COBHAM RECURSIVE SET FUNCTIONS AND WEAK SET THEORIES

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The Cobham recursive set functions (CRSF) provide a notion of polynomial time computation over general sets. In this chapter, we determine a subtheory $\mathrm{KP}_1^{\mathrm{u}}$ of Kripke-Platek set theory whose Σ_1-definable functions are precisely CRSF. The theory $\mathrm{KP}_1^{\mathrm{u}}$ is based on the $\in$-induction scheme for Σ_1-formulas whose leading existential quantifier satisfies certain boundedness and uniqueness conditions. Dropping the uniqueness condition and adding the axiom of global choice results in a theory $\mathrm{KPC}_1^{\preccurlyeq}$ whose Σ_1-definable functions are CRSF^C, that is, CRSF relative to a global choice function C. We further show that the addition of global choice is conservative over certain local choice principles.

1. Introduction

Barwise begins his chapter on admissible set recursion theory with: "There are many equivalent definitions of the class of recursive functions on the natural numbers. [...] As the various definitions are lifted to domains other than the integers (e.g., admissible sets) some of the equivalences break down. This break-down provides us with a laboratory for the study of recursion theory." ([5, p.153])

Let us informally distinguish two types of characterization of the computable functions or subsets thereof, namely, *recursion theoretic* and *definability theoretic* ones. Recursion theoretically, the computable functions on ω are those obtainable from certain simple initial functions by means of composition, primitive recursion and the μ-operator. As a second example, the primitive recursive functions are similarly defined but without the μ-operator. A third example is the recursion theoretic definition of the polynomial time functions by Cobham recursion [13] or by Bellantoni-Cook safe-normal recursion [9]. Definability theoretically, the computable functions are those that are Σ_1-definable in the true theory of arithmetic. A more relevant example of a definability theoretic definition is the clas-

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sic theorem of Parsons and Takeuti (see [12]) that the primitive recursive functions are those that are Σ_1-definable in the theory $\mathrm{I}\Sigma_1$; namely, one additionally requires that this theory proves the totality and functionality of the defining Σ_1-formula. Analogously, the polynomial time functions have a definability theoretic definition as the Σ^b_1-definable functions of S^1_2 [11]. For more definability theoretic definitions of weak subrecursive classes, see Cook-Nguyen [15].

Admissible set recursion theory provides a definability theoretic generalization of computability: one considers functions which are Σ_1-definable (in the language of set theory) in an admissible set, that is, a transitive standard model of Kripke-Platek set theory KP. Recall that KP consists of the axioms for Extensionality, Union, Pair, Δ_0-Separation, Δ_0-Collection and $\in$-Induction for all formulas $\varphi(x, \vec{w})$:

$$\forall y\, (\forall u{\in}y\, \varphi(u, \vec{w}) \to \varphi(y, \vec{w})) \to \varphi(x, \vec{w}).$$

To some extent this generalization of computability extends to the recursion theoretic view. By the Σ-Recursion Theorem ([5, Chapter I, Theorem 6.4]) the Σ_1-definable functions of KP are closed under $\in$-recursion. This implies that the *primitive recursive set functions* (PRSF) of [20] are all Σ_1-definable in KP. By definition, a function on the universe of sets is in PRSF if it is obtained from certain simple initial functions by means of composition and $\in$-recursion. Hence, PRSF is a recursion theoretic generalization of the primitive recursive functions. Paralleling Parson's theorem, Rathjen [22] showed that this generalization extends to the definability theoretic view: PRSF contains precisely those functions that are Σ_1-definable in KP_1, the fragment of KP where $\in$-Induction is adopted for Σ_1-formulas only. One can thus view PRSF as a reasonable generalization of primitive recursive computation to the universe of sets.

It is natural to wonder whether one can find a similarly good analogue of polynomial time computation on the universe of sets. In [7] we proposed such an analogue, following Cobham's [13] characterization of the polynomial time computable functions on ω as those obtained from certain simple initial functions, including the smash function $\#$, by means of composition and *limited recursion on notation*. Limited recursion on notation restricts both the depth of the recursion and the size of values. Namely, a recursion on notation on x has depth roughly $\log x$; being limited means that all values are required to be bounded by some smash term $x\#\cdots\#x$.

In [7] a smash function for sets is introduced. The role of recursion on notation is taken by $\in$-recursion, and being limited is taken to mean being in a certain sense embeddable into some #-term. In this way, [7] defines the class of *Cobham recursive set functions* (CRSF), a recursion theoretic generalization of polynomial time computation from ω to the universe of sets. This chapter extends the analogy to the definability theoretic view.

A definability theoretic characterization of polynomial time on ω has been given by Buss (cf. [12]). It is analogous to Parsons' theorem, with $\mathrm{I}\Sigma_1$ replaced by S^1_2 and Σ_1 replaced by a class Σ^b_1 of "bounded" Σ_1-formulas. The theory S^1_2 has a language including the smash function # and is based on a restricted form of induction scheme for Σ^b_1-formulas, in which the depth of an induction is similar to the depth of a recursion on notation.

Both directions in Buss' characterization hold in a strong form. First, S^1_2 *defines* polynomial time functions in the sense that one can conservatively add Σ^b_1-defined symbols and prove Cook's PV_1 [14], a theory based on the equations that can arise from derivations of functions in Cobham's calculus. Second, polynomial time functions *witness* simple theorems of S^1_2. More precisely, if S^1_2 proves $\exists y\, \varphi(y, \vec{x})$ with φ in Δ^b_0, that is, a "bounded" Δ_0-formula, then $\forall \vec{x}\, \varphi(f(\vec{x}), \vec{x})$ is true for some polynomial time computable f, even provably in PV_1 and S^1_2.

In this chapter, we analogously replace Rathjen's theory KP_1 and Σ_1-definability with a theory $\mathrm{KP}^{\preccurlyeq}_1$ and $\Sigma^{\preccurlyeq}_1$-definability; here $\Sigma^{\preccurlyeq}_1$-formulas are "bounded" Σ_1-formulas, defined using set smash and the embeddability notion $\preccurlyeq$ of [7]. The theory $\mathrm{KP}^{\preccurlyeq}_1$ has a finite language containing, along with $\in$, some basic CRSF functions including the set smash and has $\in$-Induction restricted to $\Sigma^{\preccurlyeq}_1$-formulas.

As we shall see, $\mathrm{KP}^{\preccurlyeq}_1$ defines CRSF analogously to the first part of Buss' characterization. An analogy of the second part, the witnessing theorem, would state that whenever $\mathrm{KP}^{\preccurlyeq}_1$ proves $\exists y\, \varphi(y, \vec{x})$ for a Δ_0-formula φ, then $\varphi(f(\vec{x}), \vec{x})$ is provable in ZFC (or ideally in a much weaker theory) for some f in CRSF. But this fails: a function witnessing $\exists y\, (x \neq 0 \to y \in x)$ would be a global choice function, and this is not available in $\mathrm{KP}^{\preccurlyeq}_1$.

We discuss two ways around this obstacle. If we add the axiom of global choice we get a theory $\mathrm{KPC}^{\preccurlyeq}_1$ and indeed can prove a witnessing theorem as desired (Theorem 6.13). The functions Σ_1-definable in $\mathrm{KPC}^{\preccurlyeq}_1$ are precisely those that are CRSF with a global choice function as an additional initial function (Corollary 6.2). Thus, Buss' theorems for S^1_2 and polynomial time on ω have full analogues on universes of sets equipped with global choice, if we consider the global choice function as a feasible function in such a universe.

Our second way around the obstacle is to further weaken the induction scheme, the crucial restriction being that the witness to the existential quantifier in a $\Sigma_1^{\preccurlyeq}$-formula is required to be unique. The resulting theory $\mathrm{KP}_1^{\mathrm{u}}$ still defines CRSF in the strong sense that one can conservatively add $\Sigma_1^{\preccurlyeq}$-defined function symbols and prove $\mathrm{T}_{\mathrm{crsf}}$, an analogue of PV_1 containing the equations coming from derivations in the CRSF calculus (Theorem 5.2). We prove a weak form of witnessing (Theorem 6.10): if $\mathrm{KP}_1^{\mathrm{u}}$ proves $\exists y\, \varphi(y, \vec{x})$ for φ a Δ_0-formula, then $\mathrm{T}_{\mathrm{crsf}}$ proves $\exists y \in f(\vec{x})\, \varphi(y, \vec{x})$ for some f in CRSF. This suffices to infer a definability theoretic characterization of CRSF on an arbitrary universe of sets: the Σ_1-definable functions of $\mathrm{KP}_1^{\mathrm{u}}$ are precisely those in CRSF (Corollary 6.1). We do not know whether this holds for $\mathrm{KP}_1^{\preccurlyeq}$.

In conclusion, we address the question on how much stronger $\mathrm{KPC}_1^{\preccurlyeq}$ is compared to $\mathrm{KP}_1^{\preccurlyeq}$. We show that the difference can be encapsulated in certain local choice principles.

The outline of the chapter is as follows. Section 2 recalls the development of CRSF from [7]. For the formalizations in this chapter we will use a slightly different, but equivalent, definition of CRSF which we describe in Proposition 2.9. Section 3 defines the three theories $\mathrm{KP}_1^{\preccurlyeq}$, $\mathrm{KP}_1^{\mathrm{u}}$ and $\mathrm{T}_{\mathrm{crsf}}$ mentioned above. They extend a base theory T_0 that we "bootstrap" in Section 4, in particular deriving various lemmas which allow us to manipulate embedding bounds. Section 5 proves the Definability Theorem 5.2 for $\mathrm{KP}_1^{\mathrm{u}}$. A technical difficulty is that $\mathrm{KP}_1^{\mathrm{u}}$ is too weak to eliminate $\Sigma_1^{\preccurlyeq}$-defined function symbols in the way this is usually done in developments of KP or S_2^1 (see Section 5.2). Section 6 proves the Witnessing Theorems 6.10 and 6.13 for $\mathrm{KP}_1^{\mathrm{u}}$ and $\mathrm{KPC}_1^{\preccurlyeq}$, and the Corollaries 6.1 and 6.2 on definability theoretic characterizations of CRSF. This is done via a modified version of Avigad's model-theoretic approach to witnessing [4] (see Section 6.1). Our proof gives some insight about CRSF: roughly, its definition can be given using only a certain simple form of embedding (see Section 6.4). Section 7 proves the conservativity of global choice over certain local choice principles (Theorem 7.2). Here, we use a class forcing as in [17] to construct a generic expansion of any (possibly non-standard) model of our set theory. Some extra care is needed since our set theory is rather weak.

Several related recursion theoretic notions of polynomial time set functions have been described earlier by other authors. The characterization of polynomial time by Turing machines has been generalized in Hamkins and Lewis [18] to allow binary input strings of length ω. We refer to [8] for some comparison with CRSF. Yet another characterization of polynomial time comes from the Immerman-Vardi Theorem from descriptive complexity theory (cf. [16]). Following this, Sazonov [23] gives a theory operating with terms allowing for least fixed-point constructs to capture polynomial time computations on (binary encodings of) Mostowski graphs of hereditarily finite sets. Not all of Sazonov's set functions are CRSF [7]. But under a suitable encoding of binary strings by hereditarily finite sets, CRSF does capture polynomial time [7, Theorems 30, 31].

Arai [2] gives a different such class of functions. His *Predicatively Computable Set Functions* (PCSF) form a subclass of the *Safe Recursive Set Functions* (SRSF) from [6]. SRSF is defined in analogy to Bellantoni and Cook's recursion theoretic characterization of polynomial time [9], different from Cobham's. Bellantoni and Cook's functions have two sorts of arguments, called "normal" and "safe", and recursion on notation is allowed to recurse only on normal arguments, while values obtained by such recursions are safe. Similarly, SRSF and PCSF contain two-sorted functions. It is shown in [7] that CRSF coincides with the functions having only normal arguments in PCSF^+ (from [2]), a slight strengthening of PCSF. In a recent manuscript [1], Arai gives a definability theoretic characterization of PCSF^ι, a class of set functions intermediate between PCSF and PCSF^+. He proves a weak form of witnessing akin to ours. He uses two-sorted set-theoretic proof systems whose normal sort ranges over a transitive substructure of the universe, and which contains an inference rule ensuring closure of this substructure under certain definable functions. Like $\mathrm{KP}_1^{\mathrm{u}}$, these systems contain a form of "unique" Σ_1-Induction. As in our setting, eliminating defined function symbols is problematic; the final system in [1] is a union of a hierarchy of systems, each level introducing infinitely many function symbols. Thus, dealing with similar problems, Arai's solution is quite different from the one presented here; as is his proof, which is based on cut-elimination.

2. Cobham Recursive Set Functions

In this section we review some definitions and results from [7]. In later sections, many of these results will be formalized in suitable fragments of KP.

As mentioned in the introduction, [7] generalizes Cobham's recursion theoretic characterization of polynomial time to arbitrary sets. We recall Cobham's characterization. On ω the smash $x\#y$ is defined as $2^{|x|\cdot|y|}$ where

$|x| := \lceil \log(x+1) \rceil$ is the *length* (of the binary representation) of x. We have successor functions $s_0(x) := 2x$ and $s_1(x) = 2x+1$ which add respectively 0 and 1 to the end of the binary representation of x.

Theorem 2.1: (Cobham 1965) *The* polynomial time *functions on* ω *are obtained from* initial functions, *namely, projections* $\pi^r_j(x_1,\ldots,x_r) := x_j$, *constant 0, successors* s_0, s_1 *and the smash #, by composition and* limited recursion on notation*: if* $h(\vec{x})$, $g_0(y,z,\vec{x})$, $g_1(y,z,\vec{x})$ *and* $t(y,\vec{x})$ *are polynomial time, then so is the function* $f(y,\vec{x})$ *given by*

$$f(0,\vec{x}) = h(\vec{x}),$$
$$f(s_b(y),\vec{x}) = g_b(y, f(y,\vec{x}), \vec{x}) \quad \textit{for } b \in \{0,1\} \textit{ and } s_b(y) \neq 0,$$

provided that $f(y,\vec{x}) \leqslant t(y,\vec{x})$ *holds for all* $y, \vec{x}$.

One can equivalently ask t to be built by composition from only projections, 1 and #; or just demand $|f(y,x_1,\ldots,x_k)| \leqslant p(|y|,|x_1|,\ldots,|x_k|)$ for some polynomial p.

We move to some fixed universe of sets, that is, a model of ZFC. The analogue of smash defined in [7] is best understood in terms of Mostowski graphs. The *Mostowski graph* of a set x has as vertices the elements of the transitive closure $\mathrm{tc}^+(x) := \mathrm{tc}(\{x\})$ and has a directed edge from u to v if $u \in v$. Every such graph has a unique source and a unique sink.

The *set smash* $x\#y$ replaces each vertex of x by (a copy of the graph of) y with incoming edges now going to the source of y and outgoing edges now leaving the sink of y. It can be defined using *set composition* $x \odot y$, which places a copy of x above y and identifies the source of x with the sink of y. Writing 0 for $\emptyset$,

$$x \odot y := \begin{cases} y & \text{if } x = 0, \\ \{u \odot y : u \in x\} & \text{otherwise} \end{cases}$$
$$x\#y := y \odot \{u\#y : u \in x\}.$$

The Mostowski graph of $x\#y$ is isomorphic to the graph with vertices $\mathrm{tc}^+(x) \times \mathrm{tc}^+(y)$ and directed edges from $\langle u', v' \rangle$ to $\langle u, v \rangle$ if either $u' = u \wedge v' \in v$ or $u' \in u \wedge v' = y \wedge v = 0$ (see [7, Section 2]). An isomorphism is given by $\sigma_{x,y}(u,v) := v \odot \{u'\#y : u' \in u\}$.

A *#-term* is built by composition from projections, $\#, \odot$ and the constant $1 = \{0\}$. Such terms serve as analogues of polynomial length bounds, with the bounding relation $\preccurlyeq$ defined as follows: $x \preccurlyeq y$ means that there is a *(multi-valued) embedding* that maps vertices $u \in \mathrm{tc}(x)$ to pairwise disjoint non-empty sets $V_u \subseteq \mathrm{tc}(y)$ such that whenever $u' \in u$ and $v \in V_u$, then there exists $v' \in V_{u'} \cap \mathrm{tc}(v)$. The notation $\tau(\cdot, \vec{w}) : x \preccurlyeq y$ means that $u \mapsto \tau(u, \vec{w})$ is such an embedding. Then [7] generalizes Cobham's definition as follows.

Definition 2.2: The *Cobham recursive set functions* (CRSF) are obtained from *initial functions*, namely projections, constant $0 := \emptyset$, pair $\{x, y\}$, union $\bigcup x$, set smash $x\#y$, and the conditional

$$cond_{\in}(x, y, u, v) := \begin{cases} x \text{ if } u \in v \\ y \text{ otherwise,} \end{cases}$$

by composition and *Cobham recursion*: if $g(x, z, \vec{w})$, $\tau(u, x, \vec{w})$ and $t(x, \vec{w})$ are CRSF, then so is the function $f(x, \vec{w})$ given by

$$f(x, \vec{w}) = g(x, \{f(y, \vec{w}) : y \in x\}, \vec{w})$$

provided that $\tau(\cdot, x, \vec{w}) : f(x, \vec{w}) \preccurlyeq t(x, \vec{w})$ holds for all $x, \vec{w}$.

Here the *embedding proviso* $\tau(\cdot, x, \vec{w}) : f(x, \vec{w}) \preccurlyeq t(x, \vec{w})$ ensures, intuitively, that a definition by recursion is allowed only provided that we can already bound the "structural complexity" of the defined function f. A relation is CRSF if its characteristic function is. Direct arguments show (see [7, Theorem 13]):

Proposition 2.3:

(a) (Separation) *If* $g(u, \vec{w})$ *is in* CRSF*, then so is* $f(x, \vec{w}) := \{u \in x : g(u, \vec{w}) \neq 0\}$.
(b) *The* CRSF *relations contain* $x \in y$ *and* $x = y$*, are closed under Boolean combinations and* $\in$*-bounded quantifications* $\exists u \in x$ *and* $\forall u \in x$*.*

It is then not hard to show that transitive closure $\mathrm{tc}(x)$, set composition $x \odot y$, the isomorphism $\sigma_{x,y}(u, v)$ and its inverses $\pi_{1,x,y}(z), \pi_{2,x,y}(z)$ are CRSF [7, Theorem 13]. In particular, #-terms are CRSF. Further, one can derive the following central lemma [7, Lemma 20]. It says that $\preccurlyeq$ is a pre-order and that #-terms enjoy some monotonicity properties one would expect from a reasonable analogue of "polynomial length bounds".

Lemma 2.4: *Below if* τ_0 *and* τ_1 *are in* CRSF *then* σ *can also be chosen in* CRSF*.*

(a) (Transitivity) *If* $\tau_0(\cdot, x, y, \vec{w}) : x \preccurlyeq y$ *and* $\tau_1(\cdot, y, z, \vec{w}) : y \preccurlyeq z$*, then there exists* $\sigma(u, x, y, z, \vec{w})$ *such that* $\sigma(\cdot, x, y, z, \vec{w}) : x \preccurlyeq z$*.*

(b) (Monotonocity) *Let* $t(x, \vec{w})$ *be a #-term. If* $\tau_0(\cdot, x, z, \vec{w}) : z \preccurlyeq t(x, \vec{w})$ *and* $\tau_1(\cdot, x, y, \vec{w}) : x \preccurlyeq y$*, then there exists* $\sigma(u, x, y, z, \vec{w})$ *such that* $\sigma(\cdot, x, y, z, \vec{w}) : z \preccurlyeq t(y, \vec{w})$.

Based on this lemma, a straightforward induction on the length of a derivation of a CRSF function shows [7, Theorem 17]:

Theorem 2.5: (Bounding) *For every* $f(\vec{x})$ *in* CRSF *there are a #-term* $t(\vec{x})$ *and a* CRSF *function* $\tau(u, \vec{x})$ *such that* $\tau(\cdot, \vec{x}) : f(\vec{x}) \preccurlyeq t(\vec{x})$.

In fact, in the definition of Cobham recursion one can equivalently require the function t in the embedding proviso to be a #-term [7, Theorem 21]. Using the Bounding Theorem and the Monotonicity Lemma one can obtain, similarly to Theorems 23, 29 and 30 of [7]:

Theorem 2.6:

(a) (Replacement) *If* $f(y, \vec{w})$ *is* CRSF*, then so is*

$$f\text{"}(x, \vec{w}) = \{f(y, \vec{w}) : y \in x\}.$$

(b) (Course of values recursion) *If* $g(x, z, \vec{w})$*,* $\tau(u, x, \vec{w})$ *and* $t(x, \vec{w})$ *are* CRSF*, then so is*

$$f(x, \vec{w}) := g(x, \{\langle u, f(u, \vec{w})\rangle : u \in \mathrm{tc}(x)\}, \vec{w})$$

provided $\tau(\cdot, x, \vec{w}) : f(x, \vec{w}) \preccurlyeq t(x, \vec{w})$ *holds for all* $x, \vec{w}$.

(c) (Impredicative Cobham recursion) *If* $g(x, z, \vec{w})$*,* $\tau(u, y, x, \vec{w})$ *and* $t(x, \vec{w})$ *are* CRSF*, then so is*

$$f(x, \vec{w}) = g(x, f\text{"}(x, \vec{w}), \vec{w})$$

provided $\tau(\cdot, f(x, \vec{w}), x, \vec{w}) : f(x, \vec{w}) \preccurlyeq t(x, \vec{w})$ *holds for all* $x, \vec{w}$.

Closure under replacement (a) implies that $x \times y$ is CRSF [7, Theorem 14]. Impredicative Cobham recursion (c) is, intuitively, somewhat circular in that the embedding τ may use as a parameter the set $f(x, \vec{w})$ whose existence it is supposed to justify.

We introduce a variant definition of CRSF that uses *syntactic Cobham recursion*. The name "syntactic" indicates that it does not have an embedding proviso, but rather constructs a new function from any CRSF functions g, τ and #-term t. We also allow the bound to be impredicative in the sense of (c) above.

Definition 2.7: Let $g(x, z, \vec{w})$ and $\tau(u, y, x, \vec{w})$ be functions and $t(x, \vec{w})$ a #-term. Then *syntactic Cobham recursion* gives the function $f(x, \vec{w})$ defined by

$$f(x, \vec{w}) = \begin{cases} g(x, f``(x, \vec{w}), \vec{w}) & \text{if } \tau \text{ is an embedding into } t \text{ at } x, \vec{w} \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

where the condition "τ is an embedding into t at $x, \vec{w}$" stands for

$$\tau(\cdot, g(x, f``(x, \vec{w}), \vec{w}), x, \vec{w}) : g(x, f``(x, \vec{w}), \vec{w}) \preccurlyeq t(x, \vec{w}). \tag{2.2}$$

Note that Proposition 2.3 implies that the condition (2.2) is a CRSF relation, cf. (3.2) in Section 3.2.

Proposition 2.8: *The* CRSF *functions are precisely those obtained from the initial functions by composition and syntactic Cobham recursion.*

Proof: Since the condition (2.2) is a CRSF relation, (2.1) can be written $f(x, \vec{w}) = g'(x, f``(x, \vec{w}), \vec{w})$ for some $g'$ in CRSF. The embedding proviso $\tau(\cdot, f(x, \vec{w}), x, \vec{w}) : f(x, \vec{w}) \preccurlyeq t(x, \vec{w})$ holds for all $x, \vec{w}$, since either (2.2) holds, in which case $f(x, \vec{w}) = g(x, f``(x, \vec{w}), \vec{w})$ so (2.2) gives us the embedding, or $f(x, \vec{w}) = 0$, in which case any function is an embedding of $f(x, \vec{w})$ into $t(x, \vec{w})$. We thus get that f is CRSF by impredicative Cobham recursion.

Conversely, assume $f(x, \vec{w})$ is obtained from g, τ, t by Cobham recursion, and in particular that the embedding proviso is satisfied for all $x, \vec{w}$. Then f satisfies (2.1), so f can be obtained by syntactic Cobham recursion. By [7, Theorem 21], we can assume that t is a #-term. □

The next proposition describes the definition of CRSF that we will formalize with the theory $\mathrm{T}_{\mathrm{crsf}}$ in Section 3.4. Closure under replacement and the extra initial functions are included to help with the formalization.

Proposition 2.9: *The* CRSF *functions are precisely those obtained from projection, zero, pair, union, conditional, transitive closure, cartesian product, set composition and set smash functions by composition, replacement and syntactic Cobham recursion.*

Remark 2.10: For an arbitrary function $g(\vec{x})$, let CRSF^g be defined as CRSF but with $g(\vec{x})$ as additional initial function. This class might be interpreted as a set-theoretic analogue of polynomial time computations with an oracle function $g(\vec{x})$. If there is $\tau(u, \vec{x})$ in CRSF^g such that $\tau(\cdot, \vec{x}) : g(\vec{x}) \preccurlyeq t(\vec{x})$ for some #-term $t(\vec{x})$, then all results mentioned in this section "relativize", that is, hold true with CRSF replaced by CRSF^g.

3. Theories for CRSF

3.1. *The language* $\mathbf{L_0}$ *and theory* $\mathbf{T_0}$

The language L_0 contains the relation symbol $\in$ and symbols for the following CRSF functions:

$$0,\ 1,\ \bigcup x,\ \{x,y\},\ x\times y,\ \mathrm{tc}(x),\ x\odot y,\ x\#y.$$

The meaning of these symbols is given by their defining axioms:

symbol	*defining axiom*
0	$u \notin 0$
1	$u \in 1 \leftrightarrow u = 0$
$\bigcup x$	$u \in \bigcup x \leftrightarrow \exists y \in x\,(u \in y)$
$\{x,y\}$	$u \in \{x,y\} \leftrightarrow u = x \vee u = y$
$x \times y$	$u \in x \times y \leftrightarrow \exists x' \in x\, \exists y' \in y\,(u = \langle x', y'\rangle)$
$\mathrm{tc}(x)$	$u \in \mathrm{tc}(x) \leftrightarrow u \in x \vee \exists y \in x\,(u \in \mathrm{tc}(y))$
$x \odot y$	$0 \odot y = y \wedge (x \neq 0 \rightarrow x \odot y = \{z \odot y : z \in x\})$
$x\#y$	$\exists w \in \mathrm{tc}^+(x\#y)\ (w = \{z\#y : z \in x\} \wedge (x\#y = y \odot w))$

The table above uses some special notations: as usual, $\{x\}$ stands for the term $\{x,x\}$, $\langle x,y\rangle$ for the term $\{\{x\},\{x,y\}\}$, $x \cup y$ for the term $\bigcup\{x,y\}$, and $x \subseteq y$ for the formula $\forall u{\in}x\,(u \in y)$. We write $\mathrm{tc}^+(x)$ for the term $\mathrm{tc}(\{x\})$. The final two lines of the table use "replacement terms". More generally, we use three types of comprehension terms:

Definition 3.1: The following notations are used for comprehension terms:

- *Proper comprehension terms*: for a formula $\varphi(u,\vec{x})$, we write

  $$z = \{u \in x : \varphi(u,\vec{x})\}$$

  for

  $$\forall u{\in}z\,(u \in x \wedge \varphi(u,\vec{x})) \wedge \forall u{\in}x\,(\varphi(u,\vec{x}) \rightarrow u \in z).$$

- *Collection terms*: for a formula $\varphi(u,v,\vec{x})$, we write

  $$z = \{v : \exists u \in x\, \varphi(u,v,\vec{x})\}$$

  for

  $$\forall v{\in}z\, \exists u \in x\, \varphi(u,v,\vec{x}) \wedge \forall v\,((\exists u \in x\, \varphi(u,v,\vec{x})) \rightarrow v \in z). \tag{3.1}$$

- *Replacement terms*: for a term $t(u,\vec{x})$, we write

  $$z = \{t(u,\vec{x}) : u \in x\}$$

for

$$z = \{v : \exists u \in x\, (v = t(u, \vec{x}))\}.$$

Such terms may not be used as arguments to functions.

