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Stackelberg pricing games with congestion effects

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Abstract

We study a Stackelberg game with multiple leaders and a continuum of followers that are coupled via congestion effects. The followers' problem constitutes a nonatomic congestion game, where a population of infinitesimal players is given and each player chooses a resource. Each resource has a linear cost function which depends on the congestion of this resource. The leaders of the Stackelberg game each control a resource and determine a price per unit as well as a service capacity for the resource influencing the slope of the linear congestion cost function. As our main result, we establish existence of pure-strategy Nash–Stackelberg equilibria for this multi-leader Stackelberg game. The existence result requires a completely new proof approach compared to previous approaches, since the leaders' objective functions are discontinuous in our game. As a consequence, best responses of leaders do not always exist, and thus standard fixed-point arguments á la Kakutani (Duke Math J 8(3):457–458, 1941) are not directly applicable. We show that the game is C-secure (a concept introduced by Reny (Econometrica 67(5):1029–1056, 1999) and refined by McLennan et al. (Econometrica 79(5):1643–1664, 2011), which leads to the existence of an equilibrium. We furthermore show that the equilibrium is essentially unique, and analyze its efficiency compared to a social optimum. We prove that the worst-case quality is unbounded. For identical leaders, we derive a closed-form expression for the efficiency of the equilibrium.

Keywords Stackelberg games · Pricing · Congestion games · Existence and uniqueness of equilibria · Price of anarchy

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1 Introduction

We consider a Stackelberg game with multiple leaders $N = \{1, \dots, n\}$, $n \geq 2$, and a continuum of followers represented by the interval $[0, 1]$[1]. The followers' problem constitutes a (nonatomic) singleton congestion game, that is, each follower chooses one resource and the cost of the resource depends on the congestion of this resource. The leaders of the game each control one resource, and decide about a *price* which is charged to the followers for the usage of the resource, as well as a *capacity* which influences the slope of the congestion cost of her resource. Concretely, each leader $i \in N$ chooses a price $p_i \in [0, C_i]$, where $C_i > 0$ is a given *price cap*, and a capacity $z_i \geq 0$. The resulting *effective cost function* of leader i's resource - from the perspective of a follower - is the sum of the congestion cost function $\ell_i(x_i, z_i)$ and the price p_i charged, where x_i denotes the volume of followers who have chosen leader i. We assume that the congestion cost $\ell_i(x_i, z_i)$ is infinite if no capacity is installed (i.e., $z_i = 0$), and else it depends *linearly* on the volume of followers and *inverse-linearly* on the installed capacity, that is,

$$\ell_i(x_i, z_i) := \begin{cases} \frac{a_i x_i}{z_i} + b_i, & \text{for} \quad z_i > 0, \\ \infty, & \text{for} \quad z_i = 0, \end{cases}$$

where $a_i > 0$ and $b_i \geq 0$ are given parameters for $i \in N$. As is common in the transportation science literature (see, e.g., [18,28] and the references mentioned therein), the capacity z_i is not a strict bound on the admissible flow, but instead influences the congestion dependent cost. The case $z_i = 0$ can be interpreted as if the resource controlled by leader i is not present in the followers' congestion game. Given a capacity vector $z = (z_1, \dots, z_n)$ with $\sum_{i \in N} z_i > 0$, i.e., there is at least one resource in the followers' congestion game, and a price vector $p = (p_1, \dots, p_n)$, the followers choose rationally the most attractive resource in terms of the effective cost. That is, the outcome $x \in P := \{x \in \mathbb{R}^n_{\geq 0} | \sum_{i \in N} x_i = 1\}$ of the followers' congestion game is a *Wardrop equilibrium*, described by the following Wardrop equilibrium conditions:

$$c_i(x, z, p) := \ell_i(x_i, z_i) + p_i \leq \ell_j(x_j, z_j) + p_j =: c_j(x, z, p)$$

holds for all $i, j \in N$ with $x_i > 0$. Note that for given capacities $z \neq 0$ and prices p, there is exactly one $x \in P$ satisfying the Wardrop equilibrium conditions (see, e.g., [10]). Call this flow $x = x(z, p)$ the Wardrop flow *induced by* (z, p).[2] In particular, there is a constant $K \geq 0$ such that $c_i(x, z, p) = K$ holds for each $i \in N$ with $x_i > 0$, and $c_i(x, z, p) \geq K$ holds for each $i \in N$ with $x_i = 0$. For a Wardrop flow x, call the corresponding constant K the *(routing) cost of* x. We assume that each leader $i \in N$

[1] Each follower is assumed to be infinitesimally small and represented by a number in $[0, 1]$. All results hold for arbitrary intervals $[0, d]$, $d \in \mathbb{R}_{>0}$ by a standard scaling argument.

[2] Since $x \in P$ can be interpreted as a flow in a network consisting of n parallel links, we also call x a flow.

seeks to maximize her own profit function, which is defined as

$$\Pi_i(z,p) := \begin{cases} p_i x_i(z,p) - \gamma_i z_i, & \text{for} \quad \sum_{i\in N} z_i > 0, \\ 0, & \text{else}, \end{cases}$$

where $\gamma_i > 0$ is a given installation cost parameter for leader i. This completes the description of our game, which we denote a *Stackelberg pricing game*.

Note that the introduced Stackelberg pricing game captures many aspects of realistic oligopolistic markets where firms offer a service to customers, like build-operate schemes in traffic networks or competition among WIFI-providers (see, e.g., [1,9,20,21,26,34]).

In this paper, we analyze the stable states of Stackelberg pricing games, that is, pure Nash(–Stackelberg) equilibria as defined in the following. For each leader $i \in N$, let $S_i := \{s_i = (z_i, p_i) : 0 \le z_i, 0 \le p_i \le C_i\}$ be her strategy set. A vector s consisting of strategies $s_i = (z_i, p_i) \in S_i$ for all $i \in N$ is called a strategy profile, and $S := \times_{i\in N} S_i$ denotes the set of strategy profiles. Usually, we will write a strategy profile $s \in S$ in the form $s = (z, p)$, where z denotes the vector consisting of all capacities z_i for $i \in N$, and p is the vector of prices p_i for $i \in N$. The profit of leader i for a strategy profile $s = (z, p)$ is then defined as $\Pi_i(s) := \Pi_i(z, p)$. Furthermore, we write $x(s) := x(z, p)$ for the Wardrop-flow induced by $s = (z, p)$ and $K(s) := K(z, p)$ for the routing cost of $x(s)$. For leader i, denote by $s_{-i} = (z_{-i}, p_{-i}) \in S_{-i} := \times_{j\in N\setminus\{i\}} S_j$ the vector consisting of strategies $s_j = (z_j, p_j) \in S_j$ for all $j \in N \setminus \{i\}$. We then write $(s_i, s_{-i}) = ((z_i, p_i), (z_{-i}, p_{-i}))$ for the strategy profile where leader i chooses $s_i = (z_i, p_i) \in S_i$, and the other leaders choose $s_{-i} = (z_{-i}, p_{-i}) \in S_{-i}$. Moreover, we use the simplified notation $\Pi_i((s_i, s_{-i})) = \Pi_i(s_i, s_{-i})$ and $x((s_i, s_{-i})) = x(s_i, s_{-i})$. A strategy profile $s = (s_i, s_{-i})$ is a *pure Nash(–Stackelberg) equilibrium (PNE)*, if for each leader $i \in N$:

$$\Pi_i(s_i, s_{-i}) \ge \Pi_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i.$$

For given strategies $s_{-i} \in S_{-i}$ of the other leaders, the best response correspondence of leader i is defined by

$$\mathrm{BR}_i(s_{-i}) := \arg\max\{\Pi_i(s_i, s_{-i}) | s_i \in S_i\}.$$

If s_{-i} is clear from the context, we just write BR_i instead of $\mathrm{BR}_i(s_{-i})$. Clearly, the strategy profile $s = (s_i, s_{-i})$ is a PNE if and only if $s_i \in \mathrm{BR}_i(s_{-i})$ is fulfilled for each $i \in N$.

The next subsection summarizes our results in terms of existence, uniqueness and quality of PNE for Stackelberg pricing games.

1.1 Our results and proof techniques

As our main result, we show existence of PNE for the introduced Stackelberg pricing game. This result requires a completely new proof approach compared to previous approaches, since the leaders' profit functions are not continuous. Therefore, best responses do not always exist and standard fixed-point arguments à la Kakutani are not directly applicable. For the existence proof, we first completely characterize in Theorem 1 the continuity of the joint profit function (consisting of all leaders' profits). Using this, we completely characterize the structure of best response correspondences of leaders; including the possibility of non-existence of a best response (Theorem 2). We then establish the existence of equilibria (Theorem 4) using the concept of C-security introduced by [29], which in turn resembles ideas of [31]. A game is C-*secure* at a given strategy profile, if each player has a pure strategy guaranteeing a certain utility value, even if the other players play some perturbed strategy within a (small enough) neighborhood, and furthermore, for each slightly perturbed strategy profile, there is a player whose perturbed strategy can in some sense be strictly separated from her securing strategies. Intuitively, the concept of securing strategies means that those strategies are robust to other players' small deviations. The result of [29] states that a game with compact, convex strategy sets and bounded profit functions admits an equilibrium, if every non-equilibrium profile is C-secure. It is important to note that the concept of C-security does not rely on quasi-concavity or continuity of profit functions. With our characterization of best response correspondences at hand, we show that the considered Stackelberg pricing game fulfills the conditions of [29] and thus admits PNE.

As our second main result, we show that the equilibrium is essentially unique (Theorem 5). While the general proof approach is related to that of Johari et al. [21] (see also the related work in the following subsection), our model allows for price caps thus requiring additional ideas. In particular, the set of leaders having positive capacity needs to be decomposed, where the decomposition is related to the property whether the price of a leader is equal to its cap, or strictly smaller.

We finally study the efficiency of the unique equilibrium compared to a natural benchmark, in which we relax the equilibrium conditions of the leaders, but not the equilibrium conditions of the followers. We show that the unique equilibrium might be arbitrarily inefficient (Theorem 6), by presenting a family of instances such that the quality of the equilibrium gets arbitrarily bad. Furthermore, for instances with identical leaders, we derive a closed-form expression for the equilibrium quality (Theorem 7).

1.2 Related work

Johari et al. [21] study existence, uniqueness and worst-case quality of PNE assuming that the demand of the followers is *elastic*: The volume of followers participating in the congestion game decreases with increasing combined cost of congestion and price. As a consequence of the elastic demand assumption, best responses of leaders do always exist in their model, and Kakutani's fixed point theorem can be applied to show existence of PNE. As already noted, this is not possible for the model with inelastic demand that we consider in this paper. We discuss in detail the motivation for assuming inelastic demand in Sect. 1.3. Johari et al. also consider the case of inelastic follower's demand, assuming *homogeneous* leaders (that is, all leaders have the same parameters). As shown in their paper, homogeneity (together with some assumptions on the congestion costs) implies that there is only one symmetric equilibrium candidate profile. For this specific symmetric strategy profile, they directly prove stability using concavity arguments. This proof technique is clearly not applicable in the general non-homogeneous case (with fixed demand). Further differences between our model and Johari et al. are that they do not consider price caps, but on the other hand, allow more general congestion cost functions.

Acemoglu et al. [1] study a model in which the capacities represent "hard" capacities bounding the admissible customer volume for a leader. They observe that equilibria do not exist in this model. Subsequently they study a model in which capacities and prices are not chosen simultaneously anymore, but the leaders first determine capacities, and only after the chosen capacities became apparent, set prices. For this model, they investigate existence and worst-case quality of equilibria (see also [23] for earlier work on the two-stage model).

Schmand et al. [33] study a network investment game in which the leaders invest in edges of a series-parallel graph (but do not directly set prices).

Further related models are used in the papers of Harks et al. [20] and Correa et al. [9]. There, leaders do not choose capacities, but only prices, and the prices are upper-bounded by caps (equal for all leaders in [20], leader-specific in [9]). The two papers consider the problem of a system designer who chooses the cap(s) in order to minimize total congestion.

There are also numerous works analyzing Stackelberg games with a *single* leader. For example, Labbé et al. [24] study a model where a single leader sets prices in order to maximize her profit in a subsequent network routing game without congestion effects. Situations where the leader determines capacities or prices in order to reduce the total congestion (plus investments for the case of capacities) of the resulting Wardrop equilibria are for example studied in [18,28] for setting capacities, and [4,35] for setting prices. Castiglioni et al. [7] and Marchesi et al. [27] consider Stackelberg games with an underlying (atomic) congestion game, where the single leader participates in the same congestion game than the followers.

Finally, network design problems with congestion dependent costs have also been studied from a purely optimization point of view, for example in the context of energy-efficient networks [3,14]. There one wants to minimize the congestion-dependent cost under certain network connectivity requirements.

1.3 Motivation for inelastic demand

We focus on the case of inelastic demand, that is, there is a *fixed* volume of followers in the congestion game. This assumption is made by many works in the transportation science and algorithmic game theory literature (see, e.g., [6,11–13,17,35] and [2,8,16,32], respectively) and usually considered as a fundamental base case. As already noted, in terms of equilibrium existence, the case of inelastic demand is much more complicated for our model compared to the seemingly more general case of elastic demand: best responses do not always exist in the inelastic case, putting standard fixed-point approaches out of reach.

Besides this theoretical aspect, we think that a thorough analysis of the inelastic demand case will be helpful in understanding realistic demand scenarios which are not captured by the literature for the elastic demand. To make this point clear it is worth recalling the underlying assumptions made in the literature for the elastic demand case. In Johari et al. [21], the demand function can be described by a differentiable, strictly decreasing and concave function. Liu et al. [26] assume instead of concavity that $d \cdot B(d)$ is concave, where $B(d)$ denotes the inverse demand function.[3] The strict monotonicity of the demand function implies that every follower particle has its own unique valuation w.r.t. participating in the game. These assumptions do not cover the realistic case that commuters stick to a travel mode (private car) for some cost range, and only if the travel cost exceeds that of an alternative travel mode (public transport), they switch mode. A demand function of this type would be piecewise constant, or piecewise linear, but in either case not strictly decreasing, and perhaps not even continuous.

Further examples of a rather inelastic demand appear for higher value travel (as business or commute travel, in particular "urban peak-period trips"), in case the public transport alternatives are sparse (rural areas) or if commuters with higher income are considered, see also Litman [25], where various factors influencing travel demand are discussed.

1.4 Outline of the paper

The following sections contain the technical presentation of our results. Concretely, we characterize in Sects. 2 and 3 the continuity of the leaders' profit functions, as well as the best response correspondences. These results are then used to show existence (Sect. 4) and uniqueness (Sect. 5) of PNE. Finally, we analyze the quality of PNE in Sect. 6.

2 Continuity of the profits

In this section, we prove a fundamental result about the continuity of the profits. We will use Theorem 1, which completely characterizes the strategy profiles s having the property that all profit functions Π_i, $i \in N$, are continuous at s, several times during the rest of the paper.

Theorem 1 *Let $s = (z, p) \in S$. Then: The profit function Π_i is continuous at $s = (z, p)$ for all $i \in N$ if and only if $z \neq 0$ or $(z, p) = (0, 0)$.*

Proof We start with the strategy profile $s = (z, p) = (0, 0)$ and show that for each leader i, her profit function Π_i is continuous at s. Let $i \in N$. For $\varepsilon > 0$, define $\delta := \min\{\varepsilon/(2\gamma_i), \varepsilon/2\} > 0$, and let $s' \in S$ with $||s'|| = ||s' - s|| < \delta$ (where $|| \cdot ||$ denotes the Euclidean norm). If $z'_i = 0$, then $\Pi_i(s') = \Pi_i(s) = 0$. Otherwise, the following holds, showing that Π_i is continuous at s:

$$|\Pi_i(s') - \Pi_i(s)| = |x_i(s')p'_i - \gamma_i z'_i| \leq p'_i + \gamma_i z'_i \leq ||s'|| + \gamma_i ||s'|| < \delta(1 + \gamma_i) \leq \varepsilon$$

[3] Note that for a strictly decreasing demand function, there is a well-defined inverse demand function.

Now consider $s = (z, p) \in S$ with $z \neq 0$. We again need to show that all profit functions Π_i, $i \in N$, are continuous at s. Since $z \neq 0$, we get that $N^+ := \{j \in N : z_j > 0\} \neq \emptyset$. Furthermore, for $\delta_1 > 0$ sufficiently small, $N^+ \subseteq \{j \in N : z'_j > 0\} =: N^+(s')$ holds for all $s' = (z', p') \in S$ with $||s - s'|| < \delta_1$. Write $s' = (s'_1, s'_2) \in S$, where s'_1 denotes the strategies of the leaders in N^+, and s'_2 denotes the strategies of the leaders in $N \setminus N^+$. Now let $i \in N$. We need to show that Π_i is continuous at $s = (z, p)$. For all $s' \in S$ with $||s - s'|| < \delta_1$, leader i's profit is $\Pi_i(s') = x_i(s')p'_i - \gamma_i z'_i$. Thus it is sufficient to show that x_i is continuous at $s = (s_1, s_2)$, where s_1 and s_2 denote the strategies of the leaders in N^+ and $N \setminus N^+$, respectively. The idea of the proof is to show that, for a slightly perturbed strategy profile (s'_1, s'_2), the difference between $x_i(s)$ and $x_i(s'_1, s_2)$, as well as the difference between $x_i(s'_1, s_2)$ and $x_i(s'_1, s'_2)$, is small. In the following, let $s' \in S$ with $||s - s'|| < \delta_1$. It is well known ([4], compare also [10]) that $x(s')$ is the unique optimal solution of the following optimization problem $\mathrm{Q} = \mathrm{Q}(s')$:

$$(\mathrm{Q}) \quad \min \sum_{j \in N} \int_0^{x_j} \left(\ell_j(t, z'_j) + p'_j\right) \mathrm{d}t$$
$$\text{s.t.} \sum_{j \in N} x_j = 1, \; x_j \geq 0 \; \forall j \in N.$$

Furthermore, $x_j(s') = 0$ for $j \notin N^+(s')$. Therefore, the values $(x_j(s'))_{j \in N^+(s')}$ are the unique optimal solution of

$$\max - \sum_{j \in N^+(s')} (a_j/(2z'_j)) \cdot x_j^2 + (b_j + p'_j)x_j)$$
$$\text{s.t.} \sum_{j \in N^+(s')} x_j = 1, \; x_j \geq 0 \; \forall j \in N^+(s').$$

By Berge's theorem of the maximum [5], for all $\varepsilon > 0$ there is $0 < \delta_2 = \delta_2(\varepsilon) < \delta_1$ such that $||x(s) - x(s'_1, s_2)|| < \varepsilon$ for all $(s'_1, s_2) \in S$ with $||s - (s'_1, s_2)|| < \delta_2$. That is, x is continuous at s if we only allow changes in s_1, but not in s_2. Furthermore, if $q(s')$ denotes the optimal objective function value of $\mathrm{Q}(s')$, and if only changes in s_1 are allowed, q is also continuous in s, i.e., for all $\varepsilon > 0$ there is $0 < \delta_3 = \delta_3(\varepsilon) < \delta_1$ such that $|q(s) - q(s'_1, s_2)| < \varepsilon$ for all $(s'_1, s_2) \in S$ with $||s - (s'_1, s_2)|| < \delta_3$. We now distinguish between $z_i > 0$ and $z_i = 0$.

