $\mathfrak{A} \subseteq \mathfrak{B} \models T$. We have to show that $\mathfrak{A}$ is existentially closed in $\mathfrak{B}$. Let $\mathfrak{B}_A \models \varphi$ for φ an existential $L(A)$-sentence. We have to show $\mathfrak{A}_A \models \varphi$. Choose $\mathfrak{B} \subseteq \mathfrak{A}^* \models T^*$. Then $\mathfrak{B}_A \subseteq \mathfrak{A}^*_A$. Since φ is existential, $\mathfrak{A}^*_A \models \varphi$. By model completeness, $\mathfrak{A} \preccurlyeq \mathfrak{A}^*$. Hence $\mathfrak{A}_A \models \varphi$.

We now show that every $\mathfrak{B} \in \mathcal{E}(T)$ models T^*. Since $\mathfrak{B} \models T$ there is $\mathfrak{B} \subseteq \mathfrak{A} \models T^*$. By the lemma, $\mathfrak{A} \models T$. Hence $\mathfrak{B}$ is existentially closed in $\mathfrak{A}$. By Lemma 4.41, $\mathfrak{B} \models T^*_{\forall\exists}$. But $T^*_{\forall\exists} \equiv T^*$ by Corollary 4.44.

$\Leftarrow$: since structures in $\mathcal{E}(T)$ model T, we have $T^* \vdash T$. That T^* is model complete thus follows from Robinson's test. A model of T^* embeds into a model of T, namely itself. A model of T embeds into a model of T^* by the next lemma. □

Lemma 4.52. *If T is an $\forall\exists$-theory, then every model of T has an extension in $\mathcal{E}(T)$.*

Proof. We claim that every $\mathfrak{A} \models T$ has an extension $\mathfrak{A}^* \models T$ such that

$$\begin{array}{l}\mathfrak{A}^*_A \text{ satisfies every existential } L(A)\text{-sentence } \varphi \text{ that is true}\\ \text{in some extension that models } T.\end{array} \tag{$*$}$$

The claim implies the lemma as follows. Given $\mathfrak{A}_0 \models T$, let $\mathfrak{A}_1 := \mathfrak{A}^*_0, \mathfrak{A}_2 := \mathfrak{A}^*_1, \ldots$. Then $\mathfrak{A} := \bigcup_n \mathfrak{A}_n \models T$ since T is $\forall\exists$. Assume $\mathfrak{A}_A \subseteq \mathfrak{B}_A \models \varphi$ where φ is existential. Choose $n \in \mathbb{N}$ such that φ is an $L(A_n)$-sentence. Then $(\mathfrak{A}_{n+1})_{A_n} \subseteq \mathfrak{B}_{A_n} \models \varphi$. By $(*)$, $(\mathfrak{A}_{n+1})_{A_n} \models \varphi$. Since φ is existential and $\mathfrak{A}_{n+1} \subseteq \mathfrak{A}$ we have $\mathfrak{A}_A \models \varphi$.

To prove the claim choose a list $(\varphi_\alpha)_{\alpha\in\kappa}$ of all existential $L(A)$-sentences – for a suitable cardinal κ. Define a chain $(\mathfrak{A}_\alpha)_{\alpha\in\kappa}$ of L-structures as follows. Set $\mathfrak{A}_0 := \mathfrak{A}$. For α a limit, set $\mathfrak{A}_\alpha := \bigcup_{\beta<\alpha} \mathfrak{A}_\beta$. For $\mathfrak{A}_{\alpha+1}$ choose an extension of $\mathfrak{A}_\alpha$ such that $(\mathfrak{A}_\alpha)_A \models T \cup \{\varphi_\alpha\}$; if there is none, set $\mathfrak{A}_{\alpha+1} := \mathfrak{A}_\alpha$.

Set $\mathfrak{A}^* := \bigcup_{\alpha\in\kappa} \mathfrak{A}_\alpha$. We show $\mathfrak{A}^*$ satisfies our claim. $\mathfrak{A}^* \models T$ because by construction all $\mathfrak{A}_\alpha$ model T and T is $\forall\exists$. To verify $(*)$, let $\alpha \in \kappa$ and assume $\mathfrak{A}^*_A \subseteq \mathfrak{B}_A \models T \cup \{\varphi_\alpha\}$ for some $\mathfrak{B}$. Then $(\mathfrak{A}_\alpha)_A \subseteq \mathfrak{B}_A \models T \cup \{\varphi_\alpha\}$. By construction, $(\mathfrak{A}_{\alpha+1})_A \models T \cup \{\varphi_\alpha\}$. Since φ_α is existential and $\mathfrak{A}_{\alpha+1} \subseteq \mathfrak{A}^*_A$, we have $\mathfrak{A}^*_A \models \varphi_\alpha$. □

Since T and $T_\forall$ have the same model-companions, we get for arbitrary T:

Corollary 4.53. *T^* is a model companion of T if and only if T^* axiomatizes $\mathcal{E}(T_\forall)$.*

Example 4.54 (Existentially closed fields)**.** Let T be the theory of fields. Once you observe in algebra that every non-constant polynomial over a given field has a root in some field extension, you see that fields in $\mathcal{E}(T)$ are algebraically closed (see Proposition 4.26). Then the above implies that every field has an algebraically closed extension. We shall see later (Example 6.28) that, in fact, ACF is the model-companion of T, so axiomatizes $\mathcal{E}(T)$.

4.5.1 Groups and rings are not companionable

Proposition 4.55. *The theory of groups does not have a model companion.*

Proof. We use the following group theoretic result: if $\mathfrak{A}$ is a group and $a, b \in A$ have the same order, then a, b are conjugate in some group $\mathfrak{B} \supseteq \mathfrak{A}$.

Assume T^* is a model companion of the theory of groups T. Then

$$T^* \cup \{\neg nc = 0 \wedge \neg nd = 0 \mid n > 0\} \vdash \exists x\ x + c + (-x) = d.$$

where c, d are new constants. Indeed: if $\mathfrak{A}$ models the l.h.s., $\mathfrak{A}\upharpoonright L_{Gr} \in \mathcal{E}(T)$ and $c^{\mathfrak{A}}, b^{\mathfrak{A}} \in A$ have infinite order. By the result above, $\exists x\ x + c + (-x) = d$ is true in some $\mathfrak{A} \subseteq \mathfrak{B} \models T$. Since $\mathfrak{A}\upharpoonright L_{Gr}$ is existentially closed in $\mathfrak{B}\upharpoonright L_{Gr}$, we have $\mathfrak{A} \models \exists x\ x + c + (-x) = d$.

By compactness, there is n_0 such that $T^* \cup \{\neg nc = 0 \wedge \neg nd = 0 \mid n < n_0\}$ proves $\exists x\ x + c + (-x) = d$. This is false: let $\mathfrak{B}$ be a group with elements b, b' of orders $n_0, n_0 + 1$. By Lemma 4.52 there is $\mathfrak{B} \subseteq \mathfrak{C} \in \mathcal{E}(T)$, so $\mathfrak{C} \models T^*$. Let $\mathfrak{C}'$ expand $\mathfrak{C}$ interpreting c, d by b, b'. Then $\mathfrak{C}'$ satisfies $\neg nc = 0 \wedge \neg nd = 0$ for all $n < n_0$ but not $\exists x\ x + c + (-x) = d$. □

In Section 6.3.1 we shall see that the theories of torsion-free and of ordered abelian groups do have model companions, namely the theories of divisible such groups.

Proposition 4.56 (Cherlin)**.** *The theory of (commutative unitary) rings does not have a model companion.*

Recall that an element $a \in A$ of a ring $\mathfrak{A}$ is *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$; it is *idempotent* if $a^2 = a \neq 0$. We need an algebraic lemma:

Lemma 4.57. *Let $\mathfrak{A}$ be a ring and $a \in A$. Then a is not nilpotent if and only if it divides an idempotent $b \in B$ for some ring $\mathfrak{B} \supseteq \mathfrak{A}$.*

Proof. $\Leftarrow$: if $a | b = b^2$ then $a^n | b^n = b \neq 0$, so $a^n \neq 0$ for all $n \in \mathbb{N}$.

$\Rightarrow$: let $\mathfrak{B} := \mathfrak{A}[x]/((ax)^2 - ax)$ and $b := ax \mod ((ax)^2 - ax)$. Clearly, $a | b = b^2$ in $\mathfrak{B}$. We claim $b \neq 0$. Otherwise there is $P(x) := \sum_i a_i x^i \in \mathfrak{A}[x]$, say of degree d, such that in $\mathfrak{A}[x]$

$$ax = P(x) \cdot ((ax)^2 - ax) = a_0 a^2 x^2 - a_0 a x + a_1 a^2 x^3 - a_1 a x^2 + \cdots + a_d x^{d+2} - a_d a x^{d+1}$$

This implies $-a = a_0 a$, $a_0 a^2 = a_1 a$, $a_1 a^2 = a_2 a, \ldots, a_{d-1} a^2 = a_d a$, $a_d a^2 = 0$, hence $-a^2 = a_1 a$, $-a^3 = a_2 a, \ldots, -a^{d+1} = a_d a, -a^{d+2} = 0$, so a is nilpotent. □

Proof of Proposition 4.56. Assume T^* is a model companion of the theory of rings. Then

$$T^* \cup \{\neg c^n = 0 \mid n \in \mathbb{N}\} \vdash \exists xy (x^2 = x \wedge \neg x = 0 \wedge c \cdot y = x)$$

by the lemma. By compactness, there is $n_0 \in \mathbb{N}$ such that $T^* \cup \{\neg c^n = 0 \mid n < n_0\}$ proves the r.h.s.. and thus $\neg c^{n_0} = 0$. This is false. □

In Section 6.3.1 we shall see that the theory of integral domains does have a model companion, namely *ACF*.

Chapter 5

Types

Let L be a language, T an L-theory and $\mathfrak{A}$ an L-structure.

5.1 Realizing types

Definition 5.1. A *(partial) n-type* of T is a set $p = p(x_0, \dots, x_{n-1})$ of L-formulas $\varphi(x_0, \dots, x_{n-1})$ such that $T \cup p$ is consistent, that is, there exists $\mathfrak{B} \models T$ and $\bar{b} \in B^n$ such that $\mathfrak{B} \models p[\bar{b}]$; we say $\bar{b}$ *realizes* p in $\mathfrak{B}$. A model where p is not realized is said to *omit* p. p is *complete* if $\varphi(\bar{x}) \in p$ or $\neg\varphi(\bar{x}) \in p$ for all L-formulas $\varphi(\bar{x})$.

An *n-type of* $\mathfrak{A}$ is an n-type of $Th(\mathfrak{A})$. An *n-type of* $\mathfrak{A}$ *over* $X \subseteq A$ is an n-type of $\mathfrak{A}_X$. The complete *n-type of* $\bar{a} \in A^n$ *over* X is

$$\mathrm{tp}_{\mathfrak{A}}(\bar{a}/X) := \{\varphi(\bar{x}) \mid \varphi(\bar{x}) \text{ is an } L(X)\text{-formula such that } \mathfrak{A}_X \models \varphi[\bar{a}]\}.$$

By a *type* we mean an *n-type* for some $n \in \mathbb{N}$.

Remark 5.2. Let $n \in \mathbb{N}$.

1. An n-type $p(\bar{x})$ of $\mathfrak{A}$ is also an n-type of $\mathfrak{A}_X$ for all $X \subseteq A$.

   Indeed, by Lemma 4.30, $p(\bar{x})$ is consistent with $D_e(\mathfrak{A}) = Th(\mathfrak{A}_A) \supseteq Th(\mathfrak{A}_X)$.

2. $p(\bar{x})$ is an n-type of $\mathfrak{A}$ over X if and only if $\mathfrak{A}_X \models \exists\bar{x}(\varphi_0(\bar{x}) \wedge \dots \wedge \varphi_\ell(\bar{x}))$ for all $\ell \in \mathbb{N}$ and $\varphi_0(\bar{x}), \dots, \varphi_\ell(\bar{x}) \in p(\bar{x})$.

   Indeed: $\Leftarrow$ follows from compactness. $\Rightarrow$: by assumption, $Th(\mathfrak{A}_X)$ is consistent with $\psi := \exists\bar{x}(\varphi_0(\bar{x}) \wedge \dots \wedge \varphi_{\ell-1}(\bar{x}))$; since $Th(\mathfrak{A}_X)$ is complete, $Th(\mathfrak{A}_X) \vdash \psi$, so $\mathfrak{A}_X \models \psi$.

3. Every n-type $p(\bar{x})$ of T is contained in a complete n-type.

   Indeed: $p(\bar{x}) \subseteq \mathrm{tp}_{\mathfrak{B}}(\bar{b})$ for $\bar{b}$ realizing $p(\bar{x})$ in $\mathfrak{B} \models T$.

Exercise 5.3 (Types versus ultrafilters)**.** Let $n \in \mathbb{N}$. Recall the n-th Lindenbaum algebra $\mathfrak{L}_n(T)$ from Example 2.2 (4). There is a bijection from the set of complete n-types of T onto the set of ultrafilters in $\mathfrak{L}_n(T)$ given by

$$p \mapsto F_p := \{\varphi(\bar{x})/T \mid \varphi(\bar{x}) \in p(\bar{x})\}.$$

Lemma 5.4. *Let P be a set of types of $\mathfrak{A}$. Then there exists $\mathfrak{B} \succcurlyeq \mathfrak{A}$ that realizes all types in P; moreover, $|B| \leqslant \max\{|A|, |P|, |L|\}$.*

Proof. We first show that every n-type $p(\bar{x})$ of $\mathfrak{A}$ is realized in some elementary extension of $\mathfrak{A}$. Indeed: for new constants $\bar{c}$ the theory $p(\bar{c})$ is consistent with $Th(\mathfrak{A})$. By Lemma 4.30, $p(\bar{c}) \cup D_e(\mathfrak{A})$ has a model $\mathfrak{C}$, say $\bar{d}$ interpret $\bar{c}$ in $\mathfrak{C}$. By Lemma 4.28, there is $\pi : \mathfrak{B} \cong \mathfrak{C}\restriction L$ for some $\mathfrak{B} \succcurlyeq \mathfrak{A}$. Then $\pi^{-1}(\bar{d})$ realizes $p(\bar{x})$ in $\mathfrak{B}$.

Given P, choose for every $p(\bar{x}) \in P$ an own tuple of constants $\bar{c}^p$. As above, it suffices to show that $\bigcup_{p \in P} p(\bar{c}^p) \cup Th(\mathfrak{A})$ is consistent. By compactness, we can assume $P = \{p_0, \dots, p_k\}$ is finite. Choose $\mathfrak{A}_0 \succcurlyeq \mathfrak{A}$ realizing p_0, then $\mathfrak{A}_1 \succcurlyeq \mathfrak{A}_0$ realizing p_1, etc.. Then $\mathfrak{A}_k$ realizes all p_i.

For the moreover-part, let $\mathfrak{B}' \succcurlyeq \mathfrak{A}$ realize all types in P. By Theorem 4.35 there is $\mathfrak{B} \preccurlyeq \mathfrak{B}'$ that contains A and these realizations and has the claimed size. Further, $\mathfrak{A} \subseteq \mathfrak{B} \preccurlyeq \mathfrak{B}'$ and $\mathfrak{A} \preccurlyeq \mathfrak{B}'$ imply $\mathfrak{A} \preccurlyeq \mathfrak{B}$. □

Examples 5.5.

1. Let $\mathfrak{Q}, \mathfrak{R}$ be the natural orders on $\mathbb{Q}$ and $\mathbb{R}$. A *cut* in $\mathfrak{Q}$ is pair (L, R) with $L, R \subseteq \mathbb{Q}$ such that $\mathbb{Q} = L \cup R$ and $q < q'$ for all $q \in L, q' \in R$. Let $p_{(L,R)}(x)$ be the set of $\{<\} \cup \{c_q \mid q \in \mathbb{Q}\}$-formulas $c_q < x \wedge x < c_q$ for $q \in L, q' \in R$. This is a 1-type of $\mathfrak{Q}$ over $\mathbb{Q}$ and $\mathfrak{Q}_\mathbb{Q}$ omits it. If L has no maximum and R no minimum, then $\mathfrak{R}_\mathbb{Q}$ realizes it – $\mathfrak{R}$ is the order on the reals.

   Indeed: given finitely many formulas $c_{q_i} < x \wedge x < c_{q_i'}$ from $p_{(L,R)}(x)$ choose $q \in \mathbb{Q}$ between $\max_i q_i$ and $\min_i q_i'$; then q satisfies the given formulas in $\mathfrak{Q}_\mathbb{Q}$, so the given finite set is consistent with $Th(\mathfrak{Q}_\mathbb{Q})$. If L has no maximum and R no minimum, then $\inf R \in \mathbb{R}$ realizes $p_{(L,R)}(x)$ in $\mathfrak{R}_\mathbb{Q}$.