We use collection terms only in contexts where we have

$$\forall u{\in}x\, \exists! v\, \varphi(u, v, \vec{x}),$$

so that (3.1) is equivalent to

$$\forall v{\in}z\, \exists u \in x\, \varphi(u, v, \vec{x}) \wedge \forall u{\in}x\, \exists v \in z\, \varphi(u, v, \vec{x}).$$

In particular, this is the case for replacement terms. These restrictions ensure that our formulas are $\Delta_0(\mathrm{L}_0)$ whenever φ is, provided that the comprehension terms have the uniqueness property. As usual, we write $\exists! y\, \varphi(y, \vec{x})$ for $\exists^{\leqslant 1} y\, \varphi \wedge \exists y\, \varphi$, where $\exists^{\leqslant 1} y\, \varphi$ stands for

$$\forall y, y'\, (\varphi(y, \vec{x}) \wedge \varphi(y', \vec{x}) \to y = y').$$

Instead of Δ_0 etc. we use more precise notation, making the language explicit:

Definition 3.2: Let L be a language containing L_0. Then $\Delta_0(L)$ is the set of L-formulas φ in which all quantifiers are $\in$-bounded, that is, of the form $\exists x \in t$ or $\forall x{\in}t$ where t is an L-term not involving x. We refer to t as an *$\in$-bounding term in φ.*

The classes $\Sigma_1(L)$ and $\Pi_1(L)$ contain the formulas obtained from $\Delta_0(L)$-formulas by respectively existential and universal quantification, and $\Pi_2(L)$ contains those obtained from $\Sigma_1(L)$-formulas by universal quantification.

We define our basic theory, which the other theories we consider will extend.

Definition 3.3: The theory T_0 consists of

- the defining axioms for the symbols in L_0
- the *Extensionality* axiom: $x \neq y \to \exists u \in x\, (u \notin y) \vee \exists u \in y\, (u \notin x)$
- the *Set Foundation* axiom: $x \neq 0 \to \exists y \in x\, \forall u{\in}y\, (u \notin x)$
- the tc-*Transitivity* axiom: $y \in \mathrm{tc}(x) \to y \subseteq \mathrm{tc}(x)$
- the $\Delta_0(\mathrm{L}_0)$-*Separation* scheme: $\exists z\, (z = \{u \in x : \varphi(u, \vec{w})\})$ for $\varphi \in \Delta_0(\mathrm{L}_0)$.

Lemma 3.4: *The theory* T_0 *proves the* $\Delta_0(\mathrm{L}_0)$*-Induction scheme*

$$\forall y\, (\forall u{\in}y\, \varphi(u, \vec{w}) \to \varphi(y, \vec{w})) \to \varphi(x, \vec{w}) \quad \text{for } \varphi \in \Delta_0(\mathrm{L}_0).$$

Proof: $\Delta_0(\mathrm{L}_0)$-Induction is logically equivalent to $\Delta_0(\mathrm{L}_0)$-*Foundation*

$$\varphi(x, \vec{w}) \to \exists y\, (\varphi(y, \vec{w}) \wedge \forall u{\in}y\, \neg\varphi(u, \vec{w}))$$

for φ in $\Delta_0(\mathrm{L}_0)$ which is derived in T_0 as follows. Assume $\varphi(x, \vec{w})$ and use $\Delta_0(\mathrm{L}_0)$-Separation to get the set $z = \{y \in \mathrm{tc}^+(x) : \varphi(y, \vec{w})\}$. Then $x \in \mathrm{tc}^+(x)$ by the defining axiom for tc, so $x \in z \neq 0$. Choose y as the $\in$-minimal element in z according to Set Foundation. Then $\varphi(y, \vec{w})$ and, if $u \in y$, then $u \notin z$, and thus $\neg\varphi(u, \vec{w})$ because $u \in y \subseteq \mathrm{tc}^+(x)$ by tc-Transitivity. □

Remark 3.5: It is for the sake of the previous lemma that the tc-Transitivity axiom is included in T_0. In fact, this axiom is equivalent to $\Delta_0(\mathrm{L}_0)$-Induction with respect to the remaining axioms of T_0.

3.2. *Embeddings*

An *embedding* of a set x into a set y is an injective multifunction τ from $\mathrm{tc}(x)$ to $\mathrm{tc}(y)$ which respects the $\in$-ordering on $\mathrm{tc}(x)$ in a certain sense. There are several variants of embeddings, depending on how τ is defined.

Definition 3.6: A function symbol $\tau(u, \vec{w})$ is a *strongly uniform embedding* (with parameters $\vec{w}$) of x in y if the following $\Delta_0(\mathrm{L}_0 \cup \{\tau\})$-formula holds (where for the sake of readability we suppress the parameters $\vec{w}$):

$$\begin{aligned} &\forall u{\in}\mathrm{tc}(x)\, (\tau(u) \subseteq \mathrm{tc}(y)) \\ &\wedge\ \forall u{\in}\mathrm{tc}(x)\, (\tau(u) \neq 0) \\ &\wedge\ \forall u, u'{\in}\mathrm{tc}(x)\, (u \neq u' \to (\tau(u) \text{ and } \tau(u') \text{ are disjoint})) \\ &\wedge\ \forall u{\in}\mathrm{tc}(x)\, \forall u'{\in}u\, \forall v{\in}\tau(u)\, \exists v' \in \tau(u')\, (v' \in \mathrm{tc}(v)). \end{aligned} \tag{3.2}$$

The last conjunct is read as "for all $u, u' \in \mathrm{tc}(x)$, if $u' \in u$ then for every v in the image of u there is some v' in the image of u' with $v' \in \mathrm{tc}(v)$." Note that the "identity" multifunction $u \mapsto \{u\}$ is an embedding of x in x; we will call an embedding of this form *the identity embedding*.

We abbreviate (3.2) by $\tau(\cdot, \vec{w}) : x \preccurlyeq y$. We next introduce terminology for embeddings whose graphs are given by formulas and embeddings whose graphs are given by sets.

Definition 3.7: Given a formula $\varepsilon(u, v, \vec{w})$, we define $\varepsilon(\cdot, \cdot, \vec{w}) : x \preccurlyeq y$ to be condition (3.2) with $v \in \tau(u, \vec{w})$ replaced by $v \in \mathrm{tc}(y) \wedge \varepsilon(u, v, \vec{w})$. More precisely, $\varepsilon(\cdot,\cdot,\vec{w}) : x \preccurlyeq y$ means:

$$\begin{aligned}&\forall u{\in}\mathrm{tc}(x)\,\exists v \in \mathrm{tc}(y)\,\varepsilon(u,v,\vec{w})\\ &\quad\wedge\ \forall u,u'{\in}\mathrm{tc}(x)\,\forall v{\in}\mathrm{tc}(y)\,(u \neq u' \to \neg\varepsilon(u,v,\vec{w}) \vee \neg\varepsilon(u',v,\vec{w}))\\ &\quad\wedge\ \forall u{\in}\mathrm{tc}(x)\,\forall u'{\in}u\,\forall v{\in}\mathrm{tc}(y)\,(\varepsilon(u,v,\vec{w}) \to \exists v' \in \mathrm{tc}(v)\,\varepsilon(u',v',\vec{w})).\end{aligned} \tag{3.3}$$

This kind of embedding is called a *weakly uniform embedding.*

Definition 3.8: For a set e, we write $e : x \preccurlyeq y$ if $e \subseteq \mathrm{tc}(x) \times \mathrm{tc}(y)$ and $\varepsilon(\cdot,\cdot,e) : x \preccurlyeq y$ holds when $\varepsilon(u,v,e)$ is the formula $\langle u,v\rangle \in e$. We write simply $x \preccurlyeq y$ to abbreviate $\exists e\,(e : x \preccurlyeq y)$. This is called a *nonuniform embedding.*

Note that $e : x \preccurlyeq y$ is $\Delta_0(\mathrm{L}_0)$. More generally, if ε is a $\Delta_0(L)$-formula in a language $L \supseteq \mathrm{L}_0$ then $\varepsilon(\cdot,\cdot,\vec{w}) : x \preccurlyeq y$ is also $\Delta_0(L)$.

Definition 3.9: We say that a theory *defines a* $\Delta_0(\mathrm{L}_0)$*-embedding* $x \preccurlyeq y$ if there is a $\Delta_0(\mathrm{L}_0)$-formula $\varepsilon(u,v,x,y)$ such that the theory proves $\varepsilon(\cdot,\cdot,x,y) : x \preccurlyeq y$.

The next lemma is useful for constructing embeddings. We state it for nonuniform embeddings, but there are analogous versions for strongly and weakly uniform embeddings. Say that two embeddings $e : x \preccurlyeq z$ and $f : y \preccurlyeq z$ are *compatible* if their union is still an injective multifunction, that is, if it satisfies the disjointness condition of (3.2). In particular, embeddings with disjoint ranges are automatically compatible.

Lemma 3.10: *Provably in* T_0*, if two embeddings* $e : x \preccurlyeq z$ *and* $f : y \preccurlyeq z$ *are compatible, then* $e \cup f : x \cup y \preccurlyeq z$*.*

Proof: It follows from the axioms that $u \in \mathrm{tc}(x{\cup}y)$ if and only if $u \in \mathrm{tc}(x)$ or $u \in \mathrm{tc}(y)$. The proof is then immediate. □

3.3. *The theories* $\mathbf{KP}_1^{\preccurlyeq}$ *and* $\mathbf{KP}_1^{\mathbf{u}}$

This section defines theories $\mathrm{KP}_1^{\preccurlyeq}$ and $\mathrm{KP}_1^{\mathrm{u}}$ that, intuitively, are to Rathjen's KP_1 as S^1_2 is to $\mathrm{I}\Sigma_1$. The role of "sharply bounded" quantification in S^1_2 is now played by $\in$-bounded quantification. The analogue of a "bounded" quantifier in our context is one where the quantified variable is embeddable in a #-term:

Definition 3.11: A #*-term* is a $\{1,\odot,\#\}$-term.

Saying that a set is embeddable in a #-term $t(x)$ is analogous to saying that a number/string has length at most $p(|x|)$ for some polynomial p. When we write a #-term, we will use the convention that the # operation takes precedence over $\odot$, and otherwise we omit right-associative parentheses. So for example $1 \odot x\#y \odot z$ is read as $1 \odot ((x\#y) \odot z)$.

Definition 3.12: Let L be a language containing L_0. The class $\Sigma_1^{\preccurlyeq}(L)$ consists of L-formulas of the form

$$\exists x{\preccurlyeq}t(\vec{x})\,\varphi(x,\vec{x})$$

where t is a #-term not involving x and φ is $\Delta_0(L)$. Here $\exists x{\preccurlyeq}t\,\varphi$ stands for $\exists x\,(x{\preccurlyeq}t \wedge \varphi)$. Recall that $x \preccurlyeq y$ denotes a nonuniform embedding, i.e., it stands for $\exists e\,(e : x \preccurlyeq y)$. Hence a $\Sigma_1^{\preccurlyeq}(L)$-formula is also a $\Sigma_1(L)$-formula. (See also Lemma 4.13.)

Note that the term $\preccurlyeq$-bounding the leading existential quantifier in a $\Sigma_1^{\preccurlyeq}(L)$-formula is required to be a #-term while the $\in$-bounding terms in the $\Delta_0(L)$-part can be arbitrary L-terms.

Definition 3.13: The theory $\mathrm{KP}_1^{\preccurlyeq}$ consists of T_0 without tc-Transitivity, that is, the defining axioms for the symbols in L_0, Extensionality, Set Foundation and $\Delta_0(\mathrm{L}_0)$-Separation, together with the two schemes:

- $\Delta_0(\mathrm{L}_0)$-*Collection:*

$$\forall u{\in}x\,\exists v\,\varphi(u,v,\vec{w}) \rightarrow \exists y\,\forall u{\in}x\,\exists v \in y\,\varphi(u,v,\vec{w}) \quad \text{for } \varphi \in \Delta_0(\mathrm{L}_0),$$

- $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$-*Induction:*

$$\forall y\,(\forall u{\in}y\,\varphi(u,\vec{w}) \rightarrow \varphi(y,\vec{w})) \rightarrow \varphi(x,\vec{w}) \quad \text{for } \varphi \in \Sigma_1^{\preccurlyeq}(\mathrm{L}_0).$$

We omitted tc-Transitivity from the definition of $\mathrm{KP}_1^{\preccurlyeq}$ because it is not one of the usual axioms for Kripke-Platek set theories. However, tc-Transitivity can be proven by $\Delta_0(\mathrm{L}_0)$-Induction from the rest of the axioms of T_0. Since $\Delta_0(\mathrm{L}_0)$-Induction is contained in $\mathrm{KP}_1^{\preccurlyeq}$, it follows that tc-Transitivity is a consequence of $\mathrm{KP}_1^{\preccurlyeq}$. Thus $\mathrm{KP}_1^{\preccurlyeq}$ contains T_0. The same holds for the theory $\mathrm{KP}_1^{\mathrm{u}}$ defined next.

Our goal is to $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$-define all CRSF functions in $\mathrm{KP}_1^{\preccurlyeq}$ in the following sense. Fix a universe of sets V (a model of ZFC); of course, we may view V as interpreting L_0. Let T be a theory and Φ a set of formulas. A function $f(\vec{x})$ over V is Φ-*definable in* T if there is $\varphi(y,\vec{x}) \in \Phi$ such that $\mathrm{V} \models \forall\vec{x}\,\varphi(f(\vec{x}),\vec{x})$ and T proves $\exists! y\,\varphi(y,\vec{x})$.

In fact, we will show that an apparently weaker theory $\mathrm{KP}_1^{\mathrm{u}}$ is sufficient for this purpose. $\mathrm{KP}_1^{\mathrm{u}}$ is defined in the same way as $\mathrm{KP}_1^{\preccurlyeq}$, except that the induction scheme is restricted to $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$-formulas of a special form, where the witness to the leading existential quantifier is required to be *u*nique and *u*niformly embeddable into a #-term (hence the superscript u). We will see later (in Lemma 4.13) that it is only the uniqueness requirement that distinguishes this from $\mathrm{KP}_1^{\preccurlyeq}$.

Definition 3.14: The theory $\mathrm{KP}_1^{\mathrm{u}}$ consists of T_0 without tc-Transitivity, that is, the defining axioms for the symbols in L_0, Extensionality, Set Foundation and $\Delta_0(\mathrm{L}_0)$-Separation, together with $\Delta_0(\mathrm{L}_0)$-Collection and the scheme:

- *Uniformly Bounded Unique* $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$*-Induction*

$$\forall u\, \exists^{\leqslant 1} v\, \varphi(u, v, \vec{w}) \wedge \forall y\, \big(\forall u{\in}y\, \exists v\, \varphi^{\varepsilon,t}(u, v, \vec{w}) \to \exists v\, \varphi^{\varepsilon,t}(y, v, \vec{w})\big) \\ \to \exists v\, \varphi^{\varepsilon,t}(x, v, \vec{w})$$

where $\varphi^{\varepsilon,t}(u, v, \vec{w})$ abbreviates the formula

$$\varphi(u, v, \vec{w}) \wedge \varepsilon(\cdot, \cdot, v, u, \vec{w}) : v \preccurlyeq t(u, \vec{w})$$

and the scheme ranges over $\Delta_0(\mathrm{L}_0)$-formulas φ, ε and #-terms t.

3.4. *The language* $\mathbf{L_{crsf}}$ *and theory* $\mathbf{T_{crsf}}$

Our final main theory, $\mathrm{T}_{\mathrm{crsf}}$, is an analogue of the bounded arithmetic theory PV_1. $\mathrm{T}_{\mathrm{crsf}}$ has a function symbol for every CRSF function, and Π_1 axioms describing how the CRSF functions are defined from each other. By comparison, $\mathrm{KP}_1^{\preccurlyeq}$ and $\mathrm{KP}_1^{\mathrm{u}}$ are analogues of S_2^1. One of our main results is Theorem 6.9, which states that a definitional expansion of $\mathrm{KP}_1^{\mathrm{u}}$ is $\Pi_2(\mathrm{L}_{\mathrm{crsf}})$-*conservative* over $\mathrm{T}_{\mathrm{crsf}}$, that is, every $\Pi_2(\mathrm{L}_{\mathrm{crsf}})$-sentence provable in the former theory is also provable in the latter. Theorem 6.12 states an analogous result for $\mathrm{KP}_1^{\preccurlyeq}$, but only with the addition of a global choice function to $\mathrm{T}_{\mathrm{crsf}}$.

Definition 3.15: The language $\mathrm{L}_{\mathrm{crsf}}$ consists of $\in$ and the function symbols listed below. The theory $\mathrm{T}_{\mathrm{crsf}}$ contains the axioms of Extensionality, Set Foundation and tc-Transitivity, together with a defining axiom for each function symbol of $\mathrm{L}_{\mathrm{crsf}}$, as follows.

- $\mathrm{L}_{\mathrm{crsf}}$ contains the function symbols from L_0, and $\mathrm{T}_{\mathrm{crsf}}$ contains their defining axioms.

- $\mathrm{L_{crsf}}$ contains the function symbols $proj_i^n$ for $1 \leqslant i \leqslant n$ and $cond_\in(x, y, u, v)$ with defining axioms $proj_i^n(x_1, \ldots, x_n) = x_i$ and
$$cond_\in(x, y, u, v) = \begin{cases} x & \text{if } u \in v \\ y & \text{otherwise.} \end{cases}$$
- (Closure under composition) For all function symbols $h, g_1, \ldots, g_k \in \mathrm{L_{crsf}}$ of suitable arities, $\mathrm{L_{crsf}}$ contains the function symbol $f_{h,\vec{g}}$ with defining axiom
$$f_{h,\vec{g}}(\vec{x}) = h(g_1(\vec{x}), \ldots, g_k(\vec{x})).$$
- (Closure under replacement) For all function symbols $f \in \mathrm{L_{crsf}}$, $\mathrm{L_{crsf}}$ contains the function symbol f" with defining axiom
$$f\text{"}(x, \vec{z}) = \{f(y, \vec{z}) : y \in x\}.$$
- (Closure under syntactic Cobham recursion) Suppose g, τ are function symbols in $\mathrm{L_{crsf}}$ and t is a #-term. Let us write "τ is an embedding into t at $x, \vec{w}$" for the $\Delta_0(\mathrm{L_{crsf}})$-formula
$$\tau(\cdot, g(x, f\text{"}(x, \vec{w}), \vec{w}), x, \vec{w}) : g(x, f\text{"}(x, \vec{w}), \vec{w}) \preccurlyeq t(x, \vec{w}).$$
Then $\mathrm{L_{crsf}}$ contains the function symbol $f = f_{g,\tau,t}$ with defining axiom
$$f(x, \vec{w}) = \begin{cases} g(x, f\text{"}(x, \vec{w}), \vec{w}) & \text{if } \tau \text{ is an embedding into } t \text{ at } x, \vec{w} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.16: *The universe* V *of sets can be expanded uniquely to a model of* $\mathrm{T_{crsf}}$*. The* $\mathrm{L_{crsf}}$*-function symbols then name exactly the* CRSF *functions as defined in Proposition 2.9.*

Because of closure under composition, every $\mathrm{L_{crsf}}$-term is equivalent to an $\mathrm{L_{crsf}}$-function symbol, provably in $\mathrm{T_{crsf}}$. Hence we will not always be careful to distinguish between terms and function symbols in $\mathrm{L_{crsf}}$.

Lemma 3.17: *For every function symbol* $f \in \mathrm{L_{crsf}}$*, there is a function symbol* $g \in \mathrm{L_{crsf}}$ *such that* $\mathrm{T_{crsf}}$ *proves* $g(x, \vec{w}) = \{y \in x : f(y, \vec{w}) \neq 0\}$*.*

Proof: Using $cond_\in$ we may construct a function symbol $h(y, \vec{w})$ which takes the value $\{y\}$ if $f(y, \vec{w}) \notin \{0\}$ and the value 0 otherwise. We put $g(x, \vec{w}) = \bigcup h\text{"}(x, \vec{w})$. □

The next lemma is proved as in the development of CRSF in [7, Theorem 13]. Note that we do not need recursion to prove either Lemma 3.17 or Lemma 3.18.

Lemma 3.18: *Every $\Delta_0(\mathrm{L_{crsf}})$-formula is provably equivalent in $\mathrm{T_{crsf}}$ to a formula of the form $f(\vec{x}) \neq 0$ for some $\mathrm{L_{crsf}}$-function symbol f. It follows that the $\mathrm{L_{crsf}}$ functions are closed under $\Delta_0(\mathrm{L_{crsf}})$-Separation provably in $\mathrm{T_{crsf}}$; that is, for every $\Delta_0(\mathrm{L_{crsf}})$-formula $\varphi(y, \vec{w})$ there is an $\mathrm{L_{crsf}}$-function symbol f such that $\mathrm{T_{crsf}}$ proves $f(x, \vec{w}) = \{y \in x : \varphi(y, \vec{w})\}$.*

Corollary 3.19: *The theory $\mathrm{T_{crsf}}$ extends $\mathrm{T_0}$.*

4. Bootstrapping

4.1. *Bootstrapping the defining axioms*

We first derive some simple consequences of the defining axioms, namely basic properties of tc, a description of the Mostowski graph of $x \odot y$, injectivity of $\odot$ in its first argument, and associativity of $\odot$.

Lemma 4.1: *The theory $\mathrm{T_0}$ proves*

(a) $x \subseteq y \to \mathrm{tc}(x) \subseteq \mathrm{tc}(y), \ \ \mathrm{tc}(x) = \mathrm{tc}(\mathrm{tc}(x))$,
(b) $u \in \mathrm{tc}(x \odot y) \leftrightarrow (u \in \mathrm{tc}(y) \vee \exists u' \in \mathrm{tc}(x)\, (u = u' \odot y))$,
(c) $x \neq x' \to x \odot y \neq x' \odot y$,
(d) $x \odot (y \odot z) = (x \odot y) \odot z$.

Proof: We omit the proof of (a).

For (b) argue in $\mathrm{T_0}$ as follows. If $x = 0$, the claim follows from the $\odot$-axiom, so assume $x \neq 0$. We prove $(\to)$ by $\Delta_0(\mathrm{L_0})$-Induction (recall Lemma 3.4), so assume it to hold for all $x' \in x$. Let $u \in \mathrm{tc}(x \odot y)$. By the tc,$\odot$-axioms, either $u \in \mathrm{tc}(x' \odot y)$ for some $x' \in x$ or $u \in x \odot y$. In the first case our claim follows by induction noting $\mathrm{tc}(x') \subseteq \mathrm{tc}(x)$ by (a). In the second case, $u = u' \odot y$ for some $u' \in x$ by the $\odot$-axiom.

Conversely, we first show $u \in \mathrm{tc}(y) \to u \in \mathrm{tc}(x \odot y)$. By $\Delta_0(\mathrm{L_0})$-Induction we can assume this holds for all $z \in x$. Assume $u \in \mathrm{tc}(y)$ and let $z \in x$ be arbitrary. Then $u \in \mathrm{tc}(z \odot y)$ by induction. But $z \odot y \in x \odot y$, so $\mathrm{tc}(z \odot y) \subseteq \mathrm{tc}(x \odot y)$ by (a).

Finally, we show $u \in \mathrm{tc}(x) \to u \odot y \in \mathrm{tc}(x \odot y)$. We assume this for all $z \in x$. Let $u \in \mathrm{tc}(x)$. If $u \in x$, then $u \odot y \in x \odot y \subseteq \mathrm{tc}(x \odot y)$. Otherwise $u \in \mathrm{tc}(z)$ for some $z \in x$. Then $u \odot y \in \mathrm{tc}(z \odot y)$ by induction; but $\mathrm{tc}(z \odot y) \subseteq \mathrm{tc}(x \odot y)$ by $z \odot y \in x \odot y$ and (a).

For (c) argue in $\mathrm{T_0}$ as follows. Suppose there are y, x_0, x_0' such that $x_0 \neq x_0'$ and $x_0 \odot y = x_0' \odot y$. It is easy to derive $\forall x\, (x \in \mathrm{tc}^+(x))$, so the set

$$z := \{x \in \mathrm{tc}^+(x_0) : \exists x' \in \mathrm{tc}^+(x_0')\, (x \neq x' \wedge x \odot y = x' \odot y)\}$$

is non-empty because it contains x_0. The set exists by $\Delta_0(\mathrm{L}_0)$-Separation. By Foundation, it contains an $\in$-minimal element x_1. Choose $x'_1 \in \mathrm{tc}^+(x'_0)$ with $x_1 \neq x'_1$ and $x_1 \odot y = x'_1 \odot y$.

We claim that x_1, x'_1 are both non-empty. Assume otherwise, say, $x'_1 = 0$ and hence $x_1 \neq 0$. An easy $\Delta_0(\mathrm{L}_0)$-Induction shows $x \notin \mathrm{tc}(x)$ and $(x \neq 0 \rightarrow 0 \in \mathrm{tc}(x))$ for all x. Then $y \notin y = x'_1 \odot y$ and $y = 0 \odot y \in \mathrm{tc}(x_1 \odot y)$ by (b). This contradicts $x_1 \odot y = x'_1 \odot y$.

Choose x_2 such that either $x_2 \in x_1 \wedge x_2 \notin x'_1$ or $x_2 \in x'_1 \wedge x_2 \notin x_1$. Assume the former (the latter case is similar). By the $\odot$-axiom, $x_2 \odot y \in x_1 \odot y = x'_1 \odot y$. Since $x'_1 \neq 0$ the $\odot$-axiom gives $x'_2 \in x'_1$ such that $x_2 \odot y = x'_2 \odot y$. As $x_2 \notin x'_1$ we have $x_2 \neq x'_2$. By (a), $x_2 \in x_1 \subseteq \mathrm{tc}^+(x_0)$ and $x'_2 \in x'_1 \subseteq \mathrm{tc}^+(x'_0)$. Thus $x_2 \in z$, contradicting the minimality of x_1.

For (d), an easy induction shows that $u \odot 0 = u$ for all u. Item (d) is then true immediately if any of x, y or z is 0. Otherwise it follows by induction on x, using the $\odot$-axiom. □

We write $2^\odot := 1 \odot 1$, $3^\odot := 1 \odot 1 \odot 1$, etc. Notice that, in T_0, $1 \odot x = \{x\}$. We give an example of how we can now begin to build useful embeddings.

Example 4.2: There is a #-term $t_{\mathrm{pair}}(x, y)$ such that T_0 defines a $\Delta_0(\mathrm{L}_0)$-embedding from $\langle x, y\rangle$ into $t_{\mathrm{pair}}(x, y)$.