First consider $z_i = 0$, i.e., $i \notin N^+$, and let $\varepsilon > 0$. Note that $x_i(s) = 0$, thus we need to find $\delta > 0$ such that $|x_i(s) - x_i(s')| = x_i(s') < \varepsilon$ for all $s' \in S$ with $||s - s'|| < \delta$. To this end, define

$$\delta = \delta(i, \varepsilon) := \begin{cases} \delta_3(1), & \text{if } q(s) + 1 \leq b_i \varepsilon, \\ \min\{\delta_3(1), \frac{a_i \varepsilon^2}{2(q(s)+1-b_i\varepsilon)}\}, & \text{else,} \end{cases}$$

and let $s' \in S$ with $||s - s'|| < \delta$. In particular, $|z_i - z'_i| = z'_i < \delta$. Furthermore, $q(s') \leq q(s'_1, s_2) \leq q(s) + 1$ holds since $||s - (s'_1, s_2)|| \leq ||s' - s|| < \delta \leq \delta_3(1)$. If $z'_i = 0$, we immediately get $x_i(s') = 0 < \varepsilon$. Thus assume $z'_i > 0$ and assume, by contradiction, that $x_i(s') \geq \varepsilon$. Then, by definition of δ, we get the following contradiction:

$$q(s) + 1 \geq q(s') \geq \frac{a_i}{2z'_i} x_i(s')^2 + (b_i + p'_i) x_i(s') \geq \frac{a_i}{2z'_i}\varepsilon^2 + b_i\varepsilon$$
$$> \frac{a_i}{2\delta}\varepsilon^2 + b_i\varepsilon \geq q(s) + 1.$$

Therefore, $x_i(s') < \varepsilon$ holds, showing that x_i is continuous at s if $i \notin N^+$.

Now consider the case $i \in N^+$, i.e. $z_i > 0$. For $\varepsilon > 0$, we need to find $\delta > 0$ such that $|x_i(s) - x_i(s')| < \varepsilon$ for all $s' \in S$ with $||s - s'|| < \delta$. To this end, define

$$\delta := \min \left\{ \min \left\{ \delta\left(j, \frac{\varepsilon}{2n}\right) : j \notin N^+ \right\}, \delta_2\left(\frac{\varepsilon}{2}\right) \right\}$$

and let $s' \in S$ with $||s - s'|| < \delta$. In particular, $||s - (s'_1, s_2)|| < \delta \leq \delta_2(\frac{\varepsilon}{2})$, thus $|x_i(s) - x_i(s'_1, s_2)| < \varepsilon/2$. Furthermore, since $\delta \leq \delta(j, \frac{\varepsilon}{2n})$, we get $x_j(s') \leq \frac{\varepsilon}{2n}$ for all $j \notin N^+$. If $x_j(s') = 0$ for all $j \notin N^+$, we get $x_i(s') = x_i(s'_1, s_2)$ and thus $|x_i(s) - x_i(s')| = |x_i(s) - x_i(s'_1, s_2)| < \varepsilon/2 < \varepsilon$, as desired. Otherwise, there is $j \notin N^+$ with $0 < x_j(s') \leq \frac{\varepsilon}{2n}$. In particular, $z'_j > 0$. We now use a result about the sensitivity of Wardrop flows [15, Theorem 2]. They show that if the followers are not able to choose leader j's resource anymore (we say that leader j is deleted from the followers' game), the resulting change in the Wardrop flow can be bounded by the flow that j received. More formally, if $x \in [0, 1]^n$ is the Wardrop flow for the game with leaders N, and $x' \in [0, 1]^{n-1}$ is the Wardrop flow if leader j is deleted from the followers' game, then $|x_k - x'_k| \leq x_j$ for all $k \in N \setminus \{j\}$. Obviously, changing leader j's capacity from $z'_j > 0$ to $z_j = 0$ has the same effect on the Wardrop flow as deleting leader j. Therefore, if we change, one after another, the capacities of all leaders $j \notin N^+$ having $z'_j > 0$ to $z_j = 0$, we get $|x_i(s') - x_i(s'_1, s_2)| \leq (n-1)\varepsilon/(2n) < \varepsilon/2$ (note that the flow values for $j \notin N^+$ are always upper-bounded by $\varepsilon/(2n)$ due to our choice of δ and the analysis of the case $z_i = 0$). Using this, we now get the desired inequality:

$$|x_i(s) - x_i(s')| \leq |x_i(s') - x_i(s'_1, s_2)| + |x_i(s) - x_i(s'_1, s_2)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Altogether we showed that all profit functions Π_i are continuous at $s = (z, p)$ if $z \neq 0$.

To complete the proof, it remains to show that if all profit functions Π_i, $i \in N$, are continuous at $s = (z, p)$, then $z \neq 0$ or $(z, p) = (0, 0)$ holds. We show this by contraposititon, thus assume that $s = (z, p)$ fulfills $z = 0$ and $p \neq 0$. We need to show that there is a leader i such that Π_i is not continuous at s. To this end, let $i \in N$ with $p_i > 0$. Define the sequence of strategy profiles s^n by $(z^n_j, p^n_j) := (z_j, p_j)$ for all $j \neq i$, and $(z^n_i, p^n_i) := (1/n, p_i)$. Obviously, $s^n \to s$ for $n \to \infty$. But for the profits, we get

$$\Pi_i(s^n) = x_i(s^n) p^n_i - \gamma_i z^n_i = p_i - \gamma_i/n \to_{n\to\infty} p_i > 0.$$

Since $\Pi_i(s) = 0$, this shows that Π_i is not continuous at s. □

3 Characterization of best responses

The aim of this section is to derive a complete characterization of the best-response correspondences of the leaders. We will make use of this characterization in all our main results, i.e., existence, uniqueness and quality of PNE. Given a leader $i \in N$ and fixed strategies $s_{-i} = (z_{-i}, p_{-i}) \in S_{-i}$ for the other leaders, we characterize the set $\mathrm{BR}_i = \mathrm{BR}_i(s_{-i})$ of best responses of leader i to s_{-i}. To this end, we distinguish between the two cases that $z_{-i} = 0$ (Sect. 3.1) and $z_{-i} \neq 0$ (Sect. 3.2). Section 3.3 then contains the derived complete characterization. In Sect. 3.4, we discuss how our results about the best responses influence the applicability of Kakutani's fixed point theorem.

3.1 The case $z_{-i} = 0$

In this subsection, assume that the strategies $s_{-i} = (z_{-i}, p_{-i})$ of the other leaders fulfill $z_{-i} = 0$. Under this assumption, leader i does not have a best response to s_{-i}:

Lemma 1 *If $z_{-i} = 0$, then $BR_i(z_{-i}, p_{-i}) = \emptyset$.*

Proof Whenever leader i chooses a strategy (z_i, p_i) with $z_i > 0$, then $x_i = 1$ holds for the induced Wardrop-flow x, thus leader i's profit is $p_i - \gamma_i z_i$. On the other hand, any strategy (z_i, p_i) with $z_i = 0$ yields a profit of 0. Thus, leader i's profit depends solely on her own strategy (z_i, p_i), and can be stated as follows:

$$\Pi_i(z_i, p_i) := \begin{cases} p_i - \gamma_i z_i, & \text{for} \quad z_i > 0, \\ 0, & \text{for} \quad z_i = 0. \end{cases}$$

Obviously, $\Pi_i(z_i, p_i) < C_i$ holds for each $(z_i, p_i) \in S_i$, i.e. for $z_i \geq 0$ and $0 \leq p_i \leq C_i$. On the other hand, by $(z_i, p_i) = (\varepsilon, C_i)$ for $\varepsilon > 0$, leader i gets a profit of $C_i - \gamma_i \cdot \varepsilon$ arbitrarily near to C_i, that is, $\sup\{\Pi_i(z_i, p_i) : (z_i, p_i) \in S_i\} = C_i$. This shows $\mathrm{BR}_i(z_{-i}, p_{-i}) = \emptyset$. □

3.2 The case $z_{-i} \neq 0$

In this subsection, assume that the strategies $s_{-i} = (z_{-i}, p_{-i})$ of the other leaders fulfill $z_{-i} \neq 0$. For a strategy $s_i = (z_i, p_i)$ of leader i, write $\Pi_i(z_i, p_i) := \Pi_i(s_i, s_{-i})$ for leader i's profit function, $x(z_i, p_i) := x(s_i, s_{-i})$ for the Wardrop-flow induced by (s_i, s_{-i}) and $K(z_i, p_i) := K(s_i, s_{-i})$ for the corresponding routing cost.

For $(z_i, p_i) \in S_i$, leader i's profit is $\Pi_i(z_i, p_i) = x_i(z_i, p_i)p_i - \gamma_i z_i$. It is clear that each strategy (z_i, p_i) with $z_i = 0$ yields $x_i(z_i, p_i) = 0$, and thus $\Pi_i(z_i, p_i) = 0$. On the other hand, each strategy (z_i, p_i) with $z_i > C_i/\gamma_i$ yields negative profit since $\Pi_i(z_i, p_i) = x_i(z_i, p_i)p_i - \gamma_i z_i < C_i - \gamma_i \cdot C_i/\gamma_i = 0$. Therefore, each best response (z_i, p_i) fulfills $z_i \leq C_i/\gamma_i$ since it yields nonnegative profit. Thus, BR_i can be described as the set of optimal solutions of the problem

$$\max \quad \Pi_i(z_i, p_i) \quad \text{subject to} \quad z_i \in [0, C_i/\gamma_i], \; p_i \in [0, C_i]. \tag{P$_i$}$$

Due to the theorem of Weierstrass, (P_i) has an optimal solution: The feasible set of (P_i) is compact and nonempty, and Π_i is continuous at (z_i, p_i) for all feasible (z_i, p_i) (Theorem 1). Since BR_i can be described as the set of optimal solutions of (P_i), we get $\mathrm{BR}_i \neq \emptyset$.

Note that (P_i) is a *bilevel* optimization problem (since $x(z_i, p_i)$ can be described as the optimal solution of a minimization problem [4]), and these problems are known to be notoriously hard to solve. The characterization of BR_i that we derive here has the advantage that it only uses *ordinary* optimization problems, namely the following two (1-dimensional) optimization problems in the variable $K \in \mathbb{R}$,

$$\begin{aligned} \max\ & f_i^1(K) := \overline{x}_i(K)(K - b_i - 2\sqrt{a_i\gamma_i}) \\ \text{s.t.}\ & 2\sqrt{a_i\gamma_i} + b_i \leq K \\ & K \leq \sqrt{a_i\gamma_i} + b_i + C_i \\ & \overline{x}_i(K) > 0, \end{aligned} \tag{P_i^1}$$

$$\begin{aligned} \max\ & f_i^2(K) := \overline{x}_i(K)\left(C_i - \frac{a_i\gamma_i}{K - b_i - C_i}\right) \\ \text{s.t.}\ & \sqrt{a_i\gamma_i} + b_i + C_i < K \\ & a_i\gamma_i/C_i + b_i + C_i \leq K \\ & \overline{x}_i(K) > 0, \end{aligned} \tag{P_i^2}$$

where

$$\overline{x}_i(K) := 1 - \sum_{j \in N(K)} \frac{(K - b_j - p_j)z_j}{a_j}$$

with $N^+ := \{j \in N \setminus \{i\} : z_j > 0\}$ and $N(K) := \{j \in N^+ : b_j + p_j < K\}$.

Note that $\overline{x}_i : \mathbb{R} \to \mathbb{R}$ is a continuous function which is equal to 1 for $K \leq \min\{b_j + p_j : j \in N^+\}$, and strictly decreasing for $K \geq \min\{b_j + p_j : j \in N^+\} \geq 0$ with $\lim_{K\to\infty} \overline{x}_i(K) = -\infty$. Therefore, there is a unique constant $K_i^{\max} > 0$ with the property $\overline{x}_i(K_i^{\max}) = 0$. Obviously, $\overline{x}_i(K) > 0$ if and only if $K < K_i^{\max}$. Furthermore, the function $\overline{x}_i$ is closely related to Wardrop-flows, as described in the following lemma.

Lemma 2 *1. If $(z_i, p_i) \in S_i$ with $x_i(z_i, p_i) > 0$, then $\overline{x}_i(K) = x_i(z_i, p_i)$ for $K := K(z_i, p_i)$.*
2. If $K \geq 0$ with $\overline{x}_i(K) > 0$, and $(z_i, p_i) \in S_i$ fulfills $z_i > 0$ and $a_i\overline{x}_i(K)/z_i + b_i + p_i = K$, then $x_i(z_i, p_i) = \overline{x}_i(K)$ and $K(z_i, p_i) = K$.

Proof We start with statement 1. of the lemma, so let $(z_i, p_i) \in S_i$ with $x_i(z_i, p_i) > 0$. By definition of $K(z_i, p_i) =: K$, we get $\ell_j(x_j(z_i, p_i), z_j) + p_j = K$ for all $j \in N$ with $x_j(z_i, p_i) > 0$, and $\ell_j(x_j(z_i, p_i), z_j) + p_j \geq K$ for all $j \in N$ with $x_j(z_i, p_i) = 0$. Since

$$\ell_j(x_j(z_i, p_i), z_j) + p_j = \begin{cases} a_j x_j(z_i, p_i)/z_j + b_j + p_j, & \text{for} \quad j \in N \text{ with } z_j > 0, \\ \infty, & \text{for} \quad j \in N \text{ with } z_j = 0, \end{cases}$$

we get that $\{j \in N : x_j(z_i, p_i) > 0\} = \{j \in N^+ : b_j + p_j < K\} \cup \{i\} = N(K) \cup \{i\}$. Therefore, for each $j \in N(K)$, we get $a_j x_j(z_i, p_i)/z_j + b_j + p_j = K$, which is equivalent to $x_j(z_i, p_i) = (K - b_j - p_j)z_j/a_j$. Using $\sum_{j \in N} x_j(z_i, p_i) = 1$ yields

$$x_i(z_i, p_i) = 1 - \sum_{j \in N \setminus \{i\}} x_j(z_i, p_i) = 1 - \sum_{j \in N(K)} (K - b_j - p_j)z_j/a_j = \overline{x}_i(K).$$

Now we turn to statement 2. of the lemma, so let $K \geq 0$ with $\overline{x}_i(K) > 0$ and let $(z_i, p_i) \in S_i$ be a strategy with $z_i > 0$ and $a_i \overline{x}_i(K)/z_i + b_i + p_i = K$. Consider $x \in [0, 1]^n$ defined by

$$x_j := \begin{cases} \overline{x}_i(K), & j = i, \\ (K - b_j - p_j)z_j/a_j, & j \in N^+ \text{ with } b_j + p_j < K, \\ 0, & j \in N^+ \text{ with } b_j + p_j \geq K \text{ or } j \in N \setminus (N^+ \cup \{i\}). \end{cases}$$

It is clear that $x_j > 0$ holds for all $j \in N^+$ with $b_j + p_j < K$, and $x_i = \overline{x}_i(K) > 0$. Furthermore, the definition of $\overline{x}_i(K)$ yields $\sum_{j \in N} x_j = 1$. Finally, x fulfills the Wardrop conditions:

$$c_j(x, z, p) = \begin{cases} a_i \overline{x}_i(K)/z_i + b_i + p_i = K, & j = i, \\ a_j(K - b_j - p_j)z_j/(a_j z_j) + b_j + p_j = K, & j \in N^+ \text{ with } b_j + p_j < K, \\ b_j + p_j \geq K, & j \in N^+ \text{ with } b_j + p_j \geq K, \\ \infty > K, & j \in N \setminus (N^+ \cup \{i\}). \end{cases}$$

The uniqueness of the Wardrop-flow now implies $x = x(z_i, p_i)$, and $K(z_i, p_i) = K$ follows from $x_i(z_i, p_i) = \overline{x}_i(K) > 0$ and $K(z_i, p_i) = c_i(x, z, p) = K$, completing the proof. □

In the following lemmata, we analyze the connection between (P_i^1) and (P_i^2) and the optimal solutions of (P_i).

Lemma 3 *1. If K is feasible for problem (P_i^1), the tuple $(z_i, p_i) := (\sqrt{a_i/\gamma_i} \cdot \overline{x}_i(K), K - \sqrt{a_i \gamma_i} - b_i)$ is feasible for (P_i), and fulfills $z_i > 0$ and $\Pi_i(z_i, p_i) = f_i^1(K)$.*
2. If K is feasible for problem (P_i^2), the tuple $(z_i, p_i) := (a_i \overline{x}_i(K)/(K - b_i - C_i), C_i)$ is feasible for (P_i), and fulfills $z_i > 0$ and $\Pi_i(z_i, p_i) = f_i^2(K)$.