2. Let $\mathfrak{Z}$ be the natural order on $\mathbb{Z}$. Let $p(x_0, x_1)$ contain the formula $\exists y_0 \cdots \exists y_\ell\, x_0 < y_0 < \cdots < y_\ell < x_1$ for every $\ell \in \mathbb{N}$. Then $p(x_0, x_1)$ is a 2-type of $\mathfrak{Z}$ (over $\emptyset$). $\mathfrak{Z}$ omits it, in $\mathfrak{Z} + \mathfrak{Z}$ e.g. $((0,0),(1,0))$ realizes it.

3. Consider the theory of ordered fields; write n for $1 + \cdots + 1$ (n times). Let $p(x)$ contain $n < x$ for all $n \in \mathbb{N}$. Then $p(x)$ is a 1-type of the theory that is omitted e.g. in the ordered field of reals, and generally in *archimedian* ordered fields. Fields realizing the type have *infinitesimal elements*, i.e., elements realizing the 1-type containing $0 < x \wedge n \cdot x < 1$ for all $n \in \mathbb{N}$.

4. Let $\mathfrak{A}$ be a field. Recall, we let $\langle\!\langle B \rangle\!\rangle^\mathfrak{A}$ denote the subfield generated by $B \subseteq A$ in $\mathfrak{A}$ (Section 3.1.3). Let $p(x)$ be the set of $L(B)$-formulas

   $$t_d = 0 \;\vee\; \neg\, t_d \cdot x^d + \cdots + t_1 \cdot x + t_0 = 0$$

   where $d \in \mathbb{N}, d > 0$ and the t_i are $L(B)$-terms.

   Then $p(x)$ is realized by exactly those $a \in A$ that are transcendental over $\langle\!\langle B \rangle\!\rangle^\mathfrak{A}$. If $\mathfrak{A}$ is infinite, then $p(x)$ is a 1-type of $\mathfrak{A}$ over B.

   Indeed: note $L(B)$-terms correspond to polynomials over the ring $\langle B \rangle^\mathfrak{A}$; an element that is not a root of any such polynomial, is not a root of any polynomial over the field $\langle\!\langle B\rangle\!\rangle^{\mathfrak{A}}$. Every finite subset of $p(x)\cup Th(\mathfrak{A}_B)$ is consistent because given finitely many non-constant polynomials over $\langle\!\langle B\rangle\!\rangle^{\mathfrak{A}}$ we find $a\in A$ outside the union of their roots, a finite set.

Exercise 5.6. Let $\kappa>\aleph_0$ be a cardinal and $n\in\mathbb{N}$. If $\mathfrak{A}$ is a countable L-structure with at least κ many complete n-types, then it has at least κ many pairwise non-isomorphic countable elementary extensions.

Exercise 5.7. The standard L_{Ring}-structure $\mathfrak{N}$ on $\mathbb{N}$ has $2^{\aleph_0}$ many complete 1-types.

Hint: for every set X of primes define a 1-type "x is divisible by the primes in X but not by the primes outside X".

Definition 5.8. Let κ be a cardinal. $\mathfrak{A}$ is *κ-saturated* if for every $X\subseteq A$ with $|X|<\kappa$ every 1-type of $\mathfrak{A}$ over X is realized in $\mathfrak{A}_X$.

Notation: for $\bar a=(a_0,\dots,a_{n-1})\in A^n$ let $(\mathfrak{A},\bar a)$ be the the $L\cup\{c_i\mid i<n\}$-expansion of $\mathfrak{A}$ interpreting constants $c_i\notin L$ by a_i.

We shall be mainly concerned with $\aleph_0$-saturation. Clearly, $\mathfrak{A}$ is $\aleph_0$-saturated if and only if for all (finite) tuples $\bar a$ from A the expansion $(\mathfrak{A},\bar a)$ realizes all its 1-types.

Example 5.9. The rational order $\mathfrak{Q}$ is $\aleph_0$-saturated.

Proof. Let $p(x)$ be a 1-type of $(\mathfrak{Q},\bar q)$. By Lemma 5.4, there is a countable $(\mathfrak{Q}',\bar q)\succcurlyeq(\mathfrak{Q},\bar q)$ realizing $p(x)$, say by q'. Further, $\bar q\mapsto\bar q\in I_{Sk}(\mathfrak{Q}',\mathfrak{Q}):\mathfrak{Q}'\cong_p\mathfrak{Q}$ (Lemma 3.11). By Theorem 3.8, there is $\pi:\mathfrak{Q}'\cong\mathfrak{Q}$ fixing $\bar q$. Then $\pi(q')$ realizes $p(x)$ in $(\mathfrak{Q},\bar q)$. □

Exercise 5.10. Finite structures are $\aleph_0$-saturated. Infinite $\emptyset$-structures are $\aleph_0$-saturated.

Lemma 5.11. *Let $\mathfrak{A}$ be $\aleph_0$-saturated. For every $n\in\mathbb{N}$ and every tuple $\bar a$ from A, $(\mathfrak{A},\bar a)$ realizes all its n-types.*

Proof. Assume the lemma holds for n-types, and let $p(\bar x,x)$ be a complete $(n+1)$-type of $(\mathfrak{A},\bar a)$. Let $p'(\bar x)\subseteq p(\bar x,x)$ be the subset of formulas in variables $\bar x$. This is an n-type of $(\mathfrak{A},\bar a)$, so realized, say by $\bar b\in A^n$.

Let $\bar c$ be the constants naming $\bar b$ in $(\mathfrak{A},\bar a\bar b)$ and consider $p(\bar c,x)$. This is a 1-type of $(\mathfrak{A},\bar a\bar b)$. Indeed: let $\varphi_0(\bar c,x),\dots,\varphi_{\ell-1}(\bar c,x)\in p(\bar c,x)$; then $\exists x\bigwedge_{i<\ell}\varphi_i(\bar x,x)\in p'(\bar x)$ since $p(\bar x,x)$ is complete, so $(\mathfrak{A},\bar a\bar b)\models\exists x\bigwedge_{i<\ell}\varphi_i(\bar c,x)$.

If b realizes $p(\bar c,x)$ in $(\mathfrak{A},\bar a\bar b)$, then $\bar b b$ realizes $p(\bar x,x)$ in $(\mathfrak{A},\bar a)$. □

Theorem 5.12. *Every L-structure has an $\aleph_0$-saturated elementary extension.*

Proof. Let $\mathfrak{A}$ be an L-structure. By Lemma 5.4, there is an elementary chain $\mathfrak{A}_0:=\mathfrak{A}\preccurlyeq\mathfrak{A}_1\preccurlyeq\mathfrak{A}_2\preccurlyeq\cdots$ such that $(\mathfrak{A}_{n+1})_{A_n}$ realizes all 1-types of $(\mathfrak{A}_n)_{A_n}$ for all $n\in\mathbb{N}$. Then $\mathfrak{B}:=\bigcup_n\mathfrak{A}_n$ is $\aleph_0$-saturated: given a finite $X\subseteq B$ and a 1-type p of $\mathfrak{B}$ over X, choose $n\in\mathbb{N}$ such that $X\subseteq A_n$. By Tarskis's lemma 4.39, $\mathfrak{A}_n\preccurlyeq\mathfrak{B}$, so p is a 1-type of $\mathfrak{A}_n$ over $X\subseteq A_n$, so realized in $(\mathfrak{A}_{n+1})_{A_n}$, say by a. By $\mathfrak{A}_{n+1}\preccurlyeq\mathfrak{B}$, a realizes p in $\mathfrak{B}_X$. □

Exercise 5.13. Prove a version of the above for κ-saturation.

The following implies Lemma 3.20:

Remark 5.14. Every $\aleph_0$-saturated $\mathfrak{A} \vDash ACF$ is "large" in the sense of Lemma 3.20.

Proof. Let $\mathfrak{B} \subseteq \mathfrak{A}$ be a finitely generated subfield of $\mathfrak{A}$, say $\mathfrak{B} = \langle\!\langle B \rangle\!\rangle^{\mathfrak{A}}$ for finite $B \subsetneq A$. Then $\mathfrak{A}$ realizes the 1-type $p(x)$ over B from Remark 5.2 (4). Any realization of it is transcendental over $\mathfrak{B}$. □

5.1.1 $\aleph_1$-saturation of ultraproducts

Theorem 5.15. *Assume L is at most countable, $(\mathfrak{A}_i)_{i\in\mathbb{N}}$ is a family of L-structures and F a free ultrafilter on $\mathbb{N}$. Then $\mathfrak{A} := \prod_F \mathfrak{A}_i$ is $\aleph_1$-saturated.*

Proof. Let $X \subseteq A$ be at most countable, say listed by $a_0^F, a_1^F, \ldots$ possibly with repetitions. Write $\mathfrak{A}_X$ as $(\prod_F \mathfrak{A}_i, a_0^F, a_1^F, \ldots)$. Then $\mathfrak{A}_X$ is $\prod_F (\mathfrak{A}_i, a_0(i), a_1(i), \ldots)$. Thus, it suffices to show that $\mathfrak{A}$ realizes all 1-types $p(x)$ of $\mathfrak{A}$ (over $\varnothing$).

Let $\varphi_0(x), \varphi_1(x), \ldots$ list $p(x)$ and set $\psi_n(x) := \bigwedge_{i\leqslant n} \varphi_i(x)$ for $n \in \mathbb{N}$. By Remark 5.2 (2) we have $\mathfrak{A} \vDash \exists x \psi_n(x)$, so $Y_n := \{i \mid \mathfrak{A}_i \vDash \exists x \psi_n(x)\} \in F$ by Łos. Set

$$X_n := Y_n \cap \{i \mid i \geqslant n\}.$$

Since F is free, $X_n \in F$. Further $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ and $\bigcap_n X_n = \varnothing$.

For $i \in X_0$ let $n(i) \in \mathbb{N}$ be maximal such that $i \in X_{n(i)}$. Let $a \in \prod_i A_i$ be a function that maps $i \in X_0$ to some $a(i) \in A_i$ such that $\mathfrak{A}_i \vDash \psi_{n(i)}[a(i)]$. Then for all $i \in X_0$

$$X_{n(i)} \subseteq \big\{ j \mid \mathfrak{A}_j \vDash \psi_{n(i)}[a(j)] \big\}.$$

Indeed: if $j \in X_{n(i)}$ then $n(j) \geqslant n(i)$, so $\mathfrak{A}_j \vDash \psi_{n(j)}[a(j)]$ implies $\mathfrak{A}_j \vDash \psi_{n(i)}[a(j)]$.

Given $n \in \mathbb{N}$ we have to show $\mathfrak{A} \vDash \varphi_n[a^F]$, that is, by Łos, $\{j \mid \mathfrak{A}_j \vDash \varphi_n[a(j)]\} \in F$. By the above, it suffices to show that there exists $i \in X_0$ such that $n \leqslant n(i)$. But if $n(i) < n$ for all $i \in X_0$, then $i \notin X_{n(i)+1} \supseteq X_n$, so $X_n = \varnothing$, contradicting $X_n \in F$. □

5.2 Homogeneity

Definition 5.16. Let $\mathfrak{A}, \mathfrak{B}$ be L-structures. A partial function f from A to B is *elementary* if $\mathrm{tp}_{\mathfrak{A}}(\bar{a}) = \mathrm{tp}_{\mathfrak{B}}(f(\bar{a}))$ for all tuples $\bar{a}$ from $dom(f)$.

Let $I_e(\mathfrak{A}, \mathfrak{B})$ be the set of finite such functions, that is,

$$I_e(\mathfrak{A}, \mathfrak{B}) = \big\{ \bar{a} \mapsto \bar{b} \mid \mathrm{tp}_{\mathfrak{A}}(\bar{a}) = \mathrm{tp}_{\mathfrak{B}}(\bar{b}) \big\}.$$

$\mathfrak{A}$ is $\aleph_0$-*homogenous* if $I_e(\mathfrak{A}, \mathfrak{A}) : \mathfrak{A} \cong_p \mathfrak{A}$, equivalently, $I_e(\mathfrak{A}, \mathfrak{A})$ satisfies (Forth).

Remark 5.17.

1. $I_e(\mathfrak{A},\mathfrak{B}) \neq \varnothing$ if and only if $\varnothing \in I_e(\mathfrak{A},\mathfrak{B})$, if and only if $\mathfrak{A} \equiv \mathfrak{B}$.
2. If $\mathfrak{B}$ is $\aleph_0$-saturated, then $I_e(\mathfrak{A},\mathfrak{B})$ has (Forth). In particular, $\aleph_0$-saturated structures are $\aleph_0$-homogenous.

   Indeed: given $\bar{a} \mapsto \bar{b} \in I_e(\mathfrak{A},\mathfrak{B})$ and $a \in A$, then $\mathrm{tp}_{(\mathfrak{A},\bar{a})}(a)$ is a 1-type of $(\mathfrak{B},\bar{b})$. Let $b \in B$ realize it. Then $(\mathfrak{A},\bar{a}a) \equiv (\mathfrak{B},\bar{b}b)$, i.e., $\bar{a}a \mapsto \bar{b}b \in I_e(\mathfrak{A},\mathfrak{B})$.
3. If $\mathfrak{A},\mathfrak{B}$ are $\aleph_0$-saturated, then: $\mathfrak{A} \equiv \mathfrak{B} \iff \mathfrak{A} \cong_p \mathfrak{B}$.

   Indeed, $\Rightarrow$ follows from the previous two points, and $\Leftarrow$ is Theorem 3.9.
4. If $\mathfrak{A},\mathfrak{B}$ realize the same types and $\mathfrak{B}$ is $\aleph_0$-homogenous, $I_e(\mathfrak{A},\mathfrak{B}) \neq \varnothing$ has (Forth).

   Indeed, $\varnothing \in I_e(\mathfrak{A},\mathfrak{B})$ as $\mathfrak{A},\mathfrak{B}$ realize the same 0-types. Let $\mathrm{tp}_{\mathfrak{A}}(\bar{a}) = \mathrm{tp}_{\mathfrak{B}}(\bar{b})$ and $a \in A$. Choose $\bar{c}c$ from B with $\mathrm{tp}_{\mathfrak{B}}(\bar{c}c) = \mathrm{tp}_{\mathfrak{A}}(\bar{a}a)$. Then $\mathrm{tp}_{\mathfrak{B}}(\bar{c}) = \mathrm{tp}_{\mathfrak{B}}(\bar{b})$. Since $\mathfrak{B}$ is $\aleph_0$-homogeneous, there is $b \in B$ such that $\mathrm{tp}_{\mathfrak{B}}(\bar{b}b) = \mathrm{tp}_{\mathfrak{B}}(\bar{c}c) = \mathrm{tp}_{\mathfrak{A}}(\bar{a}a)$.
5. If $\mathfrak{A},\mathfrak{B}$ are $\aleph_0$-homogenous and realize the same types, then $\mathfrak{A} \cong_p \mathfrak{B}$.

Corollary 5.18. *A consistent L-theory T is complete if and only if any two $\aleph_0$-saturated models of T are partially isomorphic.*

Proof. $\Rightarrow$: by Remark 5.17 (3). $\Leftarrow$: given $\mathfrak{A},\mathfrak{B} \vDash T$, choose $\aleph_0$-saturated elementary extensions $\mathfrak{A}^*,\mathfrak{B}^*$. Then $\mathfrak{A}^* \cong_p \mathfrak{B}^*$ by assumption. By Theorem 3.9, $\mathfrak{A} \equiv \mathfrak{A}^* \equiv \mathfrak{B}^* \equiv \mathfrak{B}$. □

Corollary 5.19. *Assume $\mathfrak{A}$ is countable and $\aleph_0$-homogenous. If $\bar{a},\bar{b}$ are tuples from $\mathfrak{A}$ with $\mathrm{tp}_A(\bar{a}) = \mathrm{tp}_{\mathfrak{A}}(b)$, then there is an automorphism π of $\mathfrak{A}$ with $\pi(\bar{a}) = \bar{b}$.*

Proof. By Remark 5.17 (4), $\bar{a} \mapsto \bar{b} \in I_e(\mathfrak{A},\mathfrak{A}) : \mathfrak{A} \cong_p \mathfrak{A}$. Apply Theorem 3.8. □

The following clarifies the relationship between saturation and homogeneity.