Proof: We put $t_{\mathrm{pair}}(x, y) := 4^\odot \odot x \odot 1 \odot y$. Consider the relations

$$\begin{aligned} e &:= \{\langle u, u\rangle : u \in \mathrm{tc}^+(y)\} \\ f &:= \{\langle u, u \odot 1 \odot y\rangle : u \in \mathrm{tc}^+(x)\} \\ g &:= \{\langle \{x\}, 2^\odot \odot x \odot 1 \odot y\rangle\} \\ h &:= \{\langle \{x, y\}, 3^\odot \odot x \odot 1 \odot y\rangle\}. \end{aligned}$$

Then $e : \{y\} \preccurlyeq t_{\mathrm{pair}}(x, y)$ and $f : \{x\} \preccurlyeq t_{\mathrm{pair}}(x, y)$, and these two embeddings are compatible since they have disjoint ranges. So $e \cup f : \{x, y\} \preccurlyeq t_{\mathrm{pair}}(x, y)$ (appealing to Lemma 3.10), hence $e \cup f \cup h : \{\{x, y\}\} \preccurlyeq t_{\mathrm{pair}}(x, y)$. On the other hand $f \cup g : \{\{x\}\} \preccurlyeq t_{\mathrm{pair}}(x, y)$. These are compatible, so $e \cup f \cup g \cup h : \{\{x\}, \{x, y\}\} \preccurlyeq t_{\mathrm{pair}}(x, y)$, as required. All these embeddings can be expressed straightforwardly in T_0 by $\Delta_0(\mathrm{L}_0)$-formulas. □

4.2. *Adding $\in$-bounded functions*

We give a small expansion T_0^+ of T_0.

Definition 4.3: Let L_0^+ be the language obtained from L_0 by adding a relation symbol $R(\vec{x})$ for every $\Delta_0(\mathrm{L}_0)$-formula $\varphi(\vec{x})$, and a function symbol

$f(\vec{x})$ for every $\Delta_0(\mathrm{L}_0)$-formula $\psi(y,\vec{x})$ such that T_0 proves $\exists! y \in t(\vec{x})\, \psi(y,\vec{x})$ for some L_0-term $t(\vec{x})$.

The theory T_0^+ has language L_0^+ and is obtained from T_0 by adding for every relation symbol $R(\vec{x})$ in L_0^+ as above the defining axiom $R(\vec{x}) \leftrightarrow \varphi(\vec{x})$, and for every function symbol $f(\vec{x})$ in L_0^+ as above the defining axiom $\psi(f(\vec{x}),\vec{x})$.

Proposition 4.4: *T_0^+ is a conservative extension of T_0. Every $\Delta_0(\mathrm{L}_0^+)$-formula is T_0^+-provably equivalent to a $\Delta_0(\mathrm{L}_0)$-formula. In particular, T_0^+ proves $\Delta_0(\mathrm{L}_0^+)$-Induction and $\Delta_0(\mathrm{L}_0^+)$-Separation.*

We omit the proof. The language L_0^+ and the theory T_0^+ are introduced mainly for notational convenience. Interesting functions often do not have $\in$-bounded values.

Lemma 4.5: *Every function symbol introduced in L_0^+ has a copy in $\mathrm{L}_{\mathrm{crsf}}$ for which $\mathrm{T}_{\mathrm{crsf}}$ proves the defining axiom.*

Proof: Suppose T_0 proves $\exists! y \in t(\vec{x})\, \psi(y,\vec{x})$. Then using Lemma 3.18 we can compute y in $\mathrm{T}_{\mathrm{crsf}}$ as $\bigcup\{y \in t(\vec{x}) : \psi(y,\vec{x})\}$. ☐

Example 4.6: The language L_0^+ contains the relation symbol $\mathit{IsPair}(x)$ with defining axiom $\exists u, v \in \mathrm{tc}(x)\, (x = \langle u, v\rangle)$, the function symbol $\mathit{cond}_\in(x,y,u,v)$ with defining axiom as in Definition 3.15, and function symbols $\pi_1(x)$, $\pi_2(x)$ and $w\text{'}x$ such that T_0^+ proves $\pi_1(\langle x_1, x_2\rangle) = x_1$, $\pi_2(\langle x_1, x_2\rangle) = x_2$ and

$$w\text{'}x = \begin{cases} y \text{ if } y \text{ is unique with } \langle x, y\rangle \in w \\ 0 \text{ otherwise.} \end{cases}$$

We now formalize the graph isomorphism for # mentioned in Section 2. We introduce $\#\text{''}(u,y)$ below as an auxiliary function to formulate the defining axiom for $\sigma_{x,y}(u,v)$.

Lemma 4.7: *There are function symbols $\#\text{''}(u,y)$, $\sigma_{x,y}(u,v)$, $\pi_{1,x,y}(w)$ and $\pi_{2,x,y}(w)$ in L_0^+ such that T_0^+ proves*

$$\#\text{''}(u,y) = \{u'\#y : u' \in u\},$$

$$\sigma_{x,y}(u,v) = \begin{cases} v \odot \#\text{''}(u,y) & \textit{if } u \in \mathrm{tc}^+(x) \textit{ and } v \in \mathrm{tc}^+(y) \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover, T_0^+ proves that

(a) $\sigma_{x,y}$ *is injective on arguments* $u \in \mathrm{tc}^+(x), v \in \mathrm{tc}^+(y)$.
(b) Every $w \in \mathrm{tc}^+(x\#y)$ *has a* $\sigma_{x,y}$*-preimage* $(\pi_{1,x,y}(w), \pi_{2,x,y}(w))$.
(c) For all $u, u' \in \mathrm{tc}^+(x)$ *and* $v, v' \in \mathrm{tc}^+(y)$,

$$\sigma_{x,y}(u', v') \in \sigma_{x,y}(u, v) \leftrightarrow (u' = u \wedge v' \in v) \vee (u' \in u \wedge v' = y \wedge v = 0).$$

Proof: The functions $\#"(u, y), \sigma_{x,y}(u, v)$ have obvious defining axioms. Concerning bounding terms, from the $\#$-axiom we get that $\#"(u, y) \in \mathrm{tc}^+(u\#y)$ and

$$x\#y = y \odot \#"(x, y). \tag{4.1}$$

By induction on x, using Lemma 4.1(b), we get that if $u \in \mathrm{tc}^+(x)$ then $u\#y \in \mathrm{tc}^+(x\#y)$. Another use of Lemma 4.1(b) shows that if $v \in \mathrm{tc}^+(y)$ then $v \odot \#"(u, y) \in \mathrm{tc}^+(u\#y)$. Hence $\sigma_{x,y}(u, v) \in \mathrm{tc}^+(x\#y)$.

Observe that for $u \in \mathrm{tc}^+(x), v \in \mathrm{tc}^+(y)$,

$$\begin{aligned} v \neq 0 \rightarrow \sigma_{x,y}(u, v) &= \{\sigma_{x,y}(u, v') : v' \in v\}, \\ \sigma_{x,y}(u, 0) &= \{\sigma_{x,y}(u', y) : u' \in u\}. \end{aligned} \tag{4.2}$$

The first line is from the definition. For the second, note $z \in \sigma_{x,y}(u, 0)$ is equivalent to $z \in \#"(u, y)$, and hence to $z = u'\#y$ for some $u' \in u$; but $u'\#y = \sigma_{x,y}(u', y)$ by (4.1).

For (a), let $u, \tilde{u}, \ldots$ range over $\mathrm{tc}^+(x)$ and $v, \tilde{v}, \ldots$ range over $\mathrm{tc}^+(y)$. We claim that $\sigma_{x,y}(u, v) = \sigma_{x,y}(\tilde{u}, \tilde{v})$ implies $u = \tilde{u}$ and $v = \tilde{v}$. By Lemma 4.1(c) it suffices to show it implies $u = \tilde{u}$. Assume otherwise. By $\Delta_0(\mathrm{L}_0^+)$-Foundation choose u $\in$-minimal such that there exist $\tilde{u}, v, \tilde{v}$ with $\sigma_{x,y}(u, v) = \sigma_{x,y}(\tilde{u}, \tilde{v})$ and $u \neq \tilde{u}$; then choose $\tilde{u}$ $\in$-minimal such that there are $v, \tilde{v}$ with this property, and so on for $v, \tilde{v}$. We distinguish two cases, as in (4.2).

First suppose $v \neq 0$. Then there is $v' \in v$ such that $\sigma_{x,y}(u, v') \in \sigma_{x,y}(\tilde{u}, \tilde{v})$. If $\tilde{v} \neq 0$, then $\sigma_{x,y}(u, v') = \sigma_{x,y}(\tilde{u}, v'')$ for some $v'' \in \tilde{v}$, and this contradicts the choice of v. If $\tilde{v} = 0$, then $\sigma_{x,y}(u, v') = \sigma_{x,y}(u', y)$ for some $u' \in \tilde{u}$, and this contradicts the choice of $\tilde{u}$.

Now suppose $v = 0$. If $\tilde{v} = 0$, then $\{\sigma_{x,y}(u', y) : u' \in u\} = \{\sigma_{x,y}(u'', y) : u'' \in \tilde{u}\}$, so for each $u' \in u$ there is $u'' \in \tilde{u}$ such that $\sigma_{x,y}(u', y) = \sigma_{x,y}(u'', y)$, so then $u' = u''$ by choice of u; thus $u \subseteq \tilde{u}$. Similarly $\tilde{u} \subseteq u$, contradicting our assumption $u \neq \tilde{u}$. If $\tilde{v} \neq 0$, then for each $u' \in u$ there is $v' \in \tilde{v}$ such that $\sigma_{x,y}(u', y) = \sigma_{x,y}(\tilde{u}, v')$, so $u' = \tilde{u}$ by choice of u. Thus $u = 0$ or $u = \{\tilde{u}\}$; the latter is impossible by choice of u, so $u = 0$; then $\sigma_{x,y}(\tilde{u}, \tilde{v}) = \sigma_{x,y}(u, v) = \sigma_{x,y}(0, 0) = 0$, so $\tilde{v} = 0$, a contradiction.

For (b) we will show surjectivity; $\pi_{1,x,y}(w)$ and $\pi_{2,x,y}(w)$ can then easily be constructed. So let $w \in \mathrm{tc}^+(x\#y)$. If $w = x\#y$, put $u := x, v := y$. Otherwise $w \in \mathrm{tc}(x\#y) = \mathrm{tc}(y \odot \#"(x,y))$ by (4.1). By Lemma 4.1(c) we have two cases. If $w = v' \odot \#"(x,y)$ for some $v' \in y$, put $u := x, v := v'$. If $w \in \mathrm{tc}(\#"(x,y))$, then $w \in \mathrm{tc}^+(x'\#y)$ for some $x' \in x$ and, using $\Delta_0(\mathrm{L}_0^+)$-Induction on x, we find $u \in \mathrm{tc}^+(x') \subseteq \mathrm{tc}^+(x)$ and $v \in \mathrm{tc}^+(y)$ with $w = \sigma_{x',y}(u,v)$. Since $\sigma_{x,y}(u,v)$ does not depend on x, we have $w = \sigma_{x',y}(u,v) = \sigma_{x,y}(u,v)$.

Claim (c) follows by (4.2). □

4.3. *Monotonicity lemma*

We can now formally derive non-uniform and weakly uniform versions of the Monotonicity Lemma 2.4, meaning "monotonicity of #-terms with respect to embeddings." Note that Lemma 4.8 includes as a special case the transitivity of embeddings,

$$z \preccurlyeq x \wedge x \preccurlyeq y \to z \preccurlyeq y. \tag{4.3}$$

Lemma 4.8: (Monotonicity) *For all #-terms $t(x,\vec{w})$ the theory T_0 proves*

$$z \preccurlyeq t(x,\vec{w}) \wedge x \preccurlyeq y \to z \preccurlyeq t(y,\vec{w}). \tag{4.4}$$

Moreover, for all $\Delta_0(\mathrm{L}_0)$-formulas ε_0, ε_1 there is a $\Delta_0(\mathrm{L}_0)$-formula ε_2 such that T_0 proves

$$\begin{aligned} &\varepsilon_0(\cdot,\cdot,x,z,\vec{w}) : z \preccurlyeq t(x,\vec{w}) \ \wedge\ \varepsilon_1(\cdot,\cdot,x,y,\vec{w}) : x \preccurlyeq y \\ &\to\ \varepsilon_2(\cdot,\cdot,x,y,z,\vec{w}) : z \preccurlyeq t(y,\vec{w}). \end{aligned}$$

Proof: We only verify the first statement; the second follows by inspection of the proof. We proceed by induction on t. We work in T_0^+, which is sufficient by Proposition 5.5.

If $t(x,\vec{w})$ is 1 or a variable distinct from x, then there is nothing to show. If $t(x,\vec{w})$ equals x then we have to show (4.3). So assume $e : z \preccurlyeq x$ and $f : x \preccurlyeq y$. Then

$$g := \{\langle u,w\rangle \in \mathrm{tc}(z) \times \mathrm{tc}(y) : \exists v \in \mathrm{tc}(x)\, (\langle u,v\rangle \in e \wedge \langle v,w\rangle \in f)\}$$

exists by $\Delta_0(\mathrm{L}_0)$-Separation. We claim $g : z \preccurlyeq y$. It is easy to see that $\langle u,w\rangle, \langle u',w\rangle \in g$ implies $u = u'$. Assume $u' \in u \in \mathrm{tc}(x)$ and $\langle u,w\rangle \in g$. Choose v such that $\langle u,v\rangle \in e$ and $\langle v,w\rangle \in f$. Then there is $v' \in \mathrm{tc}(v)$ such that $\langle u', v'\rangle \in e$. It now suffices to show that, generally, for all v, v', w we have

$$v' \in \mathrm{tc}(v) \wedge \langle v, w\rangle \in f \to \exists w' \in \mathrm{tc}(w)\, \langle v', w'\rangle \in f.$$

This is clear if $v' \in v$. Otherwise, $v' \in \mathrm{tc}(v'')$ for some $v'' \in v$. Then choose $w'' \in \mathrm{tc}(w)$ such that $\langle v'', w''\rangle \in f$. Appealing to $\Delta_0(\mathrm{L}_0)$-Induction, we can find $w' \in \mathrm{tc}(w'')$ such that $\langle v', w'\rangle \in f$. Then $w' \in \mathrm{tc}(w)$ by Lemma 4.1(a), as claimed.

As preparation for the induction step in our induction on t, we show

$$x \preccurlyeq x' \wedge y \preccurlyeq y' \to x \odot y \preccurlyeq x' \odot y' \wedge x\#y \preccurlyeq x'\#y'. \tag{4.5}$$

Assume $e : x \preccurlyeq x'$ and $f : y \preccurlyeq y'$. By $\Delta_0(\mathrm{L}_0)$-Separation the set

$$\begin{aligned}\{\langle u, v\rangle \in &\mathrm{tc}(x \odot y) \times \mathrm{tc}(x' \odot y') :\\ &\exists u' \in \mathrm{tc}(x)\, \exists v' \in \mathrm{tc}(x')\, (u = u' \odot y \wedge v = v' \odot y' \wedge \langle u', v'\rangle \in e)\}\end{aligned}$$

exists. We leave it to the reader to check that its union with f witnesses $x \odot y \preccurlyeq x' \odot y'$. For #, observe $e_+ := e \cup \{\langle x, x'\rangle\} : \{x\} \preccurlyeq \{x'\}$ and $f_+ := f \cup \{\langle y, y'\rangle\} : \{y\} \preccurlyeq \{y'\}$. Let g be the set containing the pairs $\langle \sigma_{x,y}(u, v), \sigma_{x',y'}(u', v')\rangle$ such that $\langle u, u'\rangle \in e_+$ and $\langle v, v'\rangle \in f_+$. This set g exists by $\Delta_0(\mathrm{L}_0^+)$-Separation. Using Lemma 4.7 it is straightforward to check that $g : x\#y \preccurlyeq x'\#y'$.

Now the induction step is easy. We are given embeddings $z \preccurlyeq t(x, \vec{w})$ and $x \preccurlyeq y$. Assume first that $t(x, \vec{w}) = t_1(x, \vec{w}) \odot t_2(x, \vec{w})$. By the identity embedding $t_1(x, \vec{w}) \preccurlyeq t_1(x, \vec{w})$, and applying the inductive hypothesis gives $t_1(x, \vec{w}) \preccurlyeq t_1(y, \vec{w})$. Similarly $t_2(x, \vec{w}) \preccurlyeq t_2(y, \vec{w})$. Applying (4.5) we get

$$t_1(x, \vec{w}) \odot t_2(x, \vec{w}) \preccurlyeq t_1(y, \vec{w}) \odot t_2(y, \vec{w}),$$

that is, $t(x, \vec{w}) \preccurlyeq t(y, \vec{w})$. This, together with (4.3), implies (4.4). The case of $t_1(x, \vec{w})\#t_2(x, \vec{w})$ is analogous. □

4.4. *Some useful embeddings*

We first show that we can embed all L_0^+-terms into #-terms.

Lemma 4.9: *The theory* T_0 *defines a* $\Delta_0(\mathrm{L}_0)$*-embedding of* $x \times y$ *into a* #*-term* $t_\times(x, y)$*.*

Proof: By Proposition 4.4 it suffices to prove this for T_0^+ and a $\Delta_0(\mathrm{L}_0^+)$-embedding. We set

$$t_\times(x, y) = (x\#y) \odot (x\#y) \odot x \odot y \odot x.$$

The formula $\varepsilon_\times(z, z', x, y)$ implements the following informal procedure on input z, z', x, y. In the description of this procedure we understand that whenever a "check" is carried out then the computation halts, and the procedure rejects or accepts depending on whether the check failed or not. For example, line 2 is reached only if $z \notin \mathrm{tc}(x)$.

It is easy to check that the condition that z, z', x, y is accepted is expressible as a $\Delta_0(\mathrm{L}_0)$-formula.

Input: z, z', x, y

1. `if` $z \in \mathrm{tc}(x)$ `then` check $z' = z$

2. `if` $z \in \mathrm{tc}(y)$ `then` check $z' = z \odot x$

3. `guess` $u \in x$

4. `if` $z = \{u\}$ `then` check $z' = u \odot y \odot x$

5. `guess` $v \in y$

6. `if` $z = \{u, v\}$ `then` check $z' = \sigma_{x,y}(u, v) \odot x \odot y \odot x$

7. `if` $z = \{\{u\}, \{u, v\}\}$ `then` check $z' = \sigma_{x,y}(u, v) \odot (x\#y) \odot x \odot y \odot x$

8. reject

It is clear that any $z \in \mathrm{tc}(x \times y)$ is mapped to at least one z'. Further, distinct $z \neq \tilde{z}$ cannot be mapped to the same $z' = \tilde{z}'$: any z' satisfies the check of at most one line and this line determines the pre-image z (Lemmas 4.7 and 4.1(c)).

Assume $z \in \tilde{z}$ and $\tilde{z}$ is mapped to $\tilde{z}'$. We have to find $z' \in \mathrm{tc}(\tilde{z}')$ such that z is mapped to z'. This is easy if $z \subset \mathrm{tc}(x) \cup \mathrm{tc}(y)$, so assume this is not the case. Then $\tilde{z}$ cannot satisfy any "if" condition before line 7. Hence $\tilde{z} = \{\{u\}, \{u, v\}\}$ for some $u \in x, v \in y$ and $\tilde{z}'$ satisfies the check in line 7. As $z \in \tilde{z}$ we have $z = \{u\}$ or $z = \{u, v\}$ and for suitable guesses in lines 3 and 5, z satisfies the "if" condition of line 4 or 6. Then choose z' satisfying the (first) corresponding check. □

Lemma 4.10: *For each* L_0^+*-term* $s(\vec{x})$ *the theory* T_0 *defines a* $\Delta_0(\mathrm{L}_0)$*-embedding of* $s(\vec{x})$ *into a* $\#$*-term* $s^\#(\vec{x})$.

Proof: This follows by an induction on $s(\vec{x})$ using Lemma 4.8 once we verify it for the base case that $s(\vec{x})$ is a function symbol in L_0^+.

For any such $s(\vec{x})$, there is an L_0-term $r(\vec{x})$ such that T_0^+ proves $s(\vec{x}) \in r(\vec{x})$. By Lemma 4.1 $\mathrm{tc}(s(\vec{x})) \subseteq \mathrm{tc}(r(\vec{x}))$, so the identity embedding (expressed by the formula $u = v$) embeds $s(\vec{x})$ in $r(\vec{x})$. By transitivity of $\preccurlyeq$ (Lemma 4.8), it thus suffices to verify the lemma for L_0-terms $r(\vec{x})$. As for L_0-terms, this follows by an induction on $r(\vec{x})$ using Lemma 4.8 once we verify it for the base case that $r(\vec{x})$ is a function symbol in L_0. The only non-trivial case now is crossproduct $\times$, and this is handled by the previous lemma. ☐

For tuples $\vec{u} = u_1, \dots, u_k$ let us abbreviate $\bigwedge_i u_i \in z$ as $\vec{u} \in z$. We show that given a family of sets parametrized by tuples $\vec{u} \in z$, where each set is uniformly embeddable in s, we can embed the whole family (if it exists as a set) in a #-term $t(z, s)$. Note that the existence of V in the lemma is automatic in the presence of the Collection scheme.

Lemma 4.11: *Let $\varphi(v, \vec{u}, \vec{w})$ and $\varepsilon(z, z', v, \vec{u}, \vec{w})$ be $\Delta_0(L_0^+)$-formulas and $s(\vec{w})$ a #-term. There is a $\Delta_0(L_0^+)$-formula $\delta(z, z', V, z, \vec{w})$ and a #-term $t(z, x)$ such that if*

$$\begin{aligned} &\forall \vec{u} {\in} z\, \exists^{\leqslant 1} v\, \varphi(v, \vec{u}, \vec{w}) \\ &\quad \wedge\ \forall \vec{u} {\in} z\, \exists v \left(\varphi(v, \vec{u}, \vec{w}) \wedge \varepsilon(\cdot, \cdot, v, \vec{u}, \vec{w}) : v \preccurlyeq s(\vec{w})\right) \\ &\quad \wedge\ V = \{v : \exists \vec{u} \in z\, \varphi(v, \vec{u}, \vec{w})\} \end{aligned} \tag{4.6}$$

then $\delta(\cdot, \cdot, V, z, \vec{w}) : V \preccurlyeq t(z, s(\vec{w}))$, provably in T_0^+.

Proof: For notational simplicity we suppress the side variables $\vec{w}$. We first consider the case in which $\vec{u}$ is a single variable u. Note that in the second line of the assumption (4.6) we may assume without loss of generality that we actually have $\varepsilon(\cdot, \cdot, v, u) : \{v\} \preccurlyeq s$, since otherwise we could modify ε so that $\varepsilon(v, s, v, u)$ holds and replace the bound s with $1 \odot s$. Now put $t(z, s) := z \# s$ and define

$$\varepsilon'(y, \tilde{y}', V, u) := \exists v \in V\, \exists y' \in \mathrm{tc}(s)\, (\varphi(v, u) \wedge \varepsilon(y, y', v, u) \wedge \tilde{y}' = \sigma_{z,s}(u, y')).$$

For each $u \in z$, if $\varphi(u, v)$ then the formula $\varepsilon'(\cdot, \cdot, V, u)$ describes an embedding of $\{v\}$ into $z \# s$ which is a copy of the embedding $\varepsilon(\cdot, \cdot, v, u)$, but with its range moved to lie entirely within the uth copy of s inside $z \# s$. These embeddings have disjoint ranges for distinct u, so as in Lemma 3.10 their union $\delta(y, \tilde{y}', V, z) := \exists u \in z\, \varepsilon'(y, \tilde{y}', V, u)$ describes an embedding of V into $z \# s$, since $y \in \mathrm{tc}(V)$ implies $y \in \mathrm{tc}(\{v\})$ for some $v \in V$.

When $\vec{u}$ is a tuple of k variables, we reduce to the first case by coding $\vec{u}$ as an ordered k-tuple in the usual way. So the quantifier $\forall \vec{u} {\in} z$ becomes $\forall u {\in} (z \times \cdots \times z)$ and we replace φ and ε with formulas accessing the values of $\vec{u}$ from u using projection functions. The first case then gives an embedding of V into $t(z \times \cdots \times z, s)$, and by Lemma 4.10 and the Monotonicity Lemma 4.8 we get an embedding of V into some #-term $t'(z, s)$. ☐

We finish this section by showing, in Lemma 4.13, that the non-uniform embedding bounding the existential quantifier in a $\Sigma_1^{\preccurlyeq}$-formula (over any language) can be replaced with a weakly uniform embedding. This will be useful when we want to show that structures satisfy $\Sigma_1^{\preccurlyeq}$-Induction. We first show a suitable embedding exists.

Lemma 4.12: *There is a* $\Delta_0(\mathrm{L}_0)$*-formula* $\varepsilon_{\mathrm{emb}}(u, v, e, x, y)$ *and a* #*-term* $t_{\mathrm{emb}}(y)$ *such that* T_0 *proves* $e : x \preccurlyeq y \to \varepsilon_{\mathrm{emb}}(\cdot, \cdot, e, x, y) : \langle e, x\rangle \preccurlyeq t_{\mathrm{emb}}(y)$.

Proof: Let $\varepsilon_\times(u, v, x, y) \in \Delta_0(\mathrm{L}_0)$ describe (in T_0) an embedding of $x \times y$ into $t_\times(x, y)$. Then $\varepsilon_\times(u, v, \mathrm{tc}(x), \mathrm{tc}(y))$ describes an embedding of $\mathrm{tc}(x) \times \mathrm{tc}(y)$ into $t_\times(\mathrm{tc}(x), \mathrm{tc}(y))$. The identity embedding embeds $\mathrm{tc}(x)$ into x. Combining these, Lemma 4.8 gives a $\Delta_0(\mathrm{L}_0)$-formula describing an embedding $\mathrm{tc}(x) \times \mathrm{tc}(y) \preccurlyeq t_\times(x, y)$. But $e : x \preccurlyeq y$ implies $e \subseteq \mathrm{tc}(x) \times \mathrm{tc}(y)$, so this formula also describes an embedding $e \preccurlyeq t_\times(x, y)$.