Proof We start with statement 1. of the lemma, thus assume that K is feasible for problem (P_i^1). Let $z_i := \sqrt{a_i/\gamma_i} \cdot \overline{x}_i(K)$ and $p_i := K - \sqrt{a_i \gamma_i} - b_i$ as stated in 1. The feasibility of K for (P_i^1) yields $\overline{x}_i(K) > 0$ and $2\sqrt{a_i \gamma_i} + b_i \leq K \leq \sqrt{a_i \gamma_i} + b_i + C_i$. From this we conclude $z_i > 0$ and $0 < p_i = K - \sqrt{a_i \gamma_i} - b_i \leq C_i$, thus $(z_i, p_i) \in S_i$. Furthermore, $a_i \overline{x}_i(K)/z_i + b_i + p_i = a_i \overline{x}_i(K)/(\sqrt{a_i/\gamma_i}\overline{x}_i(K)) + b_i + K - \sqrt{a_i \gamma_i} -$

$b_i = K$ holds, thus we get $x_i(z_i, p_i) = \overline{x}_i(K)$ from statement 2. of Lemma 2. Using this, we can now show that leader i's profit for (z_i, p_i) equals the objective function value of K for (P_i^1):

$$\Pi_i(z_i, p_i) = p_i x_i(z_i, p_i) - \gamma_i z_i = (K - \sqrt{a_i\gamma_i} - b_i) \cdot \overline{x}_i(K) - \gamma_i \cdot \sqrt{a_i/\gamma_i} \cdot \overline{x}_i(K)$$
$$= \overline{x}_i(K) \cdot (K - 2\sqrt{a_i\gamma_i} - b_i) = f_i^1(K)$$

Note that the feasibility of K for (P_i^1) yields $f_i^1(K) \geq 0$. It remains to show that (z_i, p_i) is feasible for (P_i). We already know that $z_i > 0$ and $0 < p_i \leq C_i$ holds. The remaining inequality $z_i \leq C_i/\gamma_i$ follows from the nonnegativity of $\Pi_i(z_i, p_i) = f_i^1(K) \geq 0$ and the fact that any strategy with $z_i > C_i/\gamma_i$ yields negative profit for leader i.

Now turn to statement 2. Assume that K is feasible for (P_i^2), and let $z_i := a_i\overline{x}_i(K)/(K - b_i - C_i)$ and $p_i := C_i$. The feasibility of K for (P_i^2) implies $K > \sqrt{a_i\gamma_i} + b_i + C_i > b_i + C_i$ and $\overline{x}_i(K) > 0$, thus $z_i > 0$ holds and this yields $(z_i, p_i) \in S_i$. Furthermore, $a_i\overline{x}_i(K)/z_i + b_i + p_i = K$ holds, thus we get $x_i(z_i, p_i) = \overline{x}_i(K)$ from 2. of Lemma 2. The profit of leader i thus is

$$\Pi_i(z_i, p_i) = p_i x_i(z_i, p_i) - \gamma_i z_i = C_i\overline{x}_i(K) - \frac{\gamma_i a_i \overline{x}_i(K)}{K - b_i - C_i}$$
$$= \overline{x}_i(K)(C_i - \frac{a_i\gamma_i}{K - b_i - C_i}) = f_i^2(K).$$

Note that $f_i^2(K) \geq 0$ holds due to the feasibility of K for (P_i^2). As in the proof of statement 1. of the lemma, this implies $z_i \leq C_i/\gamma_i$. Thus (z_i, p_i) is feasible for (P_i). □

In particular, Lemma 3 shows that any optimal solution of (P_i^1) or (P_i^2) yields a feasible strategy for (P_i) with the same objective fuction value. The next lemma shows that for certain optimal solutions of (P_i), the converse is also true.

Lemma 4 *Let (z_i^*, p_i^*) be an optimal solution of (P_i) and $K^* := K(z_i^*, p_i^*)$. If $z_i^* > 0$, then exactly one of the following two cases holds:*

1. *$(z_i^*, p_i^*) = (\sqrt{a_i/\gamma_i}\overline{x}_i(K^*), K^* - \sqrt{a_i\gamma_i} - b_i)$; K^* optimal for (P_i^1) with $f_i^1(K^*) = \Pi_i(z_i^*, p_i^*)$.*
2. *$(z_i^*, p_i^*) = (a_i\overline{x}_i(K^*)/(K^* - b_i - C_i), C_i)$; K^* optimal for (P_i^2) with $f_i^2(K^*) = \Pi_i(z_i^*, p_i^*)$.*

Proof Let (z_i^*, p_i^*) with $z_i^* > 0$ and K^* as in the lemma statement, and define $x^* := x(z_i^*, p_i^*)$. Since $\Pi_i(z_i^*, p_i^*) \geq 0$ and $z_i^* > 0$ holds, $0 \leq \Pi_i(z_i^*, p_i^*) = p_i^* x_i^* - \gamma_i z_i^* < p_i^* x_i^*$ follows, which implies $x_i^* > 0$ and $p_i^* > 0$. Therefore $K^* = a_i x_i^*/z_i^* + b_i + p_i^*$ holds and (z_i^*, p_i^*) is an optimal solution for the following problem (with variables z_i and p_i):

$$(\mathrm{P}) \qquad \max \quad p_i x_i^* - \gamma_i z_i \quad \text{s.t.:} \quad a_i x_i^*/z_i + b_i + p_i = K^*, \quad 0 < z_i, \quad 0 < p_i \leq C_i.$$

Note that the optimal solutions of (P) correspond to all best responses for leader i such that x^* remains the Wardrop flow. Reformulating the equality constraint in (P) yields $p_i = K^* - a_i x_i^*/z_i - b_i$. The constraints $0 < p_i \leq C_i$ then become (note that $K^* > b_i + p_i^* \geq b_i$ holds) $0 < K^* - a_i x_i^*/z_i - b_i \Leftrightarrow z_i > a_i x_i^*/(K^* - b_i)$ and $K^* - a_i x_i^*/z_i - b_i \leq C_i \Leftrightarrow 1/z_i \geq (K^* - b_i - C_i)/(a_i x_i^*)$. Thus (P) is equivalent to the following problem (with variable z_i):

$$\begin{aligned} \max \quad & \left(K^* - \frac{a_i}{z_i} x_i^* - b_i\right) \cdot x_i^* - \gamma_i z_i \\ \text{s.t.:} \quad & \frac{a_i x_i^*}{K^* - b_i} < z_i, \ \frac{K^* - b_i - C_i}{a_i x_i^*} \leq \frac{1}{z_i}. \end{aligned} \tag{P'}$$

Let f be the objective function of (P′) and consider the derivative $f'(z_i) = a_i x_i^{*2}/(z_i^2) - \gamma_i$. We get that f is strictly increasing for $0 < z_i < \sqrt{a_i/\gamma_i} \cdot x_i^*$ and strictly decreasing for $z_i > \sqrt{a_i/\gamma_i} \cdot x_i^*$. We now distinguish between the cases that $z_i = \sqrt{a_i/\gamma_i} \cdot x_i^*$ is feasible for (P′), or not. As we will see, the former case leads to statement 1. of the lemma, and the latter case to statement 2. Note that in either case, $\overline{x}_i(K^*) = x_i^*$ holds (by statement 1. of Lemma 2).

If $z_i = \sqrt{a_i/\gamma_i} \cdot x_i^*$ is feasible for (P′), it is the unique optimal solution of (P′). But since z_i^* is also optimal for (P′), we get

$$z_i^* = \sqrt{a_i/\gamma_i} \cdot \overline{x}_i(K^*) \text{ and } p_i^* = K^* - a_i \overline{x}_i(K^*)/z_i^* - b_i = K^* - \sqrt{a_i \gamma_i} - b_i.$$

For the profit of leader i, we get

$$\Pi_i(z_i^*, p_i^*) = p_i^* x_i^* - \gamma_i z_i^* = (K^* - \sqrt{a_i \gamma_i} - b_i)\overline{x}_i(K^*) - \gamma_i \sqrt{a_i/\gamma_i}\overline{x}_i(K^*) = f_i^1(K^*).$$

It remains to show that K^* is optimal for (P_i^1). For feasibility, we need $2\sqrt{a_i \gamma_i} + b_i \leq K^* \leq \sqrt{a_i \gamma_i} + b_i + C_i$ and $\overline{x}_i(K^*) > 0$. The last inequality follows directly from $\overline{x}_i(K^*) = x_i^* > 0$. Using this and $\overline{x}_i(K^*) \cdot (K^* - 2\sqrt{a_i \gamma_i} - b_i) = \Pi_i(z_i^*, p_i^*) \geq 0$ yields $K^* \geq 2\sqrt{a_i \gamma_i} + b_i$. Finally, the feasibility of $z_i^* = \sqrt{a_i/\gamma_i} \cdot x_i^*$ for (P′) implies $(K^* - b_i - C_i)/(a_i x_i^*) \leq \sqrt{\gamma_i}/(\sqrt{a_i} x_i^*)$, and thus $K^* \leq \sqrt{a_i \gamma_i} + b_i + C_i$ is fulfilled. Therefore, K^* is feasible for (P_i^1). The optimality follows from Lemma 3 and the optimality of (z_i^*, p_i^*) for (P_i).

Now turn to the case that $z_i = \sqrt{a_i/\gamma_i} \cdot x_i^*$ is not feasible for (P′). We show that statement 2. of the lemma holds. Since (P′) has an optimal solution (namely z_i^*), we get that $0 < a_i x_i^*/(K^* - b_i - C_i) < \sqrt{a_i/\gamma_i} \cdot x_i^*$ holds, and therefore $z_i = a_i x_i^*/(K^* - b_i - C_i)$ is the unique optimal solution for (P′). This shows

$$z_i^* = a_i \overline{x}_i(K^*)/(K^* - b_i - C_i) \text{ and } p_i^* = K^* - a_i \overline{x}_i(K^*)/z_i^* - b_i = C_i.$$

The profit of leader i becomes

$$\begin{aligned} \Pi_i(z_i^*, p_i^*) &= C_i \cdot \overline{x}_i(K^*) - \gamma_i \cdot \frac{a_i \overline{x}_i(K^*)}{K^* - b_i - C_i} = \overline{x}_i(K^*) \cdot \left(C_i - \frac{a_i \gamma_i}{K^* - b_i - C_i}\right) \\ &= f_i^2(K^*). \end{aligned}$$

Since $\overline{x}_i(K^*) = x_i^* > 0$ and the profit is nonnegative, $C_i - a_i\gamma_i/(K^* - b_i - C_i) \geq 0$, thus $K^* \geq a_i\gamma_i/C_i + b_i + C_i$ holds. Finally, $z_i^* = a_i x_i^*/(K^* - b_i - C_i) < \sqrt{a_i/\gamma_i}\, x_i^* \Leftrightarrow K^* > \sqrt{a_i\gamma_i} + b_i + C_i$, which completes the proof since we showed that K^* is a feasible solution of problem (P_i^2) (optimality follows from Lemma 3 and the optimality of (z_i^*, p_i^*) for (P_i)). □

In the next lemma, we analyze the existence of optimal solutions for the problems (P_i^1) and (P_i^2), as well as properties of such solutions.

Lemma 5 *1. If (P_i^1) is feasible, it has a unique optimal solution.*
2. Assume that (P_i^2) is feasible.

- *If $C_i \leq \sqrt{a_i\gamma_i}$, then ($P_i^2$) has a unique optimal solution.*
- *If $C_i > \sqrt{a_i\gamma_i}$, then (P_i^2) has at most one optimal solution.*
- *If K_2^* is optimal for (P_i^2), then $f_i^2(K_2^*) > 0$.*

3. If K_1^* is optimal for (P_i^1) and K_2^* is optimal for (P_i^2), then $f_i^1(K_1^*) < f_i^2(K_2^*)$.

Proof We start with statement 1. of the lemma, so assume that (P_i^1) is feasible. Note that the feasible set I_1 of (P_i^1) either is of the form $I_1 = [2\sqrt{a_i\gamma_i} + b_i, \sqrt{a_i\gamma_i} + b_i + C_i]$, or $I_1 = [2\sqrt{a_i\gamma_i} + b_i, K_i^{\max})$, depending on whether $\sqrt{a_i\gamma_i} + b_i + C_i < K_i^{\max}$ holds or not. Furthermore note that the objective function

$$f_i^1(K) = \left(1 - \sum_{j \in N^+ : b_j + p_j < K} \frac{(K - b_j - p_j)z_j}{a_j}\right) \cdot \left(K - b_i - 2\sqrt{a_i\gamma_i}\right)$$

of (P_i^1) is continuous (over $\mathbb{R}$). From this, we can conclude that (P_i^1) has *at least one* optimal solution: For $I_1 = [2\sqrt{a_i\gamma_i} + b_i, \sqrt{a_i\gamma_i} + b_i + C_i]$, this follows directly from the theorem of Weierstrass (f_i^1 is continuous and I_1 is nonempty and compact). For $I_1 = [2\sqrt{a_i\gamma_i} + b_i, K_i^{\max})$, the theorem of Weierstrass yields that f_i^1 attains its maximum over the closure of I_1, that is, over $[2\sqrt{a_i\gamma_i} + b_i, K_i^{\max}]$. But since $f_i^1(K_i^{\max}) = 0$ $(= f_i^1(2\sqrt{a_i\gamma_i} + b_i))$, and any $K \in (2\sqrt{a_i\gamma_i} + b_i, K_i^{\max})$ fulfills $f_i^1(K) > 0$, the maximum is not attained for $K = K_i^{\max}$, and we conclude that f_i^1 also attains its maximum over I_1. Thus (P_i^1) has an optimal solution for both cases. To complete the proof of statement 1. of the lemma, it remains to show that there is also *at most one* optimal solution. We prove this by showing the following monotonicity behaviour of f_i^1 over I_1: Either f_i^1 is strictly increasing over I_1, or strictly decreasing over I_1, or strictly increasing up to a unique point, and strictly decreasing afterwards. In all three cases, we obviously get the desired statement, namely that (P_i^1) has at most one optimal solution. To prove the described monotonicity behaviour, we distinguish between three cases according to the value of $\min\{b_j + p_j : j \in N^+\}$. The first case is $\sqrt{a_i\gamma_i} + b_i + C_i \leq \min\{b_j + p_j : j \in N^+\}$, which implies $N(K) = \{j \in N^+ : b_j + p_j < K\} = \emptyset$ and $\overline{x}_i(K) = 1$ for all $K \in I_1$. We conclude that $f_i^1(K) = K - b_i - 2\sqrt{a_i\gamma_i}$ is strictly increasing over I_1 (in particular, f_i^1 reaches its unique maximum over I_1 at $K = \sqrt{a_i\gamma_i} + b_i + C_i$). Next, consider the case that

$\min\{b_j + p_j : j \in N^+\} < 2\sqrt{a_i\gamma_i} + b_i$. This implies that $N(K) \neq \emptyset$ for all $K \in I_1$. Note that f_i^1 is twice differentiable on any open interval where $N(K)$ is constant, and the first and second derivative of f_i^1 then are

$$(f_i^1)'(K) = 1 - \sum_{j \in N(K)} \frac{(2K - b_j - p_j - b_i - 2\sqrt{a_i\gamma_i})z_j}{a_j} \quad \text{and}$$
$$(f_i^1)''(K) = -2 \sum_{j \in N(K)} \frac{z_j}{a_j}.$$

Since $N(K) \neq \emptyset$ for all $K \in I_1$, we conclude that for all $K \in I_1$ where $(f_i^1)''(K)$ exists, $(f_i^1)''(K) < 0$ holds. If $N(K)$ is constant on the complete interior of I_1 (that is, on $(2\sqrt{a_i\gamma_i} + b_i, \sqrt{a_i\gamma_i} + b_i + C_i)$ or $(2\sqrt{a_i\gamma_i} + b_i, K_i^{\max})$, depending on the two possible cases for I_1), the desired monotonicity behaviour of f_i^1 over I_1 follows. Otherwise, let $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ denote the different values of $b_j + p_j$, $j \in N^+$ which lie in the interior of I_1. Define $\alpha_0 := 2\sqrt{a_i\gamma_i} + b_i$ and $\alpha_{k+1} := \sup I_1$ (that is, $\alpha_{k+1} = \sqrt{a_i\gamma_i} + b_i + C_i$ or $\alpha_{k+1} = K_i^{\max}$). Then, $N(K)$ is constant on the intervals $(\alpha_{\ell-1}, \alpha_\ell]$ for $\ell \in \{1, \ldots, k+1\}$. For each $\ell \in \{1, \ldots, k\}$, the set $N(K)$ increases immediately after α_ℓ, that is, $N(\alpha_\ell) \subsetneq N(\alpha_\ell + \varepsilon)$ holds for any $\varepsilon > 0$. In particular, $N(\alpha_\ell + \varepsilon) = N(\alpha_\ell) \cup \{j \in N^+ : b_j + p_j = \alpha_\ell\}$ holds for all $0 < \varepsilon \leq \alpha_{\ell+1} - \alpha_\ell$. We now show that for any $\ell \in \{1, \ldots, k\}$, the slope of f_i^1 decreases at α_ℓ, whereby we mean that $(f_i^1)'_+(\alpha_\ell) < (f_i^1)'_-(\alpha_\ell)$ holds, with $(f_i^1)'_+(\alpha_\ell)$ and $(f_i^1)'_-(\alpha_\ell)$ denoting the right and left derivative of f_i^1 at α_ℓ, respectively. This implies the desired monotonicity behaviour of f_i^1 over I_1. Analyzing the left and right derivative of f_i^1 at α_ℓ yields

$$(f_i^1)'_-(\alpha_\ell) = 1 - \sum_{j \in N(\alpha_\ell)} \frac{(2\alpha_\ell - b_j - p_j - b_i - 2\sqrt{a_i\gamma_i})z_j}{a_j} \quad \text{and}$$
$$\begin{aligned}(f_i^1)'_+(\alpha_\ell) &= 1 - \sum_{j \in N(\alpha_\ell) \cup \{j \in N^+ : b_j + p_j = \alpha_\ell\}} \frac{(2\alpha_\ell - b_j - p_j - b_i - 2\sqrt{a_i\gamma_i})z_j}{a_j} \\ &= (f_i^1)'_-(\alpha_\ell) - \sum_{j \in N^+ : b_j + p_j = \alpha_\ell} \frac{(2\alpha_\ell - \alpha_\ell - b_i - 2\sqrt{a_i\gamma_i})z_j}{a_j} \\ &= (f_i^1)'_-(\alpha_\ell) - (\alpha_\ell - b_i - 2\sqrt{a_i\gamma_i}) \cdot \sum_{j \in N^+ : b_j + p_j = \alpha_\ell} \frac{z_j}{a_j}.\end{aligned}$$

Since α_ℓ lies in the interior of I_1, we get $\alpha_\ell > 2\sqrt{a_i\gamma_i} + b_i$ and therefore the desired inequality $(f_i^1)'_+(\alpha_\ell) < (f_i^1)'_-(\alpha_\ell)$, completing the proof for the case $\min\{b_j + p_j : j \in N^+\} < 2\sqrt{a_i\gamma_i} + b_i$. The remaining case is $2\sqrt{a_i\gamma_i} + b_i \leq \min\{b_j + p_j : j \in N^+\} < \sqrt{a_i\gamma_i} + b_i + C_i$, which implies that $N(K) = \emptyset$ holds in I_1 for $K \leq \min\{b_j + p_j : j \in N^+\}$, and $N(K) \neq \emptyset$ holds in I_1 for $K > \min\{b_j + p_j : j \in N^+\}$. We can now obviously combine the arguments of the other two cases to obtain the desired monotonicity behaviour of f_i^1 also for this case. This completes the proof of statement 1. of the lemma.