Proposition 5.20. *For an L-structure $\mathfrak{A}$, the following are equivalent.*

1. *$\mathfrak{A}$ is $\aleph_0$-saturated.*
2. *$\mathfrak{A}$ is $\aleph_0$-homogenous and* weakly saturated*: it realizes all its types.*
3. *$\mathfrak{A}$ is $\aleph_0$-homogenous and* $\aleph_0$-universal*: $\mathfrak{B} \to_e \mathfrak{A}$ for every countable $\mathfrak{B} \equiv \mathfrak{A}$.*

Proof. $1 \Rightarrow 3$: by Remark 5.17 (2), $\mathfrak{A}$ is $\aleph_0$-homogenous. Let $\mathfrak{B} \equiv \mathfrak{A}$ be countable, say listed by $b_0, b_1, \ldots$. By Remark 5.17 (1) and (2), $\varnothing \in I_e(\mathfrak{B},\mathfrak{A})$ has (Forth). Choose successively $a_0, a_1, \ldots$ in A such that $b_0 \mapsto a_0, b_0b_1 \mapsto a_0a_1, \ldots \in I_(\mathfrak{B},\mathfrak{A})$. Then $\mathfrak{B} \to_e \mathfrak{A}$ via $b_i \mapsto a_i$.

$3 \Rightarrow 2$: let $p(\bar{x})$ be a type of $\mathfrak{A}$. Choose a countable $\mathfrak{B} \equiv \mathfrak{A}$ realizing $p(\bar{x})$, say by $\bar{b}$. Choose $\pi : \mathfrak{B} \to_e \mathfrak{A}$. Then $\pi(\bar{b})$ realizes $p(\bar{x})$ in $\mathfrak{A}$.

$2 \Rightarrow 1$: let $p(x)$ be a 1-type of $(\mathfrak{A},\bar{a})$, say realized in $(\mathfrak{B},\bar{a}) \succcurlyeq (\mathfrak{A},\bar{a})$ by b. Choose $\bar{c}c \in A$ realizing $\mathrm{tp}_{\mathfrak{B}}(\bar{a}b)$ in $\mathfrak{A}$. Then $\bar{a} \mapsto \bar{c} \in I_e(\mathfrak{A},\mathfrak{A})$. Use (Back) to get $a \in A$ such that $\bar{a}a \mapsto \bar{c}c \in I_e(\mathfrak{A},\mathfrak{A})$. Then $\bar{a}a$ realizes $\mathrm{tp}_{\mathfrak{B}}(\bar{a}b)$ in $\mathfrak{A}$, so a realizes $p(x)$ in $(\mathfrak{A},\bar{a})$. □

Lemma 5.21. *Let L_0, L_1 be languages and $L = L_0 \cap L_1$. Let $\mathfrak{A}$ be a countable L_0-structure, and $\mathfrak{B}$ be a countable L_1-structure such that $\mathfrak{A}\upharpoonright L \equiv \mathfrak{B}\upharpoonright L$.*

Then there exist countable elementary extensions $\mathfrak{A}^* \succcurlyeq \mathfrak{A}$ and $\mathfrak{B}^* \succcurlyeq \mathfrak{B}$ such that both $\mathfrak{A}^*\upharpoonright L$ and $\mathfrak{B}^*\upharpoonright L$ are $\aleph_0$-homogenous and $\mathfrak{A}^*\upharpoonright L$ and $\mathfrak{B}^*\upharpoonright L$ are isomorphic.

Proof. Set $\mathfrak{A}_0 := \mathfrak{A}, \mathfrak{B}_0 := \mathfrak{B}$. We construct elementary chains $\mathfrak{A}_0 \preccurlyeq \mathfrak{A}_1 \preccurlyeq \mathfrak{A}_2 \preccurlyeq \cdots$ and $\mathfrak{B}_0 \preccurlyeq \mathfrak{B}_1 \preccurlyeq \mathfrak{B}_2 \preccurlyeq \cdots$ such that for all $n \in \mathbb{N}$:

1. A_n and B_n are countable;
2. (a) for all $k \in \mathbb{N}$ and $\bar{a}, \bar{a}' \in A_n^k$ and all $a \in A_n$, if $\mathrm{tp}_{\mathfrak{A}_n\upharpoonright L}(\bar{a}) = \mathrm{tp}_{\mathfrak{A}_n\upharpoonright L}(\bar{a}')$, then there exists $a' \in A_{n+1}$ such that $\mathrm{tp}_{\mathfrak{A}_{n+1}\upharpoonright L}(\bar{a}a) = \mathrm{tp}_{\mathfrak{A}_{n+1}\upharpoonright L}(\bar{a}'a')$;
   (b) every type realized in $\mathfrak{B}_n\upharpoonright L$, is realized in $\mathfrak{A}_{n+1}$;
3. (a) for all $k \in \mathbb{N}$ and $\bar{b}, \bar{b}' \in B_n^k$ and all $b \in B_n$, if $\mathrm{tp}_{\mathfrak{B}_n\upharpoonright L}(\bar{b}) = \mathrm{tp}_{\mathfrak{B}_n\upharpoonright L}(\bar{b}')$, then there exists $b' \in B_{n+1}$ such that $\mathrm{tp}_{\mathfrak{B}_{n+1}\upharpoonright L}(\bar{b}b) = \mathrm{tp}_{\mathfrak{B}_{n+1}\upharpoonright L}(\bar{b}'b')$;
   (b) every type realized in $\mathfrak{A}_n\upharpoonright L$, is realized in $\mathfrak{B}_{n+1}$.

Having $\mathfrak{A}_n, \mathfrak{B}_n$, we construct $\mathfrak{A}_{n+1}$. In 2a we want $(\mathfrak{A}_{n+1}, \bar{a}')$ realize $\mathrm{tp}_{(\mathfrak{A}_n\upharpoonright L,\bar{a})}(a)$: this is a type of $(\mathfrak{A}_n, \bar{a}')$ because $(\mathfrak{A}_n\upharpoonright L, \bar{a}) \equiv (\mathfrak{A}_n\upharpoonright L, \bar{a}')$. In 2b we want $\mathfrak{A}_{n+1}$ realize $\mathrm{tp}_{\mathfrak{B}_n\upharpoonright L}(\bar{b})$ for $\bar{b}$ from B_n: this is a type of $\mathfrak{A}_n$ because $\mathfrak{A}_n\upharpoonright L \equiv \mathfrak{A}\upharpoonright L \equiv \mathfrak{B}\upharpoonright L \equiv \mathfrak{B}_n\upharpoonright L$.

All together, these are countably many (partial) types of $(\mathfrak{A}_n)_{A_n}$. Lemma 5.4 gives a countable $\mathfrak{C} \succcurlyeq (\mathfrak{A}_n)_{A_n}$ realizing them all. Set $\mathfrak{A}_{n+1} := \mathfrak{C}\upharpoonright L_0$.

The construction $\mathfrak{B}_{n+1}$ is analogous. Given the chains, set $\mathfrak{A}^* := \bigcup_n \mathfrak{A}_n, \mathfrak{B}^* := \bigcup_n \mathfrak{B}_n$. By Tarski's lemma, $\mathfrak{A} = \mathfrak{A}_0 \preccurlyeq \mathfrak{A}^*$. We claim $\mathfrak{A}^*\upharpoonright L$ is $\aleph_0$-homogenous. Let $\bar{a} \mapsto \bar{a}' \in I_e(\mathfrak{A}^*\upharpoonright L, \mathfrak{A}^*\upharpoonright L)$ and $a \in A$. Choose $n \in \mathbb{N}$ such that $\bar{a}, \bar{a}', a$ are from A_n. By Tarski's lemma, $\mathrm{tp}_{\mathfrak{A}_n\upharpoonright L}(\bar{a}) = \mathrm{tp}_{\mathfrak{A}_n\upharpoonright L}(\bar{a}')$. By 2a there is $a' \in A_{n+1}$ such that $\mathrm{tp}_{\mathfrak{A}_{n+1}\upharpoonright L}(\bar{a}a) = \mathrm{tp}_{\mathfrak{A}_{n+1}\upharpoonright L}(\bar{a}'a')$. By Tarski's lemma, $\mathrm{tp}_{\mathfrak{A}^*\upharpoonright L}(\bar{a}a) = \mathrm{tp}_{\mathfrak{A}^*\upharpoonright L}(\bar{a}'a')$, i.e., $\bar{a}a \mapsto \bar{a}'a' \in I_e(\mathfrak{A}^*\upharpoonright L, \mathfrak{A}^*\upharpoonright L)$.

We claim $\mathfrak{A}^*$ realizes $\mathrm{tp}_{\mathfrak{B}^*\upharpoonright L}(\bar{b})$ for every $\bar{b}$ from B^*: choose $n \in \mathbb{N}$ such that $\bar{b}$ is from B_n. Since $\mathrm{tp}_{\mathfrak{B}_n\upharpoonright L}(\bar{b}) = \mathrm{tp}_{\mathfrak{B}^*\upharpoonright L}(\bar{b})$, by 2b, this type is realized in $\mathfrak{A}_{n+1}$, so also in $\mathfrak{A}^*$.

Analogously, $\mathfrak{B}^*\upharpoonright L$ is $\aleph_0$-homogenous and realizes all types realized in $\mathfrak{A}_n\upharpoonright L$. Then $\mathfrak{A}^*\upharpoonright L \cong_p \mathfrak{B}^*\upharpoonright L$ by Remark 5.17 (5). Thus $\mathfrak{A}^*\upharpoonright L \cong \mathfrak{B}^*\upharpoonright L$ by Theorem 3.8. ☐

Not all countable structures have $\aleph_0$-saturated elementary extensions (see Exercise 5.10). In contrast (for possibly uncountable L):

Corollary 5.22. *Every countable L-structure has a countable $\aleph_0$-homogenous elementary extension.*

We can now give an algebraic characterization of type equality.

Theorem 5.23. *Assume $\mathfrak{A}$ is infinite and let $\bar{a}, \bar{b}$ be tuples from $\mathfrak{A}$. Then $\mathrm{tp}_{\mathfrak{A}}(\bar{a}) = \mathrm{tp}_{\mathfrak{A}}(\bar{b})$ if and only if there are $\mathfrak{B} \succcurlyeq \mathfrak{A}$ and an automorphism π of $\mathfrak{B}$ such that $\pi(\bar{a}) = \bar{b}$.*

Proof. $\Leftarrow$ is clear. $\Rightarrow$: for a unary function symbol f define the theory

$$T := Th(\mathfrak{A}, \bar{a}\bar{b}) \cup \text{“}f \text{ is an automorphism wrt } L \text{ and maps } \bar{c} \text{ to } \bar{d}\text{”},$$

where $\bar{c}, \bar{d}$ are the constants for $\bar{a}, \bar{b}$ in $(\mathfrak{A}, \bar{a}\bar{b})$ By Lemma 4.30 it suffices to show that T is consistent. By compactness, we can assume that L is finite. Then $Th(\mathfrak{A}, \bar{a}\bar{b})$ has a countable model $(\mathfrak{A}', \bar{a}'\bar{b}')$. Then $\mathrm{tp}_{\mathfrak{A}'}(\bar{a}') = \mathrm{tp}_{\mathfrak{A}'}(\bar{b}')$ because $Th(\mathfrak{A}, \bar{a}\bar{b})$ contains $(\varphi(\bar{c}) \leftrightarrow \varphi(\bar{d}))$ for every L-formula $\varphi(\bar{x})$. By Corollary 5.22 we can assume that $\mathfrak{A}'$ is $\aleph_0$-homogenous. By Corollary 5.19, there is an automorphism π of $\mathfrak{A}'$ with $\pi(\bar{a}') = \bar{b}'$. Then the expansion of $(\mathfrak{A}', \bar{a}'\bar{b}')$ that interprets f by π models T. □

Exercise 5.24. Let κ be an infinite cardinal and assume $\mathfrak{A}, \mathfrak{B}$ are κ-saturated. Let I be the set of partial elementary functions f with $|f| < \kappa$. Then $I : \mathfrak{A} \cong_p \mathfrak{B}$ or $I = \emptyset$. If additionally $\mathfrak{A} \equiv \mathfrak{B}$ and $|A|, |B| = \kappa$, then $\mathfrak{A} \cong \mathfrak{B}$.

Exercise 5.25 (Uniqueness of ultrapowers)**.** Assume the continuum hypothesis. Assume L is countable and $\mathfrak{A}$ has cardinality $\aleph_1$. Let F, F' be free ultrafilters on $\mathbb{N}$. Then $\mathfrak{A}^{\mathbb{N}}_F \cong \mathfrak{A}^{\mathbb{N}}_{F'}$.

5.3 Omitting types

Which types of a complete theory can be omitted? Trivially, finite types cannot be omitted, or, similarly, types that are finitely axiomatized over the theory cannot be omitted. We shall see that this is the only obstacle. Let T be an L-theory.

Definition 5.26. A type $p(\bar{x})$ of T is *principal* if there exists an L-formula $\psi(\bar{x})$ such that $T \cup \{\psi(\bar{x})\}$ is consistent and $T \vdash (\psi(\bar{x}) \to \varphi(\bar{x}))$ for all $\varphi(\bar{x}) \in p(\bar{x})$.

A type is *free* if it is not principal.

Exercise 5.27. The bijection from complete n-types onto ultrafilters in $\mathfrak{L}_n(T)$ of Exercise 5.3 maps the principal complete types of T onto the principal ultrafilters in $\mathfrak{L}_n(T)$.

Proposition 5.28. *If T is complete, every model of T realizes all principal types of T.*

Proof. Let $\mathfrak{A} \vDash T$ and let $p(\bar{x})$ a principal type, say witnessed by $\psi(\bar{x})$. Since T is complete and consistent with $\exists\bar{x}\psi(\bar{x})$, it proves it, so there is $\bar{a}$ in A such that $\mathfrak{A} \vDash \psi[\bar{a}]$. Then $\mathfrak{A} \vDash \varphi[\bar{a}]$ for all $\varphi(\bar{x}) \in p(\bar{x})$, so $\bar{a}$ realizes $p(\bar{x})$. □

Theorem 5.29 (Omitting types)**.** *Assume L is at most countable and P is an at most countable set of free types of T. Then there exists a model of T that omits all types in P.*

Proof. We first consider the case that $P = \{p(\bar{x})\}$ for an n-type $p(\bar{x})$ of T. Let $L' := L \cup C$ where C is a countable set of new constants. Let $\varphi_0, \varphi_1, \ldots$ list all L'-sentences, and let $\bar{c}_0, \bar{c}_1, \ldots$ list C^n. We construct a sequence of L'-sentences $\chi_0, \chi_1, \ldots$ such that $(\chi_{i+1} \to \chi_i)$ is valid and $T \cup \{\chi_i\}$ is consistent for all $i \in \mathbb{N}$. Set $\chi_0 := \forall x\, x = x$ and assume χ_{2i} is defined.

We define χ_{2i+1}: set $\chi'_{2i+1} := (\chi_i \wedge \varphi_i)$ if this is consistent with $T \cup \{\chi_{2i}\}$, and otherwise $\chi'_{2i+1} := \chi_i$. In the first case and if $\varphi_i = \exists x\psi(x)$ for some L'-formula $\psi(x)$, set $\chi_{2i+1} := (\chi'_{2i+1} \wedge \psi(c))$ for some constant $c \in C$ that does not appear in χ'_{2i}; otherwise $\chi_{2i+1} := \chi'_{2i+1}$. It is straightforward to check that $T \cup \{\chi_{2i+1}\}$ is consistent.

We define χ_{2i+2}: write $\chi_{2i+1} = \chi(\bar{c}_i, \bar{c})$ for $\chi(\bar{x}, \bar{y})$ an L-formula, and $\bar{c}$ constants from C distinct from $\bar{c}_i$. Clearly, $T \cup \{\exists\bar{y}\chi(\bar{x}, \bar{y})\}$ is consistent. Since $p(\bar{x})$ is free there is $\varphi(\bar{x}) \in p(\bar{x})$ such that $T \cup \{\exists\bar{y}\chi(\bar{x}, \bar{y}), \neg\varphi(\bar{x})\}$ is consistent. We set $\chi_{2i+2} := (\chi_{2i+1} \wedge \neg\varphi(\bar{c}_i))$. Again, it is straightforward to check that $T \cup \{\chi_{2i+1}\}$ is consistent.

Let $\mathfrak{A}$ be a model of $T' := T \cup \{\chi_i \mid i \in \mathbb{N}\}$, and consider $A_0 := \{c^{\mathfrak{A}} \mid c \in C\} \subseteq A$. We use Tarski's test to verify A_0 is the universe of an elementary substructure $\mathfrak{A}_0 \preccurlyeq \mathfrak{A}$. Since every element of A_0 interprets a constant, we can restrict attention to L'-formulas. So let $a \in A$ satisfy the L'-formula $\psi(x)$ in $\mathfrak{A}$; say $\exists x\psi(x) = \varphi_i$; then φ_i is consistent with T'. But T' proves χ_{2i+1} and hence $\psi(c)$ for some $c \in C$. Thus, $\mathfrak{A} \models \psi[c^{\mathfrak{A}}]$ and $c^{\mathfrak{A}} \in A_0$.