Using Example 4.2 there is a $\Delta_0(\mathrm{L}_0)$-formula describing an embedding $\langle e, x\rangle \preccurlyeq t_{\mathrm{pair}}(e, x)$. By Lemma 4.8 and the previous paragraph, there is a $\Delta_0(\mathrm{L}_0)$-formula describing an embedding $\langle e, x\rangle \preccurlyeq t_{\mathrm{pair}}(t_\times(x, y), x)$. Since $e : x \preccurlyeq y$ it is easy to write a $\Delta_0(\mathrm{L}_0)$-formula with parameter e describing an embedding $x \preccurlyeq y$, so using Lemma 4.8 again we can replace x by y, that is, construct a $\Delta_0(\mathrm{L}_0)$-embedding $\langle e, x\rangle \preccurlyeq t_{\mathrm{emb}}(y) := t_{\mathrm{pair}}(t_\times(y, y), y)$. □

Lemma 4.13: *Let L be a language extending* L_0*. Provably in* T_0*, every* $\Sigma_1^{\preccurlyeq}(L)$*-formula* $\theta(\vec{x})$ *is equivalent to a formula of the form*

$$\exists v \left(\varphi(v, \vec{x}) \wedge \varepsilon(\cdot, \cdot, v, \vec{x}) : v \preccurlyeq t(\vec{x})\right) \tag{4.7}$$

where φ *is a* $\Delta_0(L)$*-formula,* ε *is a* $\Delta_0(\mathrm{L}_0)$*-formula and* t *is a* #*-term.*

Proof: Expanding the existential quantifier implicit in the nonuniform embedding bound, there is a $\Delta_0(L)$-formula ψ and a #-term s such that $\theta(\vec{x})$ has the form

$$\exists w \exists e \left(e : w \preccurlyeq s(\vec{x}) \wedge \psi(w, \vec{x})\right).$$

By Lemma 4.12, T_0 proves

$$e : w \preccurlyeq s(\vec{x}) \to \varepsilon_{\mathrm{emb}}(\cdot, \cdot, e, w, s(\vec{x})) : \langle e, w\rangle \preccurlyeq t_{\mathrm{emb}}(s(\vec{x})).$$

Hence $\theta(\vec{x})$ is equivalent to

$$\exists \langle e, w\rangle \left(e : w \preccurlyeq s(\vec{x}) \wedge \psi(w, \vec{x}) \wedge \varepsilon_{\mathrm{emb}}(\cdot, \cdot, e, w, s(\vec{x})) : \langle e, w\rangle \preccurlyeq t_{\mathrm{emb}}(s(\vec{x}))\right).$$

For clarity we have written this rather informally. Strictly speaking, $\langle e, w \rangle$ should be a single variable v, and e and w should be respectively $\pi_1(v)$ and $\pi_2(v)$; then apply Proposition 4.4. □

5. Definability

This section develops $\mathrm{KP}_1^{\mathrm{u}}$ with the goal of proving that it $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$-defines all CRSF functions. The Definability Theorem 5.2 below states this in a syntactic manner, without reference to the universe of sets.

Definition 5.1: A $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$-*expansion of* $\mathrm{KP}_1^{\mathrm{u}}$ is obtained from $\mathrm{KP}_1^{\mathrm{u}}$ by adding a set of formulas of the following forms:

- $\varphi(f(\vec{x}), \vec{x})$ where $f(\vec{x})$ is a function symbol outside L_0 and $\varphi(y, \vec{x})$ is $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$ such that $\mathrm{KP}_1^{\mathrm{u}}$ proves $\exists! y\, \varphi(y, \vec{x})$
- $R(\vec{x}) \leftrightarrow \varphi(\vec{x})$ where $R(\vec{x})$ is a relation symbol outside L_0 and $\varphi(\vec{x})$ is $\Delta_0(\mathrm{L}_0)$.

For example, it is not hard to give a $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$-definition of a proper comprehension term for a formula $\varphi \in \Delta_0(\mathrm{L}_0)$ as a new function symbol.

Theorem 5.2: *(Definability) There is a* $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$*-expansion of* $\mathrm{KP}_1^{\mathrm{u}}$ *which contains all function symbols in* $\mathrm{L}_{\mathrm{crsf}}$ *in its expanded language, and proves all axioms of* $\mathrm{T}_{\mathrm{crsf}}$.

A $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$-expansion of $\mathrm{KP}_1^{\mathrm{u}}$ is an expansion by definitions, and hence is conservative over $\mathrm{KP}_1^{\mathrm{u}}$. Thus Theorem 5.2 immediately implies that every CRSF function is denoted by a symbol in the language, and hence is $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$-definable in $\mathrm{KP}_1^{\mathrm{u}}$. Note that such an expansion does not include the axiom schemes of $\mathrm{KP}_1^{\mathrm{u}}$ for formulas in the expanded language. (Definition 3.14 describes the axiom schemes of $\mathrm{KP}_1^{\mathrm{u}}$.) The lack of these schemes is the main technical difficulty in proving the Theorem 5.2. Some further comments can be found in Section 5.2, where this difficulty is tackled. In Section 5.5 we describe a particular well-behaved expansion which proves all these axiom schemes in the expanded language.

We will prove Theorem 5.2 indirectly. We first define an expansion $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ which includes a function symbol for every function definable in $\mathrm{KP}_1^{\mathrm{u}}$ by a particular kind of $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$-formula. We will then show that these function symbols contain the basic functions from $\mathrm{L}_{\mathrm{crsf}}$ and satisfy the right closure properties.

Remark 5.3: For the results in this section about $\mathrm{KP}_1^{\mathrm{u}}$ and its expansions, we do not need the full strength of the collection scheme in $\mathrm{KP}_1^{\mathrm{u}}$. Every instance of $\Delta_0(\mathrm{L}_0)$-Collection we use is an instance of the apparently weaker scheme

- *Uniformly Bounded* $\Delta_0(\mathrm{L}_0)$*-Replacement*

$$\forall u{\in}x\, \exists^{\leqslant 1} v\, \varphi(u,v,\vec{x}) \wedge \forall u{\in}x\, \exists v\, \varphi^{\varepsilon,t}(u,v,\vec{x}) \\ \to \exists y\, (y = \{v : \exists u \in x\, \varphi(u,v,\vec{x})\})$$

where $\varphi^{\varepsilon,t}(u,v,\vec{x})$ abbreviates the formula

$$\varphi(u,v,\vec{x}) \wedge \varepsilon(\cdot,\cdot,v,u,\vec{x}) : v \preccurlyeq t(u,\vec{x})$$

and the scheme ranges over $\Delta_0(\mathrm{L}_0)$-formulas φ, ε and #-terms t.

5.1. *The definitional expansion* $\mathbf{KP}_1^{\mathbf{u}} + \mathbf{L_{def}}$

It will be convenient to have the L_0^+ relation and function symbols available, so the first step in the expansion is a small one, allowing this.

Definition 5.4: $\mathrm{KP}_1^{\mathrm{u}+}$ is the $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$-expansion of $\mathrm{KP}_1^{\mathrm{u}}$ which adds the defining axioms for all symbols in $\mathrm{L}_0^+ \setminus \mathrm{L}_0$.

We will use the following proposition without comment. (Cf. Proposition 4.4.)

Proposition 5.5: $\mathrm{KP}_1^{\mathrm{u}+}$ *is a conservative extension of* $\mathrm{KP}_1^{\mathrm{u}}$*. Furthermore* $\mathrm{KP}_1^{\mathrm{u}+}$ *proves the axiom schemes of* $\mathrm{KP}_1^{\mathrm{u}}$ *with* L_0^+ *replacing* L_0*.*

We now expand $\mathrm{KP}_1^{\mathrm{u}+}$ with functions symbols $f(\vec{x})$ with $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0^+)$-definitions of a special kind. The existentially quantified witness v in such a definition is not only bounded by a #-term $t(\vec{x})$, but weakly uniformly bounded. Moreover, the witness v is uniquely described by a $\Delta_0(\mathrm{L}_0^+)$-formula $\varphi(v,\vec{x})$. Intuitively, this formula says "v is a computation of the value of f on input $\vec{x}$". The "output" function e is a very simple function that *e*xtracts the value $e(v) = f(\vec{x})$ from the computation v.

Definition 5.6: A *good definition* is a tuple

$$(\varphi(v,\vec{x}), \varepsilon(z,z',v,\vec{x}), e(v), t(\vec{x}))$$

where φ, ε are $\Delta_0(\mathrm{L}_0^+)$-formulas, $e(v)$ is an L_0^+-term and $t(\vec{x})$ is a #-term such that $\mathrm{KP}_1^{\mathrm{u}+}$ proves

(Witness Existence) $\exists v\, \varphi(v, \vec{x})$
(Witness Uniqueness) $\exists^{\leqslant 1} v\, \varphi(v, \vec{x})$
(Witness Embedding) $\varphi(v, \vec{x}) \to \varepsilon(\cdot, \cdot, v, \vec{x}) : v \preccurlyeq t(\vec{x})$.

Definition 5.7: The theory $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ is obtained from $\mathrm{KP}_1^{\mathrm{u}+}$ by adding for every such good definition a function symbol $f(\vec{x})$ along with the defining axiom

$$\exists v\, (\varphi(v, \vec{x}) \wedge f(\vec{x}) = e(v)). \tag{5.1}$$

We then speak of a *good definition of f*. The language $\mathrm{L}_{\mathrm{def}}$ of $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ consists of L_0^+ together with all such function symbols.

It is obvious that $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ is a conservative extension of $\mathrm{KP}_1^{\mathrm{u}+}$. Again, we stress that we do not adopt the axiom schemes of $\mathrm{KP}_1^{\mathrm{u}}$ for the language $\mathrm{L}_{\mathrm{def}}$. For example, by definition, $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ has just $\Delta_0(\mathrm{L}_0)$-Separation, not $\Delta_0(\mathrm{L}_{\mathrm{def}})$-Separation.

Theorem 5.8: $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ *proves that* $\mathrm{L}_{\mathrm{def}}$ *satisfies the closure properties of* $\mathrm{L}_{\mathrm{crsf}}$ *from Definition 3.15. That is, (a) closure under composition, (b) closure under replacement and (c) closure under syntactic Cobham recursion.*

Statement (a) is proved in Theorem 5.10, statement (b) in Theorem 5.15 (see Example 5.16) and statement (c) in Theorem 5.17.

The Definability Theorem 5.2 follows easily from Theorem 5.8. We first observe that all function symbols in $\mathrm{L}_{\mathrm{def}}$ are $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$-definable in $\mathrm{KP}_1^{\mathrm{u}}$, since we can replace all L_0^+ symbols in (5.1) by their $\Delta_0(\mathrm{L}_0)$ definitions and appeal to the conservativity of $\mathrm{KP}_1^{\mathrm{u}+}$ over $\mathrm{KP}_1^{\mathrm{u}}$. We can then go through the function symbols in $\mathrm{L}_{\mathrm{crsf}}$ one-by-one and show that each one has a corresponding symbol in $\mathrm{L}_{\mathrm{def}}$ (see also Section 5.5 below). Notice that this gives us more than just that every CRSF function f is $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$-definable in $\mathrm{KP}_1^{\mathrm{u}}$. In particular, we have that the witness v is unique, which we will use later in Corollary 6.17. Put differently, the value $f(\vec{x})$ is $\Delta_0(\mathrm{L}_0)$-definable from a set v which is $\Delta_0(\mathrm{L}_0)$-definable from the arguments $\vec{x}$.

A first step in the proof of Theorem 5.8 is to show that we can treat the language $\mathrm{L}_{\mathrm{def}}$ uniformly, in that every function symbol in it has a good definition.

Lemma 5.9: *For every* $f(\vec{x})$ *in* L_0^+ *there exists a good definition* $(\varphi(v,\vec{x}), \varepsilon(u,v,\vec{x}), e(v), t(\vec{x}))$ *such that* $\mathrm{KP}_1^{\mathrm{u}+}$ *proves (5.1).*

Proof: For $\varphi(v,\vec{x})$ choose $v = f(\vec{x})$, for $e(v)$ choose v, and for ε and t use Lemma 4.10. □

It is straightforward to show part (a) of Theorem 5.8, that $\mathrm{L}_{\mathrm{def}}$ is closed under composition. In this proof, and in the rest of the section, we will make frequent appeals to the Monotonocity Lemma 4.8 and will simply say "by monotonicity".

Theorem 5.10: *For all* n*-ary function symbols* $h(x_1,\dots,x_n)$ *in* $\mathrm{L}_{\mathrm{def}}$ *and* m*-ary function symbols* $g_i(y_1,\dots,y_m)$ *for* $i = 1,\dots,n$ *in* $\mathrm{L}_{\mathrm{def}}$ *there is an* m*-ary function symbol* $f(\vec{y})$ *in* $\mathrm{L}_{\mathrm{def}}$ *such that* $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ *proves*

$$f(\vec{y}) = h(g_1(\vec{y}),\dots,g_n(\vec{y})).$$

Proof: For notational simplicity, assume $n = 1$. Let $h(x)$ and $g(\vec{y})$ be function symbols in $\mathrm{L}_{\mathrm{def}}$ with good definitions $(\varphi_h, \varepsilon_h, e_h, t_h)$ and $(\varphi_g, \varepsilon_g, e_g, t_g)$. Set

$$\begin{aligned}
\psi(v, v_g, v_h, \vec{y}) &:= (v = \langle v_h, v_g\rangle \wedge \varphi_g(v_g,\vec{y}) \wedge \varphi_h(v_h, e_g(v_g))),\\
\varphi_f(v,\vec{y}) &:= \exists v_h, v_g \in \mathrm{tc}(v)\, \psi(v, v_g, v_h, \vec{y}),\\
e_f(v) &:= e_h(\pi_1(v)).
\end{aligned}$$

We claim that there are ε_f, t_f such that $(\varphi_f, \varepsilon_f, e_f, t_f)$ is a good definition, i.e., such that $\mathrm{KP}_1^{\mathrm{u}+}$ proves

$$\psi(v, v_g, v_h, \vec{y}) \to \varepsilon_f(\cdot,\cdot,v,\vec{y}) : \langle v_h, v_g\rangle \preccurlyeq t_f(\vec{y}).$$

Argue in $\mathrm{KP}_1^{\mathrm{u}+}$. Assume $\psi(v, v_g, v_h, \vec{y})$. By Lemma 4.10 and monotonicity, we have $e_g(v_g) \preccurlyeq e_g^{\#}(t_g(\vec{y}))$ for some $\#$-term $e_g^{\#}$. By monotonicity, $v_h \preccurlyeq t_h(e_g(v_g))$ implies that $v_h \preccurlyeq t_h(e_g^{\#}(t_g(\vec{y})))$. Using the term t_{pair} from Example 4.2, $t_f(\vec{y}) := t_{\mathrm{pair}}\big(t_h(e_g^{\#}(t_g(\vec{y}))),\, t_g(\vec{y})\big)$ is as desired. It is easy to find a formula ε_f as desired. □

Before proving parts (b) and (c) of Theorem 5.8 we need a technical lemma. We return to the proof of part (b) in Section 5.3.

5.2. *Elimination lemma*

Recall that the axioms of $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ do not include the axiom schemes of $\mathrm{KP}_1^{\mathrm{u}}$ in the language $\mathrm{L}_{\mathrm{def}}$ but only in the language L_0. However, in order to prove closure under syntactic Cobham recursion, Theorem 5.8 (c), we will need some version of these schemes.

In the usual development of full Kripke Platek set theory KP (e.g., [5, Chapter I]), one shows that Σ_1-expansions prove each scheme for formulas mentioning new symbols from their bigger language L, for example $\Delta_0(L)$-Separation. This is done in two steps. First, one shows that occurrences of new Σ_1-defined symbols can be eliminated in a way that transforms $\Delta_0(L)$-formulas into Δ_1-formulas. Second, one proves Δ_1-Separation in KP. An analogous procedure is employed in bounded arithmetic when developing S^1_2 (cf. [12]).

For our weak theory $\mathrm{KP}^{\mathrm{u}+}_1$ the situation is more subtle. The following lemma gives a version of the elimination step, just good enough for our purposes: it eliminates new function symbols by $\in$-bounding quantifiers, with the help of an auxiliary parameter V. Intuitively, this V is a set collecting enough computations of new functions to evaluate the given formula; it is uniquely determined by a simple formula and weakly uniformly bounded. The precise statement needs the following auxiliary notion.

Definition 5.11: We write $\Delta^+_0(\mathrm{L_{def}})$ for the class of $\Delta_0(\mathrm{L_{def}})$-formulas all of whose $\in$-bounding terms are L^+_0-terms.

Lemma 5.12: (Elimination) *For every $\varphi(\vec{x}) \in \Delta^+_0(\mathrm{L_{def}})$ there are $\Delta_0(\mathrm{L}^+_0)$-formulas $\varphi_{\mathrm{equ}}(\vec{x}, V)$, $\varphi_{\mathrm{aux}}(\vec{x}, V)$, $\varphi_{\mathrm{emb}}(z, z', \vec{x}, V)$ and a $\#$-term $t_\varphi(\vec{x})$ such that $\mathrm{KP}^{\mathrm{u}}_1 + \mathrm{L_{def}}$ proves*

$$
\begin{aligned}
&\exists^{\leqslant 1} V\, \varphi_{\mathrm{aux}}(\vec{x}, V)\\
&\exists V\, (\varphi_{\mathrm{aux}}(\vec{x}, V) \wedge \varphi_{\mathrm{emb}}(\cdot, \cdot, \vec{x}, V) : V \preccurlyeq t_\varphi(\vec{x}))\\
&\varphi_{\mathrm{aux}}(\vec{x}, V) \to (\varphi(\vec{x}) \leftrightarrow \varphi_{\mathrm{equ}}(\vec{x}, V)).
\end{aligned}
\tag{5.2}
$$

Proof: This is proved by induction on $\varphi(\vec{x})$. The base case for atomic $\varphi(\vec{x})$ is the most involved and is proved by induction on the number of occurrences of symbols in φ from $\mathrm{L_{def}} \setminus \mathrm{L}^+_0$. If this number is 0, there is not much to be shown. Otherwise one can write

$$\varphi(\vec{x}) = \psi(\vec{x}, f(\vec{s}(\vec{x}))),$$

where $\psi(\vec{x}, y)$ has one fewer occurrence of symbols from $\mathrm{L_{def}} \setminus \mathrm{L}^+_0$, the symbol $f(\vec{z})$ is from $\mathrm{L_{def}} \setminus \mathrm{L}^+_0$, and $\vec{s}(\vec{x})$ is a tuple of L^+_0-terms.

Let $(\varphi_f, \varepsilon_f, e_f, t_f)$ be a good definition of $f(\vec{z})$. By Lemma 4.10 and monotonicity, we have $\#$-terms $e^{\#}_f(v), \vec{s}^{\#}(\vec{x})$ and $\Delta_0(\mathrm{L}_0)$-formulas $\varepsilon_0, \varepsilon_1$

such that $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ proves

$$\begin{aligned}&\varphi_f(v, \vec{s}(\vec{x})) \to\\ &\quad \varepsilon_0(\cdot,\cdot,v,\vec{x}) : v \preccurlyeq t_f(\vec{s}^{\#}(\vec{x})) \wedge \varepsilon_1(\cdot,\cdot,v,\vec{x}) : e_f(v) \preccurlyeq e_f^{\#}(t_f(\vec{s}^{\#}(\vec{x}))).\end{aligned}$$

By induction, there are ψ_{equ}, ψ_{aux}, $\psi_{\mathrm{emb}}, t_\psi$ such that $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ proves

$$\begin{aligned}&\exists^{\leqslant 1} W\, \psi_{\mathrm{aux}}(\vec{x}, e_f(v), W)\\ &\exists W\, \big(\psi_{\mathrm{aux}}(\vec{x}, e_f(v), W) \wedge \psi_{\mathrm{emb}}(\cdot,\cdot,\vec{x}, e_f(v), W) : W \preccurlyeq t_\psi(\vec{x}, e_f(v))\big)\\ &\psi_{\mathrm{aux}}(\vec{x}, e_f(v), W) \to (\psi(\vec{x}, e_f(v)) \leftrightarrow \psi_{\mathrm{equ}}(\vec{x}, e_f(v), W)).\end{aligned}$$

Define $\varphi_{\mathrm{aux}}(\vec{x}, V) := \exists W, v \in \mathrm{tc}(V)\,(\chi(\vec{x}, V, W, v))$ where

$$\chi(\vec{x}, V, W, v) := (V = \langle W, v\rangle \wedge \psi_{\mathrm{aux}}(\vec{x}, e_f(v), W) \wedge \varphi_f(v, \vec{s}(\vec{x}))).$$

Monotonicity lets us construct from ψ_{emb} a $\Delta_0(\mathrm{L}_0^+)$-formula ε_2 such that $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ proves

$$\begin{aligned}&\chi(\vec{x}, V, W, v) \to\\ &\varepsilon_2(\cdot,\cdot,\vec{x}, V) : W \preccurlyeq t_\psi(\vec{x}, e_f^{\#}(v)) \wedge \varepsilon_0(\cdot,\cdot,\pi_2(V), \vec{x}) : v \preccurlyeq t_f(\vec{s}^{\#}(\vec{x})).\end{aligned}$$

Using the term t_{pair} from Example 4.2 we define

$$t_\varphi(\vec{x}) := t_{\mathrm{pair}}(t_\psi(\vec{x}, e_f^{\#}(v)), t_f(\vec{s}^{\#}(\vec{x})))$$

and get a $\Delta_0(\mathrm{L}_0^+)$-formula φ_{emb} such that $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ proves

$$\chi(\vec{x}, V, W, v) \to \varphi_{\mathrm{emb}}(\cdot,\cdot,\vec{x}, V) : V \preccurlyeq t_\varphi(\vec{x}).$$

Finally, we set

$$\varphi_{\mathrm{equ}}(\vec{x}, V) := \psi_{\mathrm{equ}}(\vec{x}, e_f(\pi_2(V)), \pi_1(V)).$$

It is easy to verify (5.2). This completes the proof for the case that $\varphi(\vec{x})$ is atomic.

The induction step is easy if $\varphi(\vec{x})$ is a negation or a conjunction. We consider the case that $\varphi(\vec{x}) = \forall u{\in}s(\vec{x})\, \psi(u, \vec{x})$ for some L_0^+-term $s(\vec{x})$. By induction, there are ψ_{equ}, ψ_{aux}, $\psi_{\mathrm{emb}}, t_\psi$ such that $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ proves

$$\begin{aligned}&\forall u{\in}s(\vec{x})\, \exists^{\leqslant 1} W\, \psi_{\mathrm{aux}}(u, \vec{x}, W)\\ &\forall u{\in}s(\vec{x})\, \exists W\, \big(\psi_{\mathrm{aux}}(u, \vec{x}, W) \wedge \psi_{\mathrm{emb}}(\cdot,\cdot,u,\vec{x}, W) : W \preccurlyeq t_\psi(u, \vec{x})\big)\\ &\forall u{\in}s(\vec{x})\, \forall W\, \big(\psi_{\mathrm{aux}}(u, \vec{x}, W) \to (\psi(u, \vec{x}) \leftrightarrow \psi_{\mathrm{equ}}(u, \vec{x}, W))\big).\end{aligned}$$

By monotonicity and Lemma 4.10 there is a #-term $s^{\#}(\vec{x})$ such that T_0^+ defines a $\Delta_0(\mathrm{L}_0)$-embedding of u into $s^{\#}(\vec{x})$ when $u \in s(\vec{x})$, and hence without loss of generality we can replace the bound $t_\psi(u,\vec{x})$ above with $t_\psi(s^\#(\vec{x}),\vec{x})$. By $\Delta_0(\mathrm{L}_0)$-Collection, $\mathrm{KP}^{\mathrm{u}}_1 + \mathrm{L}_{\mathrm{def}}$ proves that the set

$$V = \{W : \exists u \in s(\vec{x})\, \psi_{\mathrm{aux}}(u,\vec{x},W)\} \tag{5.3}$$

exists, and by Lemma 4.11 it also $\Delta_0(\mathrm{L}_0)$-defines an embedding of V into some #-term $t(\vec{x})$. For $\varphi_{\mathrm{emb}}(z,z',\vec{x},V)$ we choose a formula describing this embedding and we set $t_\varphi(\vec{x}) = t(\vec{x})$. Define $\varphi_{\mathrm{aux}}(\vec{x},V)$ to be a $\Delta_0(\mathrm{L}_0^+)$-formula expressing (5.3); this is $\Delta_0(\mathrm{L}_0^+)$ because witnesses in ψ_{aux} are unique – recall the discussion of "collection terms" in Section 3.3. Define $\varphi_{\mathrm{equ}}(\vec{x},V)$ to be the $\Delta_0(\mathrm{L}_0^+)$-formula

$$\forall u{\in}s(\vec{x})\, \exists W \in V\, (\psi_{\mathrm{equ}}(u,\vec{x},W) \wedge \psi_{\mathrm{aux}}(u,\vec{x},W)).$$

It is straightforward to verify (5.2) in $\mathrm{KP}^{\mathrm{u}}_1 + \mathrm{L}_{\mathrm{def}}$. □

One can bootstrap the Elimination Lemma to yield a bigger auxiliary set V such that $\varphi_{\mathrm{equ}}(\vec{x},V)$ is equivalent to $\varphi(\vec{x})$ simultaneously for all tuples $\vec{x}$ taken from a given set. As we shall use this stronger version too, we give details. Recall $\vec{x} \in z$ stands for $\bigwedge_i x_i \in z$.

Lemma 5.13: *For every $\varphi(\vec{x},\vec{y}) \in \Delta_0^+(\mathrm{L}_{\mathrm{def}})$ and every L_0^+-term $s(\vec{y})$ there are $\Delta_0(\mathrm{L}_0^+)$-formulas $\varphi^s_{\mathrm{equ}}(\vec{x},\vec{y},U)$, $\varphi^s_{\mathrm{aux}}(\vec{y},U)$, $\varphi^s_{\mathrm{emb}}(z,z',\vec{y},U)$ and a #-term $t^s_\varphi(\vec{y})$ such that $\mathrm{KP}^{\mathrm{u}}_1 + \mathrm{L}_{\mathrm{def}}$ proves*

$$\begin{aligned}
&\exists^{\leqslant 1} U\, \varphi^s_{\mathrm{aux}}(\vec{y},U)\\
&\exists U\, (\varphi^s_{\mathrm{aux}}(\vec{y},U) \wedge \varphi^s_{\mathrm{emb}}(\cdot,\cdot,\vec{y},U) : U \preccurlyeq t^s_\varphi(\vec{y}))\\
&\varphi^s_{\mathrm{aux}}(\vec{y},U) \to \forall \vec{x}{\in}s(\vec{y})\, (\varphi(\vec{x},\vec{y}) \leftrightarrow \varphi^s_{\mathrm{equ}}(\vec{x},\vec{y},U)).
\end{aligned}$$

Proof: Using Lemma 5.12, choose φ_{equ}, φ_{aux}, φ_{emb} and t_φ for which $\mathrm{KP}^{\mathrm{u}}_1 + \mathrm{L}_{\mathrm{def}}$ proves

$$\begin{aligned}
&\forall \vec{x}{\in}s(\vec{y})\, \exists^{\leqslant 1} V\, \varphi_{\mathrm{aux}}(\vec{x},\vec{y},V)\\
&\forall \vec{x}{\in}s(\vec{y})\, \exists V\, (\varphi_{\mathrm{aux}}(\vec{x},\vec{y},V) \wedge \varphi_{\mathrm{emb}}(\cdot,\cdot,\vec{x},\vec{y},V) : V \preccurlyeq t_\varphi(\vec{x},\vec{y}))\\
&\forall \vec{x}{\in}s(\vec{y})\, \big(\varphi_{\mathrm{aux}}(\vec{x},\vec{y},V) \to (\varphi(\vec{x},\vec{y}) \leftrightarrow \varphi_{\mathrm{equ}}(\vec{x},\vec{y},V))\big).
\end{aligned}$$

From the first and second lines, exactly as in the universal quantification step in the proof of Lemma 5.12, we get a $\Delta_0(\mathrm{L}_0^+)$-formula $\varphi^s_{\mathrm{emb}}(z,z',\vec{y},U)$ and a #-term $t_\varphi(\vec{y})$ such that $\mathrm{KP}^{\mathrm{u}}_1 + \mathrm{L}_{\mathrm{def}}$ proves the existence of a set U with

$$U = \{V : \exists \vec{x} \in s(\vec{y})\, \varphi_{\mathrm{aux}}(\vec{x},\vec{y},V)\} \quad \text{and} \quad \varphi^s_{\mathrm{emb}}(\cdot,\cdot,\vec{y},U) : U \preccurlyeq t_\varphi(\vec{y}). \tag{5.4}$$

For φ^s_{aux} take a $\Delta_0(\mathrm{L}_0^+)$-formula expressing the first conjunct of (5.4), and for φ^s_{equ} take the $\Delta_0(\mathrm{L}_0^+)$-formula $\exists V \in U\, (\varphi_{\mathrm{aux}}(\vec{x},\vec{y},V) \wedge \varphi_{\mathrm{equ}}(\vec{x},\vec{y},V))$. □

As a simple application of Lemma 5.12 we derive a separation scheme.

