SS23 Mathematical Logic

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Very short history of logic:

Aristoteles, Frege, Cantor, Russel, Hilbert, Gödel, Turing

1 Propositional logic

1.1 Syntax and semantics

Propositional formulas

Lemma on unique readability

Semantics

Coincidence lemma

Validity, satisfiablilty, logical consequence, logical equivalence

1.2 Compactness theorem

Topological proof based on Tychonoff's theorem

Combinatorial proof with maximal finitely satisfiable sets

1.3 Sequent calculus

Formal propositional LK-proofs

Soundness and inversion principle

Completeness of propositional LK

2 First-order logic

2.1 Structures

Languages, structures, examples

2.2 Syntax

Part I: terms and unique readability

Part II: formulas and unique readability

2.3 Semantics

Part I: values of terms

Coincidence lemma for terms

Part II: truth values of formulas

First coincidence lemma

Second coincidence lemma

Elementary equivalence

Isomorphism lemma

2.4 Validity

validity and tautologyhood, satisfiability, logical consequence, logical equivalence

Equality axioms, Modus ponens, $\exists$-introduction

2.5 Substitutions

Substitution lemma

$\exists$-axioms

2.6 Hilbert calculus

Formal proofs in the Hilbert calculus

Propositional reasoning, $\forall$-axioms, $\forall$-introduction.

2.7 Gödel's completeness theorem

Statement weak and strong form

Proofs in theories, deductive closure

Proof of the completeness theorem

Henkin closure
Henkin term structure

2.8 Sequent calculus

Formal (first-order) *LK*-proofs

Second proof of the completeness theorem:

Completeness for equality free case: sets with Henkin properties

Completeness general case: congruence relations and factor structures

2.9 Corollaries

Deductive completeness

Compactness theorem

Löwenheim-Skolem: downwards

Löwenheim-Skolem: upwards

3 Computability

3.1 Register machines

Machines and computations

Flow-diagrams

3.2 Recursive functions

Recursive and primitive recursive functions

Recursive functions are computable

(Primitive) recursive relations, closure properties

3.3 Kleene normal form

Sequence coding

Gödel numbers of machines

Kleene normal form: computable functions are recursive

Universal computable partial function

s-m-n theorem and Kleene's fixed point theorem

3.4 Church Turing thesis and squeezing arguments

Church-Turing thesis on computability

Kreisel's squeezing argument for provability

3.5 The Ackermann-Péter function

Péter's definition

Knuth arrows

Ackerman-Péter function is recursive and not primitive recursive

3.6 Elimination of recursion

Cantor pairing

Gödel's β-function

Chracterization of recursive functions without primitive recursion

3.7 Recursively enumerable sets

Definition and closure properties

Universal r.e. set

4 Arithmetic

4.1 Definability

Δ_0- and Σ_1-formulas

Characterization of recursive functions and r.e. relations by Σ_1-definability

4.2 Gödelization

Gödel numbers of formulas

Recursivity of various syntactical operations

4.3 Tarski's undefinability of truth

Tarski's undefinability of truth

Axiomatizable theories, deductively complete theories, independent sentences

Gödel's first incompleteness theorem for true theories

4.4 Gödel's first incompleteness theorem

Representations of functions and relations in theories

Theories that admit representations

Fixed-point lemma of Gödel and Carnap

Gödel's first incompleteness theorem: general form

4.5 Gödel's second incompleteness theorem

Löb conditions

Löb's theorem

Gödel's second incompleteness theorem: general form

4.6 Robinson's Q

Definition as Shoenfield's version

Σ_1-completeness of Q

4.7 End-extensions

Models of Q are end-extensions of the standard model

Second proof of Σ_1-completeness of Q

4.8 Incompleteness: concrete form

Q admits representations

Gödel's first incompleteness theorem: concrete form

Church-Turing undecidability of Hilbert's Entscheidungsproblem

4.9 Gödel- and Rosser-sentences

Independence of Gödel-sentences for ω-consistent theories

Independence of Rosser-sentences for consistent theories