Thus, $\mathfrak{A}_0 \models T'$. We claim $\mathfrak{A}_0$ omits $p(\bar{x})$. Otherwise, there is $j \in \mathbb{N}$ such that $\bar{c}_j^{\mathfrak{A}_0}$ realizes $p(\bar{x})$ in $\mathfrak{A}_0$. But T' proves χ_{2j+2} and hence $\neg\varphi(\bar{c}_j)$ for some $\varphi(\bar{x}) \in p(\bar{x})$.

This finishes the proof for singleton P. For $P = \{p_0, p_1, \ldots\}$ proceed similarly using for each $i \in \mathbb{N}$ a list of tuples $\bar{c}_0^i, \bar{c}_1^i, \ldots$ from C, of length according to p_i. Modify χ_{2i+2}: add $\neg\varphi(\bar{c}_{i_1}^{i_0})$ for some $\varphi(\bar{x}) \in p_{i_0}$: here, $i \mapsto (i_0, i_1)$ is some surjection from $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$. □

Exercise 5.30. If $p(x)$ is a free type of $\mathfrak{A}$, then there is an elementary extension of $\mathfrak{A}$ with infinitely many realizations of $p(x)$.

Hint: it suffices to show that $Th(\mathfrak{A}) \cup \{p(c_i) \mid i \in \mathbb{N}\} \cup \{\neg c_i = c_j \mid i \neq j\}$ is consistent. To this end show that every $\varphi(x) \in p(\bar{x})$ is contained in infinitely many types of $\mathfrak{A}$.

5.3.1 McDowell-Specker

Let $L = L_{Ring} \cup \{<\}$. *Peano arithmetic PA* is the L-theory given by some base theory (like Robinson's Q) and induction for all formulas, i.e., the universal closure of

$$\varphi(0, \bar{x}) \wedge \forall y(\varphi(y, \bar{x}) \to \varphi(y+1, \bar{x})) \ \to \varphi(x, \bar{x}).$$

We write $\exists x{<}y\ \varphi$ for $\exists x(x < y \wedge \varphi)$ and $\exists^\infty x\varphi$ for $\forall u\exists x(u < x \wedge \varphi)$ for some u not free in φ. It is known that PA proves the following version of the pigeonhole principle for every formula $\varphi(x, y, \bar{x})$:

$$\exists^\infty x\exists y{<}z\ \varphi(x, y, \bar{x}) \ \to \exists y{<}z\exists^\infty x\varphi(x, y, \bar{x}).$$

Theorem 5.31 (McDowell-Specker)**.** *Every countable model $\mathfrak{A}$ of PA has a proper elementary end extension, i.e., $\mathfrak{B} \succcurlyeq \mathfrak{A}$ such that $B \setminus A \neq \emptyset$ and $a <^{\mathfrak{B}} b$ for $a \in A, b \in B \setminus A$.*

Proof via omitting types. Let c be a new constant, and consider the $L(A) \cup \{c\}$-theory $T := D_e(\mathfrak{A}) \cup \{c_a < c \mid a \in A\}$. By Lemma 4.30 its models have L-reducts isomorphic to proper elementary extensions of $\mathfrak{A}$. It thus suffices to find a model of T that omits

$$p_a(x) := \{x < c_a\} \cup \{\neg c_{a'} = x \mid a' \in A\}.$$

for every $a \in A$. Hence, it suffices to show that $p_a(x)$ is free. Assume it is principal, say witnessed by $\psi(x, c)$ where $\psi(x, y)$ is an $L(A)$-formula. Then T proves, for all $a' \in A$

$$(\psi(x, c) \to x < c_a), \quad (\psi(x, c) \to \neg x = c_{a'}).$$

Since $T \cup \{\psi(x,c)\}$ is consistent, there is $\mathfrak{B} \models T$ and $b \in B$ satisfying $\psi(x,c)$ in $\mathfrak{B}$. We can assume $\mathfrak{A} \preccurlyeq \mathfrak{B}$. Since $\mathfrak{B} \models \exists y{>}c_{a'} \exists x{<}c_a\ \psi(x,y)$ for all $a' \in A$, we have $\mathfrak{A} \models \forall z \exists y{>}z \exists x{<}c_a\ \psi(x,y)$, that is, $\mathfrak{A} \models \exists^\infty y \exists x{<}c_a\ \psi(x,y)$. By the pigeonhole principle, $\mathfrak{A} \models \exists x{<}c_a \exists^\infty y\ \psi(x,y)$. Choose $a_0 <^{\mathfrak{A}} a$ such that $\mathfrak{A} \models \exists^\infty y\ \psi(c_{a_0}, y)$.

We get the desired contradiction, showing that $T \cup \{\psi(c_{a_0}, c)\}$ is consistent. Otherwise, by compactness, there is $a_1 \in A$ such that $D_e(\mathfrak{A}) \cup \{c_{a_1} < c\} \vdash \neg\psi(c_{a_0}, c)$. Then $D_e(\mathfrak{A}) \vdash (y > c_{a_1} \to \neg\psi(c_{a_0}, y))$, contradicting the choice of a_0. □

Proof via definable ultrapowers. Recall Exercise 4.33. It is straightforward to check that all models of PA have definable Skolem functions. Let $\mathfrak{B}$ be the definable ultrapower of $\mathfrak{A}$ modulo F (chosen below). $\mathfrak{A}$ is isomorphic to the elementary substructure of $\mathfrak{B}$ with universe $\{f_a^F \mid a \in A\}$; here, f_a is the function constantly a. It suffices to find an ultrafilter F on A such that for all $a \in A$ and all $b <^{\mathfrak{B}} f_a^F$ there is $a' \in A$ such that $b = f_{a'}^F$.

Let D denote the set of functions from A to A that are definable in $\mathfrak{A}_A$. Note $D \times A$ is countable. Let $(f_0, a_0), (f_1, a_1), \dots$ list $D \times A$. We set $X_0 := A$ and define unbounded sets $X_0 \supseteq X_1 \supseteq \cdots$, each definable in $\mathfrak{A}_A$. Assume X_i is defined. Consider the function $a \mapsto \min\{f_i(a), a_i\}$. By the pigeonhole principle, it is constant on an unbounded subset of X_i, say equal to $\tilde{a}_i \leqslant^{\mathfrak{A}} a_i$. Set $X_{i+1} := \{a \in X_i \mid f_i(a) = \tilde{a}_i\}$.

We choose for F an ultrafilter containing every X_i and verify our claim above. Assume $b <^{\mathfrak{B}} f_a^F$ for $b \in B, a \in A$. Say $b = f^F$ for $f \in D$. By Łos, $X := \{a \mid f(a) <^{\mathfrak{A}} a\} \in F$. Then $a \neq 0$. Choose $i \in \mathbb{N}$ such that $f_i = f$ and $a_i +^{\mathfrak{A}} 1 = a$. Then $\tilde{a}_i <^{\mathfrak{A}} a$ and $X \cap X_i \subseteq \{a \mid f(a) = f_{\tilde{a}_i}(a)\}$. Hence, $f^F = f_{\tilde{a}_i}^F$. □

5.4 Countable models

Assume L is countable and T is a complete L-theory with an infinite model, and hence, without finite models. This section proves three results relating properties of the Lindenbaum algebras $\mathfrak{L}_n(T)$ to (special) countable models of T. In particular, we ask for 'large' countable models realizing many types ($\aleph_0$-saturated ones), and 'small' countable models realizing as few as possible – these are the following:

Lemma 5.32. *For every L-structure $\mathfrak{A}$ the following are equivalent.*

1. *$\mathfrak{A}$ is* prime*: $\mathfrak{A} \to_e \mathfrak{B}$ for every $\mathfrak{B} \equiv \mathfrak{A}$.*
2. *$\mathfrak{A}$ is countable and* atomic*: every type realized in $\mathfrak{A}$ is principal.*

Proof. $1 \Rightarrow 2$: by Löwenheim-Skolem-Tarski, $Th(\mathfrak{A})$ has countable models, so $\mathfrak{A}$ is countable. If $\bar{a}$ realizes a free type $p(\bar{x})$ in $\mathfrak{A}$, Omitting Types gives a countable $\mathfrak{B} \equiv \mathfrak{A}$ that omits $p(\bar{x})$. Then $\pi : \mathfrak{A} \to_e \mathfrak{B}$ would imply that $\pi(\bar{a})$ realizes $p(\bar{x})$ in $\mathfrak{B}$, so $\mathfrak{A} \not\to_e \mathfrak{B}$.

$2 \Rightarrow 1$: let $\mathfrak{B} \equiv \mathfrak{A}$. We claim that $I_e(\mathfrak{A}, \mathfrak{B})$ has (Forth). Let $\bar{a} \mapsto \bar{b} \in I_e(\mathfrak{A}, \mathfrak{B})$ and $a \in A$. Then $(\mathfrak{A}, \bar{a}) \equiv (\mathfrak{B}, \bar{b})$, so $\mathrm{tp}_{(\mathfrak{A},\bar{a})}(a)$ is a type of $(\mathfrak{B}, \bar{b})$. It is principal because $\mathrm{tp}_{\mathfrak{A}}(\bar{a}a)$ is. By Proposition 5.28, it is realized in $(\mathfrak{B}, \bar{b})$, say by $b \in B$. Then $\bar{a}a \mapsto \bar{b}b \in I_e(\mathfrak{A}, \mathfrak{B})$.

Let $a_0, a_1, \dots$ list A; using (Forth) choose successively $b_0, b_1, \dots$ in B such that $a_0 \mapsto b_0, a_0a_1 \mapsto b_0b_1, \dots \in I_e(\mathfrak{A}, \mathfrak{B})$; then $\mathfrak{A} \to_e \mathfrak{B}$ for π via $a_i \mapsto b_i$. □

Corollary 5.33. *T has at most one prime model up to isomorphism.*

Proof. Let $\mathfrak{A}, \mathfrak{B} \models T$ be countable and atomic. By the proof above $I_e(\mathfrak{A}, \mathfrak{B}) : \mathfrak{A} \cong_p \mathfrak{B}$. Hence $\mathfrak{A} \cong \mathfrak{B}$ by Theorem 3.8. □

Theorem 5.34. *T has a prime model if and only if for all $n \in \mathbb{N}$, $\mathfrak{L}_n(T)$ is atomic.*

Proof. Let $\mathfrak{A} \models T$ be prime, so countable and atomic, and $\varphi(\bar{x})/T$ be non-zero in $\mathfrak{L}_n(T)$. Then $T \cup \{\exists \bar{x} \varphi(\bar{x})\}$ is consistent. Since T is complete, $T \vdash \exists \bar{x} \varphi(\bar{x})$, so $\mathfrak{A} \models \varphi[\bar{a}]$ for some $\bar{a} \in A^n$. Since $p(\bar{x}) \coloneqq \mathrm{tp}_{\mathfrak{A}}(\bar{a})$ is principal, so is the ultrafilter F_p in $\mathfrak{L}_n(T)$ (see Exercise 5.27). Let $\psi(\bar{x})/T$ be the atom determining it. Then $\psi/T \leqslant \varphi/T$.

Conversely, we show there is a countable atomic model of T. For $n \in \mathbb{N}$ such that $\mathfrak{L}_n(T)$ is infinite, let F_n be a free ultrafilter (Exercise 2.26). Then F_n contains $\neg\varphi/T$ for all atoms φ/T of $\mathfrak{L}_n(T)$. Then the set $p_n(\bar{x})$ of $\neg\varphi(\bar{x})$ for all atoms φ/T is a subset of the (complete) type corresponding to F_n according to Exercise 5.3, so is a (partial) n-type of T. We show it is is free. Let $\psi(\bar{x})$ be an L-formula consistent with T, i.e., $\psi/T \neq 0$. Since $\mathfrak{L}_n(T)$ is atomic, there is an atom $\varphi(\bar{x})/T \leqslant \psi(\bar{x})/T$, i.e., $T \vdash \varphi(\bar{x}) \to \psi(\bar{x})$. Then $T \nvdash \psi(\bar{x}) \to \neg\varphi(\bar{x})$ as otherwise $\varphi(\bar{x})/T = 0$, contradicting $\varphi(\bar{x})/T$ being an atom.

By Omitting Types there is $\mathfrak{A} \models T$ that omits all these $p_n(\bar{x})$. We can assume $\mathfrak{A}$ is countable. To see it is atomic, let $\bar{a} \in A^n$. If $\mathfrak{L}_n(T)$ is finite, all ultrafilters are principal, so $\mathrm{tp}_{\mathfrak{A}}(\bar{a})$ is principal. If $\mathfrak{L}_n(T)$ is infinite, then $\mathfrak{A}$ omits $p_n(\bar{x})$, so $\mathfrak{A} \models \varphi[\bar{a}]$ for some atom $\varphi(\bar{x})/T$ of $\mathfrak{L}_n(T)$, so $\mathrm{tp}_{\mathfrak{A}}(\bar{a})$ is principal. □

Theorem 5.35. *T has a countable $\aleph_0$-saturated model if and only if for all $n \in \mathbb{N}$, $\mathfrak{L}_n(T)$ has at most countably many ultrafilters.*

Proof. $\Rightarrow$: by Lemma 5.11 and Exercise 5.3. $\Leftarrow$: by Exercise 5.3 T has at most countably many (n-)types (any n). By Lemma 5.4 there is a countable $\mathfrak{A} \models T$ that realizes all types of $T = Th(\mathfrak{A})$ (recall T is complete). By Corollary 5.22 there is a countable $\mathfrak{B} \succcurlyeq \mathfrak{A}$ that is $\aleph_0$-homogenous. Note $\mathfrak{B}$ realizes all its types. By Proposition 5.20, $\mathfrak{B}$ is $\aleph_0$-saturated. □

By Corollary 5.18 and Theorem 3.8, T has at most one countable $\aleph_0$-saturated model up to isomorphism. By Exercise 2.27 the previous two results imply:

Corollary 5.36. *If T has a countable $\aleph_0$-saturated model, then it has a prime model.*

5.4.1 Ryll-Nardzewski

Definition 5.37. Let κ be an infinite cardinal. Then T is *κ-categorical* if any two models of T of size κ are isomorphic.

Examples 5.38. By Corollaries 3.12 and 3.16 the theory of dense linear orders without endpoints and the theory of atomless Boolean algebras are $\aleph_0$-categorical. By Theorem 3.46, the theory of the random graph is $\aleph_0$-categorical.

Theorem 5.39 (Ryll-Nardzewski)**.** *The following are equivalent.*

1. *T is $\aleph_0$-categorical.*
2. *$\mathfrak{L}_n(T)$ is finite for all $n \in \mathbb{N}$.*
3. *All models of T are $\aleph_0$-saturated.*

Proof. $1 \Rightarrow 2$: if $\mathfrak{L}_n(T)$ is infinite, then, by Exercise 2.26, there is a free ultrafilter in $\mathfrak{L}_n(T)$. By Exercises 5.3 and 5.27, T has a free n-type $p(\bar{x})$. By Omitting Types there is a countable model of T that omits $p(\bar{x})$. By Lemma 5.4 there is a countable model of T that realizes $p(\bar{x})$. These two models are not isomorphic, so T is not $\aleph_0$-categorical.

$2 \Rightarrow 3$: let $\mathfrak{A} \models T$, $m \in \mathbb{N}$, $\bar{a} \in A^m$ and $p(x)$ be a 1-type of $(\mathfrak{A}, \bar{a})$. By Proposition 5.28, it suffices to show $p(x)$ is principal. Let $M \in \mathbb{N}$ be the size of $\mathfrak{L}_{m+1}(T)$, that is, up to T-provable equivalence, there are M pairwise non-equivalent L-formulas $\varphi(y_0, \ldots, y_{m-1}, x)$. Every formula in $p(x)$ can be written $\varphi(\bar{c}, x)$ for an L-formula $\varphi(\bar{y}, x)$ and $\bar{c}$ the constants for $\bar{a}$. Thus, up to equivalence in $(\mathfrak{A}, \bar{a})$, $p(x)$ contains at most M many formulas. Let $\psi(x)$ be their conjunction. Clearly, $(\mathfrak{A}, \bar{a}) \models \forall x(\psi(x) \to \varphi(x))$ for all $\varphi(x) \in p(x)$.

$3 \Rightarrow 1$: by Corollary 5.18 and Theorem 3.8. □

Corollary 5.40. *Let $\bar{a}$ be a tuple from A. Then $Th(\mathfrak{A})$ is $\aleph_0$-categorical if and only if $Th(\mathfrak{A}, \bar{a})$ is $\aleph_0$-categorical.*

Exercise 5.41. Let A be a set and G a group of permutations of A. G is *oligomorphic* if for all $n \in \mathbb{N}$ the action $(g, \bar{a}) \mapsto g(\bar{a})$ of G on A^n has finitely many orbits.