Corollary 5.14: *The theory* $\mathrm{KP}^{\mathrm{u}}_1 + \mathrm{L}_{\mathrm{def}}$ *proves* $\Delta^+_0(\mathrm{L}_{\mathrm{def}})$*-Separation.*

Proof: Let $\varphi(u, x, \vec{w})$ be a $\Delta^+_0(\mathrm{L}_{\mathrm{def}})$-formula. We want to show that $\{u \in x : \varphi(u, x, \vec{w})\}$ exists. Choose $\varphi^x_{\mathrm{aux}}, \varphi^x_{\mathrm{equ}}$ according to the previous lemma, substituting $x, \vec{w}$ for $\vec{y}$, substituting u for $\vec{x}$, and substituting x for $s(\vec{y})$. Choose U such that $\varphi^x_{\mathrm{aux}}(x, \vec{w}, U)$. Then the set $\{u \in x : \varphi(u, x, \vec{w})\}$ equals $\{u \in x : \varphi^x_{\mathrm{equ}}(u, x, \vec{w}, U)\}$, so exists by $\Delta_0(\mathrm{L}^+_0)$-Separation. □

5.3. *Closure under replacement*

The following theorem is crucial. It provides a formalized version of Theorem 2.6(a) showing, more generally, that $\mathrm{KP}^{\mathrm{u}}_1 + \mathrm{L}_{\mathrm{def}}$ can handle comprehension terms coming from Replacement. Similar terms are basic computation steps in Sazonov's term calculus [23] and in the logic of Blass et al. [10]. Recall that $\vec{x} \in u$ stands for $\bigwedge_i x_i \in u$.

Theorem 5.15: *Let* $\theta(u, \vec{y}, \vec{x})$ *be a* $\Delta^+_0(\mathrm{L}_{\mathrm{def}})$*-formula and* $g(\vec{y}, \vec{x})$ *a function symbol in* $\mathrm{L}_{\mathrm{def}}$*. Then there exists a function symbol* $f(u, \vec{y})$ *in* $\mathrm{L}_{\mathrm{def}}$ *such that* $\mathrm{KP}^{\mathrm{u}}_1 + \mathrm{L}_{\mathrm{def}}$ *proves*

$$f(u, \vec{y}) = \{g(\vec{y}, \vec{x}) : \theta(u, \vec{y}, \vec{x}) \wedge \vec{x} \in u\}.$$

Proof: For notational simplicity we assume $\vec{y}$ is the empty tuple. It is sufficient to prove the theorem for g such that $\mathrm{KP}^{\mathrm{u}}_1 + \mathrm{L}_{\mathrm{def}}$ proves $g(\vec{x}) \neq 0$. We first show that $\mathrm{KP}^{\mathrm{u}}_1 + \mathrm{L}_{\mathrm{def}}$ proves the existence of

$$z := \{g(\vec{x}) : \theta(u, \vec{x}) \wedge \vec{x} \in u\}$$

and furthermore describes an embedding of z into $t_1(u)$ for a suitable #-term t_1.

Let $(\varphi_g, \varepsilon_g, e_g, t_g)$ be a good definition of g and choose $\theta_{\mathrm{equ}}, \theta_{\mathrm{aux}}, \theta_{\mathrm{emb}}, t_\theta$ for θ according to the Elimination Lemma 5.12. Argue in $\mathrm{KP}^{\mathrm{u}}_1 + \mathrm{L}_{\mathrm{def}}$. For every $\vec{x} \in u$ there exists a unique w such that

$$\exists y, v_g, V \in \mathrm{tc}(w)\ \psi(w, y, v_g, V, u, \vec{x}),$$

where $\psi(w, y, v_g, V, u, \vec{x})$ expresses that $w = \langle\langle y, v_g\rangle, V\rangle$ where either $(y = g(\vec{x}) \wedge \theta(u, \vec{x}))$ or $(y = 0 \wedge \neg\theta(u, \vec{x}))$, and the computations of g and θ are witnessed by v_g and V. Formally, $\psi(w, y, v_g, V, u, \vec{x})$ is the following

$\Delta_0(\mathrm{L}_0^+)$-formula:

$$w = \langle\langle y, v_g\rangle, V\rangle \wedge \theta_{\mathrm{aux}}(u, \vec{x}, V) \wedge \varphi_g(v_g, \vec{x}) \\ \wedge \ \big((y = e_g(v_g) \wedge \theta_{\mathrm{equ}}(u, \vec{x}, V)) \vee (y = 0 \wedge \neg\theta_{\mathrm{equ}}(u, \vec{x}, V))\big).$$

As in the proof of Lemma 5.12, from θ_{emb} and ε_g we can construct a $\Delta_0(\mathrm{L}_0)$-formula ε and #-term $t_2(u, \vec{x})$ such that $\varepsilon(\cdot, \cdot, w, u, \vec{x}) : w \preccurlyeq t_2(u, \vec{x})$ for this w. By Collection the set

$$W = \{w : \exists \vec{x} \in u \, \exists y, v_g, V \in \mathrm{tc}(w) \ \psi(w, y, v_g, V, u, \vec{x})\},$$

exists, and by Lemma 4.11 we have $\varepsilon'(\cdot, \cdot, W, u) : W \preccurlyeq t_3(u)$ for a suitable $\Delta_0(\mathrm{L}_0)$-formula ε' and #-term $t_3(u)$. The definition of W above can be expressed by the $\Delta_0(\mathrm{L}_0^+)$-formula (as discussed in Section 3.3)

$$\forall w \in W \, \exists \vec{x} \in u \, \exists y, v_g, V \in \mathrm{tc}(w) \ \psi \ \wedge \ \forall \vec{x} \in u \, \exists w \in W \, \exists y, v_g, V \in \mathrm{tc}(w) \ \psi. \tag{5.5}$$

We see that z exists by $\Delta_0(\mathrm{L}_0)$-Separation:

$$z = \{y \in \mathrm{tc}(W) : \exists w \in W \, (y = \pi_1(\pi_1(w)) \wedge y \neq 0)\}. \tag{5.6}$$

Note $\varepsilon'(\cdot, \cdot, W, u) : z \preccurlyeq t_3(u)$ since z is a subset of $\mathrm{tc}(W)$. Recalling t_{pair} from Example 4.2, we construct a good definition $(\theta_f, \varepsilon_f, e_f, t_f)$ of $f(u)$:

$$\begin{aligned} \theta_f(v, u) &:= \exists W, z \in \mathrm{tc}(v) \, (v = \langle W, z\rangle \wedge (5.5) \text{ and } (5.6) \text{ hold}) \\ e_f(v) &:= \pi_2(v) \\ t_f(u) &:= t_{\mathrm{pair}}(t_3(u), t_3(u)), \end{aligned}$$

and ε_f such that $\varepsilon_f(\cdot, \cdot, v, u) : v \preccurlyeq t_f(u)$ for the unique v with $\theta_f(v, u)$. □

We can now show that, in $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$, weakly uniform embeddings (given by $\Delta_0(\mathrm{L}_0)$-formulas) and strongly uniform embeddings (given by function symbols) are closely related. For suppose we are given a $\Delta_0(\mathrm{L}_0)$-embedding $\varepsilon(\cdot, \cdot, \vec{x}) : s(\vec{x}) \preccurlyeq t(\vec{x})$. Then a function τ satisfying $\tau(z, \vec{x}) = \{z' \in \mathrm{tc}(t(\vec{x})) : \varepsilon(z, z', \vec{x})\}$ is in $\mathrm{L}_{\mathrm{def}}$ by Theorem 5.15, and we have $\tau(\cdot, \vec{x}) : s(\vec{x}) \preccurlyeq t(\vec{x})$. On the other hand, suppose $\tau \in \mathrm{L}_{\mathrm{def}}$ and $\tau(\cdot, \vec{x}) : s(\vec{x}) \preccurlyeq t(\vec{x})$. If we define $\varepsilon(z, z', \vec{x})$ as $z' \in \tau(z, \vec{x})$, then $\varepsilon(\cdot, \cdot, \vec{x}) : s(\vec{x}) \preccurlyeq t(\vec{x})$. The embedding ε is $\Delta_0^+(\mathrm{L}_{\mathrm{def}})$ rather than $\Delta_0(\mathrm{L}_0)$, but using the Elimination Lemma 5.12 we can find an equivalent $\Delta_0(\mathrm{L}_0)$-embedding, at the cost of involving a unique, bounded parameter V.

We will use constructions like this in the next subsection, where we need to show, using induction with bounds given by weakly uniform embeddings, that $\mathrm{L}_{\mathrm{def}}$ is closed under Cobham recursion where the bound is given by a strongly uniform embedding.

Example 5.16: Let $f(x, \vec{w})$ be a function symbol in $\mathrm{L_{def}}$. Then $\mathrm{L_{def}}$ contains a function symbol $f\text{"}(x, \vec{w})$ such that $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L_{def}}$ proves

$$f\text{"}(x, \vec{w}) = \{f(u, \vec{w}) : u \in x\}.$$

Furthermore, $\mathrm{L_{def}}$ contains the function symbols $x \cap y$ and $x \setminus y$ and $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L_{def}}$ proves the usual defining axioms for them.

5.4. *Closure under syntactic Cobham recursion*

We are ready to verify statement (c) of Theorem 5.8, that $\mathrm{L_{def}}$ is closed under syntactic Cobham recursion.

Theorem 5.17: *For all function symbols $g(x, z, \vec{w})$ and $\tau(u, v, x, \vec{w})$ in $\mathrm{L_{def}}$ and all #-terms $t(x, \vec{w})$ there is a function symbol $f(x, \vec{w})$ in $\mathrm{L_{def}}$ such that $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L_{def}}$ proves*

$$f(x, \vec{w}) = \begin{cases} g(x, f\text{"}(x, \vec{w}), \vec{w}) & \text{if } \tau \text{ is an embedding into } t \text{ at } x, \vec{w} \\ 0 & \text{otherwise,} \end{cases} \tag{5.7}$$

where "τ is an embedding into t at $x, \vec{w}$" stands for the $\Delta_0(\mathrm{L_{def}})$-formula

$$\tau(\cdot, g(x, f\text{"}(x, \vec{w}), \vec{w}), x, \vec{w}) \,:\, g(x, f\text{"}(x, \vec{w}), \vec{w}) \preccurlyeq t(x, \vec{w}).$$

Proof: Let g, τ, t be as stated. For notational simplicity we assume $\vec{w}$ is the empty tuple. We are looking for a good definition $(\varphi_f, \varepsilon_f, e_f, t_f)$ of the function $f(x)$, that is, for a good definition for which $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L_{def}}$ proves (5.7) for the associated function symbol $f(x)$ in $\mathrm{L_{def}}$.

We intend to let $\varphi_f(v, x)$ say that v encodes the course of values of f, namely the set of all pairs $\langle u, f(u)\rangle, u \in \mathrm{tc}^+(x)$. More precisely, we will express this by writing a $\Delta_0^+(\mathrm{L_{def}})$-formula $\psi(w, x)$ which asserts that the values in a sequence w are recursively computed by g, and then applying the Elimination Lemma 5.12 to get the required $\Delta_0(\mathrm{L}_0^+)$-formula $\varphi_f(v, x)$. Hence the witness v will consist of w plus some parameters needed for the elimination of $\mathrm{L_{def}}$-symbols.

By Theorem 5.15 there is a binary function symbol $w\text{"}y$ in $\mathrm{L_{def}}$ such that $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L_{def}}$ proves $w\text{"}y = \{w\text{'}z : z \in y\}$. We define an auxiliary formula

$$\xi(w, y) := \tau(\cdot, g(y, w\text{"}y), y) : g(y, w\text{"}y) \preccurlyeq t(y).$$

We then let $\psi(w, x)$ express that w is a function with domain $\mathrm{tc}^+(x)$ such that

$$\forall y{\in}\mathrm{tc}^+(x)\, \big((\xi(w, y) \wedge w\text{'}y = g(y, w\text{"}y)) \;\vee\; (\neg\xi(w, y) \wedge w\text{'}y = 0)\big).$$

Claim 1. There is a $\Delta_0^+(\mathrm{L_{def}})$-formula δ and a #-term s such that $\mathrm{KP}_1^\mathrm{u} + \mathrm{L_{def}}$ proves

$$\psi(w,x) \to \delta(\cdot,\cdot,w,x) : w \preccurlyeq s(x).$$

Proof of Claim 1. Argue in $\mathrm{KP}_1^\mathrm{u} + \mathrm{L_{def}}$. Suppose that $\psi(w,x)$ holds. Then for all $y \in \mathrm{tc}^+(x)$ we have $\tau(\cdot, w\text{'}y, y) : w\text{'}y \preccurlyeq t(y)$. Let $\delta_0(z,z',w,y)$ be the formula $z' \in \tau(z, w\text{'}y, y)$. Then we have a weakly uniform $\Delta_0^+(\mathrm{L_{def}})$-embedding $\delta_0(\cdot,\cdot,w,y) : w\text{'}y \preccurlyeq t(y)$ for all $y \in \mathrm{tc}^+(x)$, and by the proof of the Monotonicity Lemma 4.8, adapted for $\Delta_0^+(\mathrm{L_{def}})$-formulas, we can construct a $\Delta_0^+(\mathrm{L_{def}})$-embedding $\delta_1(\cdot,\cdot,w,y,x) : w\text{'}y \preccurlyeq t(x)$.

Using t_pair from Example 4.2 we can find $\delta_2 \in \Delta_0^+(\mathrm{L_{def}})$ and a #-term t' such that

$$\delta_2(\cdot,\cdot,w,y,x) : \langle y, w\text{'}y\rangle \preccurlyeq t'(x)$$

for $y \in \mathrm{tc}^+(x)$. Writing $\varphi(v,y,w)$ for the formula $v = \langle y, w\text{'}y\rangle$, we have $\exists! v\, \varphi(v,y,w)$ and an embedding of v for each y, so can apply Lemma 4.11, adapted for $\Delta_0^+(\mathrm{L_{def}})$-embeddings, to combine these into a single embedding of w into $t''(\mathrm{tc}^+(x), t'(x))$ for a #-term t''. As usual by Lemma 4.10 and monotonicity we can replace this bound with a #-term $s(x)$. ⊣

We can begin to construct a good definition of f. Since ψ is a $\Delta_0^+(\mathrm{L_{def}})$-formula there exist $\Delta_0(\mathrm{L}_0^+)$-formulas $\psi_\mathrm{equ}(w,x,V)$, $\psi_\mathrm{aux}(w,x,V)$, $\psi_\mathrm{emb}(z,z',w,x,V)$ and a #-term $t_\psi(w,x)$ satisfying the Elimination Lemma 5.12 for ψ. Choose δ and s to satisfy Claim 1. By Lemma 5.13, since δ is a $\Delta_0^+(\mathrm{L_{def}})$-formula there exist $\Delta_0(\mathrm{L}_0^+)$-formulas $\delta^r_\mathrm{equ}(z,z',w,x,U)$, $\delta^r_\mathrm{aux}(w,x,U)$, $\delta^r_\mathrm{emb}(z,z',w,x,U)$ and a #-term $t^r_\delta(w,x)$ such that $\mathrm{KP}_1^\mathrm{u} + \mathrm{L_{def}}$ proves

$$\exists^{\leqslant 1} U\, \delta^r_\mathrm{aux}(w,x,U)$$
$$\exists U\, (\delta^r_\mathrm{aux}(w,x,U) \ \wedge\ \delta^r_\mathrm{emb}(\cdot,\cdot,w,x,U) : U \preccurlyeq t^r_\delta(w,x))$$
$$\delta^r_\mathrm{aux}(w,x,U) \to \forall z,z' {\in} r(w,x)\, (\delta(z,z',w,x) \leftrightarrow \delta^r_\mathrm{equ}(z,z',w,x,U)),$$

where $r(w,x)$ is the term $\mathrm{tc}(\{w, s(x)\})$. From the third line it follows that

$$\delta^r_\mathrm{aux}(w,x,U) \to (\delta(\cdot,\cdot,w,x) : w \preccurlyeq s(x) \leftrightarrow \delta^r_\mathrm{equ}(\cdot,\cdot,w,x,U) : w \preccurlyeq s(x)).$$

Now define

$$\varphi_f(v,x) := \exists w,V,U \in \mathrm{tc}(v)$$
$$\big(v = \langle w, \langle V,U\rangle\rangle \wedge \psi_\mathrm{aux}(w,x,V) \wedge \psi_\mathrm{equ}(w,x,V) \wedge \delta^r_\mathrm{aux}(w,x,U)\big)$$
$$e_f(v) := \pi_1(v)\text{'}top(\pi_1(v)),$$

where $top(w)$ is an L_0^+-term that recovers x from w, as the unique member x' of the domain of w such that $\mathrm{tc}(x')$ contains all other members of the domain of w.

We obtain ε_f and t_f from the following claim. Recall that the properties of a good definition of an $\mathrm{L}_{\mathrm{def}}$ symbol must be provable in $\mathrm{KP}_1^{\mathrm{u}+}$.

Claim 2. There is a $\Delta_0(\mathrm{L}_0^+)$-formula ε_f and a #-term t_f such that $\mathrm{KP}_1^{\mathrm{u}+}$ proves

$$\varphi_f(v,x) \to \varepsilon_f(\cdot,\cdot,v,x) : v \preccurlyeq t_f(x).$$

Proof of Claim 2. By conservativity we can argue in $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$. Assume $\varphi_f(v,x)$ and write $v = \langle w, \langle V, U\rangle\rangle$. We have $\Delta_0(\mathrm{L}_0^+)$-formulas ψ_{emb}, δ^r_{emb} and δ^r_{equ} such that

- $\psi_{\mathrm{emb}}(\cdot,\cdot,w,x,V) : V \preccurlyeq t_\psi(w,x)$,
- $\delta^r_{\mathrm{emb}}(\cdot,\cdot,w,x,U) : U \preccurlyeq t^r_\delta(w,x)$,
- $\delta^r_{\mathrm{equ}}(\cdot,\cdot,w,x,U) : w \preccurlyeq s(x)$,

where $t_\psi(w,x)$, $t^r_\delta(w,x)$ and $s(x)$ are #-terms. Define

$$t_f(x) := t_{\mathrm{pair}}\Big(s(x),\ t_{\mathrm{pair}}\big(t_\psi(s(x),x), t^r_\delta(s(x),x)\big)\Big)$$

and use monotonicity to get a $\Delta_0(\mathrm{L}_0^+)$-formula $\varepsilon_f(z,z',v,x)$ describing an embedding of v into $t_f(x)$. ⊣

We must show that $(\varphi_f, \varepsilon_f, e_f, t_f)$ is a good definition. Claim 2 gives (Witness Embedding) and the next two claims show (Witness Uniqueness) and (Witness Existence).

Claim 3. The tuple $(\varphi_f, \varepsilon_f, e_f, t_f)$ satisfies (Witness Uniqueness).

Proof of Claim 3. It suffices to prove in $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ that

$$\psi(w,x) \wedge \psi(\tilde{w},x) \to w = \tilde{w}$$

since uniqueness of V and U is then guaranteed by ψ_{aux} and δ^r_{aux}. So suppose $\psi(w,x)$, $\psi(\tilde{w},x)$ and $w \neq \tilde{w}$. Then the set $\{y \in \mathrm{tc}^+(x) : w\text{'}y \neq \tilde{w}\text{'}y\}$ is nonempty. By set foundation it contains an $\in$-minimal element y_0. Then $w\text{''}y_0 = \tilde{w}\text{''}y_0$ since $w, \tilde{w}$ both have domain $\mathrm{tc}^+(x) \supseteq y_0$. It follows that $g(y_0, w\text{''}y_0) = g(y_0, \tilde{w}\text{''}y_0)$ and $\xi(w,y_0) \leftrightarrow \xi(\tilde{w},y_0)$. Since $\psi(w,x)$ and $\psi(\tilde{w},x)$, we get $w\text{'}y_0 = \tilde{w}\text{'}y_0$, a contradiction. ⊣

The proof of the next claim is the only place where we use the full strength of induction available in $\mathrm{KP}_1^{\mathrm{u}}$.

Claim 4. The tuple $(\varphi_f, \varepsilon_f, e_f, t_f)$ satisfies (Witness Existence).

Proof of Claim 4. Again we will work in $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ and appeal to conservativity. We will use uniformly bounded unique $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$-Induction to prove $\exists v\, \varphi_f(v, x)$. We already know by (Witness Uniqueness) that $\exists^{\leqslant 1} v\, \varphi_f(v, x)$. Furthermore by Claim 2, the witness v is automatically uniformly bounded by the embedding ε_f. It thus suffices to show

$$\forall y{\in}x\, \exists v\, \varphi_f(v, y) \;\rightarrow\; \exists v\, \varphi_f(v, x).$$

Suppose the antecedent holds. By $\Delta_0(\mathrm{L}_0)$-Collection and $\Delta_0(\mathrm{L}_0^+)$-Separation the set

$$W := \{\pi_1(v) : \exists y \in x\, \varphi_f(v, y)\}$$

exists. For each $y \in x$ this contains exactly one w_y such that $\psi(w_y, y)$, that is, such that w_y is a function with domain $\mathrm{tc}^+(y)$ which recursively applies g. By the same argument as in the proof of Claim 3, any two such functions agree on arguments where they are both defined. Hence, $w := \bigcup W$ is a function with domain $\mathrm{tc}(x)$, and we put $w' := w \cup \{\langle x, y \rangle\}$ where $y = g(x, w\text{"}x)$ if $\xi(x, w)$, and $y = 0$ otherwise. Then $\psi(w', x)$ holds. Furthermore $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ proves that there exist U and V such that $\psi_{\mathrm{aux}}(w', x, V)$ and $\delta^r_{\mathrm{aux}}(w', x, U)$. This yields $\varphi_f(v, x)$ for $v = \langle w', \langle V, U \rangle\rangle$. ⊣

We have shown $(\varphi_f, \varepsilon_f, e_f, t_f)$ is a good definition. Let f be the symbol in $\mathrm{L}_{\mathrm{def}}$ associated to this definition. To conclude the proof we verify the conclusion of the theorem, that is, that $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$ proves

$$f(x) = \begin{cases} g(x, f\text{"}(x)) & \text{if } \tau(\cdot, g(x, f\text{"}(x)), x) : g(x, f\text{"}(x)) \preccurlyeq t(x) \\ 0 & \text{otherwise.} \end{cases}$$

Argue in $\mathrm{KP}_1^{\mathrm{u}} + \mathrm{L}_{\mathrm{def}}$. The witness v for $f(x)$ has the form $\langle w, \langle V, U \rangle\rangle$ such that $\psi(w, x)$ and $f(x) = e_f(v) = w\text{'}x$. From $\psi(w, x)$ we get

$$w\text{'}x = \begin{cases} g(x, w\text{"}x) & \text{if } \tau(\cdot, g(x, w\text{"}x), x) : g(x, w\text{"}x) \preccurlyeq t(x) \\ 0 & \text{otherwise.} \end{cases}$$

It now suffices to verify $f\text{"}(x) = w\text{"}x$. This follows from $f(y) = w\text{'}y$ for every $y \in \mathrm{tc}^+(x)$ which is seen similarly as in the proofs of Claims 3 and 4. □

This completes the proof of Theorem 5.8 and thus of Theorem 5.2.

5.5. *The expanded theories*

We fix a fragment of $\mathrm{KP}^{\mathrm{u}}_1 + \mathrm{L}_{\mathrm{def}}$ whose language is exactly $\mathrm{L}_{\mathrm{crsf}}$. We show it proves the schemes in the language $\mathrm{L}_{\mathrm{crsf}}$.

Definition 5.18: $\mathrm{KP}^{\mathrm{u}}_1(\mathrm{L}_{\mathrm{crsf}})$ is the theory in the language $\mathrm{L}_{\mathrm{crsf}}$ which consists of $\mathrm{KP}^{\mathrm{u}}_1$ together with, for each function symbol f in $\mathrm{L}_{\mathrm{crsf}} \setminus \mathrm{L}_0$, a defining axiom for f from $\mathrm{KP}^{\mathrm{u}}_1 + \mathrm{L}_{\mathrm{def}}$ chosen in such a way that $\mathrm{KP}^{\mathrm{u}}_1(\mathrm{L}_{\mathrm{crsf}})$ proves the defining axiom for f from $\mathrm{T}_{\mathrm{crsf}}$.

Since $\mathrm{KP}^{\mathrm{u}}_1(\mathrm{L}_{\mathrm{crsf}})$ is an expansion by definitions of $\mathrm{KP}^{\mathrm{u}}_1$, we have:

Proposition 5.19: $\mathrm{KP}^{\mathrm{u}}_1(\mathrm{L}_{\mathrm{crsf}})$ *is conservative over* $\mathrm{KP}^{\mathrm{u}}_1$ *and proves* $\mathrm{T}_{\mathrm{crsf}}$.

We will later show that $\mathrm{KP}^{\mathrm{u}}_1(\mathrm{L}_{\mathrm{crsf}})$ is $\Pi_2(\mathrm{L}_{\mathrm{crsf}})$-conservative over $\mathrm{T}_{\mathrm{crsf}}$ (Theorem 6.9). Observe that, by Lemma 3.18, $\mathrm{KP}^{\mathrm{u}}_1(\mathrm{L}_{\mathrm{crsf}})$ proves $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$-Separation.