Now we turn to statement 2., thus we assume that (P_i^2) is feasible. Let I_2 denote the feasible set of (P_i^2). Then, either $I_2 = [a_i\gamma_i/C_i + b_i + C_i, K_i^{\max})$ or $I_2 = (\sqrt{a_i\gamma_i} + b_i + C_i, K_i^{\max})$ holds, depending on whether $C_i < \sqrt{a_i\gamma_i}$ holds or not. But in both cases, I_2 is an interval with *positive* length, so that there exists $K \in I_2$ with $K > a_i\gamma_i/C_i + b_i + C_i$, which implies $f_i^2(K) > 0$. Therefore, if (P_i^2) has an optimal solution K_2^*, we get $f_i^2(K_2^*) > 0$. Furthermore note that the objective function

$$f_i^2(K) = \left(1 - \sum_{j\in N(K)} \frac{(K - b_j - p_j)z_j}{a_j}\right) \cdot \left(C_i - \frac{a_i\gamma_i}{K - b_i - C_i}\right)$$

of (P_i^2) is continuous over $(b_i + C_i, \mathbb{R})$. Using this, we can show that (P_i^2) has *at least one* optimal solution, if $C_i \le \sqrt{a_i\gamma_i}$ holds: Due to the theorem of Weierstrass, f_i^2 attains its maximum over $[a_i\gamma_i/C_i + b_i + C_i, K_i^{\max}]$, the closure of I_2. But since $f_i^2(a_i\gamma_i/C_i + b_i + C_i) = 0 = f_i^2(K_i^{\max})$, and $f_i^2(K) > 0$ for any $K \in (a_i\gamma_i/C_i + b_i + C_i, K_i^{\max})$, the maximum is not attained at $K = a_i\gamma_i/C_i + b_i + C_i$ or $K = K_i^{\max}$, which shows that f_i^2 also attains its maximum over I_2. Thus, if $C_i \le \sqrt{a_i\gamma_i}$ holds, (P_i^2) has at least one optimal solution. To complete the proof of statement 2, it remains to show that (P_i^2) has *at most one* optimal solution (in the general case). As in the proof of statement 1 of the lemma, we prove this by showing that f_i^2 exhibits a certain monotonicity behaviour over I_2, namely: Either f_i^2 is strictly decreasing over I_2, or strictly increasing up to a unique point, and strictly decreasing afterwards. Note that f_i^2 cannot be strictly increasing over I_2, due to the continuity of f_i^2 and the fact that $f_i^2(K_i^{\max}) = 0 < f_i^2(K)$ holds for any K in the interior of I_2. The remaining proof is very similar to the proof of statement 1. First, f_i^2 is twice differentiable on any open interval where $N(K)$ is constant. First and second derivative of f_i^2 then are

$$(f_i^2)'(K) = -\sum_{j\in N(K)} \frac{z_j}{a_j} \cdot \left(C_i - \frac{a_i\gamma_i}{K - b_i - C_i}\right) + \overline{x}_i(K) \cdot \frac{a_i\gamma_i}{(K - b_i - C_i)^2} \quad \text{and}$$

$$(f_i^2)''(K) = -\frac{2a_i\gamma_i}{(K - b_i - C_i)^2} \cdot \left(\sum_{j\in N(K)} \frac{z_j}{a_j} + \frac{\overline{x}_i(K)}{K - b_i - C_i}\right).$$

Since $\overline{x}_i(K) > 0$ for all $K \in I_2$, we conclude that for all $K \in I_2$ where $(f_i^2)''(K)$ exists, $(f_i^2)''(K) < 0$ holds. If $N(K)$ is constant on the complete interior of I_2, the desired monotonicity behaviour of f_i^2 over I_2 follows, otherwise let $\beta_1 < \beta_2 < \cdots < \beta_k$ denote the different values of $b_j + p_j$, $j \in N^+$ which lie in the interior of I_2. We show that the slope of f_i^2 decreases at β_ℓ, i.e. $(f_i^2)'_+(\beta_\ell) < (f_i^2)'_-(\beta_\ell)$ holds, which implies the desired monotonicity behaviour of f_i^2 over I_2. Analyzing the left and right derivative yields

$$(f_i^2)'_-(\beta_\ell) = -\sum_{j\in N(\beta_\ell)} \frac{z_j}{a_j} \cdot \left(C_i - \frac{a_i\gamma_i}{\beta_\ell - b_i - C_i}\right) + \overline{x}_i(\beta_\ell) \cdot \frac{a_i\gamma_i}{(\beta_\ell - b_i - C_i)^2}$$

and

$$
\begin{aligned}
(f_i^2)'_+(\beta_\ell) = &- \sum_{j \in N(\beta_\ell) \cup \{j \in N^+ : b_j + p_j = \beta_\ell\}} \frac{z_j}{a_j} \cdot \left(C_i - \frac{a_i \gamma_i}{\beta_\ell - b_i - C_i} \right) \\
&+ \bar{x}_i(\beta_\ell) \cdot \frac{a_i \gamma_i}{(\beta_\ell - b_i - C_i)^2} \\
&= (f_i^2)'_-(\beta_\ell) - \sum_{j \in N^+ : b_j + p_j = \beta_\ell} \frac{z_j}{a_j} \cdot \left(C_i - \frac{a_i \gamma_i}{\beta_\ell - b_i - C_i} \right)
\end{aligned}
$$

Since β_ℓ lies in the interior of I_2, we get $C_i - a_i\gamma_i/(\beta_\ell - b_i - C_i) > 0$, and thus the desired inequality $(f_i^2)'_+(\beta_\ell) < (f_i^2)'_-(\beta_\ell)$, completing the proof of statement 2. of the lemma.

Finally we show statement 3. Assume that K_1^* and K_2^* are the optimal solutions of (P_i^1) and (P_i^2). Since (P_i^1) and (P_i^2) have to be feasible, $\sqrt{a_i\gamma_i} \le C_i$ and $\sqrt{a_i\gamma_i} + b_i + C_i < K_i^{\max}$ holds. This implies that the feasible set of (P_i^1) is $I_1 = [2\sqrt{a_i\gamma_i} + b_i, \sqrt{a_i\gamma_i} + b_i + C_i]$ and the feasible set of (P_i^2) is $I_2 = (\sqrt{a_i\gamma_i} + b_i + C_i, K_i^{\max})$. Let $\bar{K} := \sqrt{a_i\gamma_i} + b_i + C_i$. Then, $f_i^1(\bar{K}) = f_i^2(\bar{K})$ holds. If additionally the slope of f_i^1 in $\bar{K}$ is greater than or equal to the slope of f_i^2 in $\bar{K}$, whereby we mean that $(f_i^1)'_-(\bar{K}) \ge (f_i^2)'_+(\bar{K})$ holds, we get $f_i^1(K_1^*) = f_i^1(\bar{K}) < f_i^2(K_2^*)$ from our analysis of f_i^1 and f_i^2 in the proofs of the statements 1. and 2. (note that f_i^2 is strictly increasing on $(\sqrt{a_i\gamma_i} + b_i + C_i, K_2^*]$). The remaining inequality for the slopes follows from

$$
\begin{aligned}
(f_i^1)'_-(\bar{K}) &= 1 - \sum_{j \in N(\bar{K})} \frac{(2\bar{K} - b_j - p_j - b_i - 2\sqrt{a_i\gamma_i}) z_j}{a_j} \\
&= 1 - \sum_{j \in N(\bar{K})} \frac{(2C_i + b_i - b_j - p_j) z_j}{a_j} \qquad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
(f_i^2)'_+(\bar{K}) = &- \sum_{j \in N(\bar{K}) \cup \{j \in N^+ : b_j + p_j = \bar{K}\}} \frac{z_j}{a_j} \left(C_i - \frac{a_i\gamma_i}{\bar{K} - b_i - C_i} \right) \\
&+ \bar{x}_i(\bar{K}) \frac{a_i\gamma_i}{(\bar{K} - b_i - C_i)^2} \\
= &- \sum_{j \in N(\bar{K}) \cup \{j \in N^+ : b_j + p_j = \bar{K}\}} \frac{z_j}{a_j} \cdot (C_i - \sqrt{a_i\gamma_i}) \\
&+ 1 - \sum_{j \in N(\bar{K})} \frac{(\bar{K} - b_j - p_j) z_j}{a_j} \\
= &\, 1 - \sum_{j \in N(\bar{K})} \frac{(\bar{K} - b_j - p_j + C_i - \sqrt{a_i\gamma_i}) z_j}{a_j} \\
&- \sum_{j \in N^+ : b_j + p_j = \bar{K}} \frac{z_j (C_i - \sqrt{a_i\gamma_i})}{a_j}
\end{aligned}
$$

Table 1 Characterization of BR_i

$\{(z_i, p_i)\} = \mathrm{BR}_i$	Conditions
$\emptyset$	$z_{-i} = 0$
$\{(0, p_i) : 0 \leq p_i \leq C_i\}$	$z_{-i} \neq 0$, (P_i^1) and (P_i^2) infeasible
$\left\{\left(\frac{a_i \cdot \bar{x}_i(K_2^*)}{K_2^* - b_i - C_i}, C_i\right)\right\}$	$z_{-i} \neq 0$, (P_i^2) has an optimal solution
$\left\{\left(\sqrt{a_i/\gamma_i} \cdot \bar{x}_i(K_1^*), K_1^* - \sqrt{a_i\gamma_i} - b_i\right)\right\}$	$z_{-i} \neq 0$, (P_i^1) feasible, (P_i^2) has no optimal solution

$$
\begin{aligned}
&= 1 - \sum_{j \in N(\bar{K})} \frac{(2C_i + b_i - b_j - p_j)z_j}{a_j} - \sum_{j \in N^+ : b_j + p_j = \bar{K}} \frac{z_j(C_i - \sqrt{a_i\gamma_i})}{a_j} \\
&= (f_i^1)'_-(\bar{K}) - \sum_{j \in N^+ : b_j + p_j = \bar{K}} \frac{z_j(C_i - \sqrt{a_i\gamma_i})}{a_j} \leq (f_i^1)'_-(\bar{K}),
\end{aligned}
$$

where the inequality follows from $\sqrt{a_i\gamma_i} \leq C_i$. □

3.3 The characterization

The following theorem provides a complete characterization of the best response correspondence. We will make use of this characterization several times during the rest of the paper.

Theorem 2 *For a leader $i \in N$ and fixed strategies $s_{-i} = (z_{-i}, p_{-i}) \in S_{-i}$ of the other leaders, the set $BR_i = BR_i(s_{-i})$ of best responses of leader i to s_{-i} is given as indicated in Table 1, where the first column contains BR_i and the second column contains the conditions on s_{-i} under which BR_i has the stated form. For $j = 1, 2$, K_j^* denotes the unique optimal solution of problem (P_i^j), if this problem has an optimal solution.*

Furthermore, if $BR_i(s_{-i})$ consists of a unique *best response $s_i = (z_i, p_i)$ of leader i to s_{-i}, we get $z_i > 0$ and $\Pi_i(s_i, s_{-i}) > 0$.*

Proof Note that if $\mathrm{BR}_i(s_{-i})$ consists of a *unique* best response $s_i = (z_i, p_i)$ of leader i to s_{-i}, then $z_i > 0$ and $\Pi_i(s_i, s_{-i}) > 0$ hold: Otherwise, any strategy $(z_i', p_i') \in \{0\} \times [0, C_i]$ is a best response, too, contradicting the uniqueness assumption.

Now turn to the proof of the characterization. We show that the case distinction covers all possible cases, and that the given representation for BR_i is correct for each case. If $z_{-i} = 0$, Lemma 1 shows $\mathrm{BR}_i = \emptyset$. For the rest of the proof, assume $z_{-i} \neq 0$. Then, leader i has at least one best response to s_{-i}, since BR_i can be described as the set of optimal solution of the problem (P_i), and this problem has an optimal solution (as shown in the beginning of Sect. 3.2). If (P_i^1) and (P_i^2) are both infeasible, Lemma 4 implies that each best response (z_i, p_i) fulfills $z_i = 0$. Therefore, $\mathrm{BR}_i = \{0\} \times [0, C_i]$. For the remaining proof, assume that at least one of (P_i^1) and (P_i^2) is feasible.

First consider the case that (P_i^2) has an optimal solution. It follows from 2. of Lemma 5 that the solution is unique, and, if K_2^* denotes this solution, that $f_i^2(K_2^*) > 0$. Let (z_i, p_i) be an arbitrary best response of player i to s_{-i}. We need to show that $(z_i, p_i) = (a_i \bar{x}_i(K_2^*)/(K_2^* - b_i - C_i), C_i)$ holds. First, statement 2. of Lemma 3 shows that $\Pi_i(z_i, p_i) \geq f_i^2(K_2^*) > 0$. Thus $z_i > 0$ holds, since $z_i = 0$ yields a profit of 0. Note that either (P_i^1) is infeasible, or it has a unique optimal solution K_1^* with $f_i^1(K_1^*) < f_i^2(K_2^*) \leq \Pi_i(z_i, p_i)$ (see 1. and 3. of Lemma 5). In both cases, Lemma 4 yields $(z_i, p_i) = (a_i \bar{x}_i(K_2^*)/(K_2^* - b_i - C_i), C_i)$.

Now assume that (P_i^2) does not have an optimal solution (either (P_i^2) is infeasible, or it is feasible, but the maximum is not attained). We first show that (P_i^1) is feasible. If (P_i^2) is infeasible, (P_i^1) is feasible since we assumed that at least one of the two problems is feasible. Otherwise (P_i^2) is feasible, but does not have an optimal solution. Then, $C_i > \sqrt{a_i \gamma_i}$ follows from 2. of Lemma 5, and $\sqrt{a_i \gamma_i} + b_i + C_i < K_i^{\max}$ follows since (P_i^2) is feasible. Together, $2\sqrt{a_i \gamma_i} + b_i < \sqrt{a_i \gamma_i} + b_i + C_i < K_i^{\max}$ holds, showing that (P_i^1) is feasible. By 1. of Lemma 5 we then get that (P_i^1) has a unique optimal solution K_1^*. Furthermore, each best response (z_i, p_i) with $z_i > 0$ fulfills $(z_i, p_i) = (\sqrt{a_i/\gamma_i} \cdot \bar{x}_i(K_1^*), K_1^* - \sqrt{a_i \gamma_i} - b_i)$ (see Lemma 4). To complete the proof, we need to show that there is no best response (z_i, p_i) with $z_i = 0$. This follows from Lemma 3 if $f_i^1(K_1^*) > 0$. Thus it remains to show that $f_i^1(K_1^*) > 0$ holds. Assume, by contradiction, that $f_i^1(K_1^*) = 0$. This implies that $K_1^* = 2\sqrt{a_i \gamma_i} + b_i$ is the only feasible solution for (P_i^1), which in turn yields $\sqrt{a_i \gamma_i} = C_i$ and $\sqrt{a_i \gamma_i} + b_i + C_i < K_i^{\max}$. But this implies that (P_i^2) is feasible and has an optimal solution (by 2. of Lemma 5), contradicting our assumption that (P_i^2) does not have an optimal solution. □

3.4 Discussion

We now briefly discuss consequences of the characterization of the best reponses with respect to applying Kakutani's fixed point theorem (see [22]). Kakutani's theorem in particular requires that for each leader i and each vector $s_{-i} = (z_{-i}, p_{-i})$ of strategies of the other leaders, the set $\mathrm{BR}_i(s_{-i})$ of best responses is nonempty and convex. But as we have seen in Lemma 1, the set $\mathrm{BR}_i(s_{-i})$ can be empty, namely if $z_{-i} = 0$. On the other hand, a profile with $z_{-i} = 0$ for some leader i will of course never be a PNE.

A first natural approach to overcome the problem of empty best responses is the following. Given a strategy profile $s = (z, p)$ such that $z_{-i} = 0$ for some leader i, redefine, for each such leader i, the set $\mathrm{BR}_i(s_{-i})$ by some *suitable* nonempty convex set. "Suitable" here means that the correspondence BR_i has a closed graph, and at the same time, s must not be a fixed point of the global best response correspondence BR (where $\mathrm{BR}(s) := \{s' \in S : s_i' \in \mathrm{BR}_i(s_{-i}) \text{ for each } i \in N\}$). But unfortunately, these two goals are not compatible: For the strategy profile $s = (z, p)$ with $(z_i, p_i) = (0, C_i)$ for all leaders i, the closed graph property requires $(0, C_i) \in \mathrm{BR}_i(s_{-i})$ for all i, which implies that s is a fixed point of BR.

Another intuitive idea is to consider a game in which each leader has an initial capacity of some $\varepsilon > 0$. If this game has a PNE for each ε, the limit for ε going to zero should be a PNE for our original Stackelberg pricing game. For the game with at least ε capacity, one can also characterize the best response correspondences (now by optimal solutions of *three* optimization problems), but the main problem is that an analogue of Lemma 5 may not hold anymore. As a consequence, it is not clear if the best responses are always convex, and we again do not know how to apply Kakutani's theorem. Instead, we show existence of PNE by using a result of McLennan et al. [29], see Sect. 4.[4]

4 Existence of equilibria

In this section, we show that each Stackelberg pricing game has a PNE. A frequently used tool to show existence of PNE is Kakutani's fixed point theorem. But, as discussed in Sect. 3.4, we cannot directly apply this result to show existence of PNE. Furthermore, the existence theorem of [31] can also not be used, since a Stackelberg pricing game is not quasiconcave in general (it is not difficult to construct instances which are not quasiconcave). Instead, we turn to another existence result due to McLennan et al. [29]. They introduced a concept called *C-security* and they showed that if the game is C-secure at each strategy profile which is not a PNE, then a PNE exists. Informally, the game is C-secure at a strategy profile s if there is a vector $\alpha \in \mathbb{R}^n$ satisfying the following two properties: First, each leader i has some securing strategy for α_i which is robust to small deviations of the other leaders, i.e. leader i always achieves a profit of at least α_i by playing this strategy even if the other leaders slightly deviate from their strategies in s_{-i}. The second property requires for each slightly perturbed strategy profile s' resulting from s, that there is at least one leader i such that her perturbed strategy s_i' can (in some sense) be strictly separated from all strategies achieving a profit of α_i, so in particular from all her securing strategies. One can think of leader i being "not happy" with her perturbed strategy s_i' since she could achieve a higher profit. This already indicates the connection between a strategy profile which is not a PNE, and C-security. We will see that for certain strategy profiles, α_i can be chosen as the profit that leader i gets by playing a best response to s_{-i}.[5] Then, leader i's securing strategies for α_i are related to her set of best responses, and we need to "strictly separate" these best responses from s_i. At this point, our characterization of best responses in Theorem 2 becomes useful.