Show that T is $\aleph_0$-categorical if and only if every countable model of T has an oligomorphic automorphism group.

5.4.2 Vaught's never two

Theorem 5.42 (Vaught)**.** *There is no complete theory in an at most countable language with exactly two countable models, up to isomorphism.*

Proof. Assume T is such a theory, say in the at most countable language L. By Exercise 5.6 it has at most countably many n-types, for all $n \in \mathbb{N}$. By Exercise 5.3, $\mathfrak{L}_n(T)$ has at most countably many ultrafilters. By Theorem 5.35, T has a countable $\aleph_0$-saturated model $\mathfrak{B}$. By Corollary 5.36, T has a prime model $\mathfrak{A}$.

By Ryll-Nardzewski there is $n \in \mathbb{N}$ such that $\mathfrak{L}_n(T)$ is infinite. As seen in the proof, T has a free n-type $p(\bar{x})$. By Lemma 5.32, $p(\bar{x})$ is omitted in $\mathfrak{A}$ and realized in $\mathfrak{B}$, say by $\bar{b} \in B^n$. In particular, $\mathfrak{A} \ncong \mathfrak{B}$. By Corollary 5.40, $T' := Th(\mathfrak{B}, \bar{b})$ is not $\aleph_0$-categorical. By Ryll-Nardzewski, T' has a countable model that is not $\aleph_0$-saturated. Write this model as $(\mathfrak{C}, \bar{c})$ for $\mathfrak{C}$ and L-structure. Then $\mathfrak{C}$ is not $\aleph_0$-saturated, so $\mathfrak{C} \ncong \mathfrak{B}$. Further, $\mathfrak{C} \ncong \mathfrak{A}$ because $\bar{c}$ realizes $p(\bar{x})$ in $\mathfrak{C}$. Thus, $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are pairwise non-isomorphic models of T. □

Exercise 5.43 (Ehrenfeucht's example)**.** Define T to be the theory of dense linear orders without endpoints plus an increasing sequence denoted by constants $c_0, c_1, \ldots$. Show T is complete and has exactly three countable models (up to $\cong$): the sequence is cofinal, or has a supremum, or is not cofinal without supremum.

Chapter 6

Quantifier elimination

Let L be a language and $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be L-structures.

6.1 Craig interpolation

Theorem 6.1 (Craig's interpolation theorem). *Let L_0, L_1 be languages, $\varphi_0(\bar{x})$ an L_0-formula and $\varphi_1(\bar{x})$ an L_1-formula. If the implication $(\varphi_0 \to \varphi_1)$ is valid, then it has an* interpolant*: an $L_0 \cap L_1$-formula $\psi(\bar{x})$ such that both $(\varphi_0 \to \psi)$ and $(\psi \to \varphi_1)$ are valid.*

Proof. Replacing $\bar{x}$ by constants we can assume φ_0, φ_1 are sentences. We can further assume L_0, L_1 are finite. Write $L := L_0 \cap L_1$. Recall Lemma 3.29 and write $\tau^k_{\mathfrak{A}}$ for $\tau^k_{\mathfrak{A},\bar{a}}$ where $\bar{a}$ is the empty tuple. For $k > 0$ set

$$\psi_k \;:=\; \bigvee_{\mathfrak{A} \models \varphi_0} \tau^k_{\mathfrak{A}\restriction L}.$$

where $\mathfrak{A}$ ranges over L_0-structures. Note $(\varphi_0 \to \psi_k)$ is valid. By compactness, it suffices to show $\{\psi_k \mid k > 0\} \vdash \varphi_1$. Let $\mathfrak{B}$ be an L_1-structure satisfying all ψ_k. By Löwenheim-Skolem-Tarski we can assume that $\mathfrak{B}$ is at most countable. We have to show $\mathfrak{B} \models \varphi_1$.

For every $k > 0$ choose an L_0-structure $\mathfrak{A}_k \models \varphi_0$ and $\mathfrak{B} \models \tau^k_{\mathfrak{A}_k\restriction L}$. By Lemmas 3.28 and 3.29, $\mathfrak{B}\restriction L \equiv_k \mathfrak{A}_k\restriction L$. Since $\mathfrak{B} \models \tau^k_{\mathfrak{B}\restriction L}$, we have $\mathfrak{A}_k\restriction L \models \tau^k_{\mathfrak{B}\restriction L}$. But k was arbitrary and $(\psi_k \to \psi_\ell)$ is valid for $k \geqslant \ell > 0$ (because $(\tau^k_{\mathfrak{C}} \to \tau^\ell_{\mathfrak{C}})$ is valid for all $\mathfrak{C}$ by Lemma 3.29). Thus,

$$\{\varphi_0\} \cup \{\tau^k_{\mathfrak{B}\restriction L} \mid k > 0\}$$

is consistent by compactness. Let $\mathfrak{A}$ be an at most countable model of this L_0-theory.

Then $\mathfrak{A}\restriction L \equiv_k \mathfrak{B}\restriction L$ for all $k > 0$, so $\mathfrak{A}\restriction L \equiv \mathfrak{B}\restriction L$ (Remark 3.26 (1)). Then there exist $\mathfrak{A}^* \succcurlyeq \mathfrak{A}$ and $\mathfrak{B}^* \succcurlyeq \mathfrak{B}$ such that $\mathfrak{A}^*\restriction L \cong \mathfrak{B}^*\restriction L$: if $\mathfrak{A}$ is infinite, then so is $\mathfrak{B}$ and we apply Lemma 5.21; if $\mathfrak{A}$ is finite, then $\mathfrak{A}\restriction L \cong \mathfrak{B}\restriction L$ follows from $\mathfrak{A}\restriction L \equiv \mathfrak{B}\restriction L$ (recall Exercise 3.31).

Say, $\pi : \mathfrak{A}^*\restriction L \cong \mathfrak{B}^*\restriction L$. Define an $L_0 \cup L_1$-expansion $\mathfrak{C}$ of $\mathfrak{B}^*$ such that $\pi : \mathfrak{A}^* \cong \mathfrak{C}\restriction L_0$. Since $\mathfrak{A} \preccurlyeq \mathfrak{A}^*$ we have $\mathfrak{A}^* \models \varphi_0$, so $\mathfrak{C} \models \varphi_0$. Since $(\varphi_0 \to \varphi_1)$ is valid, $\mathfrak{C} \models \varphi_1$. Since $\mathfrak{B} \preccurlyeq \mathfrak{B}^* = \mathfrak{C}\restriction L_1$ we have $\mathfrak{B} \models \varphi_1$. $\square$

Corollary 6.2 (Robinson's joint consistency lemma)**.** *Let L_0, L_1 be languages, T_0 an L_0-theory and T_1 an L_1-theory. If $T_0 \cup T_1$ is inconsistent, then there exists an $L_0 \cap L_1$-sentence ψ such that $T_0 \vdash \psi$ and $T_1 \vdash \neg\psi$.*

Proof. By compactness, $X_0 \cup X_1$ is inconsistent for some finite $X_0 \subseteq T_0, X_1 \subseteq T_1$. Then $(\bigwedge X_0 \to \neg \bigwedge X_1)$ is valid. An interpolant satisfies our claim □

Corollary 6.3 (Beth's definability theorem)**.** *Let $R \in L$ be a relation symbol and T an L-theory. Assume T* implicitly defines *R: every $L \smallsetminus \{R\}$-structure has at most one L-expansion to a model of T. Then T* explicitly defines *R: there is an $L \smallsetminus \{R\}$-formula $\varphi(\bar{x})$ such that $T \vdash (R\bar{x} \leftrightarrow \varphi(\bar{x}))$.*

Proof. Let L_0 be L plus new constants $\bar{c}$ and $L_1 := (L_0 \smallsetminus \{R\}) \cup \{R'\}$ where R' is a copy of R. Let T' be obtained from T by replacing R by R'. Let $T_0 := T \cup \{R\bar{c}\}$ and $T_1 := T' \cup \{\neg R'\bar{c}\}$. Since T implicitly defines R, $T_0 \cup T_1$ is inconsistent. Choose ψ according to joint consistency. Then $\psi = \varphi(\bar{c})$ for some $L \smallsetminus \{R\}$-formula $\varphi(\bar{x})$. Then $T_0 \vdash \psi$ implies $T \vdash (R\bar{x} \to \varphi(\bar{x}))$. And $T_1 \vdash \neg\psi$ implies $T' \vdash (\neg R'\bar{x} \to \neg\varphi(\bar{x}))$, so $T \vdash (\varphi(\bar{x}) \to R\bar{x})$. □

Exercise 6.4. Formulate and prove a version of the above for a function symbol.

6.2 Expressivity of first-order logic

Infinitary logic extends the syntax of first-order logic by declaring $\bigwedge \Phi$ an *infinitary L-formula* if Φ is a nonempty set of infinitary L-formulas. Such a formula is declared true in an L-structure $\mathfrak{A}$ under an assignment β in $\mathfrak{A}$ if $\mathfrak{A} \vDash \varphi[\beta]$ for all $\varphi \in \Phi$. We write $\bigvee \Phi$ for $\neg \bigwedge \neg\Phi$ where $\neg\Phi := \{\neg\varphi \mid \varphi \in \Phi\}$.

For an L-theory T and an infinitary L-formula φ, $T \vdash \varphi$ means $\mathfrak{A} \vDash \varphi[\beta]$ for all L-structures $\mathfrak{A} \vDash T$ and all assignments β in $\mathfrak{A}$.

Lemma 6.5. *Let T be an L-theory, φ an L-formula, I a nonempty set and for every $i \in I$ let Φ^i be a nonempty set of L-formulas. Assume*

$$T \vdash \varphi \leftrightarrow \textstyle\bigvee_{i \in I} \bigwedge \Phi^i.$$

Then there exist a finite $\varnothing \neq I_0 \subseteq I$ and for every $i \in I_0$ a finite $\varnothing \neq \Phi^i_0 \subseteq \Phi^i$ such that

$$T \vdash \varphi \leftrightarrow \textstyle\bigvee_{i \in I_0} \bigwedge \Phi^i_0.$$

Proof. Let $i \in I$. By assumption, $T \cup \Phi^i \vdash \varphi$. By compactness, $T \cup \Phi^i_0 \vdash \varphi$ for some finite $\varnothing \neq \Phi^i_0 \subseteq \Phi^i$. Hence $T \vdash \varphi_i \to \varphi$ for $\varphi_i := \bigwedge \Phi^i_0$.

It thus suffices to show $T \vdash \varphi \to \bigvee_{i \in I_0} \varphi_i$ for some finite $\varnothing \neq I_0 \subseteq I$. But this follows from compactness: by assumption, $T \cup \{\varphi\} \cup \{\neg\varphi_i \mid i \in I\}$ is inconsistent. □

Definition 6.6. Let $n \in \mathbb{N}$ and Φ be a set of L-formulas $\varphi(x_0, \dots, x_{n-1})$. The *$\Phi$-type of $\bar{a} \in A^n$ in $\mathfrak{A}$* is

$$\Phi\text{-tp}_{\mathfrak{A}}(\bar{a}) := \{\varphi \in \Phi \cup \neg\Phi \mid \mathfrak{A} \vDash \varphi[\bar{a}]\}$$

For Φ the set of atomic (quantifier free) L-formulas, $\Phi\text{-tp}_{\mathfrak{A}}(\bar{a})$ is the *atomic (quantifier free) type of $\bar{a}$ in $\mathfrak{A}$*.

Remark 6.7. Let $\langle\Phi\rangle$ be the set of Boolean combinations of formulas from Φ, i.e., the smallest superset of Φ closed under $\neg, \wedge$. Then for all $\bar{a} \in A^n, \bar{b} \in B^n$:

$$\Phi\text{-tp}_{\mathfrak{A}}(\bar{a}) = \Phi\text{-tp}_{\mathfrak{B}}(\bar{b}) \iff \langle\Phi\rangle\text{-tp}_{\mathfrak{A}}(\bar{a}) = \langle\Phi\rangle\text{-tp}_{\mathfrak{B}}(\bar{b}).$$

Exercise 6.8. The following are equivalent:

1. $\bar{a}$ in $\mathfrak{A}$ has the same atomic type as $\bar{b}$ in $\mathfrak{B}$.
2. $\bar{a}$ in $\mathfrak{A}$ has the same quantifier free type as $\bar{b}$ in $\mathfrak{B}$.
3. There exists $\pi : \langle\bar{a}\rangle^{\mathfrak{A}} \cong \langle\bar{b}\rangle^{\mathfrak{B}}$ with $\pi(\bar{a}) = \bar{b}$.

Moreover, given $\bar{a}, \bar{b}$ there is at most one π as in (3).

Lemma 6.9. *Let T be an L-theory, $\psi(\bar{x})$ an L-formula and Φ a set of L-formulas $\varphi(\bar{x})$. Then $\psi(\bar{x})$ is T-provably equivalent to a formula in $\langle\Phi\rangle$, i.e.,*

$$T \vdash \psi \leftrightarrow \varphi$$

for some $\varphi \in \langle\Phi\rangle$, if and only if for all $\mathfrak{A}, \mathfrak{B} \vDash T$ and all $\bar{a}, \bar{b}$:

$$\text{if } \bar{a} \in \psi(\mathfrak{A}) \text{ and } \Phi\text{-tp}_{\mathfrak{A}}(\bar{a}) = \Phi\text{-tp}_{\mathfrak{B}}(\bar{b}), \text{ then } \bar{b} \in \psi(\mathfrak{B}).$$

Proof. Assume $T \vdash \psi \leftrightarrow \varphi$ for some $\varphi \in \langle\Phi\rangle$ and $\Phi\text{-tp}_{\mathfrak{A}}(\bar{a}) = \Phi\text{-tp}_{\mathfrak{B}}(\bar{b})$ where $\mathfrak{A}, \mathfrak{B} \vDash T$. Then $\langle\Phi\rangle\text{-tp}_{\mathfrak{A}}(\bar{a}) = \langle\Phi\rangle\text{-tp}_{\mathfrak{B}}(\bar{b})$. If $\bar{a} \in \psi(\mathfrak{A})$, then $\bar{a} \in \varphi(\mathfrak{A})$, so $\varphi \in \langle\Phi\rangle\text{-tp}_{\mathfrak{A}}(\bar{a})$, so $\mathfrak{B} \vDash \varphi[\bar{b}]$, so $\bar{b} \in \psi(\mathfrak{B})$. Conversely, assume the r.h.s.. By Lemma 6.5 it suffices to show

$$T \vdash \ \psi(\bar{x}) \ \leftrightarrow \ \bigvee \left\{ \bigwedge \Phi\text{-tp}_{\mathfrak{A}}(\bar{a}) \mid \mathfrak{A} \vDash T, \bar{a} \in \psi(\mathfrak{A}) \right\}.$$

Let $\mathfrak{B} \vDash T$ and $\bar{b} \in B^n$. If $\bar{b} \in \psi(\mathfrak{B})$, then $\bigwedge \Phi\text{-tp}_{\mathfrak{B}}(\bar{b})$ is a disjunct of the r.h.s.. Conversely, if $\bar{b}$ satisfies the r.h.s., there is $\mathfrak{A} \vDash T$ and $\bar{a} \in \psi(\mathfrak{A})$ such that $\Phi\text{-tp}_{\mathfrak{B}}(\bar{b}) = \Phi\text{-tp}_{\mathfrak{A}}(\bar{a})$; by assumption, $\mathfrak{B} \vDash \psi[\bar{b}]$. □

Theorem 6.10 (Hauptsatz on expressivity)**.** *Let T be an L-theory and for every $n \in \mathbb{N}$ let $\Phi_n \neq \varnothing$ be a set of L-formulas $\varphi(x_0, \ldots, x_{n-1})$. The following are equivalent.*

1. *Every L-formula $\psi(x_0, \ldots, x_{n-1})$ is T-provably equivalent to some $\varphi \in \langle\Phi_n\rangle$.*
2. *If $\mathfrak{A}, \mathfrak{B} \vDash T$, $n \in \mathbb{N}$ and $\bar{a} \in A^n$ has the same Φ_n-type in $\mathfrak{A}$ as $\bar{b} \in B^n$ in $\mathfrak{B}$, then*

$$\text{tp}_{\mathfrak{A}}(\bar{a}) = \text{tp}_{\mathfrak{B}}(\bar{b}).$$

3. *If $\mathfrak{A}, \mathfrak{B} \vDash T$ are $\aleph_0$-saturated, then $I = \varnothing$ or $I : \mathfrak{A} \cong_p \mathfrak{B}$ where*

$$I := \bigcup_{n \in \mathbb{N}} \left\{ \bar{a} \mapsto \bar{b} \mid \bar{a} \in A^n, \bar{b} \in B^n, \Phi_n\text{-tp}_{\mathfrak{A}}(\bar{a}) = \Phi_n\text{-tp}_{\mathfrak{B}}(\bar{b}) \right\}.$$

Proof. $1 \Leftrightarrow 2$ follows from the previous lemma: note $\mathrm{tp}_{\mathfrak{A}}(\bar{a}) = \mathrm{tp}_{\mathfrak{B}}(\bar{b})$ if and only if $\mathrm{tp}_{\mathfrak{A}}(\bar{a}) \subseteq \mathrm{tp}_{\mathfrak{B}}(\bar{b})$. For $2 \Rightarrow 3$ note (2) implies $I = I_e$; apply Remark 5.17 (1) and (2).