Lemma 5.20: *Let the theory T consist of* $\mathrm{T}_{\mathrm{crsf}}$ *together with the axiom schemes of* $\mathrm{KP}^{\mathrm{u}}_1$ *expanded to the language* $\mathrm{L}_{\mathrm{crsf}}$*, that is, the* $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$*-Collection and uniformly bounded unique* $\Sigma_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}})$*-Induction schemes, where the uniform embeddings may be given by* $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$*-formulas. Then* $\mathrm{KP}^{\mathrm{u}}_1(\mathrm{L}_{\mathrm{crsf}})$ *is equivalent to T.*

Proof: Consider a model M of T; we must show $M \vDash \mathrm{KP}^{\mathrm{u}}_1(\mathrm{L}_{\mathrm{crsf}})$. The reduct of M to L_0 is a model of $\mathrm{KP}^{\mathrm{u}}_1$, and Definition 5.18 is an extension by definitions. Thus M can be expanded to a model $\tilde{M}$ of $\mathrm{KP}^{\mathrm{u}}_1(\tilde{\mathrm{L}}_{\mathrm{crsf}})$, where $\tilde{\mathrm{L}}_{\mathrm{crsf}}$ is a disjoint copy of $\mathrm{L}_{\mathrm{crsf}}$ containing a function symbol $\tilde{f}$ for every function symbol f in $\mathrm{L}_{\mathrm{crsf}}$. The $\tilde{\mathrm{L}}_{\mathrm{crsf}}$ functions satisfy the defining axioms from $\mathrm{T}_{\mathrm{crsf}}$; we claim that this implies that the $\tilde{\mathrm{L}}_{\mathrm{crsf}}$ functions are identical to the original functions from $\mathrm{L}_{\mathrm{crsf}} \setminus \mathrm{L}_0$ in M, and thus $M \vDash \mathrm{KP}^{\mathrm{u}}_1(\mathrm{L}_{\mathrm{crsf}})$. The claim is immediate in the case of functions defined by composition or replacement, but for recursion we need to appeal to induction in M. Suppose $f = f_{g,\tau,s}$ is defined by syntactic Cobham recursion, where $g = \tilde{g}$ and $\tau = \tilde{\tau}$ in $\tilde{M}$ and s is a #-term. We have in $\tilde{M}$, where for clarity we suppress the side variables $\vec{w}$, both

$$\forall x\, \tilde{f}(x) = \begin{cases} g(x, \tilde{f}\text{''}(x)) & \text{if } \tau \text{ is an embedding into } s \text{ at } x \\ 0 & \text{otherwise} \end{cases}$$

and the same formula with f and f'' in place of $\tilde{f}$ and $\tilde{f}$''. The function $\tilde{f}$ has a good definition in the sense of Definition 5.6, so there are $\Delta_0(\mathrm{L}_0^+)$-

formulas φ, ε, an L_0^+-term e and a #-term t such that for all $x \in \tilde{M}$,

$$\begin{aligned} \tilde{M} \models \quad & \exists! v\, \varphi(x,v) \\ & \wedge\ \forall v\, (\varphi(x,v) \to \varepsilon(\cdot,\cdot,x,v) : v \preccurlyeq t(x)) \\ & \wedge\ \big(\tilde{f}(x) = f(x) \leftrightarrow \exists v\, (\varphi(x,v) \wedge e(v) = f(x))\big). \end{aligned}$$

Hence $\tilde{f}(x) = f(x)$ can be expressed as a uniformly bounded $\Sigma_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}})$-formula, for which witnesses are unique. Therefore we can prove it holds for all x by induction in M, as the induction step follows immediately from the recursive equations for $\tilde{f}$ and f.

For the other direction, suppose $M \models \mathrm{KP}_1^{\mathrm{u}}(\mathrm{L}_{\mathrm{crsf}})$. We must show that M satisfies the induction and collection schemes of T.

Suppose $\varphi(x,y)$ and $\varepsilon(z,z',x,y)$ are $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$-formulas and $t(x)$ is a #-term, all with parameters from M, and that $M \models \forall x\, \exists^{\leq 1} y\, \varphi(x,y)$. Let $\varphi^{\varepsilon,t}(x,y)$ abbreviate the formula $\varphi(x,y) \wedge \varepsilon(\cdot,\cdot,x,y) : y \preccurlyeq t(x)$. We will find $\Delta_0(\mathrm{L}_0^+)$-formulas $\tilde{\varphi}(x,w)$, $\tilde{\varepsilon}'(z,z',x,w)$ and a #-term $\tilde{t}(x)$, with the same, unwritten, parameters, such that

$$M \models \forall x\, \exists^{\leq 1} w\, \tilde{\varphi}(x,w)\ \wedge\ \forall x\, \big(\exists y\, \varphi^{\varepsilon,t}(x,y) \leftrightarrow \exists w\, \tilde{\varphi}^{\tilde{\varepsilon},\tilde{t}}(x,w)\big) \qquad (5.8)$$

from which it follows that M satisfies uniformly bounded unique $\Sigma_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}})$-Induction.

Let $\chi_\varepsilon(u,x,y)$ express $u = \{\langle z,z'\rangle \in \mathrm{tc}(y) \times \mathrm{tc}(t(x)) : \varepsilon(z,z',x,y)\}$, which implies in M that $u : y \preccurlyeq t(x) \leftrightarrow \varepsilon(\cdot,\cdot,x,y) : y \preccurlyeq t(x)$. By Lemma 3.18 there is $f \in \mathrm{L}_{\mathrm{crsf}}$ such that

$$M \models (\varphi^{\varepsilon,t}(x,y) \wedge \chi_\varepsilon(u,x,y)) \leftrightarrow f(x,y,u) \neq 0.$$

The function f has a good definition in the sense of Definition 5.6. Therefore there are $\Delta_0(\mathrm{L}_0^+)$-formulas ψ, δ, an L_0^+-term e and a #-term s such that for all $x, y \in M$,

$$\begin{aligned} M \models \quad & \exists! v\, \psi(x,y,u,v) \\ & \wedge\ \forall v\, (\psi(x,y,u,v) \to \delta(\cdot,\cdot,x,y,u,v) : v \preccurlyeq s(x,y,u)) \\ & \wedge\ \big((\varphi^{\varepsilon,t}(x,y) \wedge \chi_\varepsilon(u,x,y)) \leftrightarrow \exists v\, (\psi(x,y,u,v) \wedge e(v) \neq 0)\big). \end{aligned}$$

We can now define

$$\tilde{\varphi}(x,w) := \exists y,u,v \in \mathrm{tc}(w)\, \big(w = \langle\langle u,y\rangle, v\rangle \wedge \psi(x,y,u,v) \wedge e(v) \neq 0\big).$$

Then $\tilde{\varphi}$ satisfies the uniqueness condition; furthermore, the right-to-left implication in (5.8) will hold for any choice of $\tilde{\varepsilon}$ and $\tilde{t}$. For the other direction, $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$-Separation yields u satisfying $\chi_\varepsilon(u,x,y)$. To construct $\tilde{\varepsilon}$ and $\tilde{t}$ for the embedding, Lemma 4.12 gives $\varepsilon_{\rm emb} \in \Delta_0(\mathrm{L}_0)$ and a #-term $t_{\rm emb}$ such that $M \models u : y \preccurlyeq t(x) \to \varepsilon_{\rm emb}(\cdot,\cdot,u,y,t(x)) : \langle u,y\rangle \preccurlyeq t_{\rm emb}(t(x))$. Thus, as in the proof of the Elimination Lemma 5.12, using monotonicity and the term $t_{\rm pair}$ we can find a $\Delta_0(\mathrm{L}_0^+)$-formula $\tilde\varepsilon$ and a #-term $\tilde t$ such that for all $x, y, v \in M$,

$$M \models u : y \preccurlyeq t(x) \wedge \delta(\cdot,\cdot,x,y,v) : v \preccurlyeq s(x,y,u) \\ \to \tilde\varepsilon(\cdot,\cdot,x,\langle\langle u,y\rangle,v\rangle) : \langle\langle u,y\rangle,v\rangle \preccurlyeq \tilde t(x).$$

Then $\tilde\varphi$, $\tilde\varepsilon$ and $\tilde t$ satisfy (5.8).

For collection, suppose $M \models \forall x{\in}u\, \exists y\, \varphi(x,y)$ for $\varphi \in \Delta_0(\mathrm{L}_{\rm crsf})$. Then $\varphi(x,y)$ is equivalent to $f(x,y) \neq 0$ for some $f \in \mathrm{L}_{\rm crsf}$. The good definition of f gives $\psi \in \Delta_0(\mathrm{L}_0^+)$ such that $\varphi(x,y) \leftrightarrow \exists v\, \psi(x,y,v)$ in M for all x, y. By $\Delta_0(\mathrm{L}_0^+)$-Collection there is $W \in M$ such that

$$M \models \forall x{\in}u\, \exists y, v \in W\, \psi(x,y,v).$$

Thus $M \models \forall x{\in}u\, \exists y \in W\, \varphi(x,y)$ as required. □

Lemma 5.21: *Let* $\mathrm{KP}_1^{\preccurlyeq}(\mathrm{L}_{\rm crsf})$ *be the theory* $\mathrm{KP}_1^{\preccurlyeq} + \mathrm{KP}_1^{\rm u}(\mathrm{L}_{\rm crsf})$. *Then* $\mathrm{KP}_1^{\preccurlyeq}(\mathrm{L}_{\rm crsf})$ *is conservative over* $\mathrm{KP}_1^{\preccurlyeq}$, *and is equivalent to the theory consisting of* $\mathrm{T}_{\rm crsf}$ *plus the* $\Delta_0(\mathrm{L}_{\rm crsf})$*-Collection and* $\Sigma_1^{\preccurlyeq}(\mathrm{L}_{\rm crsf})$*-Induction schemes.*

Proof: By Lemma 5.20 it is sufficient to show that any model M of $\mathrm{KP}_1^{\preccurlyeq}(\mathrm{L}_{\rm crsf})$ satisfies the $\Sigma_1^{\preccurlyeq}(\mathrm{L}_{\rm crsf})$-Induction scheme. By Lemma 4.13 it is enough to show that uniformly bounded $\Sigma_1^{\preccurlyeq}(\mathrm{L}_{\rm crsf})$-Induction holds, and this follows by the same argument as in the proof of Lemma 5.20, ignoring the conditions about the witnesses y and w being unique. □

6. Witnessing

Theorem 5.2 established that every CRSF function is $\Sigma_1^{\preccurlyeq}$-definable in $\mathrm{KP}_1^{\preccurlyeq}$, and in fact already in $\mathrm{KP}_1^{\rm u}$. We would like to show that every function $\Sigma_1^{\preccurlyeq}$-definable in $\mathrm{KP}_1^{\preccurlyeq}$ is in CRSF. By analogy with bounded arithmetic, one could aim to prove that whenever $\mathrm{KP}_1^{\preccurlyeq} \vdash \exists y\, \varphi(y,\vec x)$ with $\varphi \in \Delta_0(\mathrm{L}_0)$, then $\mathrm{T}_{\rm crsf} \vdash \varphi(f(\vec x),\vec x)$ or at least $\mathrm{ZFC} \vdash \varphi(f(\vec x),\vec x)$ for some "witnessing" function $f(\vec x)$ in CRSF. As mentioned in the introduction, this fails: a witnessing function $C(x)$ for $(x \neq 0 \to \exists y\,(y \in x))$ would satisfy

$$(x \neq 0 \to C(x) \in x)$$

and not even ZFC can define such a C as a CRSF function.[d] This section shows two ways around this obstacle.

The first is to weaken the conclusion of the witnessing theorem from $\varphi(f(\vec{x}), \vec{x})$ to $\exists y \in f(\vec{x})\, \varphi(y, \vec{x})$. We prove such a witnessing theorem for $\mathrm{KP}_1^{\mathrm{u}}$ (Theorem 6.10), and this has as a corollary the following definability theoretic characterization of CRSF. We do not know whether Theorem 6.10 or Corollary 6.1 hold for $\mathrm{KP}_1^{\preccurlyeq}$ instead of $\mathrm{KP}_1^{\mathrm{u}}$.

Corollary 6.1: *A function is in* CRSF *if and only if it is* $\Sigma_1(\mathrm{L}_0)$*-definable in* $\mathrm{KP}_1^{\mathrm{u}}$.

The second is to simply add a global choice function C to CRSF as one of the initial functions, resulting in CRSF^C (Remark 2.10). In this way we are able to prove full witnessing, even for the stronger theory $\mathrm{KPC}_1^{\preccurlyeq}$ obtained by adding the axiom of global choice (Theorem 6.10). Again this has as a corollary the following definability theoretic characterization of CRSF^C where we write $\mathrm{L}_0^C := \mathrm{L}_0 \cup \{C\}$.

Corollary 6.2: *A function is in* CRSF^C *if and only if it is* $\Sigma_1(\mathrm{L}_0^C)$*-definable in* $\mathrm{KPC}_1^{\preccurlyeq}$.

We do not know whether some form of witnessing holds for $\mathrm{KP}_1^{\preccurlyeq}$ without choice. In particular, the following question is open: if $\mathrm{KP}_1^{\preccurlyeq} \vdash \exists! y\, \varphi(y, x)$, for φ a $\Delta_0(\mathrm{L}_0)$-formula, does this imply that there is a CRSF function f such that (provably in $\mathrm{KP}_1^{\preccurlyeq}$) $\forall x\, \varphi(f(x), x)$ holds?

It would also be interesting to prove a result of this type that needs only an appropriate form of local choice, rather than global choice. For example: if $\mathrm{KP}_1^{\preccurlyeq} \vdash \exists! y\, \varphi(y, x)$, for φ a $\Delta_0(\mathrm{L}_0)$-formula, does this imply that there is a CRSF function $f(x, r)$ such that (provably in $\mathrm{KP}_1^{\preccurlyeq}$) $\forall x\, \varphi(f(x, r), x)$ holds whenever r is a well-ordering of $\mathrm{tc}(x)$?

6.1. *Witnessing* $\mathbf{T_{crsf}}$ *and Herbrand saturation*

We use a method introduced by Avigad in [4] as a general tool for model-theoretic proofs of witnessing theorems, in particular subsuming Zambella's witnessing proof for bounded arithmetic [24]. A structure is *Herbrand saturated* if it satisfies every $\exists\forall$ sentence, with parameters, which is consistent with its universal diagram. To get a witnessing theorem for a theory T, one

[d]This is wellknown: otherwise ZFC would define a global well-order and thus prove $\mathrm{V} = \mathrm{HOD}$; but $\mathrm{V} \neq \mathrm{HOD}$ is relatively consistent (see e.g., [19, p.222]).

uses Herbrand saturation to show that T is $\forall\exists$-conservative over a suitable universal theory S. Since S is universal, a form of witnessing for S follows directly from Herbrand's theorem; conservativity means that this carries over to T.

We want to use this approach where T is $\mathrm{KP}_1^{\mathrm{u}}$ and S is $\mathrm{T}_{\mathrm{crsf}}$. We cannot do this directly since $\mathrm{T}_{\mathrm{crsf}}$ is not universal but, as $\mathrm{T}_{\mathrm{crsf}}$ is Π_1, it turns out that something similar works. Below we prove a version of Herbrand's theorem for $\mathrm{T}_{\mathrm{crsf}}$, in which a witness to a $\Sigma_1(\mathrm{L}_{\mathrm{crsf}})$ sentence is not necessarily equal to a term, but is always contained in some term.

Theorem 6.3: *Suppose* $\mathrm{T}_{\mathrm{crsf}} \vdash \exists y\, \varphi(y, \vec{x})$ *where* φ *is* $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$. *Then there is an* $\mathrm{L}_{\mathrm{crsf}}$ *function symbol* f *such that* $\mathrm{T}_{\mathrm{crsf}} \vdash \exists y \in f(\vec{x})\, \varphi(y, \vec{x})$.

Proof: Take a new tuple $\vec{c}$ of constants and let $P(\vec{c})$ be the theory

$$\mathrm{T}_{\mathrm{crsf}} + \{\forall y{\in}t(\vec{c})\, \neg\varphi(y, \vec{c}) : t(\vec{x}) \text{ an } \mathrm{L}_{\mathrm{crsf}}\text{-term}\}.$$

It suffices to show that $P(\vec{c})$ is inconsistent. Then $\mathrm{T}_{\mathrm{crsf}}$ proves $\bigvee_i \exists y \in t_i(\vec{x})\, \varphi(y, \vec{x})$ for finitely many terms $t_1(\vec{x}), \ldots, t_k(\vec{x})$; we can choose $f(\vec{x})$ so that $\mathrm{T}_{\mathrm{crsf}}$ proves $f(\vec{x}) = t_1(\vec{x}) \cup \cdots \cup t_k(\vec{x})$ using closure under composition.

For the sake of a contradiction assume $P(\vec{c})$ has a model M. Define

$$N := \{a \in M : M \models a \in t(\vec{c}) \text{ for some } \mathrm{L}_{\mathrm{crsf}}\text{-term } t(\vec{x})\}.$$

Note N contains each component c_i^M of $\vec{c}^M$ via the term $\{c_i\}$. We first show that N is a substructure of M. To see this, suppose g is an r-ary function symbol in $\mathrm{L}_{\mathrm{crsf}}$ and $\vec{a} \in N^r$. We must show $g(\vec{a}) \in N$. For each component a_i of $\vec{a}$ there is a term $t_i(\vec{x})$ such that $M \models a_i \in t_i(\vec{c})$. Choose a function symbol $G(z)$ in $\mathrm{L}_{\mathrm{crsf}}$ such that $\mathrm{T}_{\mathrm{crsf}}$ proves $G(z) = g(\pi_1^r(z), \ldots, \pi_r^r(z))$, where π_i^r is the standard projection function for ordered r-tuples (which is in L_0^+). Then in M we have $g(\vec{a}) \in G``(t_1(\vec{c}) \times \cdots \times t_r(\vec{c}))$.

Next we show that N is a $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$-elementary substructure of M, that is, for every $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$-formula θ and $\vec{a} \in N$, we have $N \models \theta(\vec{a}) \Leftrightarrow M \models \theta(\vec{a})$. This is proved by induction on θ, and the only non-trivial case is where $\theta(\vec{a})$ has the form $\exists u \in t(\vec{a})\, \psi(u, \vec{a})$ for some term t, and we have $M \models b \in t(\vec{a}) \wedge \psi(b, \vec{a})$ for some b in M. As N is a substructure, $t(\vec{a}) \in N$ and hence $M \models t(\vec{a}) \in s(\vec{c})$ for some term $s(\vec{x})$. Thus $M \models b \in \bigcup s(\vec{c})$, so $b \in N$. By the induction hypothesis $N \models \psi(b, \vec{a})$ which gives $N \models \theta(\vec{a})$ as required.

Thus $N \models \mathrm{T}_{\mathrm{crsf}}$ since $\mathrm{T}_{\mathrm{crsf}}$ is $\Pi_1(\mathrm{L}_{\mathrm{crsf}})$. Further, $N \models \forall y\,(\neg\varphi(y,\vec{c}))$, since in M there is no witness for $\varphi(y,\vec{c})$ inside any term in $\vec{c}$. This contradicts the assumption of the theorem. □

Corollary 6.4: *If* $\mathrm{T}_{\mathrm{crsf}} \vdash \exists! y\, \varphi(y,\vec{x})$*, where* φ *is* $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$*, then there is an* $\mathrm{L}_{\mathrm{crsf}}$ *function symbol* g *such that* $\mathrm{T}_{\mathrm{crsf}} \vdash \varphi(g(\vec{x}),\vec{x})$*.*

Proof: Appealing to Lemma 3.18, take $g(\vec{x})$ computing $\bigcup\{y \in f(\vec{x}) : \varphi(y,\vec{x})\}$ where f is given by Theorem 6.3. □

We give our version of Herbrand saturation. Let $L \supseteq \mathrm{L}_0$ be a countable language.

Definition 6.5: A structure M is $\Delta_0(L)$-*Herbrand saturated* if it satisfies every $\Sigma_2(L)$-sentence with parameters from M which is consistent with the $\Pi_1(L)$-diagram of M.

The next two lemmas do not use any special properties of the class $\Delta_0(L)$, beyond that it is closed under subformulas, negations and substitution.

Lemma 6.6: *Every consistent* $\Pi_1(L)$ *theory* T *has a* $\Delta_0(L)$*-Herbrand saturated model.*

Proof: Let L^+ be L together with names for countably many new constants. Enumerate all $\Delta_0(L^+)$-formulas as $\varphi_1, \varphi_2, \dots$. Let $T_1 = T$ and define a sequence of theories $T_1 \subseteq T_2 \subseteq \dots$ as follows: if $T_i + \exists\vec{x}\,\forall\vec{y}\,\varphi_i(\vec{x},\vec{y})$ is consistent, let $T_{i+1} = T_i + \forall\vec{y}\,\varphi_i(\vec{c},\vec{y})$ where $\vec{c}$ is a tuple of constant symbols that do not appear in T_i or φ_i. Otherwise let $T_{i+1} = T_i$. Let $T^* = \bigcup_i T_i$. By construction, T^* is consistent and $\Pi_1(L^+)$.

Let M be a model of T^* and let N be the substructure of M consisting of elements named by L^+-terms. We claim that $N \models T^*$. It is enough to show that for every $\Delta_0(L^+)$-formula φ and every tuple $\vec{a}$ from N, we have $N \models \theta(\vec{a}) \Leftrightarrow M \models \theta(\vec{a})$. We prove this by induction on θ. For the only interesting case, suppose $M \models \exists x\, \psi(\vec{a},x)$ where the inductive hypothesis holds for ψ. Since the components of $\vec{a}$ are named by terms, $\psi(\vec{a},x)$ is equivalent in M to some formula $\varphi_i(x)$ from our enumeration. But $M \models \exists x\, \varphi_i(x)$ implies that $\exists x\, \varphi_i(x)$ is consistent with T_i and hence that $\varphi_i(c)$ is in T_{i+1} for some constant c. Thus $M \models \varphi_i(c)$ and therefore $M \models \psi(\vec{a},c)$, so $N \models \psi(\vec{a},c)$ by the inductive hypothesis.

Finally, N is $\Delta_0(L)$-Herbrand saturated. For suppose that ψ is $\Delta_0(L)$ and $\exists\vec{x}\,\forall\vec{y}\,\psi(\vec{x},\vec{y},\vec{a})$ is consistent with the $\Pi_1(L)$-diagram of N, and hence with T^*. Then as above $\psi(\vec{x},\vec{y},\vec{a})$ is equivalent to $\varphi_i(\vec{x},\vec{y})$ for some i, and since $\exists\vec{x}\,\forall\vec{y}\,\varphi_i(\vec{x},\vec{y})$ is consistent with T_i it is witnessed in T_{i+1} by a tuple of constants and hence is true in N. □

Lemma 6.7: *If S, T are theories such that S is $\Pi_1(L)$ and every $\Delta_0(L)$-Herbrand saturated model of S is a model of T, then T is $\Pi_2(L)$-conservative over S.*

Proof: Suppose T proves $\forall\vec{x}\,\exists\vec{y}\,\varphi(\vec{x},\vec{y})$ but S does not, where φ is $\Delta_0(L)$. Then, letting $\vec{c}$ be a tuple of new constants, the theory $S+\forall\vec{y}\,(\neg\varphi(\vec{c},\vec{y}))$ has a $\Delta_0(L)$-Herbrand saturated model by Lemma 6.6. This contradicts the assumptions about S and T. □

We now describe the most useful property of $\Delta_0(\mathrm{L_{crsf}})$-Herbrand saturated models.

Lemma 6.8: *Suppose that $M \models \mathrm{T_{crsf}}$ is $\Delta_0(\mathrm{L_{crsf}})$-Herbrand saturated and that $\varphi(y,\vec{x},\vec{a})$ is a $\Delta_0(\mathrm{L_{crsf}})$-formula with parameters $\vec{a} \in M$ such that $M \models \forall\vec{x}\,\exists y\,\varphi(y,\vec{x},\vec{a})$. Then there exist a function $f \in \mathrm{L_{crsf}}$ and parameters $\vec{m} \in M$ such that $M \models \forall\vec{x}\,\exists y \in f(\vec{x},\vec{m})\,\varphi(y,\vec{x},\vec{a})$.*

Proof: Let T^* be the $\Pi_1(\mathrm{L_{crsf}})$-diagram of M. Then $T^* \vdash \forall\vec{x}\,\exists y\,\varphi(y,\vec{x},\vec{a})$ since otherwise $M \models \exists\vec{x}\,\forall y\,(\neg\varphi(y,\vec{x},\vec{a}))$ by Herbrand saturation. The rest of the argument is standard. By compactness, there are $\vec{b} \in M$ and θ in $\Delta_0(\mathrm{L_{crsf}})$ such that $M \models \forall\vec{z}\,\theta(\vec{a},\vec{b},\vec{z})$ and

$$\mathrm{T_{crsf}} + \forall\vec{z}\,\theta(\vec{a},\vec{b},\vec{z}) \vdash \forall\vec{x}\,\exists y\,\varphi(y,\vec{x},\vec{a})$$

where we treat $\vec{a},\vec{b}$ as constant symbols. Hence, replacing $\vec{a},\vec{b}$ with variables $\vec{u},\vec{v}$,

$$\mathrm{T_{crsf}} \vdash \exists\vec{z}\,\neg\theta(\vec{u},\vec{v},\vec{z}) \vee \exists y\,\varphi(y,\vec{x},\vec{u}). \tag{6.1}$$

Using pairing and projection functions to code tuples of sets as single sets, we can apply Theorem 6.3 to formulas with more than one unbounded existential quantifier. In particular from (6.1) we get an $\mathrm{L_{crsf}}$ function symbol f with

$$\mathrm{T_{crsf}} \vdash \exists\vec{z}\,\neg\theta(\vec{u},\vec{v},\vec{z}) \vee \exists y \in f(\vec{x},\vec{u},\vec{v})\,\varphi(y,\vec{x},\vec{u}).$$

Since $M \models \forall\vec{z}\,\theta(\vec{a},\vec{b},\vec{x})$ it follows that $M \models \forall\vec{x}\,\exists y \in f(\vec{x},\vec{a},\vec{b})\,\varphi(y,\vec{x},\vec{a})$. □

6.2. *Witnessing* $\mathbf{KP_1^u}$

We prove witnessing for $\mathrm{KP}_1^{\mathrm{u}}(\mathrm{L}_{\mathrm{crsf}})$ as a consequence of witnessing for $\mathrm{T}_{\mathrm{crsf}}$, together with the following conservativity result.

Theorem 6.9: *The theory* $\mathrm{KP}_1^{\mathrm{u}}(\mathrm{L}_{\mathrm{crsf}})$ *is* $\Pi_2(\mathrm{L}_{\mathrm{crsf}})$*-conservative over* $\mathrm{T}_{\mathrm{crsf}}$*.*

Proof: Let M be an arbitrary $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$-Herbrand saturated model of $\mathrm{T}_{\mathrm{crsf}}$. By Lemma 6.7 it is enough to show that M is a model of $\mathrm{KP}_1^{\mathrm{u}}(\mathrm{L}_{\mathrm{crsf}})$. By Lemma 5.20 it is enough to show that M satisfies $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$-Collection and uniformly bounded unique $\Sigma_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}})$-Induction.

For collection, suppose that for some $a \in M$ we have

$$M \models \forall u{\in}a\, \exists v\, \varphi(u, v),$$

where φ is $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$ with parameters. We rewrite this as

$$M \models \forall u\, \exists v\, (u \in a \to \varphi(u, v)).$$

By Lemma 6.8, for some $\mathrm{L}_{\mathrm{crsf}}$ function symbol f and tuple $\vec{b} \in M$,

$$M \models \forall u\, \exists v \in f(u, \vec{b})\, (u \in a \to \varphi(u, v)).$$

Hence if we let $c = \bigcup f\text{''}(a, \vec{b})$ we have, as required for collection,

$$M \models \forall u{\in}a\, \exists v \in c\, \varphi(u, v).$$

For induction, let $\varphi(u, v), \varepsilon(z, z', v, u) \in \Delta_0(\mathrm{L}_{\mathrm{crsf}})$ and $t(u)$ be a #-term, all possibly with parameters, and let $\varphi^{\varepsilon,t}(u, v)$ abbreviate $\varphi(u, v) \wedge \varepsilon(\cdot, \cdot, v, u) : v \preccurlyeq t(u)$. Working in M, suppose

$$\forall u\, \exists^{\leqslant 1} v\, \varphi(u, v) \wedge \forall x\, \big(\forall u{\in}x\, \exists v\, \varphi^{\varepsilon,t}(u, v) \to \exists v'\, \varphi^{\varepsilon,t}(x, v')\big).$$

Then in particular

$$\forall x, w\, \big(\forall u{\in}x\, \exists v \in w\, \varphi^{\varepsilon,t}(u, v) \to \exists v'\, \varphi^{\varepsilon,t}(x, v')\big).$$

By Lemma 6.8 there is an $\mathrm{L}_{\mathrm{crsf}}$-function symbol h and a tuple $\vec{a} \in M$ such that we can bound the witness v' as a member of $h(x, w, \vec{a})$. Then, since witnesses v to φ are unique, if we let $g(x, w, \vec{a})$ compute

$$\bigcup \{v' \in h(x, w, \vec{a}) : \varphi(x, v')\}$$

we have

$$\forall x, w\, \big(\forall u{\in}x\, \exists v \in w\, \varphi^{\varepsilon,t}(u, v) \to \varphi^{\varepsilon,t}(x, g(x, w, \vec{a}))\big). \tag{6.2}$$

We must be careful here with our parameters. We may assume without loss of generality that the so-far unwritten parameters in φ, ε and t are contained in the tuple $\vec{a}$, and further that $\varphi(u,v)$ is really $\varphi(u,v,\vec{a})$, $\varepsilon(z,z',v,u)$ is $\varepsilon(z,z',v,u,\vec{a})$ and $t(u)$ is $t(u,\vec{a})$.