We now formally describe McLennan et al.'s result in our context. First of all, note that they consider games with compact convex strategy sets and bounded profit functions. In a Stackelberg pricing game, the strategy set $S_i = \{(z_i, p_i) : 0 \leq z_i, 0 \leq p_i \leq C_i\}$ of leader i is not compact a priori. But since z_i will never be larger than C_i/γ_i in any best response, and thus in any PNE (see the discussion at the beginning of Sect. 3.2), we can redefine $S_i := \{(z_i, p_i) : 0 \leq z_i \leq C_i/\gamma_i, 0 \leq p_i \leq C_i\}$ without changing the set of PNE of the game, for any leader i. Furthermore, this also does not change the best responses, so Theorem 2 continues to hold. Using the redefined

[4] Perhaps interesting, McLennan et al. identify a non-trivial restriction of the players' non-equilibrium strategies so that they can eventually apply Kakutani's theorem.

[5] More precisely, we need to choose α_i slightly smaller than the profit of a best response.

strategies, for any leader i and any strategy profile s, the profit of leader i is bounded by $-C_i \leq \Pi_i(s) \leq C_i$. For a strategy profile $s \in S$, leader $i \in N$ and $\alpha_i \in \mathbb{R}$ let

$$B_i(s, \alpha_i) := \{s_i' \in S_i : \Pi_i(s_i', s_{-i}) \geq \alpha_i\} \text{ and } C_i(s, \alpha_i) := \operatorname{conv} B_i(s, \alpha_i),$$

where $\operatorname{conv} B_i(s, \alpha_i)$ denotes the convex hull of $B_i(s, \alpha_i)$.

Definition 1 A leader i can *secure a profit* $\alpha_i \in \mathbb{R}$ on $S' \subseteq S$, if there is some $s_i \in S_i$ such that $s_i \in B_i(s', \alpha_i)$ for all $s' \in S'$. We say that leader i can *secure* α_i *at* $s \in S$, if she can secure α_i on $U \cap S$ for some open set U with $s \in U$.

Definition 2 The game is *C-secure on* $S' \subseteq S$, if there is an $\alpha \in \mathbb{R}^n$ such that the following conditions hold:

(i) Every leader i can secure α_i on S'.
(ii) For any $s' \in S'$, there exists some leader i with $s_i' \notin C_i(s', \alpha_i)$.

The game is *C-secure at* $s \in S$, if it is C-secure on $U \cap S$ for some open set U with $s \in U$.

Theorem 3 (Proposition 2.7 in [29]) *If the game is C-secure at each* $s \in S$ *that is not a PNE, then the game has a PNE.*

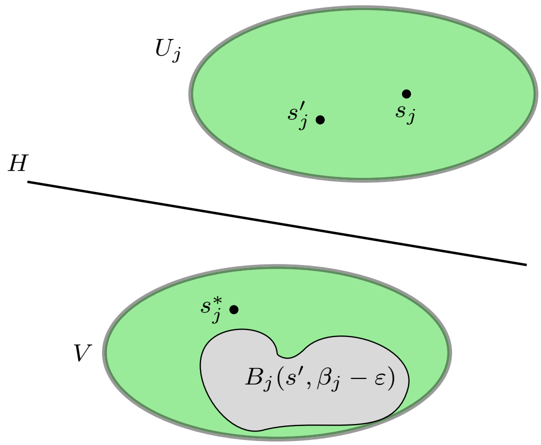
We now turn to Stackelberg pricing games and show the existence of a PNE by using Theorem 3, i.e., we show that if a given strategy profile $s = (z, p)$ is not a PNE, then the game is C-secure at s. To this end, we distinguish between the two cases that there are at least two leaders i with $z_i > 0$ (Lemma 6), or not (Lemma 7). Both lemmata together then imply the desired existence result. Note that the mentioned case distinction is equivalent to the case distinction that each leader i has a best response for s_{-i}, or there is at least one leader i with $\mathrm{BR}_i(s_{-i}) = \emptyset$ (see Theorem 2).

We start with the case that all best responses exist. The proof of the following lemma follows an argument in [29, p. 1647f] where McLennan et al. show that Theorem 3 implies the existence result of [30].

Lemma 6 *Let* $s = (z, p) \in S$ *be a strategy profile which is not a PNE. Assume that there are at least two leaders* $i \in N$ *such that* $z_i > 0$ *holds. Then the game is C-secure at* s.

Proof We first introduce some notation used in this proof. Let $S' \subseteq S$ be a subset of the strategy profiles and $i \in N$. By $S_i' \subseteq S_i$, we denote the projection of S' into S_i, the set of leader i's strategies, and $S_{-i}' \subseteq S_{-i}$ denotes the projection of S' into $S_{-i} = \times_{j \in N \setminus \{i\}} S_j$, the set of strategies of the other leaders. Note that since z has at least two positive entries z_i, all strategy profiles $s' = (z', p')$ in a sufficiently small open neighbourhood of s also have at least two entries $z_i' > 0$. In the following, whenever we speak of an open set U containing s, we implicitly require U small enough to fulfill this property. Furthermore, since it is clear that we are only interested in the elements of U which are strategy profiles, we simply write U instead of $U \cap S$. Consequently, $s' \in U$ denotes a strategy profile contained in U. Now we turn to the actual proof.

Fig. 1 Illustration of the proof construction for the case $B_j(s, \beta_j) = \{s_j^*\}$. Note that it is not necessary that $s_j^* \in B_j(s', \beta_j - \varepsilon)$



Since we assumed that at least two leaders have positive capacity at $s = (z, p)$, we get that $z_{-i} \neq 0$ holds for each leader i. Thus, by Theorem 2, each leader i has a best response to s_{-i}, and the set of best responses either is a singleton, or consists of all strategies $(0, p_i)$ for $0 \leq p_i \leq C_i$. Since s is not a PNE, there is at least one leader j such that $s_j = (z_j, p_j)$ is not a best response, i.e. either $s_j \neq s_j^*$ for the unique best response s_j^*, or $z_j > 0$, and all best responses $s_j^* = (z_j^*, p_j^*)$ fulfill $z_j^* = 0$. In both cases it is clear that there is a hyperplane H which strictly separates s_j from the set of best responses to s_{-j}.

Now turn to the properties in Definition 2. For each leader i, let s_i^* be a best response of leader i to s_{-i} and $\beta_i := \Pi_i(s_i^*, s_{-i})$. We know that Π_i is continuous at (s_i^*, s_{-i}) for each leader i (Theorem 1). Therefore, for each $\varepsilon > 0$, there is an open set $U(\varepsilon) \supset \{s\}$ with $\Pi_i(s_i^*, s'_{-i}) \geq \beta_i - \varepsilon$ for each $s' \in U(\varepsilon)$ and each leader i. That is, each leader i can secure $\beta_i - \varepsilon$ on $U(\varepsilon)$. Now turn to the second property of Definition 2 and consider leader j. We show that there is an $\varepsilon > 0$ and an open set $U \subseteq U(\varepsilon)$ containing s, such that for each $s' \in U$, the hyperplane H (which strictly separates s_j and $B_j(s, \beta_j)$) also strictly separates s'_j and $B_j(s', \beta_j - \varepsilon)$, thus $s'_j \notin C_j(s', \beta_j - \varepsilon)$ (see Fig. 1 for illustration). Since each leader i can secure $\beta_i - \varepsilon$ on $U \subseteq U(\varepsilon)$, both properties of Definition 2 are fulfilled, completing the proof.

To this end, choose an open set V containing $B_j(s, \beta_j)$ such that H strictly separates s_j and V. Since $S_j \setminus V$ is a compact set and Π_j is continuous at $(\widetilde{s}_j, s_{-j})$ for all $\widetilde{s}_j \in S_j \setminus V$ (by Theorem 1), we get that $f(s_{-j}) := \max\{\Pi_j(\widetilde{s}_j, s_{-j}) : \widetilde{s}_j \in S_j \setminus V\}$ exists. Furthermore, since $B_j(s, \beta_j) \subset V$, we get $f(s_{-j}) < \beta_j$. Let $0 < \varepsilon < \beta_j - f(s_{-j})$, thus $f(s_{-j}) < \beta_j - \varepsilon$. Note that if we consider, for an open neighbourhood $\bar{U}$ of s and for fixed $s'_{-j} \in \bar{U}_{-j}$, the problem of maximizing $\Pi_j(\widetilde{s}_j, s'_{-j})$ subject to $\widetilde{s}_j \in S_j \setminus V$, Berge's theorem of the maximum ([5]) yields that $f(s'_{-j}) := \max\{\Pi_j(\widetilde{s}_j, s'_{-j}) : \widetilde{s}_j \in S_j \setminus V\}$ is a continuous function. Using the continuity of f, there is an open set $U \subseteq U(\varepsilon)$ containing s such that $f(s'_{-j}) < \beta_j - \varepsilon$ for all $s'_{-j} \in U_{-j}$. Additionally, let U be small enough such that H strictly separates U_j and V. Now we have the desired properties: For each $s' \in U$ and for each $\widetilde{s}_j \in S_j \setminus V$, we get $\Pi_j(\widetilde{s}_j, s'_{-j}) < \beta_j - \varepsilon$, thus $B_j(s', \beta_j - \varepsilon) \subset V$. Since $s'_j \in U_j$ and H strictly separates U_j and V, we get that H strictly separates s'_j and $B_j(s', \beta_j - \varepsilon)$, as desired. □

It remains to analyze the strategy profiles $s = (z, p)$ with at most one positive z_i. Note that these profiles cannot be PNE.

Lemma 7 *Let $s = (z, p) \in S$ be a strategy profile such that $z_i > 0$ holds for at most one leader i. Then the game is C-secure at s.*

Proof We distinguish between the two cases that there is a leader with positive capacity, or all capacities are zero.

In the former case, assume that $z_i > 0$ for leader i, and $z_j = 0$ for all $j \neq i$ hold. Choose $\alpha_i \in (C_i - \gamma_i z_i, C_i)$ and $0 < \varepsilon < \min\{z_i, (\alpha_i + \gamma_i z_i - C_i)/\gamma_i\}$. Then, there is an open set U containing s such that leader i can secure α_i on $U \cap S =: S'$ (note that by choosing U sufficiently small, leader i can secure each profit $< C_i$ on S') *and* $|z'_i - z_i| < \varepsilon$ holds for each $s' = (z', p') \in S'$. For $j \neq i$, set $\alpha_j := 0$. It is clear that each leader $j \neq i$ can secure $\alpha_j = 0$ on S' (by any strategy with zero capacity). In this way, property (i) of C-security is fulfilled. For property (ii), let $s' = (z', p') \in S'$. We show that $s'_i \notin C_i(s', \alpha_i)$ holds. To this end, note that any strategy $s^*_i = (z^*_i, p^*_i) \in B_i(s', \alpha_i)$, i.e., with $\Pi_i(s^*_i, s'_{-i}) \geq \alpha_i$, fulfills $z^*_i \leq z_i - \varepsilon$, since, for $z^*_i > z_i - \varepsilon > 0$, we get $\Pi_i(s^*_i, s'_{-i}) = x_i(s^*_i, s'_{-i}) p^*_i - \gamma_i z^*_i \leq C_i - \gamma_i z^*_i < C_i - \gamma_i(z_i - \varepsilon) < \alpha_i$, where the last inequality follows from the choice of ε. Clearly, any strategy in $C_i(s', \alpha_i)$, i.e., any convex combination of strategies in $B_i(s', \alpha_i)$, then also has this property. Since $z'_i > z_i - \varepsilon$, we get $(z'_i, p'_i) \notin C_i(s', \alpha_i)$, as desired. Thus we showed that the game is C-secure at s if one leader has positive capacity.

Now turn to the case that $z_i = 0$ for all $i \in N$. We distinguish between two further subcases, namely that there is a leader i with $p_i < C_i$, or all prices are at their upper bounds. In the former case, let $i \in N$ with $p_i < C_i$, and choose α_i with $p_i < \alpha_i < C_i$. There is an open set U containing s such that leader i can secure α_i on $U \cap S =: S'$ *and* $p'_i < \alpha_i$ holds for each $s' = (z', p') \in S'$. By setting $\alpha_j := 0$ for all $j \neq i$, property (i) of C-security is fulfilled. For property (ii), let $s' = (z', p') \in S'$. We show that $s'_i \notin C_i(s', \alpha_i)$ holds. Our assumptions about S' yield $p'_i < \alpha_i$. On the other hand, any strategy $s^*_i = (z^*_i, p^*_i)$ with $\Pi_i(s^*_i, s'_{-i}) \geq \alpha_i$ obviously fulfills $p^*_i > \alpha_i$. In particular, this holds for any strategy in $C_i(s', \alpha_i)$, thus showing that $s'_i \notin C_i(s', \alpha_i)$. We conclude that the game is C-secure at s for the case that all z_i are zero *and* there is a leader i with $p_i < C_i$.

It remains to consider the case that $(z_i, p_i) = (0, C_i)$ holds for all leaders i. For each leader i, choose α_i with $(1 - \frac{a_i}{2(a_i + C_i)})C_i < \alpha_i < C_i$. Note that this implies

$$1/2 < 1/2 + (C_i - \alpha_i)/a_i < \alpha_i/C_i. \tag{1}$$

There is an open set U containing s such that each leader i can secure α_i on $U \cap S =: S'$ *and* $z'_i < 1$ holds for each $s' = (z', p') \in S'$. Thus, property (i) of C-security is fulfilled. For property (ii), let $s' = (z', p') \in S'$. In the following, $s^*_i = (z^*_i, p^*_i)$ denotes a strategy of leader i with $\Pi_i(s^*_i, s'_{-i}) \geq \alpha_i > 0$. We say that s^*_i *achieves a profit of at least* α_i. Obviously $z^*_i > 0$ and $p^*_i > \alpha_i$. Furthermore, $\Pi_i(s^*_i, s'_{-i}) =$

$x_i(s_i^*, s'_{-i})p_i^* - \gamma_i z_i^* \geq \alpha_i$ implies

$$x_i(s_i^*, s'_{-i}) \geq (\alpha_i + \gamma_i z_i^*)/p_i^* > \alpha_i/C_i > 1/2, \tag{2}$$

where the last inequality is due to (1). If $z'_i = 0$ holds for a leader i, then $s'_i \notin C_i(s', \alpha_i)$ holds, since any strategy (z_i^*, p_i^*) achieving a profit of at least $\alpha_i > 0$ fulfills $z_i^* > 0$. Thus we can assume in the following that $z'_i > 0$ holds for all leaders i. Then, since $n \geq 2$, there is at least one leader i with $x_i(s') \leq 1/2$. We now show that $s'_i \notin C_i(s', \alpha_i)$ holds. If $z'_i = 0$ or $p'_i \leq \alpha_i$ holds, $s'_i \notin C_i(s', \alpha_i)$ follows, since $z_i^* > 0$ and $p_i^* > \alpha_i$ hold for any strategy (z_i^*, p_i^*) achieving a profit of at least α_i. Thus we can assume in the following that $z'_i > 0$ and $p'_i > \alpha_i$ hold. If $x_i(s') = 0$, the Wardrop equilibrium conditions yield $K(s') \leq b_i + p'_i$. Then, any strategy $\bar{s}_i = (\bar{z}_i, \bar{p}_i)$ with $\bar{p}_i \geq p'_i$ yields $x(\bar{s}_i, s'_{-i}) = x(s')$, and thus $x_i(\bar{s}_i, s'_{-i}) = 0$ and $\Pi_i(\bar{s}_i, s'_{-i}) \leq 0 < \alpha_i$ hold. Therefore, $p_i^* < p'_i$ holds for any strategy (z_i^*, p_i^*) achieving a profit of at least α_i, and $s'_i \notin C_i(s', \alpha_i)$ follows. We can thus assume in the following that $x_i(s') > 0$ holds. Summarizing, we can assume that the following inequalities are fulfilled:

$$0 < x_i(s'_i, s'_{-i}) \leq 1/2, \quad \alpha_i < p'_i \leq C_i \quad \text{and} \quad 0 < z'_i < 1. \tag{3}$$

We now show that each strategy $s_i^* = (z_i^*, p_i^*)$ with $\Pi_i(s_i^*, s'_{-i}) \geq \alpha_i$ fulfills $z_i^* > z'_i$, showing that $s'_i \notin C_i(s', \alpha_i)$ and completing the proof. Assume, by contradiction, that there is a strategy $s_i^* = (z_i^*, p_i^*)$ which achieves a profit of at least α_i and fulfills $z_i^* \leq z'_i$. For any strategy $\tilde{s}_i \in S_i$, write $x(\tilde{s}_i) := x(\tilde{s}_i, s'_{-i})$ and $K(\tilde{s}_i) := K(\tilde{s}_i, s'_{-i})$. Now consider the strategy $\tilde{s}_i := (z'_i, \alpha_i)$. Assume, for the moment, that

$$K(\tilde{s}_i) \leq K(s_i^*) < K(s'_i) \tag{4}$$

holds (we prove (4) below). Using $K(\tilde{s}_i) < K(s'_i)$ then implies $a_i x_i(\tilde{s}_i)/z'_i + b_i + \alpha_i < a_i x_i(s'_i)/z'_i + b_i + p'_i$. Reformulating this inequality and using (3) and (1) then yields

$$x_i(\tilde{s}_i) < z'_i(p'_i - \alpha_i)/a_i + x_i(s'_i) < (C_i - \alpha_i)/a_i + 1/2 < \alpha_i/C_i. \tag{5}$$

The inequality $K(\tilde{s}_i) \leq K(s_i^*)$ from (4) implies $x_j(\tilde{s}_i) \leq x_j(s_i^*)$ for all leaders $j \neq i$. Therefore, $x_i(\tilde{s}_i) \geq x_i(s_i^*)$ holds. Using $x_i(s_i^*) > \alpha_i/C_i$ from (2) now leads to $x_i(\tilde{s}_i) > \alpha_i/C_i$, which contradicts (5). To complete the proof, it remains to show (4). The property $K(s_i^*) < K(s'_i)$ holds since $K(s_i^*) \geq K(s'_i)$ would imply $x_j(s_i^*) \geq x_j(s'_i)$ for all leaders $j \neq i$, and thus $x_i(s_i^*) \leq x_i(s'_i)$, but we know from (2) and (3) that $x_i(s_i^*) > 1/2 \geq x_i(s'_i)$. To prove the other inequality in (4), assume, by contradiction, that $K(\tilde{s}_i) > K(s_i^*)$. This implies $x_j(\tilde{s}_i) \geq x_j(s_i^*)$ for all leaders $j \neq i$, and thus $x_i(\tilde{s}_i) \leq x_i(s_i^*)$. Together with $z_i^* \leq z'_i$ and $p_i^* > \alpha_i$, this leads to the following contradiction, and finally completes the proof:

$$K(\tilde{s}_i) \leq a_i x_i(\tilde{s}_i)/z'_i + b_i + \alpha_i < a_i x_i(s_i^*)/z_i^* + b_i + p_i^* = K(s_i^*) < K(\tilde{s}_i).$$

□

Using Theorem 3 together with the Lemmata 6 and 7 now yields the existence of a PNE:

Theorem 4 *Every Stackelberg pricing game has a pure Nash equilibrium.*

Note here that in any PNE (z, p), there are at least two leaders i with $z_i > 0$.