$3 \Rightarrow 2$: by Theorem 5.12 it suffices to verify (2) for $\aleph_0$-saturated $\mathfrak{A}, \mathfrak{B}$. The assumption of (2) means $\bar{a} \mapsto \bar{b} \in I$. By (3), $(\mathfrak{A}, \bar{a}) \cong_p (\mathfrak{B}, \bar{b})$, so $(\mathfrak{A}, \bar{a}) \equiv (\mathfrak{B}, \bar{b})$ by Theorem 3.9, i.e., $\mathrm{tp}_{\mathfrak{A}}(\bar{a}) = \mathrm{tp}_{\mathfrak{B}}(\bar{b})$. □

Exercise 6.11. Let I be the set of $\bar{a} \mapsto \bar{b}$ such that $\bar{a}$ has the same atomic type in $\mathfrak{A}$ as $\bar{b}$ in $\mathfrak{B}$. Show: $I : \mathfrak{A} \cong_p \mathfrak{B} \iff I_{Sk}(\mathfrak{A}, \mathfrak{B}) : \mathfrak{A} \cong_p \mathfrak{B}$.

Exercise 6.12 (Application to discrete orders)**.** For $k \in \mathbb{N}$ write a formula $\delta_k(x, y)$ such that for every discrete linear order without endpoints $\mathfrak{A}$ and all $a, b \in A$:

$$\mathfrak{A} \vDash \delta_k[a, b] \iff d^{\mathfrak{A}}(a, b) \leqslant k$$

(notation from Section 3.2.1). Let Φ_0 be the set of all sentences, and for $n > 0$ let $\Phi_n(\bar{x})$ be the set of all formulas $x_i < x_j, \delta_k(x_i, x_j)$ where $k \in \mathbb{N}$ and $i, j < n$. Verify Theorem 6.10 (3).

6.3 Quantifier elimination

Let T be an L-theory.

Definition 6.13. T *eliminates quantifiers* if every L-formula is T-provably equivalent to a quantifier free L-formula.

Remark 6.14. Assume T eliminates quantifiers. If $n > 0$ and $\varphi = \varphi(x_0, \ldots, x_{n-1})$ is T-provably equivalent to $\psi = \psi(\bar{x}, y_0, \ldots, y_{m-1})$, then also to $\psi(\bar{x}, x_0, \ldots, x_0)$ – so we can assume the formulas have the same free variables. For $n = 0$ and an L-sentence φ, we can write $\varphi = \varphi(x_0)$ and get a T-provably equivalent quantifier free formula $\psi(x_0)$. Then φ is T-provably equivalent to $\forall x_0 \psi(x_0)$. It follows that every L-formula is T-provably equivalent to a universal L-formula, so T is model complete by Theorem 4.24.

Exercise 6.15. Assume T eliminates quantifiers. Let C be a set of constants. Show that T, viewed as an $L \cup C$-theory, eliminates quantifiers.

Definition 6.16. A *simply existential* L-formula has the form $\exists x \varphi$ for φ a conjunction of literals.

Lemma 6.17. *T eliminates quantifiers if and only if every simply existential L-formula is T-provably equivalent to a quantifier free L-formula.*

Proof. The set of formulas T-provably equivalent to a quantifier free formula is clearly closed under $\wedge, \neg$. It thus suffices to show that $\exists x \varphi$ for quantifier-free φ is T-provably equivalent to a quantifier free formula. Write φ as $\bigvee_i \psi_i$ for ψ_i conjunctions of literals. Then $\exists x \varphi$ is logically equivalent to $\bigvee_i \exists x \psi_i$. Each $\exists x \psi_i$ is T-provably equivalent to a quantifier free χ_i. Thus, $\exists x \varphi$ is T-provably equivalent to $\bigvee_i \chi_i$. □

We first treat a special case.

Proposition 6.18. *Assume L is a finite relational language and T is complete with an infinite model. Then T eliminates quantifiers if and only if there exists $\mathfrak{A} \models T$ with $I_{Sk}(\mathfrak{A}, \mathfrak{A}) : \mathfrak{A} \cong_p \mathfrak{A}$. If so, then T is $\aleph_0$-categorical.*

Proof. If T eliminates quantifiers, then it is $\aleph_0$-categorical by Ryll-Nardzweski: for fixed $\bar{x}$, by assumption on L, there are only finitely many pairwise non-equivalent quantifier free formulas in $\bar{x}$. Let $\mathfrak{A} \models T$ be countable. By Corollary 5.22 and $\aleph_0$-categoricity, $\mathfrak{A}$ is $\aleph_0$-homogeneous, i.e., $I_e(\mathfrak{A}, \mathfrak{A}) : \mathfrak{A} \cong_p \mathfrak{A}$. By quantifier elimination, $I_{Sk}(\mathfrak{A}, \mathfrak{A}) = I_e(\mathfrak{A}, \mathfrak{A})$.

Conversely, assume $I_{Sk}(\mathfrak{A}, \mathfrak{A}) : \mathfrak{A} \cong_p \mathfrak{A}$. Since T is complete, it suffices to show every formula $\varphi(\bar{x})$ is in $\mathfrak{A}$ equivalent to a quantifier free formula. For a tuple $\bar{a}$ from A let $\psi_{\bar{a}}(\bar{x})$ be the conjunction of all literals satisfied by $\bar{a}$ in $\mathfrak{A}$ - these are finitely many by assumption on L. Then $\varphi(\bar{x})$ is in $\mathfrak{A}$ equivalent to the quantifier free formula

$$\psi(\bar{x}) \;:=\; \bigvee_{\bar{a} \in \varphi(\mathfrak{A})} \psi_{\bar{a}}(\bar{x})$$

Indeed, $\varphi(\mathfrak{A}) \subseteq \psi(\mathfrak{A})$ is clear. Conversely, if $\bar{b} \in \psi(\mathfrak{A})$, choose $\bar{a} \in \varphi(\mathfrak{A})$ such that $\mathfrak{A} \models \psi_{\bar{a}}[\bar{b}]$. Then $\bar{a} \mapsto \bar{b} \in I_{Sk}(\mathfrak{A}, \mathfrak{A})$. By assumption, $(\mathfrak{A}, \bar{a}) \cong_p (\mathfrak{A}, \bar{b})$, so $(\mathfrak{A}, \bar{a}) \equiv (\mathfrak{A}, \bar{b})$ by Theorem 3.8, so $\mathfrak{A} \models \varphi[\bar{b}]$. □

Theorem 6.19 (Hauptsatz on quantifier elimination)**.** *The following are equivalent.*

1. *T eliminates quantifiers.*
2. *If $\mathfrak{A}, \mathfrak{B} \models T$, and $\bar{a}, \bar{b}$ are non-empty tuples of the same atomic type in $\mathfrak{A}, \mathfrak{B}$, then $\mathrm{tp}_{\mathfrak{A}}(\bar{a}) = \mathrm{tp}_{\mathfrak{B}}(\bar{b})$.*
3. *If $\mathfrak{A}, \mathfrak{B} \models T$ are $\aleph_0$-saturated, then $I_{Sk}(\mathfrak{A}, \mathfrak{B}) = \varnothing$ or $I_{Sk}(\mathfrak{A}, \mathfrak{B}) : \mathfrak{A} \cong_p \mathfrak{B}$.*
4. *If $\mathfrak{A} \models T$ and $\mathfrak{C} \subseteq \mathfrak{A}$, then $T \cup D_a(\mathfrak{C})$ is complete.*
5. *If $\mathfrak{A}, \mathfrak{B}$ are models of T with a common substructure $\mathfrak{C} \subseteq \mathfrak{A}, \mathfrak{B}$, then every simply existential $L(C)$-sentence which is true in $\mathfrak{A}_C$ is also true in $\mathfrak{B}_C$.*

Proof. Let Φ_0 be the set of L-sentences and for $n > 0$, let Φ_n be the set of atomic L-formulas in the variables $x_0, \ldots, x_n$. Then (1), (2), (3) are equivalent to the corresponding statements of Theorem 6.10 (recall Exercise 6.11). Hence, (1)-(3) are equivalent.

$1 \Rightarrow 4$: we claim that every $\mathfrak{B} \models T \cup D_a(\mathfrak{C})$ models $Th(\mathfrak{A}_C)$. By Lemma 4.3, we can assume $\mathfrak{C} \subseteq \mathfrak{B}$. Let $\mathfrak{A}_C \models \varphi(\bar{c})$ for $\varphi(\bar{x})$ and L-formula. Choose a T-provably equivalent quantifier free $\varphi_0(\bar{x})$. Then $\mathfrak{C}_C \models \varphi_0(\bar{c})$, so $\mathfrak{B}_C \models \varphi_0(\bar{c})$, hence $\mathfrak{B}_C \models \varphi(\bar{c})$ since $\mathfrak{B} \models T$.

$4 \Rightarrow 5$: note $\mathfrak{A}_C \equiv \mathfrak{B}_C$ because both structures model $T \cup D_a(\mathfrak{C})$, complete by (4).

$5 \Rightarrow 1$: by Lemma 6.17 is suffices to show that every simply existential $\varphi(\bar{x})$ is T-provably equivalent to a quantifier free formula. We apply Lemma 6.9: assume $\mathfrak{A}, \mathfrak{B} \models T$, $\bar{a} \in \varphi(\mathfrak{A})$ and $\bar{b}$ has the same atomic type in $\mathfrak{B}$ as $\bar{a}$ in $\mathfrak{A}$. We have to show $\bar{b} \in \varphi(\mathfrak{B})$. By Exercise 6.8, there is $\pi : \langle \bar{a} \rangle^{\mathfrak{A}} \cong \langle \bar{b} \rangle^{\mathfrak{B}}$ with $\pi(\bar{a}) = \bar{b}$. We can assume π is the identity, so $\bar{b} = \bar{a}$ and $\mathfrak{C} := \langle \bar{a} \rangle^{\mathfrak{A}} \subseteq \mathfrak{B}$. By (5), $\mathfrak{A} \models \varphi[\bar{a}]$ implies $\mathfrak{B} \models \varphi[\bar{a}]$. □

The following clarifies the relationship of quantifier elimination and model completeness (cf. Definition 3.39).

Definition 6.20. T has *amalgamation* if for all $\mathfrak{A}, \mathfrak{B}, \mathfrak{B}' \vDash T$ with $\mathfrak{A} \subseteq \mathfrak{B}, \mathfrak{B}'$ there exist $\mathfrak{C} \vDash T$ and $\pi : \mathfrak{B} \to_a \mathfrak{C}$ and $\pi' : \mathfrak{B}' \to_a \mathfrak{C}$ such that $\pi \upharpoonright A = \pi' \upharpoonright A$.

Proposition 6.21. *T eliminates quantifiers if and only if T is model complete and $T_\forall$ has amalgamation.*

Proof. $\Rightarrow$: let $\mathfrak{A}, \mathfrak{B}, \mathfrak{B}' \vDash T_\forall$ with $\mathfrak{A} \subseteq \mathfrak{B}, \mathfrak{B}'$. By Lemma 4.4, there are extensions $\hat{\mathfrak{B}}, \hat{\mathfrak{B}}' \vDash T$ of $\mathfrak{B}, \mathfrak{B}'$. By Theorem 6.19, $T \cup D_a(\mathfrak{A})$ is complete. Hence, $\hat{\mathfrak{B}}_A \equiv \hat{\mathfrak{B}}'_A$. By Exercise 4.31, there is $\pi : \hat{\mathfrak{B}}_A \to_e \mathfrak{C}_A \succcurlyeq \hat{\mathfrak{B}}'_A$ for some L-structure $\mathfrak{C}$. Let $\pi' : \hat{\mathfrak{B}}'_A \to_e \mathfrak{C}_A$ be the identity. Then $\pi(a) = \pi(c_a^{\hat{\mathfrak{B}}_A}) = c_a^{\mathfrak{C}_A} = a = \pi'(a)$ for every $a \in A$.

$\Leftarrow$: We verify Theorem 6.19 (4). Given $\mathfrak{A} \subseteq \mathfrak{B} \vDash T$ we show every model of $T \cup D_a(\mathfrak{A})$ is $\equiv \mathfrak{B}_A$. By Lemma 4.3 the L-reduct of such a model is isomorphic to some $\mathfrak{B}' \vDash T$ with $\mathfrak{A} \subseteq \mathfrak{B}'$. By amalgamation, there are embeddings π, π' of $\mathfrak{B}, \mathfrak{B}'$ into some $\mathfrak{C}' \vDash T_\forall$ such that $\pi \upharpoonright A = \pi' \upharpoonright A$. By Lemma 4.4, $\mathfrak{C}'$ has an extension $\mathfrak{C} \vDash T$. Then π, π' are embeddings into $\mathfrak{C}$. Since T is model complete, π and π' are elementary. Let $C_0 \subseteq C$ be the image of π and note $\pi : \mathfrak{B}_A \to_e \mathfrak{C}_{C_0}$ and $\pi' : \mathfrak{B}'_A \to_e \mathfrak{C}_{C_0}$. Thus $\mathfrak{B}_A \equiv \mathfrak{C}_{C_0} \equiv \mathfrak{B}'_A$. □

Exercise 6.22. If $T_\forall$ has amalgamation, then also T.

Exercise 6.23 (General Nullstellensatz)**.** Assume T has amalgamation. Let $\mathfrak{A} \vDash T$ and φ be an existential $L(A)$-sentence. Let $\mathfrak{A} \subseteq \mathfrak{C} \in \mathcal{E}(T)$. Then $\mathfrak{C}_A \vDash \varphi$ if and only if there exists $\mathfrak{A} \subseteq \mathfrak{B} \vDash T$ with $\mathfrak{B}_A \vDash \varphi$.

6.3.1 Examples

We first give a direct argument without using Theorem 6.19.

Example 6.24. The theory of divisible ordered abelian groups eliminates quantifiers. It is the model companion of the theory of ordered abelian groups.

Proof. By Lemma 6.17 is suffices to show every simply existential formula is (provably in the theory) equivalent to a quantifier free one. Let $\exists y \varphi(y, \bar{x})$ be a simply existential formula. We extend the notation mx to $m \in \mathbb{Z}$ to mean $-nx$ if $m = -n$ for $n \in \mathbb{N}$.

Write $\varphi(y, \bar{x})$ as a disjunction of conjunctions of atoms of the form

$$0 = my + t, \quad 0 < my + t$$

where $m \in \mathbb{Z}$ and $t = t(\bar{x})$ is a term. This can be done: $\neg t_0 = t_1$ is equivalent to $(0 < t_0 - t_1 \vee 0 < t_1 - t_0)$ and $\neg t_0 < t_1$ to $(0 = t_0 - t_1 \vee 0 < t_0 - t_1)$; further, for every term $t(y, \bar{x})$ the theory proves $t(y, \bar{x}) = my + t'(\bar{x})$ for a certain $m \in \mathbb{Z}$ and a term $t'(\bar{x})$.

We can thus assume $\varphi(y, \bar{x})$ is a conjunction of such atoms. Let $\varphi'(y, \bar{x})$ be the subconjunction of such atoms with $m \neq 0$. It suffices to show $\exists y \varphi'(y, \bar{x})$ is equivalent to a quantifier free formula $\psi(\bar{x})$.

Assume an atom $0 = my + t$ appears, say with $m > 0$ (the case $m < 0$ is similar) let $\psi(\bar{x})$ be obtained replacing, informally speaking y by $-t/m$, i.e., every $0 = m'y + t'$ by $0 = -m't + mt'$ and every $0 < m'y + t'$ by $0 < -m't + mt'$.

Otherwise, $\varphi'(y, \bar{x})$ is a conjunction of atoms $0 < my + t$. Informally, such y has to be $> -t/m$ for the positive m, and $< -t/m$ for the negative m; it exists if and only if for every positive m and every negative m' we have $-t/m < -t/m'$, i.e., $m't < mt$. Let $\psi(\bar{x})$ be the conjunction of these atoms.

2nd statement: an ordered abelian group embeds into a divisible one. Indeed, order the divisible hull of $\mathfrak{A}$ (Exercise 4.14) setting $a/n < a'/m$ if and only if $ma <^{\mathfrak{A}} na'$. □

Exercise 6.25. The theory of divisible torsion-free abelian groups eliminates quantifiers. It is the model companion of the theory of torsion-free abelian groups.

