Now if X and Y are independent random variables, then $f(x,y) = f_X(x)f_Y(y)$, and Formula (1.88) reduces to

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\,\mathrm{d}\,x\,, \quad -\infty < z < \infty\,. \tag{1.89}$$

Further, if both X and Y are nonnegative random variables, then

$$f_Z(z) = \int_{0}^{z} f_X(x)f_Y(z-x)\,\mathrm{d}\,x\,, \quad 0 < z < \infty\,. \tag{1.90}$$

This integral is known as the *convolution* of f_X and f_Y. Thus, the pdf of the sum of two non-negative independent random variables is the convolution of their individual pdfs.

Example 1.1 If $X_1, X_2, \ldots, X_k$ are mutually independent, identically distributed exponential random variables with parameter $\lambda = k\mu$, then the random variable $X_1 + X_2 + \cdots + X_k$ has an Erlang-k distribution with parameters k and μ.

Consider an asynchronous transfer mode (ATM) switch with cell interarrival times that are independent from each other and exponentially distributed with parameter λ. Let X_i be the random variable denoting the time between the $(i-1)$st and ith arrival. Then $Z_k = X_1 + X_2 + \cdots + X_k$ is the time until the kth arrival and has an Erlang-k distribution. Another way to obtain this result is to consider N_t, the number of arrivals in the interval $(0, t]$. As pointed out earlier, N_t has a Poisson distribution with parameter λt. Since the events $[Z_k > t]$ and $[N_t < k]$ are equivalent, we have

$$P(Z_k > t) = P(N_t < k) = \sum_{j=0}^{k-1} \mathrm{e}^{-\lambda t} \left[\frac{(\lambda t)^j}{j!}\right],$$

which implies that

$$F_{Z_k}(t) = P(Z_k \le t) = 1 - \sum_{j=0}^{k-1} \frac{(\lambda t)^j}{j!}\,\mathrm{e}^{-\lambda t} = 1 - \mathrm{e}^{-k\mu t}\cdot \sum_{j=0}^{k-1} \frac{(k\mu t)^j}{j!}\,,$$

which is the Erlang-k distribution function (see Eq. (1.28)).

Example 1.2 If $X \sim EXP(\lambda_1)$, $Y \sim EXP(\lambda_2)$, X and Y are independent, and $\lambda_1 \ne \lambda_2$, then $Z = X + Y$ has a two-stage hypoexponential distribution with two phases and parameters λ_1 and λ_2. To see this, we use

$$\begin{aligned} f_Z(z) &= \int_0^z f_X(x) f_Y(z-x)\,\mathrm{d}\,x\,, \quad z > 0 \quad \text{(by Eq. (1.90))} \\ &= \int_0^z \lambda_1\,\mathrm{e}^{-\lambda_1 x}\,\lambda_2\,\mathrm{e}^{-\lambda_2(z-x)}\;\mathrm{d}\,x = \lambda_1\lambda_2\,\mathrm{e}^{-\lambda_2 z}\int_0^z \mathrm{e}^{(\lambda_2-\lambda_1)x}\;\mathrm{d}\,x \\ &= \lambda_1\lambda_2\,\mathrm{e}^{-\lambda_2 z}\left[\frac{\mathrm{e}^{-(\lambda_1-\lambda_2)x}}{-(\lambda_1-\lambda_2)}\right]_{x=0}^{x=z} \\ &= \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\,\mathrm{e}^{-\lambda_2 z} + \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\,\mathrm{e}^{-\lambda_1 z} \quad \text{(see Eq. (1.31)).} \end{aligned}$$

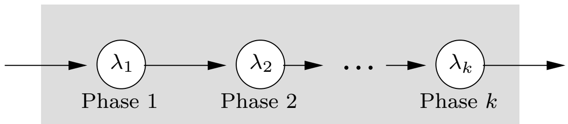

Fig. 1.16 Hypoexponential distribution as a series of exponential stages.

A more general version of this result follows: Let $Z = \sum_{i=1}^{k} X_i$, where $X_1, X_2, \ldots, X_k$ are mutually independent and X_i is exponentially distributed with parameter λ_i ($\lambda_i \neq \lambda_j$ for $i \neq j$). Then Z has a hypoexponential distribution with k phases and the density function:

$$f_Z(z) = \sum_{i=1}^{k} a_i \lambda_i\,\mathrm{e}^{-\lambda_i z}\,, \quad z > 0\,, \tag{1.91}$$

where

$$a_i = \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{\lambda_j}{\lambda_j - \lambda_i}\,, \quad 1 \leq i \leq k\,. \tag{1.92}$$

Such a phase-type distribution can be visualized as shown in Fig. 1.16 (see also Section 1.3.1.2).