5 Uniqueness of equilibria

As we have seen in the last section, a Stackelberg pricing game always has a PNE. In this section we show that this equilbrium is *essentially* unique. With *essentially* we mean that if (z, p) and (z', p') are two different PNE, and $i \in N$ is a leader such that $(z_i, p_i) \neq (z_i', p_i')$, then $z_i = z_i' = 0$ holds (and thus $p_i \neq p_i'$).

For a PNE $s = (z, p)$, denote by $N^+(z, p) := \{i \in N : z_i > 0\}$ the set of leaders with positive capacity (note that $|N^+(z, p)| \geq 2$ and $N^+(z, p) = \{i \in N : x_i(s) > 0\}$). For $i \in N^+(z, p)$, let $(\mathrm{P}_i^1)(s_{-i})$ and $(\mathrm{P}_i^2)(s_{-i})$ be the two auxiliary problems from Sect. 3.[6] By Lemma 4, the routing cost $K(z, p)$ is an optimal solution of either $(\mathrm{P}_i^1)(s_{-i})$ or $(\mathrm{P}_i^2)(s_{-i})$. We denote by $N_1^+(z, p)$ the set of leaders $i \in N^+(z, p)$ such that $K(z, p)$ is an optimal solution of $(\mathrm{P}_i^1)(s_{-i})$, and $N_2^+(z, p)$ contains the leaders $i \in N^+(z, p)$ such that $K(z, p)$ is an optimal solution of $(\mathrm{P}_i^2)(s_{-i})$. Thus $N^+(z, p) = N_1^+(z, p) \mathbin{\dot\cup} N_2^+(z, p)$. Throughout this section, we use the simplified notation $N' \setminus i$ instead of $N' \setminus \{i\}$ for any subset $N' \subseteq N$ of leaders and $i \in N'$.

Note that the proofs in this section are similar to the proofs that [21] use to derive their uniqueness results. However, since our model includes price caps, some new ideas are required, in particular the decomposition of $N^+(z, p)$ in $N_1^+(z, p) \mathbin{\dot\cup} N_2^+(z, p)$.

We first derive further necessary equilibrium conditions (by using the KKT conditions) which will become useful in the following analysis.

Lemma 8 *Let $s = (z, p)$ be a PNE with $x := x(z, p)$ and $K := K(z, p)$. Let $i \in N^+ := N^+(z, p)$. If $p_i < C_i$ holds, then $z_i = \sqrt{a_i/\gamma_i}x_i$, $p_i = \frac{x_i}{\sum_{j \in N^+ \setminus i} z_j/a_j} + \sqrt{a_i\gamma_i}$, and if $p_i = C_i$, then $z_i = \frac{a_i x_i}{K - b_i - C_i}$ and $\frac{C_i}{1 + a_i/z_i \cdot \sum_{j \in N^+ \setminus i} z_j/a_j} = \frac{\gamma_i z_i^2}{a_i x_i \sum_{j \in N^+ \setminus i} z_j/a_j}$.*

Proof Since (z, p) is a PNE, $p_j > 0$ and $x_j > 0$ holds for all $j \in N^+$, and $x_k = 0$ holds for $k \notin N^+$. Furthermore, (z_i, p_i) is a best response of leader i to s_{-i} and $K = a_j x_j/z_j + b_j + p_j$ holds for all $j \in N^+$. Altogether we get that $(z_i, p_i, (x_j)_{j \in N^+})$ is an optimal solution for the following optimization problem (with variables $(z_i', p_i', (x_j')_{j \in N^+})$):

$$\max \quad x_i' p_i' - \gamma_i z_i' \quad \text{s.t.:} \quad 0 \leq p_i' \leq C_i,\ 0 < z_i',\ \sum_{j \in N^+} x_j' = 1,\ x_j' \geq 0\ \forall j \in N^+,$$
$$a_i x_i'/z_i' + b_i + p_i' = a_j x_j'/z_j + b_j + p_j\ \forall j \in N^+ \setminus i.$$

[6] In Sect. 3, we considered *fixed* strategies s_{-i}, thus we just used (P_i^1) and (P_i^2) for the problems corresponding to s_{-i}. In this section, we need to consider different strategy profiles, thus we now write $(\mathrm{P}_i^1)(s_{-i})$ and $(\mathrm{P}_i^2)(s_{-i})$, as well as $K_i^{\max}(s_{-i})$.

It is easy to show that the LICQ holds for $(z_i, p_i, (x_j)_{j\in N^+}) = (z_i, p_i, x_i, (x_j)_{j\in N^+\setminus i})$, thus the KKT conditions are fulfilled. We get the following equations:

$$\gamma_i - a_i x_i / z_i^2 \sum_{j\in N^+\setminus i} \lambda_j = 0 \tag{KKT1}$$

$$-x_i + \mu + \sum_{j\in N^+\setminus i} \lambda_j = 0 \tag{KKT2}$$

$$-p_i + \lambda + a_i/z_i \sum_{j\in N^+\setminus i} \lambda_j = 0 \tag{KKT3}$$

$$\lambda - \lambda_j a_j / z_j = 0 \; \forall j \in N^+ \setminus i. \tag{KKT4}$$

We now distinguish between the two cases $p_i < C_i$ and $p_i = C_i$.

In the first case, $\mu = 0$ holds, and (KKT2) yields $x_i = \sum_{j\in N^+\setminus i} \lambda_j$. Using this, (KKT1) yields $z_i = \sqrt{a_i/\gamma_i} x_i$. Plugging this in (KKT3) leads to $p_i = \lambda + a_i x_i / z_i = \lambda + \sqrt{a_i \gamma_i}$. Using (KKT4), i.e., $\lambda_j = \lambda z_j / a_j$ for all $j \in N^+ \setminus i$, together with (KKT2) yields $x_i = \lambda \sum_{j\in N^+\setminus i} z_j/a_j$, or equivalently, $\lambda = \frac{x_i}{\sum_{j\in N^+\setminus i} z_j/a_j}$. This shows $p_i = \frac{x_i}{\sum_{j\in N^+\setminus i} z_j/a_j} + \sqrt{a_i\gamma_i}$, as required.

The other case is $p_i = C_i$. The formula for z_i follows from $K = a_i x_i / z_i + b_i + C_i$. Plugging $\lambda_j = \lambda z_j / a_j$ for all $j \in N^+ \setminus i$ in (KKT1) and (KKT3) yields

$$\lambda = \frac{\gamma_i z_i^2}{a_i x_i \sum_{j\in N^+\setminus i} z_j/a_j} \text{ and } \lambda = C_i - a_i/z_i \cdot \lambda \cdot \sum_{j\in N^+\setminus i} z_j/a_j \Leftrightarrow$$
$$\lambda = \frac{C_i}{1 + a_i/z_i \sum_{j\in N^+\setminus i} z_j/a_j},$$

which shows the desired equality. □

In the next lemma, we introduce two functions Γ_i^1 and Γ_i^2 for each leader i and derive useful properties of these functions.

Lemma 9 *For each $i \in N$, define*

$$\Gamma_i^1 : (\sqrt{a_i\gamma_i} + b_i, \infty) \to \mathbb{R}, \quad \Gamma_i^1(\kappa) := \frac{\sqrt{a_i\gamma_i}}{\kappa - \sqrt{a_i\gamma_i} - b_i} \quad \textit{and}$$
$$\Gamma_i^2 : (b_i + C_i, \infty) \to \mathbb{R}, \quad \Gamma_i^2(\kappa) := \frac{a_i\gamma_i/C_i}{\kappa - b_i - C_i}.$$

Furthermore, let $s = (z, p)$ be a PNE with $x := x(z, p)$, cost $K := K(z, p)$, and $N^+ := N^+(z, p)$ with $N_1^+ := N_1^+(z, p)$, $N_2^+ := N_2^+(z, p)$. Then:

1. *Γ_i^1 and Γ_i^2 are strictly decreasing functions.*
2. *If $i \in N_1^+$, then $\Gamma_i^1(K) = 1 - \frac{z_i/a_i}{\sum_{j\in N^+} z_j/a_j} < 1$.*

3. *If* $i \in N_2^+$, *then* $\Gamma_i^2(K) = 1 - \frac{z_i/a_i}{\sum_{j \in N^+} z_j/a_j} < 1$.
4. $\sum_{i \in N_1^+} \Gamma_i^1(K) + \sum_{i \in N_2^+} \Gamma_i^2(K) = |N^+| - 1$.
5. *If* $i \in N_1^+$ *and there is a (different) PNE* $s' = (z', p')$ *with* $i \in N_2^+(z', p')$, *then* $K < K' := K(z', p')$ *and* $\Gamma_i^1(K) > \Gamma_i^2(K')$.

Proof Statement 1. is clear from the definitions of Γ_i^1 and Γ_i^2, so turn to statement 2. and let $i \in N_1^+$. Lemma 4 yields $z_i = \sqrt{a_i/\gamma_i} x_i$, $p_i = K - \sqrt{a_i \gamma_i} - b_i$. Using Lemma 8, $B := \sum_{j \in N^+} z_j/a_j$, and $z_i = \sqrt{a_i/\gamma_i} x_i$, we get

$$\begin{aligned} K &= \frac{x_i}{\sum_{j \in N^+ \setminus i} z_j/a_j} + 2\sqrt{a_i \gamma_i} + b_i = \frac{x_i}{B - z_i/a_i} + 2\sqrt{a_i \gamma_i} + b_i \\ &= \frac{x_i + (B - x_i/\sqrt{a_i \gamma_i})\sqrt{a_i \gamma_i}}{B - x_i/\sqrt{a_i \gamma_i}} + \sqrt{a_i \gamma_i} + b_i = \frac{B\sqrt{a_i \gamma_i}}{B - x_i/\sqrt{a_i \gamma_i}} + \sqrt{a_i \gamma_i} + b_i. \end{aligned}$$

Using this we get statement 2.:

$$\Gamma_i^1(K) = \frac{B - x_i/\sqrt{a_i \gamma_i}}{B} = 1 - \frac{z_i/a_i}{B} < 1.$$

For statement 3., let $i \in N_2^+$. Lemmas 4 and 8 imply $\frac{C_i}{1 + a_i/z_i \cdot \sum_{j \in N^+ \setminus i} z_j/a_j} = \frac{\gamma_i z_i^2}{a_i x_i \sum_{j \in N^+ \setminus i} z_j/a_j}$. Rearranging and using the definition of B yields

$$\begin{aligned} \frac{a_i x_i}{z_i} &= \frac{\gamma_i z_i (1 + a_i/z_i \cdot \sum_{j \in N^+ \setminus i} z_j/a_j)}{C_i \cdot \sum_{j \in N^+ \setminus i} z_j/a_j} = \frac{\gamma_i z_i (1 + a_i/z_i \cdot (B - z_i/a_i))}{C_i \cdot (B - z_i/a_i)} \\ &= \frac{\gamma_i a_i B}{C_i \cdot (B - z_i/a_i)}. \end{aligned}$$

Using $K = a_i x_i/z_i + b_i + C_i$ then yields $K = \frac{\gamma_i a_i B}{C_i \cdot (B - z_i/a_i)} + b_i + C_i$, and statement 3. follows:

$$\Gamma_i^2(K) = \frac{B - z_i/a_i}{B} = 1 - \frac{z_i/a_i}{B} < 1.$$

Statement 4. now follows from the statements 2. and 3.:

$$\sum_{i \in N_1^+} \Gamma_i^1(K) + \sum_{i \in N_2^+} \Gamma_i^2(K) = \sum_{i \in N_1^+} \left(1 - \frac{z_i/a_i}{B}\right) + \sum_{i \in N_2^+} \left(1 - \frac{z_i/a_i}{B}\right) = |N^+| - 1$$

It remains to show statement 5. Let $i \in N_1^+ \cap N_2^+(z', p')$. Since $i \in N_1^+$, the cost K is in particular feasible for $(\mathrm{P}_i^1)(s_{-i})$, thus $2\sqrt{a_i \gamma_i} + b_i \leq K \leq \sqrt{a_i \gamma_i} + b_i + C_i$ holds. Analogously, using $i \in N_2^+(z', p')$, the cost K' is feasible for $(\mathrm{P}_i^2)(s'_{-i})$, therefore $\sqrt{a_i \gamma_i} + b_i + C_i < K' < K_i^{\max}(s'_{-i})$. Together we get $K \leq \sqrt{a_i \gamma_i} + b_i + C_i < K'$. It remains to show $\Gamma_i^1(K) > \Gamma_i^2(K')$. By definition of $N_2^+(z', p')$, the cost K' is an optimal solution for problem $(\mathrm{P}_i^2)(s'_{-i})$ and in particular (see 1. and 3. of Lemma 5) yields a better objective function value than $\sqrt{a_i\gamma_i} + b_i + C_i$ in $(\mathrm{P}_i^1)(s'_{-i})$ (note that $\sqrt{a_i\gamma_i}+b_i+C_i$ is feasible for $(\mathrm{P}_i^1)(s'_{-i})$ since $2\sqrt{a_i\gamma_i}+b_i \leq K \leq \sqrt{a_i\gamma_i}+b_i+C_i < K' < K_i^{\max}(s'_{-i})$). If $\overline{x}_i$ denotes the function occurring in the definitions of $(\mathrm{P}_i^1)(s'_{-i})$ and $(\mathrm{P}_i^2)(s'_{-i})$, we thus get $\overline{x}_i(\sqrt{a_i\gamma_i} + b_i + C_i) \cdot (C_i - \sqrt{a_i\gamma_i}) < \overline{x}_i(K') \cdot (C_i - a_i\gamma_i/(K' - b_i - C_i)) \leq \overline{x}_i(\sqrt{a_i\gamma_i} + b_i + C_i) \cdot (C_i - a_i\gamma_i/(K' - b_i - C_i))$, where the last inequality follows from $\sqrt{a_i\gamma_i} + b_i + C_i < K'$ and the fact that $\overline{x}_i$ is a decreasing function. Since $\overline{x}_i(\sqrt{a_i\gamma_i} + b_i + C_i) > 0$, we get

$$C_i - \sqrt{a_i\gamma_i} < C_i - \frac{a_i\gamma_i}{K' - b_i - C_i} \Leftrightarrow \frac{\sqrt{a_i\gamma_i}}{C_i} > \frac{a_i\gamma_i/C_i}{K' - b_i - C_i} = \Gamma_i^2(K').$$

Note that $2\sqrt{a_i\gamma_i} + b_i \leq K \leq \sqrt{a_i\gamma_i} + b_i + C_i$, thus $1/(K - \sqrt{a_i\gamma_i} - b_i) \geq 1/C_i$, and altogether

$$\Gamma_i^1(K) = \frac{\sqrt{a_i\gamma_i}}{K - \sqrt{a_i\gamma_i} - b_i} \geq \frac{\sqrt{a_i\gamma_i}}{C_i} > \Gamma_i^2(K').$$

□

Now we turn to the uniqueness of the equilibrium, and start with the following lemma.

Lemma 10 *For a fixed subset N^+ of the leaders and a fixed disjoint decomposition $N^+ = N_1^+ \dot{\cup} N_2^+$, there is essentially at most one PNE (z, p) such that $N^+(z, p) = N^+$, $N_1^+(z, p) = N_1^+$ and $N_2^+(z, p) = N_2^+$.*

Proof Assume that there are two PNE (z, p) and (z', p') with the described properties, i.e., $N^+(z, p) = N^+(z', p') = N^+$, $N_1^+(z, p) = N_1^+(z', p') = N_1^+$ and $N_2^+(z, p) = N_2^+(z', p') = N_2^+$. Let $x := x(z, p)$ and $x' := x(z', p')$ with costs $K := K(z, p)$ and $K' := K(z', p')$. We show that $(z_i, p_i) = (z_i', p_i')$ holds for all $i \in N^+$, showing that (z, p) and (z', p') are essentially the same.

First note that $K = K'$ holds, since $f(\kappa) := \sum_{i\in N_1^+} \Gamma_i^1(\kappa) + \sum_{i\in N_2^+} \Gamma_i^2(\kappa)$ is a strictly decreasing function in κ and $f(K) = \sum_{i\in N_1^+} \Gamma_i^1(K) + \sum_{i\in N_2^+} \Gamma_i^2(K) = |N^+| - 1 = \sum_{i\in N_1^+} \Gamma_i^1(K') + \sum_{i\in N_2^+} \Gamma_i^2(K') = f(K')$ holds from 4. in Lemma 9. This implies $p_i = p_i'$ for all $i \in N^+$, since $p_i = K - \sqrt{a_i\gamma_i} - b_i = K' - \sqrt{a_i\gamma_i} - b_i = p_i'$ holds for $i \in N_1^+$, and $p_i = C_i = p_i'$ for $i \in N_2^+$.