We now use syntactic Cobham recursion to iterate g. To use the recursion available in $\mathrm{T_{crsf}}$ we need to turn the weakly uniform embedding given by ε into a strongly uniform embedding. So let τ be an $\mathrm{L_{crsf}}$-function symbol for which $\mathrm{T_{crsf}}$ proves

$$\tau(z,v,u,\vec{w}) = \{z' \in t(u,\vec{w}) : \varepsilon(z,z',v,u,\vec{w})\}.$$

Let f be the $\mathrm{L_{crsf}}$ function symbol $f_{g,\tau,t}$ with defining axiom

$$f(u,\vec{w}) = \begin{cases} g(u, f\text{"}(u,\vec{w}),\vec{w}) & \text{if } \tau \text{ is an embedding into } t \text{ at } u,\vec{w} \\ 0 & \text{otherwise,} \end{cases}$$

where "τ is an embedding into t at $u,\vec{w}$" stands for the $\Delta_0(\mathrm{L_{crsf}})$-formula

$$\tau(\cdot, g(u,f\text{"}(u,\vec{w}),\vec{w}),u,\vec{w}) \,:\, g(u,f\text{"}(u,\vec{w}),\vec{w}) \preccurlyeq t(u,\vec{w}).$$

It suffices now to show that $\forall x\, \varphi^{\varepsilon,t}(x,f(x,\vec{a}),\vec{a})$. We will use $\Delta_0(\mathrm{L_{crsf}})$-Induction, which is available by Lemma 3.18. Suppose $\forall u{\in}x\, \varphi^{\varepsilon,t}(u,f(u,\vec{a}),\vec{a})$. Let $w = f\text{"}(x,\vec{a})$ and let $v = g(x,w,\vec{a})$. By (6.2) we have $\varphi^{\varepsilon,t}(x,v,\vec{a})$, and in particular $\varepsilon(\cdot,\cdot,v,x,\vec{a}) : v \preccurlyeq t(x,\vec{a})$. Hence also $\tau(\cdot,v,x,\vec{a}) : v \preccurlyeq t(x,\vec{a})$, that is, τ is an embedding into t at $x,\vec{a}$. From the defining axiom for f we conclude that $f(x,\vec{a}) = v$, and thus $\varphi^{\varepsilon,t}(x,f(x,\vec{a}),\vec{a})$. This completes the proof. □

From this we get witnessing for $\mathrm{KP}^{\mathrm{u}}_1(\mathrm{L_{crsf}})$, and a fortiori for $\mathrm{KP}^{\mathrm{u}}_1$:

Theorem 6.10: *Suppose* $\mathrm{KP}^{\mathrm{u}}_1(\mathrm{L_{crsf}}) \vdash \exists y\, \varphi(y,\vec{x})$ *where* φ *is* $\Delta_0(\mathrm{L_{crsf}})$*. Then there is an* $\mathrm{L_{crsf}}$*-function symbol* f *such that* $\mathrm{T_{crsf}} \vdash \exists y \in f(\vec{x})\, \varphi(y,\vec{x})$.

Proof: By Theorem 6.9, $\mathrm{T_{crsf}} \vdash \exists y\, \varphi(y,\vec{x})$. Then apply Theorem 6.3. □

Corollary 6.11: *If* $\mathrm{KP}^{\mathrm{u}}_1(\mathrm{L_{crsf}}) \vdash \exists! y\, \varphi(y,\vec{x})$ *where* φ *is* $\Sigma_1(\mathrm{L_{crsf}})$*, then there is an* $\mathrm{L_{crsf}}$*-function symbol* g *such that* $\mathrm{T_{crsf}} \vdash \varphi(g(\vec{x}),\vec{x})$.

Proof: Suppose $\mathrm{KP}^{\mathrm{u}}_1(\mathrm{L_{crsf}}) \vdash \exists! y\, \exists v\, \theta(y,v,\vec{x})$ where θ is $\Delta_0(\mathrm{L_{crsf}})$. Using Theorem 6.10 it is not hard to show that $\mathrm{T_{crsf}} \vdash \exists y,v \in f(\vec{x})\, \theta(y,v,\vec{x})$ for some $\mathrm{L_{crsf}}$-function symbol f, and from Theorem 6.9 we get $\mathrm{T_{crsf}} \vdash \exists^{\leqslant 1} y\, \theta(y,v,\vec{x})$. Thus we can define the witnessing function as $g(\vec{x}) := \bigcup\{y \in f(\vec{x}) : \exists v \in f(\vec{x})\, \theta(y,v,\vec{x})\}$. □

Together with the Definability Theorem 5.2, the above implies Corollary 6.1.

6.3. *Witnessing with global choice*

We add to our basic language L_0 and theory T_0 a symbol C for a *global choice* function, with defining axiom

$$(\mathrm{GC}): \quad C(0) = 0 \wedge (x \neq 0 \rightarrow C(x) \in x).$$

We denote the augmented language and theory by L_0^C and T_0^C. We write $\mathrm{L}_{\mathrm{crsf}}^C$ and $\mathrm{T}_{\mathrm{crsf}}^C$ for $\mathrm{L}_{\mathrm{crsf}}$ and $\mathrm{T}_{\mathrm{crsf}}$ defined using L_0^C and T_0^C in place of L_0 and T_0. The symbols in $\mathrm{L}_{\mathrm{crsf}}^C$ correspond to the functions in CRSF^C, defined like CRSF but with the global choice function $C(x)$ as an additional initial function (cf. Remark 2.10).

We write $\mathrm{KPC}_1^{\preccurlyeq}$ for the corresponding version of $\mathrm{KP}_1^{\preccurlyeq}$, that is, the theory consisting of T_0^C and $\Delta_0(\mathrm{L}_0^C)$-Collection and $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0^C)$-Induction schemes. Similarly $\mathrm{KPC}_1^{\mathrm{u}}$ consists of T_0^C and the $\Delta_0(\mathrm{L}_0^C)$-Collection and uniformly bounded unique $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0^C)$-Induction schemes (where we allow embeddings to be $\Delta_0(\mathrm{L}_0^C)$). Earlier results about the theories without choice carry over to the theories with choice as expected; there is one extra case in Lemma 4.10, taken care of by noting that the identity embedding embeds $C(x) \preccurlyeq x$. In particular, $\mathrm{KPC}_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}}^C)$ is a $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0^C)$-expansion of $\mathrm{KPC}_1^{\preccurlyeq}$ to the language $\mathrm{L}_{\mathrm{crsf}}^C$ which is equivalent to the theory consisting of $\mathrm{T}_{\mathrm{crsf}}^C$ and the schemes of $\mathrm{KPC}_1^{\preccurlyeq}$ over the language $\mathrm{L}_{\mathrm{crsf}}^C$ (cf. Lemma 5.21). Thus, Corollary 6.2 follows from Theorem 6.13 below, a witnessing theorem for $\mathrm{KPC}_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}}^C)$. We prove it by showing conservativity over $\mathrm{T}_{\mathrm{crsf}}^C$:

Theorem 6.12: *The theory* $\mathrm{KPC}_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}}^C)$ *is* $\Pi_2(\mathrm{L}_{\mathrm{crsf}}^C)$*-conservative over* $\mathrm{T}_{\mathrm{crsf}}^C$.

Proof: Let M be an arbitrary $\Delta_0(\mathrm{L}_{\mathrm{crsf}}^C)$-Herbrand saturated model of $\mathrm{T}_{\mathrm{crsf}}^C$. As before, by Lemma 6.7 it is enough to show that M is a model of $\mathrm{KPC}_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}}^C)$. By exactly the same argument as in the proof of Theorem 6.9, we get that M is a model of $\Delta_0(\mathrm{L}_{\mathrm{crsf}}^C)$-Collection.

It remains to show that $\Sigma_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}}^C)$-Induction holds in M. By Lemma 4.13, it is enough to show that uniformly bounded $\Sigma_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}}^C)$-Induction holds. That is, exactly the induction shown for the formula $\exists v\, \varphi^{\varepsilon,t}(u,v)$ in the proof of Theorem 6.9, except without the uniqueness assumption that $M \models \forall u\, \exists^{\leqslant 1} v\, \varphi(u,v)$. Working through that proof, we see that uniqueness is used only in one place, to construct an $\mathrm{L}_{\mathrm{crsf}}$-function symbol g satisfying

$$\forall x, w\, \big(\forall u{\in}x\, \exists v \in w\, \varphi^{\varepsilon,t}(u,v) \rightarrow \varphi^{\varepsilon,t}(x, g(x,w,\vec{a}))\big)$$

from an $\mathrm{L}_{\mathrm{crsf}}$-function symbol h satisfying

$$\forall x, w \left(\forall u{\in}x\, \exists v \in w\, \varphi^{\varepsilon,t}(u,v) \to \exists v' \in h(x,w,\vec{a})\, \varphi^{\varepsilon,t}(x,v'))\right).$$

In $\mathrm{L}^C_{\mathrm{crsf}}$ this can be done without the assumption, by setting

$$g(x,w,\vec{a}) = C(\{v' \in h(x,w,\vec{a}) : \varphi^{\varepsilon,t}(x,v')\}).$$

The rest of the proof goes through as before. □

Theorem 6.13: *Suppose* $\mathrm{KPC}^{\preccurlyeq}_1(\mathrm{L}^C_{\mathrm{crsf}}) \vdash \exists y\, \varphi(y,\vec{x})$ *where* φ *is* $\Delta_0(\mathrm{L}^C_{\mathrm{crsf}})$. *Then there is an* $\mathrm{L}^C_{\mathrm{crsf}}$*-function symbol* f *such that* $\mathrm{T}^C_{\mathrm{crsf}} \vdash \varphi(f(\vec{x}),\vec{x})$.

Proof: By Theorem 6.12, $\mathrm{T}^C_{\mathrm{crsf}} \vdash \exists y\, \varphi(y,\vec{x})$. Using Theorem 6.3 for $\mathrm{T}^C_{\mathrm{crsf}}$ there is g in $\mathrm{L}^C_{\mathrm{crsf}}$ such that $\mathrm{T}^C_{\mathrm{crsf}} \vdash \exists y{\in}g(\vec{x})\varphi(y,\vec{x})$. Using Lemma 3.18 for $\mathrm{T}^C_{\mathrm{crsf}}$ we find h such that $\mathrm{T}^C_{\mathrm{crsf}} \vdash h(\vec{x}) = \{y \in g(\vec{x}) : \varphi(y,\vec{x})\}$. Then choose f such that $\mathrm{T}^C_{\mathrm{crsf}} \vdash f(\vec{x}) = C(h(\vec{x}))$. □

For the theory without choice, we get a weak result in the style of Parikh's theorem [21].

Corollary 6.14: *Suppose* $\mathrm{KP}^{\preccurlyeq}_1(\mathrm{L}_{\mathrm{crsf}}) \vdash \exists y\, \varphi(y,\vec{x})$ *where* φ *is* $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$. *Then in the universe of sets we can bound the complexity of the witness* y *in the following sense: there is a #-term* t *such that* $\forall\vec{x}\,\exists y{\preccurlyeq}t(\vec{x})\,\varphi(y,\vec{x})$ *holds.*

Proof: It is easy to show that for every $\mathrm{L}^C_{\mathrm{crsf}}$-function symbol $f(\vec{x})$ there is an $\mathrm{L}^C_{\mathrm{crsf}}$-function symbol $\tau(z,\vec{x})$ and a #-term $t(\vec{x})$ such that $\tau(\cdot,\vec{x}) : f(\vec{x}) \preccurlyeq t(\vec{x})$, provably in $\mathrm{T}^C_{\mathrm{crsf}}$. For the initial symbols from L^C_0 this is by Lemma 4.10 (extended to cover C). For function symbols obtained by composition we use monotonicity, for replacement we use Lemma 4.11, and for syntactic Cobham recursion we are explicitly given such a bound.

Now suppose the assumption of the corollary holds. Then also $\mathrm{KPC}^{\preccurlyeq}_1(\mathrm{L}^C_{\mathrm{crsf}}) \vdash \exists y\, \varphi(y,\vec{x})$, hence $\mathrm{T}^C_{\mathrm{crsf}} \vdash \varphi(f(\vec{x}),\vec{x})$ for some $\mathrm{L}^C_{\mathrm{crsf}}$-function symbol f by Theorem 6.13. It follows that $\mathrm{T}^C_{\mathrm{crsf}} \vdash \exists y{\preccurlyeq}t(\vec{x})\,\varphi(y,\vec{x})$, by the previous paragraph and using $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$-Separation to get a nonuniform embedding. In ZFC, global choice can be forced without adding new sets (see for example [17]) so we can expand the universe V of sets to a model (V, C) of ZF + (GC) and in particular of $\mathrm{T}^C_{\mathrm{crsf}}$. Then $\forall\vec{x}\,\exists y{\preccurlyeq}t(\vec{x})\,\varphi(y,\vec{x})$ holds in (V, C), and thus also in V, since it does not mention the symbol C. □

6.4. *Uniform Cobham recursion*

We can use our definability and witnessing theorems to partially answer a question that arose from [7]. Namely, the embedding giving the bound on a Cobham recursion is given by a CRSF function. If we only allow simpler embeddings, given by $\Delta_0(\mathrm{L}_0)$-formulas, does the class CRSF change? We show that it does not. This is a partial answer because we only consider what happens if we make this change in our definition of CRSF from Proposition 2.9, which is slightly different from the original definition in [7].

Definition 6.15: In the universe of sets, the $\mathrm{CRSF_u}$ functions are those obtained from the projections, zero, pair, union, conditional, transitive closure, cartesian product, set composition and set smash functions by composition, replacement and "weakly uniform syntactic Cobham recursion". This is the following recursion scheme: suppose $g(x, z, \vec{w})$ is a $\mathrm{CRSF_u}$ function, $\varepsilon(z, z', y, x, \vec{w})$ is a $\Delta_0(\mathrm{L}_0)$-formula and $t(x, \vec{w})$ is a #-term. Then $\mathrm{CRSF_u}$ contains the function symbol $f = f_{g,\varepsilon,t}$ defined by

$$f(x, \vec{w}) = \begin{cases} g(x, f\text{''}(x, \vec{w}), \vec{w}) & \text{if } \varepsilon \text{ is an embedding into } t \text{ at } x, \vec{w} \\ 0 & \text{otherwise} \end{cases}$$

where the condition "ε is an embedding into t at $x, \vec{w}$" stands for

$$\varepsilon(\cdot, \cdot, g(x, f\text{''}(x, \vec{w}), \vec{w}), x, \vec{w}) : g(x, f\text{''}(x, \vec{w}), \vec{w}) \preccurlyeq t(x, \vec{w}).$$

The language $\mathrm{L_{crsfu}}$ and theory $\mathrm{T_{crsfu}}$ are defined by changing the syntactic Cobham recursion case in the definitions of $\mathrm{L_{crsf}}$ and $\mathrm{T_{crsf}}$ to match the description above.

Theorem 6.16: *The theory* $\mathrm{KP}_1^{\mathrm{u}}$ *is* $\Pi_2(\mathrm{L}_0)$*-conservative over* $\mathrm{T_{crsfu}}$.

Proof: It is straightforward to show that $\mathrm{T_{crsfu}}$ proves T_0, just as $\mathrm{T_{crsf}}$ does. Similarly the results about Herbrand saturation go through for $\mathrm{T_{crsfu}}$. By Lemma 6.7 it is enough to show that any $\Delta_0(\mathrm{L_{crsfu}})$-Herbrand saturated model of $\mathrm{T_{crsfu}}$ is a model of $\Delta_0(\mathrm{L}_0)$-Collection and uniformly bounded unique $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0)$-Induction. For this we can simply repeat the proof of Theorem 6.9 with $\mathrm{T_{crsfu}}$ in place of $\mathrm{T_{crsf}}$, observing that the proof becomes more direct, since in the application of syntactic Cobham recursion we can use the embedding ε directly without needing to construct the function symbol τ. □

Corollary 6.17: *In the universe of sets,* $\mathrm{CRSF_u} = \mathrm{CRSF}$.

Proof: It is clear that every $\mathrm{CRSF_u}$ function is CRSF. For the other direction, suppose $f(\vec{x})$ is CRSF. Then there is a good definition of f in the sense of Definition 5.6, and in particular there is a $\Delta_0(\mathrm{L}_0)$-formula $\varphi(v, \vec{x})$ and an L_0^+-term e such that $\mathrm{KP}_1^{\mathrm{u}}$ proves $\exists! v\, \varphi(v, \vec{x})$ and such that $\varphi(v, \vec{x}) \to f(\vec{x}) = e(v)$ holds in the universe for all sets $v, \vec{x}$. There is a similar version of Herbrand's theorem for $\mathrm{T_{crsfu}}$ as there is for $\mathrm{T_{crsf}}$. Combining this with Theorem 6.16 we get that there is an $\mathrm{L_{crsfu}}$ function symbol g such that $\mathrm{T_{crsfu}}$ proves $\varphi(g(\vec{x}), \vec{x})$. Hence in the universe $f(\vec{x}) = e(g(\vec{x}))$, which is a $\mathrm{CRSF_u}$ function. □

7. Partial Conservativity of Global Choice

Recall from Section 6.3 the versions of our theories with global choice (GC).

Proposition 7.1: T_0^C *is not* $\Pi_2(\mathrm{L}_0)$*-conservative over* ZF.

Proof: The theory T_0^C proves (AC) in the form: for every set x of disjoint, nonempty sets, there is a set z containing exactly one element from every member of x. Indeed, $z := \{v \in \mathrm{tc}(x) : \exists u \in x\ C(u) = v\}$ can be obtained by $\Delta_0(\mathrm{L}_0^C)$-Separation. □

In particular, the extension $\mathrm{KPC}_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}}^C)$ of $\mathrm{KP}_1^{\preccurlyeq}(\mathrm{L_{crsf}})$ is not $\Pi_2(\mathrm{L}_0)$-conservative. Informally, we ask how much stronger $\mathrm{KPC}_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}}^C)$ is compared to $\mathrm{KP}_1^{\preccurlyeq}(\mathrm{L_{crsf}})$. More formally, we aim to encapsulate the difference in some local choice principles, namely a strong form of (AC) plus a form of dependent choice.[e]

7.1. *Dependent Choice*

The class of ordinals is denoted by $\mathit{Ord}(x)$ in L_0^+ with defining axiom $\forall y {\in} x {\cup} \{x\}\, (\mathrm{tc}(y) = y)$. It is routine to verify in T_0^+ some elementary properties of ordinals, e.g., elements of ordinals are ordinals and, given two distinct ordinals, one is an element of the other. We let $\alpha, \beta, \ldots$ range over ordinals. By this we mean that $\forall \alpha \ldots$ and $\exists \alpha \ldots$ stand for $\forall \alpha\, (\mathit{Ord}(\alpha) \to \ldots)$ and $\exists \alpha\, (\mathit{Ord}(\alpha) \wedge \ldots)$ respectively.

[e]This aims at technical simplicity of the argument rather than the strongest possible result.

The scheme $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$-*Dependent Choice* gives for every $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$-formula $\varphi(x, y, \vec{x})$

$$\forall x\, \exists y\, \varphi(x, y, \vec{x}) \to \forall \alpha\, \exists z\, (Fct(z) \wedge dom(z) = \alpha \wedge \forall \beta{\in}\alpha\, \varphi(z{\restriction}\beta, z\text{'}\beta, \vec{x})). \tag{7.1}$$

We assume $\mathrm{L}_0^+ \subseteq \mathrm{L}_{\mathrm{crsf}}$ (cf. Lemma 4.5), $Fct(y)$ is a unary relation symbol in L_0^+ expressing that y is a function, and $dom(x)$, $im(x)$, $x{\restriction}y$ are function symbols in $\mathrm{L}_{\mathrm{crsf}}$ such that $\mathrm{KP}_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}})$ proves $dom(x) = \pi_1\text{''}(x)$, $im(x) = \pi_2\text{''}(x)$ and $x{\restriction}y = \{z \in x : \pi_1(z) \in y\}$.

We further consider the following strong version of (AC) that we refer to as the *well-ordering principle* (WO):

$$\forall x \exists \alpha \exists y (\text{“}y \text{ is a bijection from } \alpha \text{ onto } x\text{”} \wedge \forall \beta, \gamma{\in}\alpha\, (y\text{'}\beta \in y\text{'}\gamma \to \beta \in \gamma)).$$

The goal of this section is to prove:

Theorem 7.2: *The theory* $\mathrm{KPC}_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}}^C)$ *is conservative over* $\mathrm{KP}_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}})$ *plus* $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$*-Dependent Choice plus (WO).*

Note this just says that every $\mathrm{L}_{\mathrm{crsf}}$-formula proved by the former theory is also proved by the latter. But the former theory is not an extension of the latter:

Proposition 7.3: $\mathrm{KPC}_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}}^C)$ *does not prove (WO).*

Proof: By Theorem 6.13, if $\mathrm{KPC}_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}}^C)$ proves the existence of an ordinal α bijective to a given x, then it proves $f(x)$ is such an α for some function symbol $f(x)$ in $\mathrm{L}_{\mathrm{crsf}}^C$. Fix a universe of sets with a global choice function C, and view it as a structure interpreting $\mathrm{L}_{\mathrm{crsf}}^C$. There, $f(x)$ denotes a function in CRSF^C. By Theorem 2.5 (for CRSF^C instead CRSF, recall Remark 2.10), there is a #-term $t(x)$ such that $\alpha = f(x) \preccurlyeq t(x)$ holds for all x. Then there is a polynomial p such that the von Neumann rank $\mathrm{rk}(f(x)) = \alpha \geq |x|$ is at most $p(\mathrm{rk}(x))$ (cf. [7, Lemma 2, 4, Proposition 10]). This is false for many x. □

7.2. *The forcing*

Let M be a countable model of $\mathrm{KP}_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}})$ and $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$-Dependent Choice and (WO). We intend to produce a generic extension of M modelling (GC). Note we do *not* assume that M is standard, in particular, M possibly does not interpret $\in$ by $\in$. While the forcing frame is the class forcing commonly used to force global choice, we use a technically simplified forcing relation avoiding the use of names. This is similar to [17]. The argument that the forcing preserves $\mathrm{KP}_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}})$ needs some care since this theory and hence M is very weak.

The forcing frame $(\mathbb{P}, \leqslant^{\mathbb{P}})$ is defined as follows: $\mathbb{P} \subseteq M$ contains $p \in M$ if and only if p is a choice function in the sense of M, that is, M satisfies

$$(Fct(p) \wedge \langle 0,0\rangle \in p \wedge \forall x, y\, (\langle x,y\rangle \in p \wedge x \neq 0 \rightarrow y \in x)).$$

Further, $p \leqslant^{\mathbb{P}} q$ means $M \models q \subseteq p$. Then $(\mathbb{P}, \leqslant^{\mathbb{P}})$ is a partial order. In the following we let $p, q, r, \ldots$ range over *conditions*, i.e., elements of $\mathbb{P}$. A subset X of $\mathbb{P}$ is *dense below* p if for all $q \leqslant^{\mathbb{P}} p$ there is $r \leqslant^{\mathbb{P}} q$ such that $r \in X$. Being *dense* means being dense below $1^{\mathbb{P}} := \{\langle 0,0\rangle\}$ (calculated in M). A subset X of $\mathbb{P}$ is a *filter* if $p \cup^M q \in X$ whenever $p, q \in X$, and $q \in X$ whenever $p \leqslant^{\mathbb{P}} q$ and $p \in X$. Being *generic* means being a filter that intersects all dense subsets of $\mathbb{P}$ that are definable (with parameters) in M. The *forcing language* is $\mathrm{L}_{\mathrm{crsf}} \cup \{R\}$ for a new binary relation symbol R.

The *forcing relation* $\Vdash$ relates conditions p to sentences of the forcing language with parameters from M. It is defined as follows. For an atomic sentence φ that does not mention R we let $p \Vdash \varphi$ if and only if $M \models \varphi$. For an atomic sentence of the form Rts with closed terms t, s we let

$$p \Vdash Rts \quad \Longleftrightarrow \quad M \models (t = \{s\} \vee \langle t, s\rangle \in p).$$

We extend this definition via the recurrence:

$$\begin{aligned} p \Vdash (\varphi \wedge \psi) &\Longleftrightarrow p \Vdash \varphi \text{ and } p \Vdash \psi, \\ p \Vdash \neg\varphi &\Longleftrightarrow \text{for all } q \leqslant^{\mathbb{P}} p : q \nVdash \varphi, \\ p \Vdash \forall x\, \varphi(x) &\Longleftrightarrow \text{for all } a \in M : p \Vdash \varphi(a). \end{aligned}$$

This defines $p \Vdash \varphi$ for all sentences φ of the forcing language with parameters from M which are written using the logical symbols $\wedge, \neg, \forall$. We freely use the symbols $\vee, \rightarrow, \exists$ understanding these as classical abbreviations. Namely, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, $\exists x\, \chi(x)$ stand for $\neg(\neg\varphi \wedge \neg\psi)$, $\neg(\varphi \wedge \neg\psi)$, $\neg\forall x\, (\neg\chi(x))$ respectively. Lemma 7.4 (f) below shows that $p \Vdash \varphi$ does not depend of the choice of these abbreviations.