The back and forth systems of Sections 3.1.1-3.1.3 imply quantifier elimination:

Example 6.26. The theory of dense linear orders without endpoints eliminates quantifiers. It is the model companion of the theory of linear orders.

Proof. Lemma 3.11 verifies Theorem 6.19 (3) or Proposition 6.18. 2nd statement: Exercise 4.49. □

Example 6.27. The theory of (nontrivial) atomless Boolean algebras eliminates quantifiers. It is the model companion of the theory of nontrivial Boolean algebras.

Proof. Lemma 3.15 verifies Theorem 6.19 (3). 2nd statement: Exercise 4.6. □

Example 6.28. *ACF* eliminates quantifiers. It is the model-companion of the theory of integral domains (or fields).

Proof. We verify Theorem 6.19 (3). Let $\mathfrak{A}, \mathfrak{B} \vDash ACF$ be $\aleph_0$-saturated. If $\mathfrak{A}, \mathfrak{B}$ have distinct characteristic, then $I_{Sk}(\mathfrak{A}, \mathfrak{B}) = \varnothing$. Otherwise $I_{Sk}(\mathfrak{A}, \mathfrak{B}) : \mathfrak{A} \cong_p \mathfrak{B}$ by Lemma 3.18 and Exercise 3.19 – recall $\mathfrak{A}, \mathfrak{B}$ are "large" by Remark 5.14.

2nd statement: an integral domain embeds into its quotient field, and a field embeds into an algebraically closed one (Example 4.54). □

6.3.2 Quantifier elimination in Fraïssé limits

Corollary 6.29. *Assume L is a finite relational language and $\mathcal{K}$ a Fraïssé class of (finite) L-structures. Then the theory of the Fraïssé limit of $\mathcal{K}$ is $\aleph_0$-categorical and eliminates quantifiers.*

Proof. Let $\mathfrak{A}$ be the Fraïssé limit of $\mathcal{K}$. Then $I_{Sk}(\mathfrak{A}, \mathfrak{A}) : \mathfrak{A} \cong_p \mathfrak{A}$ by Remark 3.38 (2) and (4). Apply Proposition 6.18. □

This provides an alternative proof that the theory of dense linear orders without endpoints eliminates quantifiers (recall Examples 3.43). For atomless Boolean algebras we generalize the above to languages with function symbols:

Theorem 6.30. *Assume L is finite and $\mathcal{K}$ is a Fraïssé class of finite L-structures such that for all $n \in \mathbb{N}$ there are, up to isomorphism, only finitely many structures in $\mathcal{K}$ that are generated by n elements. Then the theory of the Fraïssé limit of $\mathcal{K}$ is $\aleph_0$-categorical and eliminates quantifiers.*

Proof. For every $n \in \mathbb{N}$ there are, up to $\cong$, only finitely many $(\mathfrak{K}, \bar{k})$ with $\mathfrak{K} \in \mathcal{K}$ and $\bar{k} \in K^n$ and $\mathfrak{K} = \langle \bar{k} \rangle^{\mathfrak{K}}$. It is not hard to write an L-formula $\psi_{\mathfrak{K},\bar{k}}(\bar{x})$ such that for every L-structure $\mathfrak{B}$ and $\bar{b} \in B^n$ we have $\mathfrak{B} \models \psi_{\mathfrak{K},\bar{k}}[\bar{b}]$ if and only if there is $f : \langle \bar{b} \rangle^{\mathfrak{A}} \cong \mathfrak{K}$ with $f(\bar{b}) = \bar{k}$. Note there is at most one isomorphism with $f(\bar{b}) = \bar{k}$. Further, $\psi_{\mathfrak{K},\bar{k}} = \psi_{\mathfrak{K}',\bar{k}'}$ if $(\mathfrak{K}, \bar{k}) \cong (\mathfrak{K}', \bar{k}')$, so there are only finitely many such formulas.

Let $\mathfrak{A}$ be the Fraïssé limit of $\mathcal{K}$ and $\varphi(x_0, \dots, x_{n-1})$ with $n > 0$ is an L-formula. Then $\varphi(\bar{x})$ is in $\mathfrak{A}$ equivalent to the quantifier free formula

$$\bigvee_{\bar{a} \in \varphi(\mathfrak{A})} \psi_{\langle \bar{a} \rangle^{\mathfrak{A}},\bar{a}}(\bar{x}).$$

Indeed, if $\bar{b} \in \psi_{\langle \bar{a} \rangle^{\mathfrak{A}},\bar{a}}(\mathfrak{A})$ and $\bar{a} \in \varphi(\mathfrak{A})$, then there is $f : \langle \bar{a} \rangle^{\mathfrak{A}} \cong \langle \bar{b} \rangle^{\mathfrak{A}}$ with $f(\bar{a}) = \bar{b}$; by ultrahomogeneity, there is an automorphism of $\mathfrak{A}$ extending f; hence $\bar{b} \in \varphi(\mathfrak{A})$.

By Remark 3.38 (3) we are left to show every $\mathfrak{B} \equiv \mathfrak{A}$ has skeleton $\mathcal{K}$ and is $\mathcal{K}$-saturated.

$Sk(\mathfrak{B}) = \mathcal{K}$: let $\mathfrak{K} \in \mathcal{K}$ and choose $\bar{k}$ generating it; then $\mathfrak{A}$ and hence $\mathfrak{B}$ satisfies $\exists \bar{x} \psi_{\mathfrak{K},\bar{k}}(\bar{x})$, so $\mathfrak{K} \in Sk(\mathfrak{B})$. Conversely, let $\bar{b} \in B^n$. With $\mathfrak{A}$ also $\mathfrak{B}$ satisfies $\forall \bar{x} \bigvee_{(\mathfrak{K},\bar{k})} \psi_{\mathfrak{K},\bar{k}}(\bar{x})$ where $(\mathfrak{K}, \bar{k})$ ranges over pairs as above with $\bar{k} \in K^n$. Hence $\langle \bar{b} \rangle^{\mathfrak{B}} \cong \mathfrak{K}$ for some $\mathfrak{K} \in \mathcal{K}$.

$\mathcal{K}$-saturation: let $f_0 : \mathfrak{K}_0 \to_a \mathfrak{B}$ and $\mathfrak{K}_0 \subseteq \mathfrak{K}_1 \in \mathcal{K}$. Choose $\bar{k}_0$ generating $\mathfrak{K}_0$ and $\bar{k}_1$ such that $\bar{k}_0\bar{k}_1$ generates $\mathfrak{K}_1$. Note $f : \mathfrak{K}_0 \to_a \mathfrak{B}$ means $\bar{b}_0 := f_0(\bar{k}_0)$ satisfies $\psi_{\mathfrak{K}_0,\bar{k}_0}(\bar{x})$ in $\mathfrak{B}$. It suffices to find $\bar{b}_1$ such that $\bar{b}_0\bar{b}_1$ satisfies $\psi_{\mathfrak{K}_1,\bar{k}_0\bar{k}_1}(\bar{x}, \bar{y})$ in $\mathfrak{B}$: then we get $f_1 : \mathfrak{K}_1 \cong \langle \bar{b}_0\bar{b}_1 \rangle^{\mathfrak{B}} \subseteq \mathfrak{B}$ with $f_1(\bar{k}_0\bar{k}_1) = \bar{b}_0\bar{b}_1$; such f_1 extends f_0. But such a $\bar{b}_1$ does indeed exist: $\mathfrak{A}$ is $\mathcal{K}$-saturated, so $\mathfrak{A}$ and hence $\mathfrak{B}$ satisfy $\forall \bar{x} \exists \bar{y} \big(\psi_{\mathfrak{K}_0,\bar{k}_0}(\bar{x}) \to \psi_{\mathfrak{K}_1,\bar{k}_0\bar{k}_1}(\bar{x}, \bar{y})\big)$. □

Corollary 6.31. *The theory of the random graph eliminates quantifiers.*

6.4 Applications to algebraically closed fields

Let $\mathfrak{A}$ be an algebraically closed field, $\bar{x} = (x_0, \dots, x_{n-1})$ a tuple of variables and $\mathfrak{A}[\bar{x}]$ the polynomial ring over $\mathfrak{A}$.

Recall the correspondence $t \mapsto P_t$ and $P \mapsto t_P$ between $L_{Ring}(A)$-terms and polynomials in $\mathfrak{A}[\bar{x}]$ in the proof of Proposition 4.26.

6.4.1 Hilbert's Nullstellensatz

We need the following result from algebra:

Theorem 6.32 (Hilbert's basis theorem)**.** *Every ideal of $\mathfrak{A}[\bar{x}]$ is finitely generated.*

Theorem 6.33 (Hilbert's Nullstellensatz - weak form)**.** *Every proper ideal I of $\mathfrak{A}[\bar{x}]$ has a common zero in $\mathfrak{A}$, i.e., there is $\bar{a} \in A^n$ such that $P(\bar{a}) = 0$ for all $P(\bar{x}) \in I$.*

Proof. Let J be a maximal ideal extending I. Let $\mathfrak{B}$ be an algebraically closed extension of the field $\mathfrak{A}[\bar{x}]/J$ (Example 4.54). Setting $b_i := x_i \mod J$ we have $P(\bar{b}) = 0$ for all $P(\bar{x}) \in J$. Let J_0 be a finite generator of J (Theorem 6.32). Then $\mathfrak{B} \models \exists \bar{x} \bigwedge_{P \in J_0} t_P(\bar{x}) = 0$. Since $\mathfrak{A} \subseteq \mathfrak{B}$ we have $\mathfrak{A} \preccurlyeq \mathfrak{B}$ by model completeness (Remark 6.14). Hence $\mathfrak{A} \models \exists \bar{x} \bigwedge_{P \in J_0} t_P(\bar{x}) = 0$. But if $P(\bar{a}) = 0$ for all $P(\bar{x}) \in J_0$, then $P(\bar{a}) = 0$ for all $P(\bar{x}) \in J$. □

Theorem 6.34 (Hilbert's Nullstellensatz - strong form)**.** *Let J be a proper ideal of $\mathfrak{A}[\bar{x}]$, $P(\bar{x}) \in \mathfrak{A}[\bar{x}]$ and assume $P(\bar{a}) = 0$ for all $\bar{a} \in A^n$ such that for all $Q(\bar{x}) \in J$ we have $Q(\bar{a}) = 0$. Then there is $k \in \mathbb{N}$ such that $P(\bar{x})^k \in J$.*

Proof by Rabinowitsch's trick. The ideal generated by $J \cup \{(1 - y \cdot P(\bar{x}))\}$ does not have a common zero in $\mathfrak{A}$. By the weak form,

$$1 = Q_0(\bar{x}, y) \cdot P_0(\bar{x}) + \dots + Q_{\ell-1}(\bar{x}, y) \cdot P_{\ell-1}(\bar{x}) + Q_\ell(\bar{x}, y) \cdot (1 - y \cdot P(\bar{x}))$$

for certain $P_i(\bar{x}) \in J$ and $Q_i(\bar{x}, y) \in \mathfrak{A}[\bar{x}, y]$. Work in the field of rational functions $\mathfrak{A}(\bar{x}, y)$ and plug $1/P(\bar{x})$ for y:

$$1 = Q_0(\bar{x}, 1/P(\bar{x})) \cdot P_0(\bar{x}) + \dots + Q_{\ell-1}(\bar{x}, 1/P(\bar{x})) \cdot P_{\ell-1}(\bar{x}).$$

Choose $k \in \mathbb{N}$ large enough such that $Q_i'(\bar{x}) := Q_i(\bar{x}, 1/P(\bar{x})) \cdot P(\bar{x})^k \in \mathfrak{A}[\bar{x}]$ for all $i < \ell$. Multiplying the equation gives $P(\bar{x})^k \in J$ because

$$P(\bar{x})^k = Q_0'(\bar{x}) \cdot P_0(\bar{x}) + \dots + Q_{\ell-1}'(\bar{x}) \cdot P_{\ell-1}(\bar{x}).$$ □

6.4.2 Definability versus constructibility

For $E \subseteq \mathfrak{A}[\bar{x}]$ and $X \subseteq A^n$ set

$$\begin{aligned} V(E) &:= \{\bar{a} \in A^n \mid P(\bar{a}) = 0 \text{ for all } P(\bar{x}) \in E\}, \\ I(X) &:= \{P(\bar{x}) \in \mathfrak{A}[\bar{x}] \mid P(\bar{a}) = 0 \text{ for all } \bar{a} \in X\}. \end{aligned}$$

Clearly, $X \subseteq V(I(X))$ and $E \subseteq I(V(E))$. Sets of the form $V(E)$ are *Zariski closed*. It is straightforward to check that $I(X)$ is a *radical* ideal of $\mathfrak{A}[\bar{x}]$, i.e., $P(\bar{x})^k \in I$ implies $P(\bar{x}) \in I$ for all $k \in \mathbb{N}$ and $P(\bar{x}) \in \mathfrak{A}[\bar{x}]$.

Remark 6.35. By Theorem 6.34, $J = I(V(J))$ for radical ideals J.

Further, for Zariski closed X, we have $X = V(I(X))$. In particular, $X \mapsto I(X)$ is injective on Zariski closed sets X. Zariski closed sets are closed under $\cap, \cup$. Indeed, one easily checks $X \cap X' = V(I(X) + I(X'))$ and $X \cup X' = V(I(X) \cap I(X'))$.

Proposition 6.36. *For $n \in \mathbb{N}$ the Zariski closed subsets of A^n are the closed subsets of some topology on A^n.*

Proof. Let $\mathcal{X} \neq \emptyset$ be a set of Zariski closed sets. We have to show that $\bigcap \mathcal{X}$ is Zariski closed. Assume otherwise and let $X_0 \in \mathcal{X}$. Since $X_0 \neq \bigcap \mathcal{X}$ there is $X \in \mathcal{X}$ such that $X_0 \notin X$. Set $X_1 := X_0 \cap X$. Continue to get $X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$. Then $I(X_1) \subsetneq I(X_1) \subsetneq \cdots$. Then the ideal $\bigcup_n I(X_n)$ is not finitely generated. This contradicts Theorem 6.32. □

Theorem 6.37. *Let $n \in \mathbb{N}$. A set $X \subseteq A^n$ is definable in $\mathfrak{A}_A$ if and only if it is* constructible*, i.e., a Boolean combination of Zariski closed sets.*

Proof. ⇒: an atomic $L_{Ring}(A)$-formula is equivalent to $t(\bar{x}) = 0$ for some $L_{Ring}(A)$-term $t(\bar{x})$, so defines $V(P_t(\bar{x}))$ in $\mathfrak{A}_A$. By quantifier-elimimation, every definable (in $\mathfrak{A}_A$) set is a Boolean combinations of such sets, hence constructible.

⇐: it suffices to show that every Zariski closed X is definable in $\mathfrak{A}_A$. But $X = V(I(X))$ and, by Theorem 6.32, $I(X)$ is generated by a finite set E of polynomials over $\mathfrak{A}$. Then $X = V(E)$ is defined by the conjunction of $t_P(\bar{x}) = 0$ for $P(\bar{x}) \in E$. □

Corollary 6.38 (Chevalley's theorem)**.** *The image of a constructible set under a polynomial function is constructible.*

Proof. Let $X \subseteq A^n$ be constructible, and $f : A^n \to A^m$ be a polynomial function, say given by $P_0(\bar{x}), \ldots, P_{m-1}(\bar{x})$. By Theorem 6.37 there is an $L_{Ring}(A)$-formula $\varphi(\bar{x})$ such that $\varphi(\mathfrak{A}) = X$. Then $f(X)$ is constructible because it is defined by

$$\exists \bar{x}(\varphi(\bar{x}) \wedge \textstyle\bigwedge_{i<m} y_i = t_{P_i}(\bar{x})).$$ □

The following mentions a concept that is central to the development of model theory beyond the limits of this course.