If $B := \sum_{j\in N^+} z_j/a_j = \sum_{j\in N^+} z_j'/a_j =: B'$ holds, we also get $z_i = z_i'$ for all $i \in N^+$, since 2. of Lemma 9 yields $z_i = (1 - \Gamma_i^1(K))a_i B = (1 - \Gamma_i^1(K'))a_i B' = z_i'$ for all $i \in N_1^+$ and 3. of Lemma 9 yields $z_i = (1 - \Gamma_i^2(K))a_i B = (1 - \Gamma_i^2(K'))a_i B' = z_i'$ for all $i \in N_2^+$.

It remains to show $B = B'$. First consider $i \in N_1^+$. Using $z_i = \sqrt{a_i/\gamma_i} \cdot x_i$ and $z_i' = \sqrt{a_i/\gamma_i} \cdot x_i'$, as well as 2. of Lemma 9, yields $x_i/B = (1 - \Gamma_i^1(K))\sqrt{a_i\gamma_i} = (1 - \Gamma_i^1(K'))\sqrt{a_i\gamma_i} = x_i'/B'$. For $i \in N_2^+$, we use $\frac{z_i}{a_i} = \frac{x_i}{K - b_i - C_i}$ and $\frac{z_i'}{a_i} = \frac{x_i'}{K' - b_i - C_i}$ and 3. of Lemma 9 to achieve $x_i/B = (1 - \Gamma_i^2(K))(K - b_i - C_i) = (1 - \Gamma_i^2(K'))(K' -$

$b_i - C_i) = x'_i/B'$. Altogether we have $x_i/B = x'_i/B'$ for all $i \in N^+$. Using that $\sum_{i \in N^+} x_i = 1 = \sum_{i \in N^+} x'_i$ yields $B = B'$. □

In the previous lemma, we showed that given a fixed subset $N^+ \subseteq N$ and a fixed disjoint decomposition $N^+ = N_1^+ \dot\cup N_2^+$, there is at most one PNE (z, p) such that $N^+(z, p) = N^+$, $N_1^+(z, p) = N_1^+$ and $N_2^+(z, p) = N_2^+$. Next, we strengthen this result by showing that for a fixed subset $N^+ \subseteq N$, there is at most one PNE (z, p) with $N^+(z, p) = N^+$ (independently of the decomposition of N^+).

Lemma 11 *For a fixed subset N^+ of the leaders, there is essentially at most one PNE (z, p) with $N^+(z, p) = N^+$.*

Proof Assume, by contradiction, that there are two essentially different PNE (z, p) and $(\overline{z}, \overline{p})$ with $N^+(z, p) = N^+ = N^+(\overline{z}, \overline{p})$. Let $N_1^+ := N_1^+(z, p)$, $N_2^+ := N_2^+(z, p)$ and $\overline{N}_1^+ := N_1^+(\overline{z}, \overline{p})$, $\overline{N}_2^+ := N_2^+(\overline{z}, \overline{p})$. Further denote $x := x(z, p)$, $K := K(z, p)$ and $\overline{x} := x(\overline{z}, \overline{p})$, $\overline{K} := K(\overline{z}, \overline{p})$.

Lemma 10 yields that the decompositions of N^+ have to be different. Without loss of generality, there is a leader $j \in N_1^+ \setminus \overline{N}_1^+$. Since $j \in \overline{N}_2^+$, statement 5. of Lemma 9 yields $K < \overline{K}$. The existence of a leader $i \in \overline{N}_1^+ \setminus N_1^+$ leads (by the same argumentation) to the contradiction $\overline{K} < K$, thus $\overline{N}_1^+ \subsetneq N_1^+$ and $N_2^+ \subsetneq \overline{N}_2^+$ hold and we can write (using 4. of Lemma 9)

$$
\begin{aligned}
|N^+| - 1 &= \sum_{i \in N_1^+} \Gamma_i^1(K) + \sum_{i \in N_2^+} \Gamma_i^2(K) = \sum_{i \in N_1^+ \setminus \overline{N}_1^+} \Gamma_i^1(K) + \sum_{i \in \overline{N}_1^+} \Gamma_i^1(K) \\
&\quad + \sum_{i \in N_2^+} \Gamma_i^2(K) \text{ and} \\
|N^+| - 1 &= \sum_{i \in \overline{N}_1^+} \Gamma_i^1(\overline{K}) + \sum_{i \in \overline{N}_2^+} \Gamma_i^2(\overline{K}) = \sum_{i \in \overline{N}_1^+} \Gamma_i^1(\overline{K}) + \sum_{i \in \overline{N}_2^+ \setminus N_2^+} \Gamma_i^2(\overline{K}) \\
&\quad + \sum_{i \in N_2^+} \Gamma_i^2(\overline{K}).
\end{aligned}
$$

Using $K < \overline{K}$ and that both $\sum_{i \in N_2^+} \Gamma_i^2(\kappa)$ and $\sum_{i \in \overline{N}_1^+} \Gamma_i^1(\kappa)$ are decreasing in κ yields

$$
\sum_{i \in N_2^+} \Gamma_i^2(K) \geq \sum_{i \in N_2^+} \Gamma_i^2(\overline{K}) \quad \text{and} \quad \sum_{i \in \overline{N}_1^+} \Gamma_i^1(K) \geq \sum_{i \in \overline{N}_1^+} \Gamma_i^1(\overline{K}).
$$

Finally, 5. of Lemma 9 yields $\Gamma_i^1(K) > \Gamma_i^2(\overline{K})$ for all $i \in N_1^+ \setminus \overline{N}_1^+ = \overline{N}_2^+ \setminus N_2^+ \neq \emptyset$, thus

$$
\sum_{i \in N_1^+ \setminus \overline{N}_1^+} \Gamma_i^1(K) > \sum_{i \in \overline{N}_2^+ \setminus N_2^+} \Gamma_i^2(\overline{K})
$$

holds. Altogether we get the following contradiction, completing the proof:

$$
\begin{aligned}
|N^+| - 1 &= \sum_{i \in N_1^+ \setminus \overline{N}_1^+} \Gamma_i^1(K) + \sum_{i \in \overline{N}_1^+} \Gamma_i^1(K) + \sum_{i \in N_2^+} \Gamma_i^2(K) \\
&> \sum_{i \in \overline{N}_1^+} \Gamma_i^1(\overline{K}) + \sum_{i \in \overline{N}_2^+ \setminus N_2^+} \Gamma_i^2(\overline{K}) + \sum_{i \in N_2^+} \Gamma_i^2(\overline{K}) = |N^+| - 1
\end{aligned}
$$

□

For the uniqueness of the PNE, it remains to show that there is at most one set N^+ such that a PNE (z, p) with $N^+(z, p) = N^+$ exists. To this end, we first show that each leader i has a threshold K_i^* such that, for any PNE (z, p), leader i has $z_i > 0$ if and only if $K(z, p) > K_i^*$.

Lemma 12 *For each $i \in N$, define*

$$
K_i^* := \begin{cases} a_i\gamma_i / C_i + b_i + C_i, & \text{if } \sqrt{a_i\gamma_i} > C_i, \\ 2\sqrt{a_i\gamma_i} + b_i, & \text{else.} \end{cases}
$$

Then, for any PNE $s = (z, p)$ and any leader $i \in N$, it holds that $z_i > 0$ iff $K(z, p) > K_i^*$.

Proof Let $s = (z, p)$ be a PNE with $x := x(z, p)$, $K := K(z, p)$, and $i \in N$.

First assume that $z_i = 0$. Since (z, p) is a PNE, the strategy $(z_i, p_i) = (0, p_i)$ is a best response of leader i to s_{-i}. As we have seen in Theorem 2, this is equivalent to the fact that both problems $(\mathrm{P}_i^1)(s_{-i})$ and $(\mathrm{P}_i^2)(s_{-i})$ are infeasible. Note that $K = K_i^{\max}(s_{-i})$ holds due to the definition of $K_i^{\max}(s_{-i})$ (cf. Sect. 3.2 and $x_j = (K - b_j - p_j)z_j/a_j$ for all $j \in \{j' \in N : x_{j'} > 0\} = \{j' \in N \setminus i : z_{j'} > 0, b_{j'} + p_{j'} < K\}$. To show $K \le K_i^*$, we have to distinguish between the two cases $\sqrt{a_i\gamma_i} > C_i$ and $\sqrt{a_i\gamma_i} \le C_i$. First consider $\sqrt{a_i\gamma_i} > C_i$, thus $K_i^* = a_i\gamma_i/C_i + b_i + C_i$. Since $(\mathrm{P}_i^2)(s_{-i})$ is infeasible and $\sqrt{a_i\gamma_i} + b_i + C_i < a_i\gamma_i/C_i + b_i + C_i$, we get the desired inequality $K = K_i^{\max}(s_{-i}) \le a_i\gamma_i/C_i + b_i + C_i = K_i^*$. Now consider $\sqrt{a_i\gamma_i} \le C_i$, i.e. $K_i^* = 2\sqrt{a_i\gamma_i} + b_i$. Since $(\mathrm{P}_i^1)(s_{-i})$ is infeasible and $2\sqrt{a_i\gamma_i} + b_i \le \sqrt{a_i\gamma_i} + b_i + C_i$, we get $K = K_i^{\max}(s_{-i}) \le 2\sqrt{a_i\gamma_i} + b_i = K_i^*$, as desired. We have seen that $z_i = 0$ implies $K \le K_i^*$, or, equivalently, $K > K_i^*$ implies $z_i > 0$.

It remains to show the other direction, i.e., $z_i > 0$ implies $K > K_i^*$. We consider the cases $\sqrt{a_i\gamma_i} > C_i$ and $\sqrt{a_i\gamma_i} \le C_i$ and use our results from Lemma 4 and Theorem 2. If $\sqrt{a_i\gamma_i} > C_i$, thus $K_i^* = a_i\gamma_i/C_i + b_i + C_i$, the cost K is an optimal solution for problem $(\mathrm{P}_i^2)(s_{-i})$ with positive objective function value (note that $(\mathrm{P}_i^1)(s_{-i})$ is infeasible), therefore $K_i^* = a_i\gamma_i/C_i + b_i + C_i < K$. In the second case, i.e., $\sqrt{a_i\gamma_i} \le C_i$ and $K_i^* = 2\sqrt{a_i\gamma_i} + b_i$, the cost K either is optimal for $(\mathrm{P}_i^1)(s_{-i})$, or optimal for $(\mathrm{P}_i^2)(s_{-i})$, and has positive objective function value in both cases. We get the desired property, since $K_i^* = 2\sqrt{a_i\gamma_i} + b_i < K$ holds for the first case and $K_i^* = 2\sqrt{a_i\gamma_i} + b_i \le \sqrt{a_i\gamma_i} + b_i + C_i < K$ holds for the second, completing the proof. □

We can now show the remaining result for the desired uniqueness of PNE.

Lemma 13 *There is at most one $N^+ \subseteq N$ such that a PNE (z,p) with $N^+(z,p) = N^+$ exists.*

Proof Assume, by contradiction, that there are different subsets N^+ und $\overline{N}^+$ with PNE (z,p) and $(\overline{z},\overline{p})$, such that $N^+(z,p) = N^+$ and $N^+(\overline{z},\overline{p}) = \overline{N}^+$. Let $N_1^+ \dot\cup N_2^+$ and $\overline{N}_1^+ \dot\cup \overline{N}_2^+$ be the decompositions of N^+ and $\overline{N}^+$, i.e., $N_1^+(z,p) = N_1^+$, $N_2^+(z,p) = N_2^+$, $N_1^+(\overline{z},\overline{p}) = \overline{N}_1^+$ and $N_2^+(\overline{z},\overline{p}) = \overline{N}_2^+$. Finally denote $x := x(z,p)$, $K := K(z,p)$ and $\overline{x} := x(\overline{z},\overline{p})$, $\overline{K} := K(\overline{z},\overline{p})$.

Using Lemma 12, we can assume w.l.o.g. that $K < \overline{K}$ and $N^+ \subsetneq \overline{N}^+$. Then, $N_2^+ \subseteq \overline{N}_2^+$ holds, since the existence of a leader $i \in N_2^+ \setminus \overline{N}_2^+$, i.e. $i \in N_2^+ \cap \overline{N}_1^+$, leads to the contradiction $\overline{K} < K$ by statement 5. of Lemma 9. Furthermore, if there is a leader $i \in N_1^+ \setminus \overline{N}_1^+$, i.e. $i \in N_1^+ \cap \overline{N}_2^+$, statement 5. of Lemma 9 yields $\Gamma_i^1(K) > \Gamma_i^2(\overline{K})$. Finally, $\Gamma_i^1(\overline{K}) < 1$ holds for all $i \in \overline{N}_1^+$, and $\Gamma_i^2(\overline{K}) < 1$ holds for all $i \in \overline{N}_2^+$ (see 2. and 3. of Lemma 9). Altogether, this leads to the following contradiction, and completes the proof (where we additionally use $K < \overline{K}$, and the statements 1. and 4. of Lemma 9):

$$
\begin{aligned}
|\overline{N}^+| - 1 &= \sum_{i \in \overline{N}_1^+} \Gamma_i^1(\overline{K}) + \sum_{i \in \overline{N}_2^+} \Gamma_i^2(\overline{K}) \\
&= \sum_{i \in \overline{N}_1^+ \cap N_1^+} \Gamma_i^1(\overline{K}) + \sum_{i \in \overline{N}_1^+ \setminus N^+} \Gamma_i^1(\overline{K}) + \sum_{i \in \overline{N}_2^+ \cap N_1^+} \Gamma_i^2(\overline{K}) + \sum_{i \in N_2^+} \Gamma_i^2(\overline{K}) \\
&\quad + \sum_{i \in \overline{N}_2^+ \setminus N^+} \Gamma_i^2(\overline{K}) \\
&< \sum_{i \in N_1^+ \cap \overline{N}_1^+} \Gamma_i^1(K) + \sum_{i \in N_1^+ \cap \overline{N}_2^+} \Gamma_i^1(K) + \sum_{i \in N_2^+} \Gamma_i^2(K) + |\overline{N}^+| - |N^+| \\
&= \sum_{i \in N_1^+} \Gamma_i^1(K) + \sum_{i \in N_2^+} \Gamma_i^2(K) + |\overline{N}^+| - |N^+| = |\overline{N}^+| - 1.
\end{aligned}
$$

□

Together with the existence result in Theorem 4, the preceding Lemmata 11 and 13 show:

Theorem 5 *Every Stackelberg pricing game has an essentially unique PNE, i.e., if (z,p) and (z',p') are different PNE and $i \in N$ with $(z_i,p_i) \neq (z_i',p_i')$, then $z_i = z_i' = 0$ holds.*

6 Quality of equilibria

In the last section, we showed that a Stackelberg pricing game has an (essentially) unique PNE. Now we analyze the quality of the PNE compared to a *social optimum*. Define the *social cost* $C(z,p)$ of a strategy profile $s = (z,p)$ as

$$C(z,p) = \begin{cases} \sum_{i \in \{1,\dots,n\}: z_i > 0} (\ell_i(x_i(s), z_i) x_i(s) + \gamma_i z_i), & \text{if } \sum_{i=1}^n z_i > 0, \\ \infty, & \text{else.} \end{cases}$$

The function $C(z, p)$ measures utilitarian social welfare over leaders and followers (the price component cancels out). Common notions to measure the quality of equilibria are the Price of Anarchy (PoA) and the Price of Stability (PoS), which are defined as the worst case ratios of the cost of a worst, respectively best, PNE, and a social optimum. Note that for Stackelberg pricing games, PoA and PoS are the same: Since a Stackelberg pricing game G has an essentially unique PNE, all PNE of G have the same social cost. If we denote this cost by $C(\mathrm{PNE}(G))$, and the minimum social cost in G (compared to all possible strategy profiles) by $\mathrm{OPT}(G)$, we thus get that $\mathrm{PoA} = \mathrm{PoS} = \sup_G C(\mathrm{PNE}(G))/\mathrm{OPT}(G)$.

In Sect. 6.1, we show that the PoA for Stackelberg pricing games is unbounded. For the proof, we use a family of instances with *heterogeneous* leaders, that is, the leaders have different parameters. In Sect. 6.2, we then turn to the *homogeneous* case, where all leaders have the same parameters. We derive a closed-form expression of the ratio $C(\mathrm{PNE}(G))/\mathrm{OPT}(G)$. In particular, this expression implies that the PoA is unbounded also for homogeneous leaders. Finally, in Sect. 6.3, we briefly discuss other definitions for the social cost of a strategy profile, and consequences for the quality of equilibria.

6.1 Unboundedness of the PoA

The following theorem shows that PoA and PoS are unbounded for Stackelberg pricing games.

Theorem 6 *PoA $=$ PoS $= \infty$. The bound is attained even for games with only two leaders.*

Proof Consider the Stackelberg pricing game G_M with $n = 2,\ a_1 = \gamma_1 = C_1 = 1, b_1 = 0$, and $a_2 = \gamma_2 = C_2 = M, b_2 = 0,$ where $M \geq 1$.