Lemma 7.4: *Let φ be a sentence of the forcing language with parameters from M.*

(a) (Conservativity) *If R does not occur in φ, then $p \Vdash \varphi$ if and only if $M \models \varphi$.*
(b) (Extension) *If $p \leqslant^{\mathbb{P}} q$ and $q \Vdash \varphi$, then $p \Vdash \varphi$.*

(c) (Stability) $p \Vdash \varphi$ *if and only if* $p \Vdash \neg\neg\varphi$*, that is, if and only if* $\{q \mid q \Vdash \varphi\}$ *is dense below* p*.*
(d) (Truth Lemma) *For every generic* G *there is* $R_G \subseteq M^2$ *such that for every* φ *we have that* $(M, R_G) \models \varphi$ *if and only if some* $p \in G$ *forces* φ*.*
(e) (Forcing Completeness) $p \Vdash \varphi$ *if and only if* $(M, R_G) \models \varphi$ *for every generic filter* G *containing* p*.*
(f) $\{\varphi \mid p \Vdash \varphi\}$ *is closed under logical consequence.*

Proof: (a) and (b) obviously hold for atomic φ; for general φ the claim follows by a straightforward induction (see, e.g., [3, Lemma 2.6]). Similarly, it suffices to show (c) for atomic φ. Assume $\varphi = Rts$ for closed terms t, s. The second equivalence is trivial. The forward direction follows from (b): if $p \Vdash \varphi$, then $\{q \mid q \Vdash \varphi\} \supseteq \{q \mid q \leqslant^{\mathbb{P}} p\}$ is dense below p. Conversely, it is enough to find given p with $p \nVdash Rts$ some $q \leqslant^{\mathbb{P}} p$ forcing $\neg Rts$. We have $M \models t \neq \{s\}$ and $M \models \langle t, s\rangle \notin p$. If $M \models t = 0$, then $M \models s \neq 0$ and no condition forces Rts, so $q := p \Vdash \neg Rts$. If $M \models t \neq 0$, then there is $a \in M$ such that $M \models (s \neq a \wedge a \in t)$ and $q := p \cup \{\langle t, a\rangle\}$ calculated in M is a condition. Then no $r \leqslant^{\mathbb{P}} q$ forces Rts.

The remaining claims can be proved by standard means. We give precise references from [3]. A generic G is generic in the sense of [3, Definition 2.9], and $M[G]$ is defined for every such G (cf. [3, Definition 2.16]). [3, Proposition 2.26] states that up to isomorphism each such model $M[G]$ has the form (M, R_G) as in (d). Then (d), (e), (f) are [3, Theorem 2.19, Corollary 2.20 (2), Corollary 2.20 (3)]. □

It is easy to see that for each $\varphi(\vec{x})$ of the forcing language the set $\{(p, \vec{a}) : p \Vdash \varphi(\vec{a})\}$ is definable in M. There is, however, no good control of the logical complexity of the defining formula. Therefore we use the following auxiliary *strong forcing* relation $\Vvdash$ between conditions and sentences of the forcing language with parameters from M. It is defined via the same recurrence as $\Vdash$ except for the negation clause. Namely, $p \Vvdash \neg\varphi$ is defined as $p \Vdash \neg\varphi$ for atomic φ and otherwise via the recursion:

$$\begin{aligned} p \Vvdash \neg(\psi \wedge \chi) &\iff p \Vvdash \neg\psi \text{ or } p \Vvdash \neg\chi, \\ p \Vvdash \neg\neg\psi &\iff p \Vvdash \psi, \\ p \Vvdash \neg\forall x\, \psi(x) &\iff \text{there is } a \in M : p \Vvdash \neg\psi(a). \end{aligned}$$

Remark 7.5: One can check that $p \Vvdash \exists x\, \varphi(x)$ if and only if there is $a \in M$ such that $p \Vvdash \varphi(a)$, and $p \Vvdash (\varphi \vee \psi)$ if and only if $p \Vvdash \varphi$ or $p \Vvdash \psi$. Here we understand $\exists, \vee$ by the particular abbreviations mentioned earlier. In this sense $\Vdash\!\!\vdash$ commutes with quantifiers and connectives $\wedge, \vee$. The price to pay for these nice properties is that $\Vdash\!\!\vdash$ does not behave like a notion of forcing. For example, let $a, b \in M, a \neq b$, and calculate $c := \{a, b\}$ in M; then $1^{\mathbb{P}} \not\Vdash\!\!\vdash Rca$ and $1^{\mathbb{P}} \not\Vdash\!\!\vdash \neg Rca$, so $1^{\mathbb{P}} \not\Vdash\!\!\vdash (Rca \vee \neg Rca)$, and hence Lemma 7.4 (f) fails for $\Vdash\!\!\vdash$.

A formula is in *negation normal form* (NNF) if negations appear only in front of atomic subformulas.

Lemma 7.6: *Let φ be a sentence of the forcing language with parameters from M.*

(a) If $p \leqslant^{\mathbb{P}} q$ and $q \Vdash\!\!\vdash \varphi$, then $p \Vdash\!\!\vdash \varphi$.
(b) If $p \Vdash\!\!\vdash \varphi$, then $p \Vdash \varphi$.
(c) Let $L \subseteq \mathrm{L}_{\mathrm{crsf}}$ and $\psi(\vec{x})$ be a $\Delta_0(\mathrm{L}_0^+ \cup L \cup \{R\})$-formula with parameters from M. Then there exists a $\Delta_0(\mathrm{L}_0 \cup L)$-formula $\widetilde{\psi}(u, \vec{x})$ with parameters from M such that $\widetilde{\psi}(u, \vec{x})$ defines $\{(p, \vec{a}) : p \Vdash\!\!\vdash \psi(\vec{a})\}$ in M.
(d) If φ is a $\Sigma_1(\mathrm{L}_{\mathrm{crsf}} \cup \{R\})$-sentence in NNF with parameters from M and $p \Vdash \varphi$, then there is $q \leqslant^{\mathbb{P}} p$ such that $q \Vdash\!\!\vdash \varphi$.

Proof: (a) and (b) are straightforward. (c) is proved by induction on ψ. We only verify the case when $\psi(\vec{x})$ equals $\neg Rts$ for terms $t = t(\vec{x}), s = s(\vec{x})$. Then define $\widetilde{\psi}(u, \vec{x})$ as

$$u \in \mathbb{P} \wedge \big((t = 0 \wedge s \neq 0) \vee (t \neq 0 \wedge s \notin t) \vee \exists x \in t\, (x \neq s \wedge \langle t, x \rangle \in u)\big).$$

Here, $u \in \mathbb{P}$ abbreviates a suitable $\Delta_0(\mathrm{L}_0)$-formula defining $\mathbb{P} \subseteq M$ in M. We have to show that for all $p \in \mathbb{P}$ and $\vec{a}$ from M:

$$M \models \widetilde{\psi}(p, \vec{a}) \iff \text{for all } q \leqslant^{\mathbb{P}} p : q \not\Vdash Rt(\vec{a})s(\vec{a}).$$

The direction from left to right is easy to see. Conversely, assume no condition $q \leqslant^{\mathbb{P}} p$ forces $Rt(\vec{a})s(\vec{a})$ and note $M \models p \in \mathbb{P}$. Arguing in M, then $t(\vec{a}) \neq \{s(\vec{a})\}$ and $\langle t(\vec{a}), s(\vec{a})\rangle \notin p$, in particular $t(\vec{a}), s(\vec{a})$ are not both 0. If $t(\vec{a}) = 0$, then $s(\vec{a}) \neq 0$ and $\widetilde{\psi}(p, \vec{a})$ is true. So suppose $t(\vec{a}) \neq 0$. Then $q := p \cup \{\langle t(\vec{a}), s(\vec{a})\rangle\} \notin \mathbb{P}$. Hence $s(\vec{a}) \notin t(\vec{a})$ or there is $a \in t(\vec{a})$ with $a \neq s(\vec{a})$ and $\langle t(\vec{a}), a\rangle \in p$. Both cases imply $\widetilde{\psi}(p, \vec{a})$.

(d). Let $\varphi(\vec{x})$ be a formula of the forcing language with parameters from M. Call $\varphi(\vec{x})$ *good* if for all $\vec{a}$ from M and $p \in \mathbb{P}$: if $p \Vdash \varphi(\vec{a})$, then there is $q \leqslant^{\mathbb{P}} p$ with $q \Vdash\!\!\vdash \varphi(\vec{a})$.

Atomic and negated atomic formulas are good, as we can take $q := p$. Good formulas are closed under conjunctions and disjunctions, and $\exists y\, \psi(y, \vec{x})$ is good whenever $\psi(y, \vec{x})$ is good: if $p \Vdash \exists y\, \psi(y, \vec{a})$, then $\bigcup_{b \in M} \{q \mid$

$q \Vdash \psi(b, \vec{a})\}$ is dense below p, so there are $q \leqslant^{\mathbb{P}} p$ and $b \in M$ such that $q \Vdash \psi(b, \vec{a})$; as $\psi(y, \vec{x})$ is good, there is $r \leqslant^{\mathbb{P}} q$ such that $r \Vvdash \psi(b, \vec{a})$ and hence $r \Vvdash \exists y\, \psi(y, \vec{a})$.

Finally, we show that for a good $\Delta_0(\mathrm{L_{crsf}} \cup \{R\})$-formula $\psi(y, \vec{x})$, also $\forall y{\in}t(\vec{x})\, \psi(y, \vec{x})$ is good, where t is a term. If $p \Vdash \forall y{\in}t(\vec{a})\, \psi(y, \vec{a})$, then by Conservativity $p \Vdash \psi(b, \vec{a})$ for all b with $M \models b \in t(\vec{a})$. As $\psi(y, \vec{x})$ is good, we find for every $q \leqslant^{\mathbb{P}} p$ and every such b some $q_b \leqslant^{\mathbb{P}} q$ such that $q_b \Vvdash \psi(b, \vec{a})$. By (WO) we find $s \in M$ which is, in the sense of M, a bijection from an ordinal α onto $t(\vec{a})$. It suffices to find $\pi \in M$ such that π is, in the sense of M, a function with domain α and such that for all $\gamma \in^M \beta \in^M \alpha$:

$$\pi\text{'}\beta \leqslant^{\mathbb{P}} \pi\text{'}\gamma \leqslant^{\mathbb{P}} p \text{ and } \pi\text{'}\beta \Vvdash \psi(s\text{'}\beta, \vec{a}). \tag{7.2}$$

More precisely, the first three ' should read $'^M$. This suffices indeed: by (a), then $q := \bigcup im(\pi)$, calculated in M, is a condition extending p such that $q \Vvdash \psi(s\text{'}\beta, \vec{a})$ for all $\beta \in^M \alpha$. Thus $q \Vvdash \psi(b, \vec{a})$ for all b with $M \models b \in t(\vec{a})$, and hence $q \Vvdash \forall y{\in}t(\vec{a})\, \psi(y, \vec{a})$.

To find such π we apply $\Delta_0(\mathrm{L_{crsf}})$-Dependent Choice in M with the following $\Delta_0(\mathrm{L_{crsf}})$-formula $\varphi(x, y)$ with parameters from M:

$$\begin{aligned}&\big(dom(x) \in \alpha \wedge p \cup \textstyle\bigcup im(x) \in \mathbb{P}\\ &\quad \to p \cup \textstyle\bigcup im(x) \subseteq y \wedge y \in \mathbb{P} \wedge \tilde{\psi}(y, s\text{'}dom(x), \vec{a})\big).\end{aligned}$$

where $\tilde{\psi}$ is as in (c). Since ψ is $\Delta_0(\mathrm{L_{crsf}} \cup \{R\})$, we have $\tilde{\psi}$ (by (c)) and hence φ in $\Delta_0(\mathrm{L_{crsf}})$. We show that M models $\forall x\, \exists y\, \varphi(x, y)$. Argue in M: given $c \in M$ with $\beta := dom(c) \in \alpha$ and $q := p \cup \bigcup im(c) \in \mathbb{P}$ a witness for y is given by q_b for $b := s\text{'}\beta$.

For α as above, choose π witnessing z in (7.1). We claim π satisfies (7.2). It suffices to show $M \models \forall \beta{\in}\alpha\, (p \cup \bigcup im(\pi{\restriction}\beta) \in \mathbb{P})$, or equivalently, $M \models \forall \gamma, \gamma'{\in}\beta\, (p \cup \pi\text{'}\gamma \cup \pi\text{'}\gamma' \in \mathbb{P})$ for all $\beta \in^M \alpha$. This follows by $\Delta_0(\mathrm{L}_0)$-Induction on β and elementary properties of ordinals. □

7.3. *Proof of Theorem 7.2*

It suffices to show that every countable model M of $\mathrm{KP}_1^{\preccurlyeq}(\mathrm{L_{crsf}})$ plus $\Delta_0(\mathrm{L_{crsf}})$-Dependent Choice plus (WO) has an expansion to a model of $\mathrm{KPC}_1^{\preccurlyeq}(\mathrm{L}^C_{\mathrm{crsf}})$. Recall, $\mathrm{KPC}_1^{\preccurlyeq}(\mathrm{L}^C_{\mathrm{crsf}})$ is a $\Sigma_1^{\preccurlyeq}(\mathrm{L}^C_0)$-expansion of $\mathrm{KPC}_1^{\preccurlyeq}$ (cf. Lemma 5.21). It thus suffices to find an expansion of M to a model of $\mathrm{KPC}_1^{\preccurlyeq}$. Using the notation of the Truth Lemma 7.4(d), for every generic G we have that R_G is the graph of a function C satisfying the axiom of global choice (GC). A $\Delta_0(\mathrm{L}_0^C)$-formula in the corresponding expansion is equivalent to a $\Delta_0(\mathrm{L}_0 \cup \{R\})$-formula. It thus suffices to show that (M, R_G) satisfies $\Sigma_1^{\preccurlyeq}(\mathrm{L}_0 \cup \{R\})$-Induction, $\Delta_0(\mathrm{L}_0 \cup \{R\})$-Separation and $\Delta_0(\mathrm{L}_0 \cup \{R\})$-Collection.

We start with Induction. So, given a $\Delta_0(\mathrm{L}_0 \cup \{R\})$-formula $\psi(y, z)$ with parameters from M, a #-term $t(y)$ with parameters from M and $b \in M$ we have to show that

$$(M, R_G) \models \forall x \left(\forall y{\in}x\, \exists z{\preccurlyeq}t(y)\, \psi(y, z) \rightarrow \exists z{\preccurlyeq}t(x)\, \psi(x, z)\right) \\ \rightarrow \exists z{\preccurlyeq}t(b)\, \psi(b, z).$$

Recall $\mathit{IsPair}(x)$ from Examples 4.6. We define

$$\psi'(y, z) := (\mathit{IsPair}(z) \wedge \pi_1(z) : \pi_2(z) \preccurlyeq t(y) \wedge \psi(y, \pi_2(z))).$$

We can assume that ψ' is in NNF. Recall $\mathrm{L}_0^+ \subseteq \mathrm{L}_{\mathrm{crsf}}$, so M interprets L_0^+. We assume

$$(M, R_G) \models \forall x\, (\forall y{\in}x\, \exists z\, \psi'(y, z) \rightarrow \exists z\, \psi'(x, z)) \tag{7.3}$$

and aim to show $(M, R_G) \models \exists z\, \psi'(b, z)$. By the Truth Lemma there exists $p \in G$ such that p forces (7.3). It suffices to show that p forces $\exists z\, \psi'(b, z)$. By Stability it suffices to find, given $p' \leqslant^{\mathbb{P}} p$, some $q \leqslant^{\mathbb{P}} p'$ forcing $\exists z\, \psi'(b, z)$.

By (WO) we find $s \in M$ such that, in the sense of M, s is a bijection from some ordinal α onto $\mathrm{tc}^+(b)$ that respects $\in$, i.e., $M \models (s\text{'}\gamma \in s\text{'}\beta \in \mathrm{tc}^+(b) \rightarrow \gamma \in \beta)$. So by Lemma 7.6(b), it suffices to find for every $\beta \in^M \alpha$ a pair $\langle q_\beta, a_\beta \rangle$ (in the sense of M) such that $q_\beta \leqslant^{\mathbb{P}} p'$ and

$$q_\beta \Vdash \psi'(s\text{'}\beta, a_\beta). \tag{7.4}$$

We intend to apply $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$-Dependent Choice with the following formula $\varphi(x, y)$:

$$\begin{aligned}
\varphi(x, y) &:= (\varphi_0(x) \rightarrow \varphi_1(x, y)),\\
\varphi_0(x) &:= \mathit{Fct}(x) \wedge \mathit{dom}(x) \in \alpha\\
&\quad\ \wedge \forall \gamma, \gamma'{\in}\mathit{dom}(x)\, (p' \cup \pi_1(x\text{'}\gamma) \cup \pi_1(x\text{'}\gamma') \in \mathbb{P})\\
&\quad\ \wedge \forall \gamma{\in}\mathit{dom}(x) \left(\mathit{IsPair}(x\text{'}\gamma) \wedge \widetilde{\psi}'(\pi_1(x\text{'}\gamma), s\text{'}\gamma, \pi_2(x\text{'}\gamma))\right),\\
\varphi_1(x, y) &:= \mathit{IsPair}(y) \wedge \forall \gamma{\in}\mathit{dom}(x)\, (p' \cup \pi_1(x\text{'}\gamma) \subseteq \pi_1(y))\\
&\quad\ \wedge \widetilde{\psi}'(\pi_1(y), s\text{'}\mathit{dom}(x), \pi_2(y)),
\end{aligned}$$

where $\widetilde{\psi}'$ is defined as in Lemma 7.6(c).

We have $\varphi \in \Delta_0(\mathrm{L}_{\mathrm{crsf}})$ by Lemma 7.6(c). We show $M \models \forall x\, \exists y\, \varphi(x, y)$. Let $c \in M$ and assume $M \models \varphi_0(c)$. We have to show $M \models \exists y\, \varphi_1(c, y)$. Compute

$$\beta := dom(c)$$
$$q := p' \cup \bigcup_{\gamma\in\beta} \pi_1(c\text{'}\gamma)$$

in M (this can be done: for $f(x)$ such that $\mathrm{KP}_1^{\preccurlyeq}(\mathrm{L}_{\mathrm{crsf}})$ proves $f(x) = \pi_1(\pi_2(x))$, we have $q = p' \cup \bigcup f\text{"}(c)$ in M). Then $q \in \mathbb{P}$ extends $(\pi_1(c\text{'}\gamma))^M$ for all $\gamma \in^M \beta$. By Lemma 7.6(a), $q \Vdash \psi'(s\text{'}\gamma, \pi_2(c\text{'}\gamma))$ for all $\gamma \in^M \beta$, and hence $q \Vdash \exists z\, \psi'(s\text{'}\gamma, z)$ for all $\gamma \in^M \beta$. This implies $q \Vdash \exists z\, \psi'(d, z)$ for all $d \in M$ with $M \models d \in s\text{'}\beta$. By Lemma 7.6 (b), we see that q forces $\forall y{\in}s\text{'}\beta\ \exists z\, \psi'(y, z)$. But $q \leqslant^{\mathbb{P}} p' \leqslant^{\mathbb{P}} p$, so by Extension q forces (7.3). Plugging $s\text{'}\beta$ for x in (7.3) and recalling Lemma 7.4 (f) we see that $q \Vdash \exists z\, \psi'(s\text{'}\beta, z)$. This is a $\Sigma_1(\mathrm{L}_{\mathrm{crsf}} \cup \{R\})$-sentence in NNF with parameters from M, so Lemma 7.6 (d) gives $q_\beta \leqslant^{\mathbb{P}} q$ and $a_\beta \in M$ such that $q_\beta \Vdash \psi'(s\text{'}\beta, a_\beta)$. Then $M \models \varphi_1(c, \langle q_\beta, a_\beta\rangle)$ and thus $M \models \exists y\, \varphi_1(c, y)$.

By Dependent Choice there is $\pi \in M$, in the sense of M a function with domain α, such that $M \models \varphi(\pi{\restriction}\beta, \pi\text{'}\beta)$ for all $\beta \in^M \alpha$.

To show (7.4) it suffices to show $M \models \varphi_0(\pi{\restriction}\beta)$ for all $\beta \in^M \alpha$, or equivalently

$$\forall \gamma, \gamma'{\in}\beta\, (p' \cup \pi_1(\pi\text{'}\gamma) \cup \pi_1(\pi\text{'}\gamma') \in \mathbb{P})$$
$$\wedge\ \forall \gamma{\in}\beta\, (\mathit{IsPair}(\pi\text{'}\gamma) \wedge \widetilde{\psi}'(\pi_1(\pi\text{'}\gamma), s\text{'}\gamma, \pi_2(\pi\text{'}\gamma)))$$

holds in M for all $\beta \in^M \alpha$. By Lemma 7.6(c), this can be written $\chi(\beta)$ for a $\Delta_0(\mathrm{L}_0^+)$-formula $\chi(x)$ with parameters from M. Since $\Delta_0(\mathrm{L}_0^+)$-Induction holds in M, it suffices to verify $M \models \chi(\beta)$ assuming $M \models \forall \gamma{\in}\beta\, \chi(\gamma)$. This is easy. Thus (M, R_G) satisfies $\Delta_0(\mathrm{L}_0 \cup \{R\})$-Induction.

We show that (M, R_G) satisfies $\Delta_0(\mathrm{L}_0 \cup \{R\})$-Collection. Let $\psi(y, z)$ be a $\Delta_0(\mathrm{L}_0 \cup \{R\})$-formula with parameters from M and $a \in M$ such that (M, R_G) satisfies

$$\varphi := \forall y{\in}a\, \exists z\, \psi(y, z).$$

By the Truth Lemma, φ is forced by some $p \in G$. Arguing as for (7.4) we can find $q \leqslant^{\mathbb{P}} p$ such that for all $b \in^M a$ there is $c \in M$ such that $q \Vdash \psi(b, c)$ (observe that the proof of (7.4) gave a descending chain of q_β's) – equivalently: $q \Vdash \varphi$. Note $q \Vdash \varphi$ if and only if

$$M \models \forall y{\in}a\, \exists z\, \widetilde{\psi}(q, y, z),$$

where $\widetilde{\psi}$ is $\Delta_0(\mathrm{L}_0)$, chosen according Lemma 7.6(c). Then $\{q : q \Vdash\!\vdash \varphi\}$ is M-definable and dense below p, so we find such q in G. Applying $\Delta_0(\mathrm{L}_0)$-Collection in M we get

$$M \models \exists V\, \forall y{\in}a\, \exists z \in V\, \widetilde{\psi}(q, y, z).$$

But for all $b, c \in M$ we have

$$(M, R_G) \models (\widetilde{\psi}(q, b, c) \to \psi(b, c)),$$

by Lemma 7.6(b) and the Truth Lemma. Thus (M, R_G) satisfies $\Delta_0(\mathrm{L}_0 \cup \{R\})$-Collection.

We show that (M, R_G) satisfies $\Delta_0(\mathrm{L}_0 \cup \{R\})$-Separation. Let $a \in M$ and $\varphi(x)$ be a $\Delta_0(\mathrm{L}_0 \cup \{R\})$-formula with parameters from M. We can assume $\varphi(x)$ is in NNF. Let $\overline{\varphi}(x)$ be logically equivalent to $\neg\varphi(x)$ and in NNF. By the Truth Lemma it suffices to show $1^{\mathbb{P}} \Vdash \exists z\, (z = \{x \in a : \varphi(x)\})$. By Stability it suffices to show that for every $p \in \mathbb{P}$ there is $q \leqslant^{\mathbb{P}} p$ such that $q \Vdash \exists z\, (z = \{x \in a : \varphi(x)\})$.

Let $p \in \mathbb{P}$ be given. We claim that it suffices to find $q \leqslant^{\mathbb{P}} p$ that *strongly decides* $\varphi(b)$ for every $b \in^M a$ in the sense that $q \Vdash\!\vdash \varphi(b)$ or $q \Vdash\!\vdash \overline{\varphi}(b)$. Indeed, such a q forces $\exists z\, (z = \{x \in a : \varphi(x)\})$. By Forcing Completeness we have to show $(M, R_{G'}) \models \exists z\, (z = \{x \in a : \varphi(x)\})$ for every generic G' containing q. But z is witnessed by $\{x \in a : \widetilde{\varphi}(q, x)\}$, a set obtainable in M by $\Delta_0(\mathrm{L}_0)$-Separation (Lemma 7.6(c)). To see this, we verify for every $b \in^M a$:

$$(M, R_{G'}) \models (\varphi(b) \leftrightarrow \widetilde{\varphi}(q, b)).$$

The direction from right to left follows from Lemma 7.6(b) and the Truth Lemma. Conversely, assuming $(M, R_{G'}) \models \varphi(b)$ the Truth Lemma gives $r \in G'$ forcing $\varphi(b)$; then $r \cup q \in G'$ since G' is a filter, so $r \cup q$ forces $\varphi(b)$ by Extension, so cannot force $\overline{\varphi}(b)$ by Lemma 7.4(f), so $q \nVdash \overline{\varphi}(b)$ by Extension, so $q \not\Vdash\!\vdash \overline{\varphi}(b)$ by Lemma 7.6(b), so $q \Vdash\!\vdash \varphi(b)$ and $(M, R_{G'}) \models \widetilde{\varphi}(q, b)$ since q strongly decides $\varphi(b)$.

Thus, given a condition p, we are looking for $q \leqslant^{\mathbb{P}} p$ that strongly decides $\varphi(b)$ for every $b \in^M a$. By (WO) choose $s \in M$ such that, in the sense of M, s is a bijection from α onto a. A condition q as desired is obtained in M as the union of a descending sequence $(q_\beta)_{\beta \in \alpha}$ with $q_0 \leqslant^{\mathbb{P}} p$ such that each q_β strongly decides $\varphi(q_\beta, s\text{'}\beta)$. To get such a sequence in M

we apply $\Delta_0(\mathrm{L}_{\mathrm{crsf}})$-Dependent Choice on the following formula $\psi(x,y)$:

$$\begin{aligned}&(\mathit{Fct}(x) \wedge \mathit{dom}(x) \in \alpha \wedge \forall \gamma, \gamma' {\in} \mathit{dom}(x)\, x\text{'}\gamma \cup x\text{'}\gamma' \in \mathbb{P} \\ &\quad \to \forall \gamma {\in} \mathit{dom}(x)\, (p \cup x\text{'}\gamma \subseteq y) \wedge (\widetilde{\varphi}(y, s\text{'}\mathit{dom}(x)) \vee \widetilde{\overline{\varphi}}(y, s\text{'}\mathit{dom}(x)))).\end{aligned} \tag{7.5}$$

A function π with domain α (in the sense of M) such that $M \models \psi(\pi{\restriction}\beta, \pi\text{'}\beta)$ for all $\beta \in^M \alpha$, is a sequence as desired. We are left to show

$$M \models \forall x\, \exists y\, \psi(x,y).$$

Let $c \in M$ satisfy the antecedent of (7.5), and compute $\beta := \mathit{dom}(c)$ and $q^0 := \bigcup \mathit{im}(c)$ in M. Then $q^0 \in \mathbb{P}$. There exists $q^1 \leqslant^{\mathbb{P}} q^0$ such that $q^1 \Vdash \varphi(s\text{'}\beta)$ or $q^1 \Vdash \overline{\varphi}(s\text{'}\beta)$. Indeed, by Stability, if $q_0 \nVdash \varphi(s\text{'}\beta)$, then there is $q^1 \leqslant^{\mathbb{P}} q^0$ such that $r \nVdash \varphi(s\text{'}\beta)$ for all $r \leqslant^{\mathbb{P}} q^1$, i.e., $q^1 \Vdash \neg\varphi(s\text{'}\beta)$ and hence $q^1 \Vdash \overline{\varphi}(s\text{'}\beta)$ by Lemma 7.4(f). Lemma 7.6(d) gives $q^2 \leqslant^{\mathbb{P}} q^1$ such that $q^2 \Vvdash \varphi(s\text{'}\beta)$ or $q^2 \Vvdash \overline{\varphi}(s\text{'}\beta)$ respectively. Then $M \models \psi(c, q^2)$.

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