Corollary 6.39. $\mathfrak{A}$ *is* strongly minimal*: a subset of A is definable in $\mathfrak{A}_A$ if and only if it is finite or cofinite.*

Proof. A finite $\{a_0, \ldots, a_{n-1}\} \subseteq A$ is defined by $\bigvee_{i<n} x = c_{a_i}$. Conversely, a definable subset of A is a Boolean combination of sets $V(P(x))$ with $P(x) \in \mathfrak{A}[x]$; but $V(P(x))$ is finite. □

6.4.3 Types versus prime ideals

Let $\mathfrak{B} \subseteq \mathfrak{A}$ be a subfield. We need an algebraic lemma:

Lemma 6.40. *For every prime ideal I of $\mathfrak{B}[\bar{x}]$, there is a prime ideal J of $\mathfrak{A}[\bar{x}]$ with*

$$J \cap \mathfrak{B}[\bar{x}] = I.$$

Proof. Let $\hat{I} = \mathfrak{A}[\bar{x}] \cdot I$ be the ideal generated by I in $\mathfrak{A}[\bar{x}]$. Let $G \subseteq A$ be a basis for $\mathfrak{A}$ as a vectorspace over $\mathfrak{B}$ with $1 \in G$. Then every $Q \in \mathfrak{A}[\bar{x}]$ can be uniquely written $\sum_{g \in G} P_g \cdot g$ with $P_g \in \mathfrak{B}[\bar{x}]$, finitely many $\neq 0$. If $Q \in \hat{I}$, all P_g are in I. If additionally $P \in \mathfrak{B}[\bar{x}]$, all P_g with $g \neq 1$ are 0, so $Q \in I$. Hence, $\hat{I} \cap \mathfrak{B}[\bar{x}] = I$.

Hence there exists ideals J of $\mathfrak{A}[\bar{x}]$ that are disjoint from $\bar{I} := \mathfrak{B}[\bar{x}] \setminus I$. By Zorn's lemma there is a maximal such ideal J. We are left to show that J is prime.

Given $P, P' \notin J$, we show $PP' \notin J$. By maximality, the ideals generated by $J \cup \{P\}$ and $J \cup \{P'\}$ intersect $\bar{I}$. Hence there are $Q_0, Q_0' \in J$ and $Q_1, Q_1' \in \mathfrak{A}[\bar{x}]$ such that $Q_0 + Q_1 \cdot P \in \bar{I}$ and $Q_0' + Q_1' \cdot P' \in \bar{I}$. Since I is prime, $\bar{I}$ is closed under multiplication, so contains

$$Q_0 Q_0' + Q_1 Q_0' \cdot P + Q_1' Q_0 \cdot P' + Q_1 Q_1 \cdot PP' = Q_0'' + Q_1'' \cdot PP'$$

for suitable $Q_0'' \in J, Q_1'' \in \mathfrak{A}[\bar{x}]$. As $J \cap \bar{I} = \varnothing$, we have $PP' \notin J$. □

Theorem 6.41. *Let $n \in \mathbb{N}$. There is a bijection from the set of complete n-types $p = p(\bar{x})$ of $\mathfrak{A}$ over B onto the set of prime ideals of $\mathfrak{B}[\bar{x}]$ given by*

$$p \mapsto I_p := \{P(\bar{x}) \mid t_P(\bar{x}) = 0 \in p\}.$$

Proof. It is straightforward to check that I_p is a prime ideal. Injectivity: if $I_p = I_q$, then p, q contain the same quantifier free formulas; by quantifier elimination, $p = q$.

Surjectivity: let I be a prime ideal of $\mathfrak{B}[\bar{x}]$. Choose J according to the previous lemma. Let $\mathfrak{C}$ be an algebraically closed field extending the quotient field of $\mathfrak{A}[\bar{x}]/J$. For $i < n$ set $c_i := x_i \mod J$. By model completeness, $\mathfrak{A} \preccurlyeq \mathfrak{C}$, so $p(\bar{x}) := \mathrm{tp}_{\mathfrak{C}}(\bar{c}/B)$ is an n-type of $\mathfrak{A}$ over B. For $P(\bar{x}) \in \mathfrak{B}[\bar{x}]$, p contains $t_P = 0$ if and only if $P(\bar{c}) = 0$ in $\mathfrak{C}$, if and only if $P(\bar{x}) \in J$, if and only if $P(\bar{x}) \in I$. Thus, $I_p = I$. □

6.5 Applications to real closed fields

6.5.1 Background in real algebra

Definition 6.42. A field $\mathfrak{A}$ is *formally real* if -1 is not a sum of squares in $\mathfrak{A}$. It is *real closed* if no proper algebraic extension is formally real. A *real closure* of $\mathfrak{A}$ is an algebraic field extension that is real closed.

Remark 6.43. Every formally real field has a real closure (apply Zorn's lemma).

We need the following result from real algebra:

Theorem 6.44 (Artin-Schreier)**.** *Let $\mathfrak{A}$ be formally real. The following are equivalent.*

1. *$\mathfrak{A}$ is real closed.*
2. *For every $a \in A$, a or $-a$ is a square in $\mathfrak{A}$ and every polynomial in $\mathfrak{A}[x]$ of odd degree has a root in $\mathfrak{A}$.*
3. *The field extension $\mathfrak{A}(\sqrt{-1})$ is algebraically closed.*

Definition 6.45. The *theory RCF of real closed fields* is the theory of ordered fields plus, for every $n \in \mathbb{N}$, the universal closures of

$$\exists y(x = y^2 \vee -x = y^2), \quad \neg\, x_0^2 + \cdots + x_n^2 = -1, \quad \exists y\; x^{2n+1} + x_{2n} \cdot y^{2n} + \cdots + x_1 \cdot y + x_0 = 0.$$

We write a model of RCF as $(\mathfrak{A}, <^{\mathfrak{A}})$ where $\mathfrak{A}$ is a real closed field. This abuse of notation is justified because the order $<^{\mathfrak{A}}$ is uniquely determined by $\mathfrak{A}$:

Lemma 6.46. *A real closed field has a unique expansion to an ordered field.*

Proof. Assume $\mathfrak{A}$ is real closed. We first claim that at most one of $a, -a$ is a sum of squares in A. Indeed, if $a, b \in A$ are sums of squares, then so is $a/b = (a/b^2) \cdot b$.

We define the order setting $a <^{\mathfrak{A}} b$ if and only if $a \neq b$ and $b - a$ is a square in $\mathfrak{A}$. This defines a linear order: for transitivity of $<^{\mathfrak{A}}$, assume $b-a$ and $c-b$ are squares; then $c-a$ is a sum of squares, so $a-c$ is not a (sum of) square, so $c-a$ is a square. The other axioms are trivial, so the expansion is an ordered field. Since in ordered fields squares $\neq 0$ are positive, uniqueness follows. □

Lemma 6.47. *Assume $\mathfrak{A}$ is real closed, $a, b \in A$, $P(x) \in \mathfrak{A}[x]$ and $P(a) < 0 < P(b)$. Then there is $a < c < b$ such that $P(c) = 0$.*

Proof. By Theorem 6.44, $\mathfrak{A}(\sqrt{-1})$ is algebraically closed. This implies that every $Q(x) \in \mathfrak{A}[x]$ is a product of polynomials of degree 1 or 2. Indeed, in $\mathfrak{A}(\sqrt{-1})$ write $Q(x)$ as a product of $(X - c)$ where $c = c_0 + c_1\sqrt{-1}$ for some $c_0, c_1 \in A$; since $\mathfrak{A}(\sqrt{-1})$ has an automorphism fixing A and mapping $\sqrt{-1}$ to $-\sqrt{-1}$, also $c' := c_0 - c_1\sqrt{-1}$ appears; hence, Q is the product of $(x-c)$ with $c_1 = 0$ and $(x-c)(x-c')$ with $c_1 \neq 0$. But $(x-c)(x-c') = x^2 - 2c_0x + c_0^2 + c_1^2$ is in $\mathfrak{A}[x]$.

Let $P(x)$ accord the assumption. We can assume $P(x)$ is irreducible (since some factor has to change signs). Then $P(x)$ has degree 1 or 2. If $P(x)$ is $x - c$, then $a < c < b$. If $P(x)$ is $x^2 - 2c_0x + c_0^2 + c_1^2$ with $c_1 \neq 0$, then $P(x) = (x - c_0)^2 + c_1^2 > 0$ for all values of x, contradicting $P(a) < 0$. □

We need another result from real algebra. Omitting superscripts, we write an ordered field $\mathfrak{A}'$ as $(\mathfrak{A}, <)$ for $\mathfrak{A} := \mathfrak{A}' \upharpoonright L_{Ring}$.

Theorem 6.48. *Let $(\mathfrak{A}, <)$ be an ordered field. Then there is a real closure $\mathfrak{B}$ of $\mathfrak{A}$ such that $(\mathfrak{A}, <) \subseteq (\mathfrak{B}, <^{\mathfrak{B}})$. Moreover, for every real closure $\mathfrak{B}'$ of $\mathfrak{A}$ with $(\mathfrak{A}, <) \subseteq (\mathfrak{B}', <^{\mathfrak{B}'})$ there is an isomorphism from $(\mathfrak{B}, <^{\mathfrak{B}})$ onto $(\mathfrak{B}', <^{\mathfrak{B}'})$ that fixes A.*

6.5.2 Quantifier elimination and Tarski-Seidenberg

Theorem 6.49. *RCF eliminates quantifiers.*

Proof. We verify Theorem 6.19 (5). Let $(\mathfrak{A}, <^{\mathfrak{A}}), (\mathfrak{B}, <^{\mathfrak{B}}) \models RCF$, $(\mathfrak{C}, <)$ a common substructure, and $\exists y \varphi(\bar{x}, y)$ be a simply existential formula. Then $\mathfrak{C}$ is an integral domain. We can assume $\mathfrak{C}$ is a field – note the order extends uniquely to the fraction field. Let $\mathfrak{C}'$ be the relative algebraic closure of $\mathfrak{C}$ in $\mathfrak{A}$: its universe consists of the elements $a \in A$ that are algebraic over $\mathfrak{C}$. Theorem 6.44 implies that $\mathfrak{C}'$ is a real closure of $\mathfrak{C}$; the unique order $<^{\mathfrak{C}'}$ agrees with $<^{\mathfrak{A}}$, so extends $<$ on C. Let $\mathfrak{C}''$ be defined analogously within $\mathfrak{B}$, in particular, $<^{\mathfrak{C}''}$ agrees $<^{\mathfrak{B}}$ and extends $<$ on C. By Theorem 6.48, $(\mathfrak{C}', <^{\mathfrak{C}'}), (\mathfrak{C}'', <^{\mathfrak{C}''})$ are isomorphic substructures of $(\mathfrak{A}, <^{\mathfrak{A}}), (\mathfrak{B}, <^{\mathfrak{B}})$ and the isomorphism fixes C. We can thus further assume that $\mathfrak{C}$ is a real closed field.

Assume $\mathfrak{A} \vDash \varphi[\bar{c}, a]$ for $\bar{c}$ from C and $a \in A$. We have to find $b \in B$ such that $\mathfrak{B} \vDash \varphi[\bar{c}, b]$. If a is algebraic over $\mathfrak{C}$ then $a \in C \subseteq B$, and we can choose $b := a$.

Assume a is transcendental over $\mathfrak{C}$. We have a closer look at φ. The theory of ordered fields proves $(\neg t = t' \leftrightarrow (0 < t - t' \vee 0 < t' - t))$ and $(\neg t < t'' \leftrightarrow (t = t' \vee 0 < t' - t))$. Hence $\varphi(\bar{x}, y)$ is equivalent to a disjunction of conjunctions of atoms of the form $t = 0, 0 < t$ for terms $t = t(\bar{x}, y)$. Choose such a disjunct satisfied by $\bar{c}a$. Since a is not algebraic over $\mathfrak{C}$, all atoms $t(\bar{x}, y) = 0$ in this disjunct are trivial in that $P_t(\bar{c}, y)$ is the zero polynomial. Deleting the equations, leaves a conjunction of $0 < t_i(\bar{x}, y)$ for certain terms t_i; let $P_i(y) := P_{t_i}(\bar{c}, y) \in \mathfrak{C}[y]$. We have to find $b \in B$ such that $P_i(b) > 0$ for all i.

Let $c_0 < \cdots < c_{\ell-1}$ list the roots of the polynomials $P_i(y)$ in $\mathfrak{C}$ (equivalently, in $\mathfrak{A}$ or $\mathfrak{B}$). Assume $c_j < a < c_{j+1}$ (the cases $a < c_0$ and $c_{\ell-1} < a$ are similar). By Lemma 6.47, $P_i(y) > 0$ for all y between c_j and c_{j+1}. Set $b := (c_j + c_{j+1})/2 \in C \subseteq B$. □

As in the previous section we get an algebraic characterization of definable sets:

Exercise 6.50. Let $(\mathfrak{A}, <^{\mathfrak{A}}) \vDash RCF$ and $n \in \mathbb{N}$. $X \subseteq A^n$ is definable in $(\mathfrak{A}, <^{\mathfrak{A}})_A$ if and only if it is *semi-algebraic*: a Boolean combination of sets $\{\bar{a} \in A^n \mid P(\bar{a}) <^{\mathfrak{A}} 0\}$ with $P(\bar{x}) \in \mathfrak{A}[\bar{x}]$.

In particular, for $n = 1$, a subset $X \subseteq A$ is definable in $\mathfrak{A}_A$ if and only if it is a finite union of points and open intervals. This means that $\mathfrak{A}$ is *o-minimal*.

Corollary 6.51 (Tarski-Seidenberg). *Let $(\mathfrak{A}, <^{\mathfrak{A}}) \vDash RCF$. If $X \subseteq A^{m+n}$ is semi-algebraic, then so is $\{\bar{a} \in A^m \mid \exists \bar{b} \in A^n : (\bar{a}, \bar{b}) \in X\}$.*

6.5.3 Hilbert's 17th problem

Lemma 6.52. *Let $\mathfrak{A}$ be a formally real field and assume $a \in A$ is not a sum of squares in $\mathfrak{A}$. Then some real closed extension $\mathfrak{B}$ of $\mathfrak{A}$ satisfies $a <^{\mathfrak{B}} 0$.*

Proof. It suffices to show that the field extension $\mathfrak{A}(\sqrt{-a})$ is formally real: then a real closed $\mathfrak{B} \supseteq \mathfrak{A}(\sqrt{-a})$ satisfies $0 <^{\mathfrak{B}} -a$ since $-a$ is a square in $\mathfrak{B}$. Otherwise, $\sqrt{-a} \notin A$ and

$$-1 = \textstyle\sum_i (a_i + b_i\sqrt{-a})^2 = \sum_i (a_i^2 + 2a_ib_i\sqrt{-a} - b_i^2 a).$$

for certain $b_i, a_i \in A$. Since $\sqrt{-a} \notin A$ the $2a_ib_i\sqrt{-a}$ cancel, so $-1 = \sum_i a_i^2 - a\sum_i b_i^2$. Then

$$a = \frac{\sum_i a_i^2 + 1}{\sum_i b_i^2} = \frac{\sum_i a_i^2 \cdot \sum_i b_i^2 + \sum_i b_i^2}{(\sum_i b_i^2)^2}$$

is a sum of squares – contradiction. □

The following answers *Hilbert's 17th problem.*

Theorem 6.53 (Artin). *Let $\mathfrak{A}$ be a real closed field and $f(\bar{x}) \in \mathfrak{A}(\bar{x})$ be a rational function. If $0 <^{\mathfrak{A}} f(\bar{a})$ for all tuples $\bar{a}$ from A, then $f(\bar{x})$ is a sum of squares in $\mathfrak{A}(\bar{x})$.*

Proof. It is not hard to verify that $\mathfrak{A}(\bar{x})$ is formally real. Assume $f(\bar{x})$ is not a sum of squares in $\mathfrak{A}(\bar{x})$. By the lemma, there is some real closure $\mathfrak{B}$ of $\mathfrak{A}(\bar{x})$ such that $f(\bar{x}) <^{\mathfrak{B}} 0$. Write $f(\bar{x}) = P(\bar{x})/Q(\bar{x})$ for polynomials $P(\bar{x}), Q(\bar{x}) \in \mathfrak{A}[\bar{x}]$. Set

$$\text{“}f(\bar{x}) < 0\text{”} \;:=\; (t_P(\bar{x}) < 0 \wedge 0 < t_Q(\bar{x})) \vee (0 < t_P(\bar{x}) \wedge t_Q(\bar{x}) < 0).$$

Then $(\mathfrak{B}, <^{\mathfrak{B}})_A \models \exists\bar{x}\text{“}f(\bar{x}) < 0\text{”}$ witnessed by $\bar{x}$ viewed as elements of $\mathfrak{A}(\bar{x})$. But $\mathfrak{A} \subseteq \mathfrak{A}(\bar{x}) \subseteq \mathfrak{B}$, so $(\mathfrak{A}, <^{\mathfrak{A}}) \subseteq (\mathfrak{B}, <^{\mathfrak{B}})$ by uniqueness of the order. By model completeness, $(\mathfrak{A}, <^{\mathfrak{A}})_A \preccurlyeq (\mathfrak{B}, <^{\mathfrak{B}})_A$, so $(\mathfrak{A}, <^{\mathfrak{A}})_A \models \exists\bar{x}\text{“}f(\bar{x}) < 0\text{”}$. Thus, $f(\bar{a}) <^{\mathfrak{A}} 0$ for some $\bar{a}$ from A. $\square$