By $z_1 = 1, p_1 = z_2 = p_2 = 0$, we get a profile with social cost 2, thus $\mathrm{OPT}(G_M) \leq 2$. We now show $C(\mathrm{PNE}(G_M)) > M$, which implies $\mathrm{PoA} = \mathrm{PoS} \geq C(\mathrm{PNE}(G_M))/\mathrm{OPT}(G_M) > M/2$. By $M \to \infty$, this yields the desired result. It remains to show $C(\mathrm{PNE}(G_M)) > M$. For fixed $M \geq 1$, let $s = (z, p)$ be a PNE of G_M with induced Wardrop flow $x := x(s)$ and cost $K := K(s)$. Note that $z_i > 0$ holds for $i \in \{1, 2\}$, since any PNE has at least two positive capacities. Lemma 4 together with Theorem 2 yields that, for each leader $i \in \{1, 2\}$, the cost K either is optimal for $(\mathrm{P}_i^1)(s_{-i})$, or optimal for $(\mathrm{P}_i^2)(s_{-i})$, and has positive objective function value in both cases. Since for each leader $i \in \{1, 2\}$, the only candidate for a feasible solution of $(\mathrm{P}_i^1)(s_{-i})$ is $2\sqrt{a_i\gamma_i} + b_i = \sqrt{a_i\gamma_i} + b_i + C_i$ and this yields an objective function value of 0, we get that K is an optimal solution of $(\mathrm{P}_i^2)(s_{-i})$. In particular this yields, by considering leader 2, that $2M = \sqrt{a_2\gamma_2} + b_2 + C_2 < K$. Furthermore we get $z_1 = \frac{x_1}{K-1}$ and $z_2 = \frac{M(1-x_1)}{K-M}$ for the capacities. Altogether, the desired inequality for $C(s) = C(\mathrm{PNE}(G_M))$ follows: $C(s) = \frac{1}{z_1}x_1^2 + \frac{M}{z_2}(1-x_1)^2 + z_1 + Mz_2 > (K-1)x_1 + (K-M)(1-x_1) \geq K - M > M$. ◻

Note that for $M > 1$, the two leaders in the proof of Theorem 6 do not have the same parameters. This raises the question whether the PoA is bounded if we restrict ourselves to *homogeneous* leaders. This question is analyzed in the next subsection.

6.2 Homogeneous leaders

In this subsection, we analyze the quality of equilibria for the case that the leaders are *homogeneous*, that is, there exist $a > 0, b \geq 0, C > 0$ and $\gamma > 0$ such that $a_i = a, b_i = b, C_i = C$ and $\gamma_i = \gamma$ for all $i \in N$. We derive the following.

Theorem 7 *Assume that G is a Stackelberg pricing game with n homogeneous leaders, i.e., there exist $a > 0, b \geq 0, C > 0$ and $\gamma > 0$ such that $a_i = a, b_i = b, C_i = C$ and $\gamma_i = \gamma$ for all $i \in N$. Then, the ratio between the equilibrium cost $C(PNE)$ and the optimal cost OPT of G is*

$$\frac{C(PNE)}{OPT} = \begin{cases} 1, & \text{if } \frac{n}{n-1}\sqrt{a\gamma} < C, \\ \frac{\frac{n}{n-1}\cdot\frac{a\gamma}{C}+b+\frac{n-1}{n}C}{2\sqrt{a\gamma}+b}, & \text{if } \frac{n}{n-1}\sqrt{a\gamma} \geq C. \end{cases}$$

Note that this shows that even for homogeneous leaders, the PoA for Stackelberg pricing games is unbounded (consider $C \to 0$). On the other hand, if the caps are large enough, one can ensure an optimal equilibrium.

Proof First, we analyze the social cost OPT of an optimal profile, and show OPT $= 2\sqrt{a\gamma} + b$. Note that given z with $\sum_{i\in N} z_i > 0$, the flow x^* defined by $x_i^* := z_i / \sum_{j\in N} z_j$ for all $i \in N^+ := \{i \in N : z_i > 0\}$, minimizes the total congestion cost $\sum_{i\in N^+} \ell_i(x_i, z_i)x_i$: Consider

$$\min \sum_{i\in N^+} (ax_i/z_i + b)\, x_i = a \cdot \sum_{i\in N^+} x_i^2/z_i + b \quad \text{s.t.} \quad \sum_{i\in N^+} x_i = 1,\ x_i \geq 0\ \forall i \in N^+.$$

For this problem, the KKT conditions are necessary and sufficient for the unique optimal solution. Therefore, x^* is optimal if and only if there exist $\lambda \in \mathbb{R}$ and $\mu_i \in \mathbb{R}$ for all $i \in N^+$, such that $2ax_i^*/z_i + \lambda - \mu_i = 0,\ \mu_i x_i^* = 0,$ and $\mu_i \geq 0$ hold for all $i \in N^+$. Setting $\lambda := -2a/\sum_{j\in N} z_j$ and $\mu_i := 0$ for all $i \in N^+$, these conditions are fulfilled, and we conclude that x^* is optimal. Furthermore, we can induce x^* as Wardrop-flow, for example by $p_i := 0$ for all $i \in N$, since then, $ax_i^*/z_i + b + p_i = a/\sum_{j\in N} z_j + b$ for all $i \in N^+$. This shows that OPT equals the optimal objective function value of the problem

$$\min\ a/\sum_{j\in N} z_j + b + \gamma \cdot \sum_{j\in N} z_j \quad \text{s.t.} \quad \sum_{j\in N} z_j > 0,\ z_j \geq 0\ \forall j \in N.$$

Since the objective function only depends on the sum of the capacities, we can also minimize $a/\zeta + b + \gamma\zeta$ for $\zeta > 0$. The optimal solution of the latter problem is $\zeta^* = \sqrt{a/\gamma}$, and the optimal objective function value is $2\sqrt{a\gamma} + b$. We have thus shown that OPT $= 2\sqrt{a\gamma} + b$.

We now analyze the cost $C(\mathrm{PNE})$. Note that if (z, p) is a PNE, $z_i > 0$ holds for each player $i \in N$ (this follows from the threshold Lemma 12). This shows that in case of homogeneous leaders, there is a unique PNE, and we denote this PNE with (z^E, p^E). The uniqueness implies that (z^E, p^E) is symmetric, that is, there exist $\zeta > 0, \rho \in (0, C]$ such that $z_i^E = \zeta$ and $p_i^E = \rho$ for all $i \in N$. This immediately implies that $x_i^E = 1/n$ holds for all $i \in N$, where x^E denotes the Wardrop flow induced by the PNE. Next, we exploit Lemma 8 to derive explicit formulas for ζ and ρ, leading to

$$(\zeta, \rho) = \begin{cases} (\sqrt{a/\gamma} \cdot \frac{1}{n}, \frac{n}{n-1} \cdot \sqrt{a\gamma}), & \text{if } \rho < C, \\ (\frac{n-1}{n^2} \cdot \frac{C}{\gamma}, C), & \text{if } \rho = C. \end{cases} \tag{6}$$

We show below that $\rho < C$ holds if and only if $n/(n-1) \cdot \sqrt{a\gamma} < C$. Using this, we get

$$(\zeta, \rho) = \begin{cases} (\sqrt{a/\gamma} \cdot \frac{1}{n}, \frac{n}{n-1} \cdot \sqrt{a\gamma}), & \text{if } \frac{n}{n-1} \cdot \sqrt{a\gamma} < C, \\ (\frac{n-1}{n^2} \cdot \frac{C}{\gamma}, C), & \text{if } \frac{n}{n-1} \cdot \sqrt{a\gamma} \geq C, \end{cases} \quad \text{and} \tag{7}$$

$$C(z^E, p^E) = \frac{a}{\zeta n} + b + n\gamma\zeta = \begin{cases} 2\sqrt{a\gamma} + b, & \text{if } \frac{n}{n-1} \cdot \sqrt{a\gamma} < C, \\ \frac{n}{n-1} \cdot \frac{a\gamma}{C} + b + \frac{n-1}{n}C, & \text{if } \frac{n}{n-1} \cdot \sqrt{a\gamma} \geq C. \end{cases}$$

Together with $\mathrm{OPT} = 2\sqrt{a\gamma} + b$, this implies the theorem.

It remains to show that $\rho < C \Leftrightarrow n/(n-1)\sqrt{a\gamma} < C$. Using (6), $\rho < C$ clearly implies $n/(n-1)\sqrt{a\gamma} < C$. We now assume that $n/(n-1)\sqrt{a\gamma} < C$. Assume, by contradiction, that $\rho = C$, and thus $\zeta = \frac{n-1}{n^2} \cdot \frac{C}{\gamma}$ by (6). If K^E denotes the routing cost induced by x^E, we get $K^E = a/(\zeta n) + b + C = an\gamma/((n-1)C) + b + C < \sqrt{a\gamma} + b + C$. By our characterization of best responses, this shows that K^E is the optimal solution of $(\mathrm{P}_i^1)(z_{-i}, p_{-i})$, where $(z_j, p_j) = (\zeta, \rho)$ holds for each $j \in N \setminus i$, and thus $\rho = K^E - \sqrt{a\gamma} - b < C$; contradiction. □

At the end of this subsection, we briefly address the natural question if one can achieve a closed-form expression of the ratio between equilibrium and optimal cost, as in Theorem 7, also for *heterogeneous* leaders, where the parameters do not have to be equal. We found that at least for two leaders, this is possible. But the resulting expression is rather complicated and hardly insightful, thus we did not include it in this paper.

6.3 Different social cost functions

To complete our analysis of the quality of PNE, we now briefly discuss other definitions of the social cost of a strategy profile. Note that the function $C(z, p)$ defined at the beginning of this section measures utilitarian social welfare over followers *and* leaders. Alternatively, one could consider *only* the followers, or *only* the leaders. But as we show below, the worst-case quality of the equilibrium is also unbounded for these cases.

For a strategy profile $s = (z, p)$ with $\sum_{i \in N} z_i > 0$, the *total cost of the followers* equals $K(s)$, the cost of the induced Wardrop flow. (If $\sum_{i \in N} z_i = 0$, i.e., there is no resource to choose for the followers, we define, as before, that the cost is ∞.) But for this alternative definition of the cost of a strategy profile, the optimal cost is not attained, and may even tend to zero: Obviously, $K(s) > b_{\min} := \min\{b_i : i \in N\}$ holds for each profile s. On the other hand, the cost of the profile s_M defined by $z_i = M$, $p_i = 0$ for some leader i with $b_i = b_{\min}$, and $z_j = p_j = 0$ else, tends to $b_{\min}$ for $M \to \infty$. In other words, in a "near-optimal" profile, the congestion effects are extinguished by very high capacities. But naturally, such capacities also induce high investment costs. To impose an upper bound on the total investment cost (and thus on the capacities) for the optimal profile does not only seem natural, but would also imply that the optimal cost is attained, and is strictly positive.[7] Assume, e.g., that there is a budget parameter $B > 0$ such that $\sum_{i \in N} \gamma_i z_i \leq B$ needs to hold for an optimal profile. With this restriction, we now analyze the equilibrium quality, and again get that PoA and PoS are unbounded: To see this, consider the family of instances $(G_M)_{M \geq 1}$ given in the proof of Theorem 6, and assume that $B = 1$. It follows from the proof of Theorem 6 that the equilibrium cost is larger than $2M$, whereas the optimal cost is at most 1, leading to a lower bound of $2M$ for the PoA. By $M \to \infty$, we get that PoA and PoS are unbounded.[8]

Alternatively, one could measure the quality of a strategy profile via the *total profits of the leaders*. Here, it is again the case that an optimal profile (maximizing the sum of the leaders' profits) does not exist: The total profit is always strictly smaller than $C_{\max} := \max\{C_i : i \in N\}$, but for $z_i = \varepsilon$, $p_i = C_i$ for some leader i with $C_i = C_{\max}$, and $z_j = p_j = 0$ else, we get a total profit arbitrarily near to $C_{\max}$ if ε tends to 0. Irrespective of whether we define OPT as $C_{\max}$, or restrict the optimal profile for example by some lower bound β on the sum of capacities $\sum_{i \in N} z_i \geq \beta$, the worst-case quality of the PNE is unbounded: To see this, consider (7), where we state the equilibrium strategies for the case that the leaders are homogeneous. Assume that we have a homogeneous instance with $\sqrt{a\gamma} \geq C$. The total profit of the PNE then is $C - (n-1)/n \cdot C = C/n$. As n gets large, this clearly tends to zero. On the other hand, the optimal total profit (C or $C - \beta\gamma$) is independent of n, and strictly positive (for $\beta < C/\gamma$). Altogether, this shows that if we consider the total profits of the leaders, the quality of the PNE may also get arbitrarily bad.

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[7] Note that in a PNE, the total investment cost is also upper-bounded since $\gamma_i z_i < C_i$ holds for each leader i.

[8] The same argumentation shows that even if we drop the prices and define the cost of a strategy profile as the total *congestion cost* of the followers, the PoA is unbounded.

References

1. Acemoglu, D., Bimpikis, K., Ozdaglar, A.: Price and capacity competition. Games Econom. Behav. **66**(B), 1–26 (2009)
2. Ackermann, H., Skopalik, A.: On the complexity of pure Nash equilibria in player-specific network congestion games. In: Deng, X., Graham, F. (eds.) Proceedings of 3rd International Workshop on Internet and Network Econom., *LNCS*, vol. 4858, pp. 419–430 (2007)
3. Andrews, M., Antonakopoulos, S., Zhang, L.: Minimum-cost network design with (dis)economies of scale. SIAM J. Comput. **45**(1), 49–66 (2016)
4. Beckmann, M., McGuire, C., Winsten, C.: Studies in the Economics and Transportation. Yale University Press, New Haven (1956)
5. Berge, C.: Topological Spaces: Including a Treatment of Multi-Valued Functions. Vector Spaces and Convexity. Dover Publications, Mineola (1963)
6. Bergendorff, P., Hearn, D., Ramana, M.: Congestion toll pricing of traffic networks. In: Pardalos, P., Hearn, D., Hager, W. (eds.) Network Optimization, Lecture Notes in Economics and Mathematical Systems, vol. 450, pp. 51–71 (1997)
7. Castiglioni, M., Marchesi, A., Gatti, N., Coniglio, S.: Leadership in singleton congestion games: what is hard and what is easy. Artif. Intell. **277**, 103177 (2019)
8. Cominetti, R., Correa, J.R., Stier-Moses, N.E.: The impact of oligopolistic competition in networks. Oper. Res. **57**(6), 1421–1437 (2009)
9. Correa, J.R., Guzmán, C., Lianeas, T., Nikolova, E., Schröder, M.: Network pricing: How to induce optimal flows under strategic link operators. In: Proceedings of the 2018 ACM Conference on Economics and Computation, Ithaca, NY, USA, pp. 375–392 (2018)
10. Correa, J.R., Stier-Moses, N.E.: Wardrop Equilibria. Wiley Encyclopedia of Operations Research and Management Science (2011)
11. Dial, R.B.: Minimal-revenue congestion pricing part I: a fast algorithm for the single-origin case. Transp. Res. **33**(1), 189–202 (1999)
12. Dial, R.B.: Network-optimized road pricing: part I: a parable and a model. Oper. Res. **47**(1), 54–64 (1999)
13. Dial, R.B.: Network-optimized road pricing: part II: algorithms and examples. Oper. Res. **47**(2), 327–336 (1999)
14. Emek, Y., Kutten, S., Lavi, R., Shi, Y.: Approximating generalized network design under (dis)economies of scale with applications to energy efficiency. J. ACM **67**(1), 7:1–7:33 (2020)
15. Englert, M., Franke, T., Olbrich, L.: Sensitivity of Wardrop equilibria. Theory Comput. Syst. **47**(1), 3–14 (2010)
16. Fabrikant, A., Papadimitriou, C., Talwar, K.: The complexity of pure Nash equilibria. In: Babai, L. (ed.) Proceedings of 36th Annual ACM Symposium Theory Computer, pp. 604–612 (2004)
17. Fleischer, L., Jain, K., Mahdian, M.: Tolls for heterogeneous selfish users in multicommodity networks and generalized congestion games. In: Proceedings of 45th Annual IEEE Symposium Foundations Computer Science, pp. 277–285 (2004)
18. Gairing, M., Harks, T., Klimm, M.: Complexity and approximation of the continuous network design problem. SIAM J. Optim. **27**(3), 1554–1582 (2017)
19. Harks, T., Schedel, A.: Capacity and price competition in markets with congestion effects. In: Proceedings of 15th Internation Conference on Web and Internet Economics, p. 341 (2019)
20. Harks, T., Schröder, M., Vermeulen, D.: Toll caps in privatized road networks. Eur. J. Oper. Res. **276**(3), 947–956 (2019)
21. Johari, R., Weintraub, G.Y., Van Roy, B.: Investment and market structure in industries with congestion. Oper. Res. **58**(5), 1303–1317 (2010)
22. Kakutani, S.: A generalization of Brouwer's fixed point theorem. Duke Math. J. **8**(3), 457–458 (1941)
23. Kreps, D.M., Scheinkman, J.A.: Quantity precommitment and Bertrand competition yield Cournot outcomes. Bell J. Econ. **14**(2), 326–337 (1983)

24. Labbé, M., Marcotte, P., Savard, G.: A Bilevel model of taxation and its application to optimal highway pricing. Manag. Sci. **44**(12), 1608–1622 (1998)
25. Litman, T.: Understanding transport demands and elasticities: how prices and other factors affect travel behavior. http://www.vtpi.org/elasticities.pdf (2019). Accessed 22 Oct 2019
26. Liu, T.L., Chen, J., Huang, H.J.: Existence and efficiency of oligopoly equilibrium under toll and capacity competition. Transp. Res. E Logist. Transp. Rev. **47**(6), 908–919 (2011)
27. Marchesi, A., Castiglioni, M., Gatti, N.: Leadership in congestion games: multiple user classes and non-singleton actions. In: Kraus, S. (ed.) Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI, pp. 485–491 (2019)
28. Marcotte, P.: Network design problem with congestion effects: a case of bilevel programming. Math. Program. Ser. A **34**, 142–162 (1986)
29. McLennan, A., Monteiro, P., Tourky, R.: Games with discontinuous payoffs: a strengthening of Reny's existence theorem. Econometrica **79**(5), 1643–1664 (2011)
30. Nishimura, K., Friedman, J.: Existence of Nash equilibrium in n person games without quasi-concavity. Int. Econ. Rev. **22**(3), 637–648 (1981)
31. Reny, P.: On the existence of pure and mixed strategy Nash equilibria in discontinuous games. Econometrica **67**(5), 1029–1056 (1999)
32. Roughgarden, T., Tardos, É.: How bad is selfish routing? J. ACM **49**(2), 236–259 (2002)
33. Schmand, D., Schröder, M., Skopalik, A.: Network investment games with wardrop followers. In: Baier, C., Chatzigiannakis, I., Flocchini, P., Leonardi, S. (eds.) 46th International Colloquium on Automata, Languages, and Programming, ICALP 2019, *LIPIcs*, vol. 132, pp. 151:1–151:14 (2019)
34. Xiao, F., Yang, H., Han, D.: Competition and efficiency of private toll roads. Transp. Res. B Methodol. **41**(3), 292–308 (2007)
35. Yang, H., Huang, H.J.: The multi-class, multi-criteria traffic network equilibrium and systems optimum problem. Transp. Res. **38**(B), 1–15 (2004)

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